# ALGEBRA QUAL PREP: RATIONAL CANONICAL FORM

TONY FENG

I haven't found any exposition of rational canonical form that I really like, so here's a summary. (You'll feel more comfortable with it after you do a couple problems.) First of all, both Jordan canonical form and rational canonical form are about the classification of matrices up to conjugation.

The punchline is that two matrices with entries in a field $k$ are conjugate (by matrices with entries in $k$) if and only if their rational canonical forms are the *same*. It's hard to tell when two matrices are conjugate, but it's easy to tell when two polynomials are the same!

Jordan canonical form is also a statement of this sort. However, Jordan canonical form applies *only* for matrices over an *algebraically closed field*. Rational canonical form applies for matrices over an *arbitrary field*. So if the problem involve fields that are not algebraically closed, that's a hint to use rational canonical form (rather than Jordan canonical form).

Rational canonical form is analogous to the classification of finitely generated torsion abelian groups (in fact, both are instances of a slightly more general theory). Any such group is isomorphic to

$$\mathbf{Z}/n_1 \oplus \mathbf{Z}/n_2 \oplus \ldots \oplus \mathbf{Z}/n_r$$

with $n_2 \mid n_1$, $n_3 \mid n_2$, etc. (So all the $n_i$ divide $n_1$.) The sequence $(n_1, n_2, \ldots)$ uniquely determines the group (up to isomorphism).

Rational canonical form says that every $m \times m$ matrix $M$ with entries in $k$ is classified up to conjugacy by a sequence of polynomials

$$a_1(T), \ldots, a_r(T)$$

with $a_{i+1}(T) \mid a_i(T)$. (In particular, everything divides $a_1(T)$.) These are the analogues of the $n_i$ above. Furthermore, $M$ is conjugate to the block-diagonal matrix whose block-diagonals are the "companion matrices" for $a_1(T), a_2(T), \ldots, a_r(T)$.

$$M \sim \begin{pmatrix} \mathrm{Comp}(a_1) & & & \\ & \mathrm{Comp}(a_2) & & \\ & & \ddots & \\ & & & \mathrm{Comp}(a_r) \end{pmatrix}$$

Think of $\mathrm{Comp}(a_r)$ as being analogous to $\mathbf{Z}/n_i$. What are the companion matrices? For a polynomial

$$f(T) = c_n T^n + c_{n-1} T^{n-1} + \ldots + c_0$$

the companion matrix is the matrix for the linear transformation "multiplication by $T$" on the vector space $k[T]/f(T)$, in terms of the natural basis $1, T, T^2, \ldots$. Note that

$$\begin{aligned} 1 &\mapsto T \\ T &\mapsto T^2 \\ T^{n-1} &\mapsto T^n = -\frac{1}{c_n}(c_0 + c_1 T + \ldots + c_{n-1} T^{n-1}) \end{aligned}$$

so the companion matrix $\mathrm{Comp}(f)$ is

$$\mathrm{Comp}(f) = \begin{pmatrix} 0 & 0 & \ldots & -c_0/c_n \\ 1 & 0 & \ldots & -c_1/c_n \\ 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ldots & -c_{n-1}/c_n \end{pmatrix}$$

Sanity check: if $M$ is an $m \times m$ matrix, then the sum of the degrees of the $a_i(T)$ has to be equal to $n$.

In practice, rational canonical form is often used to reduce a question about linear transformations to the elemental case of multiplication by $T$ on $k[T]/f(T)$. See Fall 2014 B4 for an example.