# AUTOMORPHIC FORMS AND MOTIVIC COHOMOLOGY II

LECTURES BY AKSHAY VENKATESH,
NOTES BY TONY FENG

Contents



## 1. Recap

We begin by reviewing the discussion from last time. Let $G$ be a reductive group over $\mathbf{Q}$, whose center has no split torus. Let $X$ be the locally symmetric space for $G$.

Let $\pi$ be a cohomological, cuspidal automorphic representation which is tempered at $\infty$. We assume that at almost all places its Hecke eigenvalues are in $\mathbf{Q}$. This allows us to define $H^*(X;\mathbf{Q})_\pi$.

**Remark 1.1.** By "cohomological" we mean for the trivial local system. I believe the story extends for arbitrary local systems, but I have not thought about this.

The starting point was the following numerology:

$$\dim H^{q+j}(X,\mathbf{Q})_\pi = \binom{\delta}{j} \dim H^q(X,\mathbf{Q})_\pi$$

where $\delta = \operatorname{rank} G_{\mathbf{R}} - \operatorname{rank} K_\infty$ and $2q+\delta = \dim X$. The point is that the betti numbers are binomial coefficients centered around the middle cohomology of $X$.

We proposed the following explanation of this numerology. Let $\Lambda_{\mathrm{mot}}$ be a certain motivic cohomology group; it is a $\mathbf{Q}$-vector space. Under Beilinson's conjecture $\operatorname{rank} \Lambda_{\mathrm{mot}} = \delta$.

**Conjecture 1.2.** *There is a free action of $\Lambda_{\mathrm{mot}}^\vee$ on $H^*(X,\mathbf{Q})_\pi$.*

There are complex and $p$-adic regulators

$$\begin{tikzcd} & \Lambda^\vee_{\mathrm{mot},\mathbf{C}} \\ \Lambda^\vee_{\mathrm{mot}} \arrow[ur] \arrow[dr] & \\ & \Lambda^\vee_{\mathrm{mot},\mathbf{Q}_p} \end{tikzcd}$$

*Date*: November 21, 2017.

Here $\Lambda^\vee_{\mathrm{mot},\mathbf{C}}$ is a $\mathbf{C}$-vector space and $\Lambda^\vee_{\mathrm{mot},\mathbf{Q}_p}$ is a $\mathbf{Q}_p$-vector space.

Our goal is as follows:

(1) Make an action of $\Lambda^\vee_{\mathrm{mot},\mathbf{C}}$ on $H^*(X,\mathbf{C})_\pi$, and give evidence that it preserves the $\mathbf{Q}$-structure.
(2) Make an action of $\Lambda^\vee_{\mathrm{mot},\mathbf{Q}_p}$ on $H^*(X,\mathbf{Q}_p)_\pi$, and give evidence that it preserves the $\mathbf{Q}$-structure.

What does it mean to "give evidence? The difficulty is that the motivic cohomology is hard to access. However, Beilinson's Conjecture makes a prediction about it, namely that its volume is related to special values of $L$-functions. Thus, our Conjecture plus Beilinson's Conjecture makes a concrete (unconditional) numerical prediction relating periods to special values of $L$-functions. This is what we will check; it turns out to be sort of interesting independent of its source.

From $\pi$ we can make a coadjoint motive $M/\mathbf{Q}$ with $\mathbf{Q}$-coefficients, which has the feature that the Galois action on the étale cohomology of $M$ should be equal to the Galois representation for $\pi$ composed with the $\mathrm{Ad}^*$ of the dual group (or some slight modification thereof). In particular $\dim M = \dim G$. Then $\Lambda_{\mathrm{mot}} = H^1_{\mathrm{mot}}(M,\mathbf{Q}(1))$. This is the motivic cohomology group that shows up in Beilinson's conjecture for $L(\mathrm{Ad},\pi,1)$.

**Example 1.3.** For $\mathrm{GL}_n$: if $Y$ is the $n$-dimensional motive attached to $\pi$, then $M = Y\otimes Y^\vee$ and $\Lambda_{\mathrm{mot}} = \mathrm{Ext}^1(Y,Y(1))$.

