AUTOMORPHIC FORMS AND MOTIVIC COHOMOLOGY I

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Contents

1. Introduction 1
2. An example 2
3. Which motivic cohomology? 4
4. The Betti realization 5
References 7

1. Introduction

I am going to discuss a conjecture about a "hidden action" of motivic cohomology groups on the cohomology of locally symmetric spaces.

1.1. **Notation.** Let G be a reductive algebraic group over $\mathbf{Q}$. We have a locally symmetric space

$$X = G(\mathbf{Q})\backslash G(\mathbb{A})/K_\infty^0 K.$$

1.2. **Why is $H^*(X; \mathbf{Z})$ interesting?** In the theory of automorphic forms, the basic objects are, informally speaking, modules attached to G which have a Hecke action.

The standard definition for automorphic forms are as spaces of functions on X with moderate growth condition and fixed eigenvalues over $\mathbf{C}$. However, there are other notions, such as:

- The topological cohomology $H^*(X, \mathbf{C})$.
- If X happens to be a Shimura variety, $H^*(X, \text{vector bundles})$. For example, classical modular forms can be realized as sections of line bundles over modular curves.

We don't understand all such things yet. For example, they should account for all Galois representations. But at present we don't have any construction of "automorphic forms" which see even mod p Galois representations.

We're going to focus on discussing $H^*(X, \mathbf{C})$. There are a couple nice things about it: first, it has a canonical integral structure; and second, it is defined uniformly for all groups G.

Remark 1.1. One can ask about other kinds of cohomology theories, e.g. topological K-theory. The rational structure is not so different, but the integral structure may be quite different.

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1.3. **Hecke eigensystems.** The problem which we'd like to understand is the following.

The same Hecke eigensystem can occur in multiple cohomological degrees.

Example 1.2. The first case where one sees this is in the setting of modular forms of weight 1. On the modular curve $X_1(N)$, there is a line bundle ω whose sections are $S_1(\Gamma(N))$. In higher weights this has no higher cohomology, but in weight 1 there is a lot of higher cohomology. Any Hecke eigensystem in $H^0(X_1(N), \omega)$ also occurs in $H^1(X_1(N), \omega)$. These spaces are almost dual, up to issues of cusps, which justifies the statement up to twist, but in fact it is true on the nose.

The issue highlighted in this example occurs in a lot of generality in the topological cohomology.

Let π be a cuspidal cohomological automorphic representation for G, which is tempered at ∞. (This means informally that its cohomology is as close to the middle as possible.) We define

$$H^*(X, \mathbf{C})_\pi := \{h \in H^*(X, \mathbf{C}) \colon T \text{ has same eigenvalue on } h \text{ and } \pi\}.$$

When there is a packet, we demand everything is cuspidal and tempered at ∞. We'll be sloppy about the technicalities; they are treated precisely in [PV].

Assume for simplicity that the eigenvalues lie in $\mathbf{Q}$. (Otherwise, just extend the field of coefficients throughout.)

It is a fact that (we might have to assume that $Z(G)$ has no split component)

$$\dim H^{q+j}(X, \mathbf{Q})_\pi = \begin{cases} 0 & j \notin [0, \delta], \\ (\dim H^q_\pi) \cdot \binom{\delta}{j} & j \in [0, \delta] \end{cases}$$

Here $\delta = \operatorname{rank} G(\mathbf{R}) - \operatorname{rank} K_\infty$ (not split rank) and q is such that $[q, q+\delta]$ is symmetric with respect to Poincaré duality, i.e. $2q + \delta = \dim G - \dim K_\infty$.

This comes out of a computation of $(\mathfrak{g}, K)$-cohomology, but we'll explain it in a special case that is hopefully more enlightening.

In other words, the dimensions are binomial coefficients symmetric about the middle dimension. *We would like to explain this phenomenon by constructing a graded algebra A acting on $H^*(X, \mathbf{Q})_\pi$ such that $H^*(X, \mathbf{Q})_\pi$ is free over A and generated in degree q.* The basic proposal is that A is the exterior algebra on (the dual of) a certain motivic cohomology group:

$$A = \wedge^\bullet H_{\mathcal{M}}(?)^\vee.$$

The goal is to formulate this in a more precise way, including describing which motivic cohomology group.

