ALGEBRA QUAL PREP: COMMUTATIVE RINGS

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1. SPRING 2010 MORNING 3

Since Q is maximal, B/Q is a field which is finitely generated over k as a ring (since B is). By the Nullstellensatz, this implies that B is a finite field extension of k.

Since $A/\phi^{-1}(Q) \hookrightarrow B/Q$, we deduce that $A/\phi^{-1}(Q)$ is an integral domain which is finitely over a field k. This implies that $A/\phi^{-1}(Q)$ is itself a field. To see this we just need to show that non-zero elements have inverses; multiplication by any non-zero $a \in A/\phi^{-1}(Q)$ induces an injection of finite-dimensional k-vector spaces, which must then be a bijection.

2. SPRING 2010 AFTERNOON 3

(i) Run the usual argument for the Hilbert basis theorem, but look at the leading terms. Let $Is \subset A[[x]]$ be an ideal. The leading terms of elements of I generate an ideal $\mathscr{L}$ in A; pick generators $a_1, \ldots, a_n$.

Run the following algorithm to find a finite set of generators for I: adjoin an element in I of minimal leading order, whose leading coefficient is not in the ideal generated by existing elements. If this terminates, then we are done. Otherwise, this leads to a set $f_1, \ldots, f_m, \ldots$ where the leading orders of $f_1, \ldots, f_m$ generate $\mathscr{L}$. Then an appropriate $A[[x]]$–combination of $f_1, \ldots, f_m$ equals the leading term of f_{m+1}, contradiction.

(ii) Pick $x \notin P$. By the DCC, we have $(x^n) = (x^{n+1})$ for some n. Hence $x^n = yx^{n+1}$ for some $y \in A$, i.e. $x^n(xy-1) = 0$. Since $x \notin P$, the definition of prime ideal implies $xy - 1 \in P$, i.e. x is invertible mod P. This shows that A/P is a field, so P was maximal.

Let $P_1, P_2, P_3, \ldots$ be distinct prime ideals of A. Consider descending chains

$$P_1 \supset P_1P_2 \supset P_1P_2P_3 \supset \ldots$$

By the descending chain condition, this stabilizes, so $(P_1 \ldots P_n)P_{n+1} = (P_1 \ldots P_n)$. By Nakayama's Lemma this implies that $P_1 \ldots P_n$ is 0 in $A_{P_{n+1}}$. But in fact we must have $P_1 \ldots P_n$ generates the unit ideal in $A_{P_{n+1}}$, so this is impossible.

3. FALL 2011 MORNING 3

(i) Let x be a non-unit. If $x = x_1x_2$ with neither x_1, x_2 being units, then we have a strict inclusion of ideals

$$(x) \subsetneq (x_2)$$

Applying the same reasoning to x_2, etc. we get a chain of ideals, which must terminate.

(ii) Geometric translation: $V(I)$ is contained in the union of finitely many irreducible components. First, we claim that if R is Noetherian then Spec R is a Noetherian topological space (DCC for closed subspaces).

We claim that a Noetherian space can only have finitely many irreducible components. First, it is easy to check that a closed subspace of a Noetherian space is Noetherian. We consider the set of closed subsets X which do not have this property. Suppose there is more than the empty set. By the DCC we can find a minimal element. Since it is not irreducible, it can be written as a union of two smaller closed subsets; by the minimality, these each have finitely many irreducible components, which is a contradiction.

4. Spring 2012 Morning 1

(i) Addressed in earlier question.

(ii) The assumption implies $rsr = r$, hence $r(sr-1)=0$. Now note that rs is surjective, so r is surjective, hence invertible. So $r(sr-1)=0$ implies $sr=1$.

5. Fall 2013 Morning 3

(a) We may check this locally. Locally, A is a DVR and this is obvious.

(b) We must produce something in the kernel of the map $I\otimes_A J \to IJ$. Write $I=(x,y)$, $I'=(x',y')$. Then $x\otimes y' - x'\otimes y$ is sent to 0. It is not 0 in $I\otimes_A J$ because we can define an A-bilinear form on $I\times J$ which does not vanish on it.

To do this, we define an A-linear form from $I\times J$ to $k=A/(x,y)$ which sends $B(x,y')=1$, $B(x,x')=0$, $B(y,x')=0, B(y,y')=0$ (with all other values forced by linearity).

6. Spring 2015 Afternoon 5

(a) There are $f_n \mid \ldots \mid f_2 \mid f_1$ such that

$$M = \bigoplus A/(f_i).$$

(b) Suppose

$$M \cong \bigoplus A/(f_i) \cong \bigoplus A/(f_i').$$

Since f_1 annihilates M, we have f_1 annihilates each $A/(f_i')$, hence $f_1 \mid f_1'$. Similarly $f_1' \mid f_1$, and then we win by induction.

7. Fall 2015 Morning 4

(a) We say $b\in B$ is integral over A if it satisfies a monic polynomial with coefficients in A. We say B is integral over A if every $b\in B$ is integral over A.

(b) We need T to satisfy a monic polynomial over A. Since its minimal polynomial is $a_0T^n+\ldots+a_n$, the polynomials satisfied by T are all multiples of $a_0T^n+\ldots+a_n$. Such can only have leading coefficient 1 if a_0 is a unit.