## 2. The Betti realization

2.1. **Numerical consequences of the conjecture.** Since we want to extract numerics, put a metric on everything: fix an invariant, non-degenerate symmetric bilinear form

$$\mathrm{Lie}\, G\times \mathrm{Lie}\, G\to \mathbf{Q}.$$

(If $G$ is semisimple we could take the Killing form. In general, we want the same sort of definiteness properties that the Killing form enjoys.)

This gives a Riemannian metric on $X$. This induces an inner product on the cohomology groups $H^*(X,\mathbf{R})_\pi$, by identifying them with spaces of harmonic forms. (Recall that $\pi$ is cuspidal.) This gives a hermitian inner product on $\mathfrak{a}_G$. (To remind you, $\mathfrak{a}_G$ is the Lie algebra of the split part of a fundamental Cartan, tensored with $\mathbf{C}$.)

Assume that $H^q(X,\mathbf{C})_\pi = 1$ for simplicity.

**Remark 2.1.** We use $A\sim B$ to mean that two numbers are equal up to (usually) powers of $\pi$ and algebraic numbers.

The action of $\Lambda_{\mathrm{mot},\mathbf{C}}$, which we constructed last time, came from an identification $\Lambda_{\mathrm{mot},\mathbf{C}} \xrightarrow{\sim} \mathfrak{a}_G$ and an action of $\wedge^*\mathfrak{a}_G^\vee$ on $(\mathfrak{g},K)$-cohomology.

**Fact 2.2.** *If $h\in H^q(X,\mathbf{C})_\pi$ is in minimal degree, and $\nu\in\wedge^j\mathfrak{a}_G^\vee$, then we have $\nu\cdot h\in H^{q+1}$ with*

$$||\nu\cdot h|| = ||\nu||\cdot||h||.$$

This comes down to a computation in $(\mathfrak{g},K)$-cohomology; it is not global.

In the example from last time where the action was constructed by Hodge $*$, the claim boils down to saying that the Hodge $*$'s preserve norms.

Now take $h\in H^q(X,\mathbf{Q})_\pi$, which is in minimal degree. For $\nu\in\wedge^\delta\Lambda^\vee_{\mathrm{mot}}$ ($\to\wedge^\delta\mathfrak{a}_G^\vee$), we have $h' := \nu\cdot h\in H^{q+\delta}(X,\mathbf{R})_\pi$. The Conjecture predicts that $h'\in H^{q+\delta}(X,\mathbf{Q})_\pi$.

We want to consider the (Poincaré duality) pairing $\langle h, h' \rangle$. Under the conjecture this is in $\mathbf{Q}$. Choose harmonic representatives $\omega_h$ and $\omega_{h'}$ for $h$ and $h'$. Since $\omega_h$ and $*\omega_{h'}$ lie in the same line, we have

$$\begin{aligned} \int_X \omega_h \wedge \omega_{h'} &= \int \langle \omega_h, *\omega_{h'} \rangle \\ &= ||\omega_h|| \cdot ||\omega_{h'}|| \\ &= ||h|| \cdot ||\nu \cdot h|| \\ &= ||h||^2 \cdot ||\nu||. \end{aligned}$$

So we see that the Conjecture implies that for $\nu \in \wedge^\delta \Lambda_{\mathrm{mot}}^\vee$ and $h \in H^q(X, \mathbf{Q})_\pi$,

$$||\nu|| \cdot ||h||^2 \in \mathbf{Q}. \tag{2.1.1}$$

Let's reformulate this a bit. Choose $\omega$ a harmonic representative of $h$ in $H^q(X, \mathbf{R})_\pi$. Choose a $\gamma \in H_q(X)$, with $\int_\gamma \omega \neq 0$. Then

$$h = \frac{[\omega]}{\int_\gamma \omega} \in H^q(X, \mathbf{Q})_\pi \implies ||h||^2 = \frac{\langle \omega, \omega \rangle}{(\int_\gamma \omega)^2}$$

which allows us to rearrange (2.1.1) as

$$\frac{(\int_\gamma \omega)^2}{\langle \omega, \omega \rangle} \in \mathbf{Q}||\nu||.$$