Remark 1.3. Another instance of this sort of phenomenon is that you see cohomology in several degrees in Shimura varieties because of Lefschetz operators. This only exists for non-tempered representations. You might like to produce this by cup product, but there are almost never classes in H^1. So this is of a different nature.

2. An example

We are going to work out a specific example, which is subsumed by the general theory of $(\mathfrak{g}, K)$ cohomology but where you can see things play out concretely.

Suppose $G(\mathbf{R}) = (\mathrm{PGL}_2 \mathbf{C})^3$. (This is the minimal example where things get interesting.) For example, $G = \operatorname{Res}_{E/\mathbf{Q}} \mathrm{PGL}_2$ where E is a degree 6 totally complex extension.

In general, X is a union of

$$\Gamma\backslash\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3.$$

For simplicity let's just assume it's connected, i.e. there's only one copy.

We have $\operatorname{rank} G(\mathbf{R}) = 6$, and $\operatorname{rank} K_\infty = 3$, so $\delta = 3$.

We consider the de Rham complex $\Omega^n(X)$. This is like a Kahler manifold in the sense that there are extra structures on the tangent space that can be upgraded to cohomology. At every point, we have $T_xX = T_x^{(1)} \oplus T_x^{(2)} \oplus T_x^{(3)}$. Accordingly, we can decompose differential forms as

$$\Omega^n = \bigoplus_{p+q+r=n} \Omega^{p,q,r}.$$

Now, just as in the case of Kähler manifolds, a subtler fact is that this decomposition preserves harmonic forms. Namely, if $h \in \Omega^n(X)$ is harmonic then writing $h = \sum h^{p,q,r}$ we have that $h^{p,q,r}$ is harmonic as well. This means that we get a splitting

$$H^n(X, \mathbf{C}) = \bigoplus_{p+q+r=n} H^{p,q,r}(X, \mathbf{C}).$$

We need one more fact. In usual Hodge theory you see Poincaré duality using the Hodge $*$. Here we have that for each component separately.

$$\begin{aligned} *_1 &\colon \Omega^{p,q,r} \to \Omega^{3-p,q,r} \\ *_2 &\colon \Omega^{p,q,r} \to \Omega^{p,3-q,r} \\ *_3 &\colon \Omega^{p,q,r} \to \Omega^{p,q,3-r}. \end{aligned}$$

The $*_i$ induce an isomorphism on harmonic forms, e.g.

$$*_1 \colon \mathcal{H}^{1,1,1} \xrightarrow{\sim} \mathcal{H}^{2,1,1}.$$

Writing $h^{p,q,r} = \dim H^{p,q,r}(X, \mathbf{C})$, this implies

$$\begin{aligned} &h^{1,1,1} \\ =&h^{2,1,1} = h^{1,2,1} = h^{1,1,2} \\ =&h^{2,2,1} = h^{1,2,2} = h^{2,1,2} \\ =&h^{2,2,2}. \end{aligned}$$

Letting this common value be h, we see that this contributes $3h$ to H^4, $3h$ to H^5, and h to H^6. Thus the cohomology looks as if it's a product, even though it is globally not.

Let's switch to the cocompact case (which can be effected by replacing G by an inner form). Then also

$$\begin{aligned} &h^{0,0,0} \\ =&h^{3,0,0} = h^{0,3,0} = h^{0,0,3} \\ =&h^{0,3,3} = h^{3,3,0} = h^{3,0,3} \\ =&h^{3,3,3} = 1 \end{aligned}$$

which corresponds to the trivial Hecke eigensystem π.

Putting together the $*$, we get an isomorphism

$$(*_1, *_2, *_3) \colon H^3(X, \mathbf{C})_\pi \otimes \mathbf{C}^3 \to H^4(X, \mathbf{C})_\pi.$$

This doesn't preserve the obvious rational structures. The conjecture says that it preserves $\mathbf{Q}$-structures if we put the $\mathbf{Q}$-structure on $\mathbf{C}$ coming from the image of a motivic cohomology group under a regulator map.

Remark 2.1. Where is the dependence on the metric? It is wound up in the regulator map.

It's easy to produce cycles in H^3 (via sub locally symmetric spaces). It is hard to produce things in the intermediate degrees. If you believe the conjecture then this must be the case, because it's equivalent to producing something in motivic cohomology, which should be hard.