(c) We can view $B = k[U][T]/(f)$ where

$$f(U,T) = a_0(U - T^{n+1})T^n + a_1(U - T^{n+1})T^{n-1} + \ldots + a_n(U - T^{n+1}).$$

The leading power of T $d_i(n+1)+i$ where i is such that $a_i(X)$ had the maximal degree, and i is maximal with this property. Furthermore, it will be a scalar.

8. Fall 2015 Afternoon 3

(a) Write $x = a + b\sqrt{-31} \in K$. Then we must have $\mathrm{Tr}(x) = 2a \in \mathbf{Z}$, and $\mathrm{Nm}(x) = a^2 + 31b^2 \in \mathbf{Z}$. So $a \in \frac{1}{2}\mathbf{Z}$. If $a \in \mathbf{Z}$, then $b \in \mathbf{Z}$. If $a \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$, then $b \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$. This describes $\mathbf{Z}[\alpha]$; finally we check that α is in fact integral over $\mathbf{Z}$: $\alpha^2 = -\frac{15}{2} + \frac{\sqrt{-31}}{2}$, and then $\alpha^2 - \alpha = -8$.

(b) Evidently $\mathfrak{m}$ and $\mathfrak{m}'$ contain (2).

Note that $\mathfrak{m}\mathfrak{m}' = (4, 2\alpha, 2\alpha+2, \alpha(\alpha+1)) = (2)$.

Next we find the units. From the formula $N(x + y\alpha) = x^2 + xy + 4y^2$, any unit has $y = 0$ and $x = \pm 1$. It is evident that $\mathfrak{m}$ is not principal.

9. Spring 2015 Morning 5

(a) Any such extension is the composite of two quadratic extensions, and quadratic extensions are all obtained by adjoining the square roots of non-squares.

(b) We need to know if every embedding $E \hookrightarrow \overline{K}$ lands in E, or in other words if every automorphism of L takes $\sqrt{c}$ to an element of E. Now, $\sigma(c)$ is a non-square since c is a non-square, hence $L(\sqrt{\sigma(c)}) = E$ if and only if $\sigma(c)/c$ is a square in L.

(c) The norm of c to $\mathbf{Q}(\sqrt{2})$ is $6(2-\sqrt{2})^2$. Since 6 is not a square in $\mathbf{Q}(\sqrt{2})$, c is not a square in $\mathbf{Q}(\sqrt{2}, \sqrt{3})$.

We need to examine various $\sigma(c)/c$. For σ generating $\mathrm{Gal}(L/\mathbf{Q}(\sqrt{3}))$, we find

$$\frac{\sigma(c)}{c} = \frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{6+4\sqrt{2}}{2} = 3 + 2\sqrt{2} = (1+\sqrt{2})^2.$$

For σ generating $\mathrm{Gal}(L/\mathbf{Q}(\sqrt{2}))$, we find

$$\frac{\sigma(c)}{c} = \frac{3+\sqrt{3}}{3-\sqrt{3}} = \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3} = \frac{(1+\sqrt{3})^2}{(\sqrt{2})^2}.$$

10. Spring 2016 Afternoon 4

(a) We have $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = A/\mathfrak{p} \otimes_A A_\mathfrak{p}$ is the localization of $A/\mathfrak{p}$ at the image of $A - \mathfrak{p}$, which is $(A/\mathfrak{p}) - 0$, which is the same as $\mathrm{Frac}(A/\mathfrak{p})$.

(b) The correspondence comes from $\mathfrak{m} \supset f(\mathfrak{p})$ and is disjoint from $f(A-\mathfrak{p})$. We have $B'/\mathfrak{m}' = B/\mathfrak{m} \otimes_A (A/\mathfrak{p})_\mathfrak{p}$ is the localization at a set of units, hence is isomorphic to $B/\mathfrak{m}$. Since $B/\mathfrak{m}$ is a finitely generated field over K, it is actually finite over K, hence also a finitely generated $A_\mathfrak{p}$-module.

(c) Take a finite set of generators $\{b_i\}$ of $B/\mathfrak{m}$ as an A-algebra. Since $B/\mathfrak{m}$ is finite over $A_\mathfrak{p}$, they satisfy monic polynomials of degree d_i with coefficients over $A_\mathfrak{p}$. By localizing at some t, we can assume that that these coefficients all lie in A_t. Then the finitely many monomials in the $\{b_i\}$, of degree at most d_i in b_i, generate $B/\mathfrak{m}$ as a module over A_t.

(d) Since $B/\mathfrak{m}$ is a field which is a finitely generated module over the domain $A_t/\mathfrak{p}A_t$, the latter must also be a field. To see this, we must explain why any $a \in A_t/\mathfrak{p}A_t$ has an inverse. It has an inverse in $B/\mathfrak{m}$, which satisfies a minimal monic polynomial over $A_t/\mathfrak{p}A_t$. Multiplying by a produces a monic polynomial of lower degree, unless already $a^{-1} \in A_t/\mathfrak{p}A_t$.