In summary, the conjecture implies that for $\omega$ a harmonic representative for a class in $H^q(X, \mathbf{R})_\pi$ we should have

$$\boxed{\frac{(\int_\gamma \omega)^2}{\langle \omega, \omega \rangle} \in \mathbf{Q} \cdot ||\nu||.}$$

Now, $||\nu||$ is $\mathrm{vol}(\Lambda_{\mathrm{mot}})^{-1}$ where for $\mathbf{Q}$-vector space $L$ with metric $\langle , \rangle$ on $L \otimes \mathbf{R}$ we define

$$\mathrm{vol}(L) = \sqrt{\det \langle x_i, x_j \rangle} \in \mathbf{C}^*/\mathbf{Q}^*$$

where $x_i$ is a $\mathbf{Q}$-basis. By Beilinson's conjecture this is related to $L(\mathrm{Ad}, \pi, 1)$.

This is the most accessible invariant, but more generally the conjecture tells you about the "period matrix" of $H^*(X, \mathbf{Q})_\pi$.

2.2. **Automorphic periods.** The study of quantities like

$$\frac{(\int_\gamma \omega)^2}{\langle \omega, \omega \rangle}$$

falls under the theory of automorphic periods. In many cases this theory tells you that it is the value of an $L$-function.

For $H \subset G$ a reductive subgroup, we get an inclusion of locally symmetric spaces $X_H \subset X$. There are many examples of pairs $(G, H)$ such that

$$\int_{X_H} (\text{automorphic form on } G) \sim L\text{-function}$$

where the particular $L$-function depends on the situation.

**Example 2.3.** The original example was found by Hecke. For $G = \mathrm{PGL}_2/\mathbf{Q}$ and $H$ the diagonal $\mathrm{GL}_1$, and $X_H$ is the geodesic from 0 to $i\infty$. For $f$ a weight 2 holomorphic form, we can form

$$\omega = f(z)\,dz$$

and the period integral is then

$$\frac{\left|\int_0^\infty \omega\right|^2}{\langle \omega, \omega\rangle} = \frac{L(1/2, f)^2}{L(1, \mathrm{Ad}, f)}.$$

If we can find a subgroup $H$ such that $\dim X_H = q$, then we can take $\gamma$ to be the fundamental class of $X_H$. In that case we get a formula from the theory of automorphic periods, which looks like

$$\frac{(\int_\gamma \omega)^2}{\langle \omega, \omega\rangle} = \frac{L^?(\pi)}{L(1, \mathrm{Ad}, \pi)}$$

The $L^?(\pi)$ depends on $(G, H)$. The content of the conjecture is that it should essentially equal to the $||\nu||$ which is prescribed by Beilinson's conjecture.

There is a nice class of cases, i.e. pairs $(G, H)$ where you have a period formula with $\dim X_H = q$. These are the Gross-Prasad cases:

- $G = \mathrm{SO}_n \times \mathrm{SO}_{n+1}$, $H = \mathrm{SO}_n$ (diagonal)
- $G = \mathrm{PGL}_n \times \mathrm{PGL}_{n+1}$, $H = \mathrm{GL}_n$ (diagonal)
- Restriction of scalars of the above from an imaginary quadratic field.

**Theorem 2.4** (Prasanna, V)**.** *The prediction is true in all cases, up to $\sqrt{\mathbf{Q}}$:*

$$\frac{1}{\mathrm{vol}(\Lambda_{\mathrm{mot}})} \sim_{\sqrt{\mathbf{Q}}} \frac{(\int_\gamma \omega)^2}{\langle \omega, \omega\rangle}$$

*assuming Beilinson's conjecture, Ichino-Ikeda conjecture (known in $\mathrm{GL}_n$ cases), and that archimedean integrals are equal to the expected local $L$-factor at $\infty$.*

I want to emphasize that the equality is somewhat miraculous: in each case there is a cancellation that has to happen. In fact I didn't believe it at first. The main point is that when you explicate Beilinson's conjecture for $\Lambda_{\mathrm{mot}}$ and for $L^?$, various factors cancel.