To summarize, the picture is that

> There should be an action of $\wedge^\bullet$ of (dual of) a motivic cohomology group acting on $H^*(X, \mathbf{Q})_\pi$.

The next goal is to explain which motivic cohomology group we're discussing. The whole point of my lectures is to refine the conjecture to something which is testable.

3. Which motivic cohomology?

Beilinson has a conjecture relating special values of L-functions to volumes of motivic cohomology. The motivic cohomology group in question is the one which appears in Beilinson's conjecture for $L(\mathrm{Ad}, \pi, 1)$. (This is normalized so that 1 is at the edge of critical strip; the center is $1/2$).

Conjecturally π should be associated to a Galois representation

$$\rho \colon G_{\mathbf{Q}} \to {}^L G(\overline{\mathbf{Q}}_p)$$

(or really a slight modification of the L-group ${}^L G$).

This is proven for GL_n over a CM field by Harris-Lan-Taylor-Thorne and Scholze.

We then compose this with the coadjoint representation

$$G_{\mathbf{Q}} \xrightarrow{\rho} {}^L G(\overline{\mathbf{Q}}_p) \xrightarrow{\mathrm{Ad}^*} \mathrm{Aut}(\widehat{\mathfrak{g}}^*_{\overline{\mathbf{Q}}_p}).$$

(The composite is always well-behaved; no modification of the L-group is necessary.)

We'll call the composite $\mathrm{Ad}^* \rho$. (The dualization is unnecessary for semisimple groups but necessary for tori. You can safely ignore it.)

Moreover $\mathrm{Ad}^* \rho$ should come from a weight 0 Chow motive M over $\mathbf{Q}$, in the following sense. This we don't know in any cases. (We'll assume the motive has coefficients in $\mathbf{Q}$; this is true most of the time.) Concretely, this means the following:

- We have $H_B(M, \mathbf{Q}) \cong$ inner form of $\widehat{\mathfrak{g}}_{\mathbf{Q}}^\vee$. In particular $\dim M = \dim G$.
- Under the identification $H_B(M, \overline{\mathbf{Q}}_p) \cong \widehat{\mathfrak{g}}^\vee_{\overline{\mathbf{Q}}_p}$, the Galois action is identified with $\mathrm{Ad}^* \rho$.
- Under the identification $H_B(M, \mathbf{C}) \cong \widehat{\mathfrak{g}}^*_{\mathbf{C}}$, the action of $W_{\mathbf{R}}$ comes from the L-parameter of π_∞, call it
$$\varphi \colon W_{\mathbf{R}} \to {}^L G.$$

Example 3.1. When $G = \mathrm{GL}_n$, ρ should come from an n-dimensional motive X, and $M = X^\vee \otimes X$.

We'll use Λ for the basic motivic cohomology group of interest.

$$\Lambda = H^1_{\mathrm{mot}}(M, \mathbf{Q}(1))_{\mathrm{int}}.$$

What is the subscript "int"? If one has a Chow motive, this should signify motivic cohomology classes which "extend to an integral model". Think of "int" as meaning "unramified". For Chow motives this notion is defined unconditionally by Scholl. This is meant to be

$$\Lambda = \mathrm{Ext}^1_{\mathcal{MM}}(\mathbf{Q}, M(1))_{\mathrm{int}}.$$

For GL_n, it is the same as $\mathrm{Ext}^1_{\mathcal{MM}}(X, X(1))_{\mathrm{int}}$.

There is a p-adic regulator map

$$\Lambda \to H^1_f(G_{\mathbf{Q}}, \mathrm{Ad}^* \rho(1)).$$

The Λ is what arises in Beilinson's conjecture for $L(\mathrm{Ad}, \pi, 1)$. Part of the conjecture predicts the dimension, namely $\dim \Lambda = \delta$.

Right now Λ is a $\mathbf{Q}$-vector space.

Conjecture 3.2. $\wedge^\bullet \Lambda^\vee$ *acts freely on* $H^*(X; \mathbf{Q})_\pi$ *(with generators in minimal degree).*

There are regulator maps $\Lambda \to \Lambda_{\mathbf{C}}$ and $\Lambda_{\mathbf{Q}_p}$, where $\Lambda_{\mathbf{C}}$ and $\Lambda_{\mathbf{Q}_p}$ are the targets of the regulators.