2.3. **Beilinson's conjecture for $L(\mathrm{Ad}, \pi, 1)$.** Let $M$ be the coadjoint motive from before. Beilinson's conjecture is formulated as follows. Consider $H_B(M, \mathbf{R})^{F_\infty}$ (the subspace fixed by complex conjugation). You consider the exact sequence

$$0 \to F^1 H_{\mathrm{dR}}(M) \otimes_{\mathbf{Q}} \mathbf{R} \xrightarrow{\mathrm{Re}} H_B(M, \mathbf{R})^{F_\infty} \to H_B(M, \mathbf{R})^{W_{\mathbf{R}}} \to 0.$$

Also note that $H_B(M, \mathbf{R})^{W_{\mathbf{R}}}$ gives $\mathfrak{a}_G$ after tensoring with $\mathbf{C}$.

These have rational structures. Denote by $L, M, R$ the left, middle, and right terms in the exact sequence. For $L$ the rational structure comes from algebraic de Rham cohomology, while for $M$ it comes from taking $\mathbf{Q}$-coefficients. For $R$, it comes from a regulator map from motivic cohomology: $\Lambda_{\mathrm{mot}} \to H_B(M, \mathbf{R})^{W_{\mathbf{R}}}$. Thus we have an isomorphism of determinants, but the $\mathbf{Q}$-structures don't match:

$$\det M \xrightarrow{\sim} \det L \otimes \det R.$$

Beilinson's conjecture predicts that

$$L(1, \mathrm{Ad}, \pi) \det M \to \det L \otimes \det R$$

preserves rational structure. We can use this to compute $\mathrm{vol}(\Lambda_{\mathrm{mot}})$. There is a natural way to choose metrics, which we'll omit. So taking inner products gives

$$L(1, \mathrm{Ad}, \pi)^2 \sim_{\mathbf{Q}} \mathrm{vol}(\Lambda_{\mathrm{mot}})^2 \, \mathrm{vol}(F^1 H_{\mathrm{dR}})^2$$

(since $M$ has a rational polarization) hence

$$L(1, \mathrm{Ad}, \pi) \sim_{\sqrt{\mathbf{Q}}} \mathrm{vol}(\Lambda_{\mathrm{mot}}) \cdot \mathrm{vol}(F^1 H_{\mathrm{dR}})$$

so what we need is that $L^?(\pi) \sim \mathrm{vol}(F^1 H_{\mathrm{dR}})$.

**Remark 2.5.** Note that up to $\sqrt{\mathbf{Q}^*}$, the quantity $\mathrm{vol}(F^1 H_{\mathrm{dR}})$ is an invariant of the $\mathbf{Q}$ Hodge structure. The point is that complex conjugation gives an isomorphism

$$F^1 H_{\mathrm{dR}}(M) \otimes \mathbf{C} \cong H_{\mathrm{dR}}/F^0 H_{\mathrm{dR}} \otimes \mathbf{C}.$$

The volume measures the effect of complex conjugation on rational structures.

In all the Gross-Prasad cases, $L^?$ is critical and is predicted by Deligne to be a period $C^{\pm}$ of some motive attached to $\pi$. In every case we checked that $C^{\pm} \sim_{\mathbf{Q}^*} \mathrm{vol}(F^1 H_{\mathrm{dR}})$. This is done case-by-case as an exercise in Hodge linear algebra.

### 2.4. **Refinements.**

2.4.1. *Integral structures.* We'd like to have an integral version, accounting for integral structures. This concerns $\mathrm{vol}(H^q(X, \mathbf{Z})_\pi)$. (We mean modulo torsion). I speculate that this is closely related to the height of the motive corresponding to $\pi$. One reason comes from Kato's paper generalizing Faltings heights to motives, where you see a similarity with the computations that we did.

2.4.2. *Intermediate degrees.* The conjecture makes prediction about things in intermediate degrees. It's hard to get things around the middle dimension, but there's one example. Let $G$ be an inner form of $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{PGL}_2$ where $[F : \mathbf{Q}] = 6$ and is totally complex. Here $\delta = 3, q = 3$, and $X = \Gamma \backslash \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$ (compact), which is 9-dimensional. Assume $\dim H^3(X, \mathbf{C})_\pi = 1$, and that no other $\pi$ contributes to $H^*(X)$. We have a map

$$H^3(X, \mathbf{Q})_\pi \otimes \Lambda_{\mathrm{mot}}^\vee \to H^4(X, \mathbf{Q})_\pi$$

whose image is a 3-dimensional $\mathbf{Q}$-vector space which is conjecturally equal to $H^4(X, \mathbf{Q})_\pi$.

**Theorem 2.6.** *Assuming Beilinson's conjecture, these two $\mathbf{Q}$-structures have the same volume, up to an algebraic number.*

The key point of the proof is that you can compute the analytic torsion of $X$, and it's equal to 1. This implies that

$$\prod_i (\mathrm{vol}\, H^i(X, \mathbf{Q}))^{(-1)^i} \in \mathbf{Q}^*.$$

From this it is not so hard to prove the theorem.

**Theorem 2.7** (Cheeger, Müller)**.** *For compact Riemannian $X$,*

$$\textit{Reidemeister torsion of } X = \textit{Analytic torsion of } X.$$

The analytic torsion is meant to be the alternating product of the determinants in the de Rham complex. To make sense of this, one let $\Delta_i$ be the Laplacian acting on $i$-forms and defines

$$\text{Analytic torsion} := \prod_{i=0}^{q} \det(\Delta_i)^{i(-1)^i}$$

This is $= 1$ when $\delta \geq 2$.

The Reidemeister torsion is a topological invariant, which we won't define.

This allows us to compute one more invariant of the cohomology, and it's consistent with the conjecture.

## 3. The $p$-adic realization

### 3.1. The conjecture over $\mathbf{Q}_p$ and the derived Hecke algebra.

We want to produce maps

$$H^j(X, \mathbf{Q}_p)_\pi \to H^{j+1}(X, \mathbf{Q}_p).$$

A natural thing to try is to take a cup product with something in $H^1$, but typically $H^1(X, \mathbf{Q}_p) = 0$. However, there are many classes with *torsion* coefficients, at least if you raise the level.

**Example 3.1.** The subgroup $\Gamma_0(q) \subset \mathrm{SL}_2 \mathbf{Z}$ has a map to $(\mathbf{Z}/q)^*$ sending a matrix to $a \pmod q$. If $q \equiv 1 \pmod{p^n}$, we can compose this with $\alpha \colon (\mathbf{Z}/q)^* \to \mathbf{Z}/p^n\mathbf{Z}$ to get a class $\widetilde{\alpha} \in H^1(\Gamma_0(q), \mathbf{Z}/p^n\mathbf{Z})$.

These are the same primes as appear in the Taylor-Wiles method, and in fact the whole story is closely related to the Taylor-Wiles method.

We will construct derived Hecke operators by pulling back, cupping with these sorts of classes, and then pushing back down.

Let $S = \mathbf{Z}/p^n$ be the coefficient ring. (We'll make operations here and then take an inverse limit to get operations in characteristic 0.) Let $q \equiv 1 \pmod{p^n}$. Let $K = G(\mathbf{Z}_q)$ and $U \leq K$, and denote by $X(U) \to X$ the covering corresponding to level $U$ structure.

A usual Hecke operator takes the form: for $g \in G(\mathbf{Q}_q)$, we have a correspondence

$$\begin{tikzcd} & X(K \cap gKg^{-1}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

and we define $T_g = \pi_{2*} \circ \pi_1^*$.

A *derived Hecke operator* is obtained by the following construction: given $g \in G(\mathbf{Q}_q)$, and a cohomology class $\alpha \in H^*(K \cap gKg^{-1}, S)$, we can pull $\alpha$ back to $\widetilde{\alpha} \in H^*(X(K \cap gKg^{-1}), S)$, and set $T_{g,\alpha}$ to be the composition

$$\begin{tikzcd} & H^*(X(K \cap gKg^{-1})) \arrow[r, "\smile \widetilde{\alpha}"] & H^*(X(K \cap gKg^{-1})) \arrow[dr, "(\pi_2)_*"] & \\ H^*(X) \arrow[ur, "\pi_1^*"] & & & H^*(X) \end{tikzcd}$$

It is easy to see that this commutes with Hecke operators away from $q$. It also commutes with Hecke operators at $q$.