The plan is to give evidence for Conjecture 3.2 in the following form:

(1) To construct an action of $\wedge^* \Lambda^*_{\mathbf{C}}$ on $H^*(X, \mathbf{C})_\pi$ and give evidence that it preserves a $\mathbf{Q}$-structure.
(2) To construct an action of $\wedge^* \Lambda_{\mathbf{Q}_p}$ (using the derived Hecke algebra) and give evidence that it preserves the $\mathbf{Q}$-structures.

This should work for all cohomological π, not just the tempered ones. For example, if π is trivial and $G = \mathrm{GL}_n$ then $X = \mathbf{Q} \oplus \mathbf{Q}(1) \oplus \ldots \oplus \mathbf{Q}(n-1)$, and $X \otimes X$ is a direct sum of various $\mathbf{Q}(i)$. Then $\Lambda = \bigoplus (K_i \mathbf{Z})^{??}$. We know that $H^*(X, \mathbf{Q})_\pi$ contains the stable cohomology of $\mathrm{GL}_n \mathbf{Z}$, which is (dual of) an exterior algebraic on algebraic K-theory. This conjecture should recover that relationship.

The conjecture should give a relationship between $H^*_{\mathrm{stab}}(\mathrm{GL}_n(\mathbf{Z}), \mathbf{Q})$ with $K_* \mathbf{Z}$. The problem is that the multiplicies are too high, but I know how to handle that in this case.

Remark 3.3. The exterior algebra $\wedge^*(\Lambda)$ arises naturally in another context. We mention this because it is probably key to understanding *integral* structure. Under some assumptions,

$$\wedge^* \Lambda_{\mathbf{Q}_p} \cong (\pi_* \mathcal{R}) \otimes \mathbf{Q}_p$$

where $\mathcal{R}$ is a derived version of the Galois deformation ring for $\rho_\pi \pmod p$, which I constructed in a paper with Soren Galatius [GV]. This is a pro-object in simplicial commutative rings.

There is an essential difference between Galois deformation theory on a Shimura variety and in these settings. When $\delta := \mathrm{rank}\, G - \mathrm{rank}\, K_\infty > 0$, the deformation theory is obstructed in an essential way. You can think of this as forcing the ring to live in different degrees.

We construct an action of $\pi_* \mathcal{R}$ on $H_*(X, \mathbf{Z}_p)_\pi$. (We assume that we are in a setting with no congruences.)

4. The Betti realization

4.1. **The invariant δ.** We have to understand a bit better the invariant δ. Here is a list of simple real groups with $\delta := \mathrm{rank}\, G(\mathbf{R}) - \mathrm{rank}\, K_\infty > 0$.

- SL_n for $n \geq 3$.
- $\mathrm{SO}_{p,q}$ for pq odd.
- The split form of E_6. All complex groups (restricted to $\mathbf{R}$).

- Inner twists of the above.

Harish-Chandra showed that $\delta = 0 \iff G(\mathbf{R})$ has discrete series. Equivalently, $G(\mathbf{R})$ has a compact maximal torus. So in general δ measures the extent to which these fail.

A general group has many conjugacy classes of maximal torus. There are two distinguished ones, the ones which are most and least split. The *fundamental Cartan* subgroup is the one whose split part has the minimal dimension; these are all conjugate, and the split part has dimension δ.

Example 4.1. In $\mathrm{GL}_{2n}(\mathbf{R})$, the fundamental Cartan is

$$\begin{pmatrix} \boxed{\mathbf{C}^\times} & & & \\ & \boxed{\mathbf{C}^\times} & & \\ & & \ddots & \\ & & & \boxed{\mathbf{C}^\times} \end{pmatrix}$$

Thus $\delta = n$.

Next we try to formulate the way in which δ measures the obstruction to having discrete series: "δ is the smallest dimension of a family of tempered representations". Thus when $\delta = 0$ you have discrete series, when $\delta = 1$ the tempered representations are parametrized by a vector space, the smallest of which has dimension δ. This is the "fundamental series". In Example 4.1, you can such a family is obtained by putting twists of a discrete series in each factor.