There is a nicer presentation of this. Denoting $G_q = G(\mathbf{Q}_q)$ and $K_q = G(\mathbf{Z}_q)$, the usual Hecke algebra with $S$ coefficients is

$$\mathrm{Hom}_{G_q}(S[G_q/K_q], S[G_q/K_q]).$$

This acts on $H^*(X, S)$. The derived version should then be

$$\mathrm{Ext}^*_{SG_q}(S[G_q/K_q], S[G_q/K_q])$$

(The Ext is taking place in the category $SG_q$ of smooth $G_q$-representations.) This is a graded algebra, under composition. You can also make this act on $H^*(X, S)$ in such a way that there's an element corresponding to $T_{g,\alpha}$.

You can view $\mathrm{Hom}_{G_q}(S[G_q/K_q], S[G_q/K_q])$ as a function on double cosets. There is a corresponding description of $\mathrm{Ext}^*_{SG_q}(S[G_q/K_q], S[G_q/K_q])$ as functions from double cosets to cohomology:

$$K_q g K_q \mapsto H^*(K_q \cap gK_q g^{-1}, S).$$

This acts on $H^*(X, S)$ as in the diagram on the right.

What you really want to understand is the algebra structure. For this we have a derived version of the Satake isomorphism.

**Fact 3.2** (Satake isomorphism). *If $q \equiv 1 \pmod{p^n}$, then the derived Hecke algebra of $G$ is isomorphic to $dHA(T)^W$, for $q$ larger than $|W|$.*

The dHA of $T$ is the usual Hecke algebra of $T$ tensored with $H^*(T(\mathbf{Z}_q))$. This is graded commutative, so the derived Hecke algebra of $G$ is graded commutative under the assumptions of the fact.

We want to remark that it is not clear from this definition that the dHA acts in a non-zero manner. We'll continue this story next time.

## 4. Variant: weight one forms

We want to end by discussing how to adjust this story for weight one forms. Let $X_1(N)$ be the (compactified) modular curve, thought of as a scheme over $\mathbf{Z}[1/N]$.

We have a line bundle $\omega$ corresponding to weight 1 modular forms. We want to make operations

$$H^0(X, \omega) \to H^1(X, \omega).$$

We have the same diagram as before, with $\alpha\colon (\mathbf{Z}/q)^* \to \mathbf{Z}/p^n$:

$$\begin{tikzcd} & X_0(q) \arrow[r, "\smile \widetilde{\alpha}"] \arrow[dl, "\pi_1"'] & X_0(q) \arrow[dr, "\pi_2"] & \\ X & & & X \end{tikzcd}$$

Then $\alpha$ gives a class $\widetilde{\alpha} \in H^1(\Gamma_0(q), \mathbf{Z}/p^n)$. It comes from the covering $\mathbb{H}/\Gamma_1(q) \to \mathbb{H}/\Gamma_0(q)$; it's an interesting fact that this remains étale over the cusp. So you can extend it to $\widetilde{\alpha} \in H^1(X_0(q)_{\mathbf{Z}[1/Nq]}, \mathbf{Z}/p^n)$. We want to push it into coherent cohomology. We can first restrict it to $H^1_{\text{ét}}(X_0(q)_{\mathbf{Z}/p^n}, \mathbf{Z}/p^n)$, and then map it to $H^1_{\text{ét}}(X_0(q)_{\mathbf{Z}/p^n}, \mathcal{O}) = H^1_{\text{Zar}}(X_0(q)_{\mathbf{Z}/p^n}, \mathcal{O})$. The image is what we'll call $\widetilde{\alpha}$. The same construction using the diagram gives

$$T_{q,\alpha}\colon H^0(X_{\mathbf{Z}/p^n}, \omega) \to H^1(X_{\mathbf{Z}/p^n}, \omega).$$

## References

[GV] Galatius, Soren and Venkatesh, Akshay. Derived Galois deformation rings. Preprint available at https://arxiv.org/pdf/1608.07236.pdf.

[PV] Prasanna, Kartik and Venkatesh, Akshay. Automorphic cohomology, motivic cohomology, and the adjoint L-function. Preprint available at https://arxiv.org/pdf/1609.06370.pdf.

[V] Venkatesh, Akshay.Derived Hecke algebra and cohomology of arithmetic groups. Preprint available at https://arxiv.org/pdf/1608.07234.pdf.