4.2. **The construction of the action.** Choose a fundamental Cartan subgroup T, define

$$\mathfrak{a}_G = \mathrm{Lie}(\text{split part of } T \otimes \mathbf{C}).$$

This is a $\mathbf{C}$-vector space of dimension δ. (By the way, we can take T to be the centralizer of a maximal torus K_∞.)

Remark 4.2. This can be defined "canonically". The split group of a maximal torus is not unique, because two tori can be conjugate in different ways. However, the torus quotient of a Borel is unique up to unique isomorphism.

We want to make an action of $\Lambda_{\mathbf{C}}^\vee$ on $H^*(X, \mathbf{C})_\pi$. (In fact, we will identify $\Lambda_{\mathbf{C}} = \mathfrak{a}_G$).

Fact 4.3. *If π is a cohomological tempered representation of $G_{\mathbf{R}}$, then there is a free action of $\wedge^* \mathfrak{a}_G^\vee$ on $H^*(\mathfrak{g}, K; \pi)$.*

The $H^*(\mathfrak{g}, K; \pi)$ has the same numerology as before: $\dim H^{q+j} = \binom{\delta}{j} \dim H^q$. (This is a local analog of the earlier situation.)

In the paper with Kartik, we explicitly compute this using Vogan-Zuckerman theory. I will outline a more conceptual method. Assume for simplicity that G is semisimple and simply connected (we reduce to this case). Then

$$H^*(\mathfrak{g}, K; \pi) = \mathrm{Ext}^*(1, \pi).$$

We have a canonical action of $\mathrm{Ext}^*(\pi, \pi)$ on $\mathrm{Ext}^*(1, \pi)$, so we are reduced to constructing a map

$$\wedge^* \mathfrak{a}_G^\vee \to \mathrm{Ext}^*(\pi, \pi).$$

This will come from deformating π in a way parametrized by $\mathfrak{a}_G^\vee$. In fact, we can write

$$\pi = \mathrm{Ind}_P^G \sigma$$

where P is a parabolic subgroup with Levi the centralizer of the split part of T. (Tempered cohomological representations are always of this form.) You could twist σ by a character of M. In other words, $\wedge^* \mathfrak{a}_G^\vee = \mathrm{Ext}_{\mathfrak{a}_G}(1,1)$ and parabolic induction gives a map

$$\mathrm{Ext}^*_{\mathfrak{a}_G}(1,1) \to \mathrm{Ext}^*(\pi,\pi).$$

Anyway, we have constructed an action of $\wedge^* \mathfrak{a}_G^\vee$ on $H^*(X,\mathbf{C})_\pi$. We will next identify $\mathfrak{a}_G$ with $\Lambda_{\mathbf{C}} \cong \widehat{\mathfrak{g}}^\vee_{\mathbf{C}}$.

Recall that the Chow motive M corresponds to $\mathrm{Ad}^* \rho$ in such a way that the $W_{\mathbf{R}}$-action on $H_B^*(M,\mathbf{C})$ transports to the $W_{\mathbf{R}}$-action on $\widehat{\mathfrak{g}}^\vee$ coming from the L-parameter of π_∞. There is a map from Λ to a real Deligne cohomology group, which we will use to make the desired identification. We explain this map now.

Given a class in Λ, it defines an extension of real Hodge structures

$$0 \to H_B(M(1),\mathbf{R}) \to Y \to \mathbf{R} \to 0.$$

Thus Y is a real mixed Hodge structure. Choose $y \in F^0 Y_{\mathbf{C}}$ lifting $1 \in \mathbf{R}$. This is well-defined up to $F^0 H_B(M(1),\mathbf{R})$. This y determines the Hodge filtration, because

$$F^i Y_{\mathbf{C}} = \begin{cases} F^i H_B & i \geq 1 \\ F^i H_B + \mathbf{C} y & i \leq 0 \end{cases}$$

Now consider the map sending y to its imaginary part $y - \overline{y} \in H_B(M(1), i\mathbf{R})$. We can modify it by anything in $F^0 H_B(M(1),\mathbf{R})$, so $y - \overline{y}$ really lives in

$$\frac{H_B(M(1), i\mathbf{R})}{\mathrm{Imaginary}(F^0 H_B(M(1), i\mathbf{R}))} = \frac{H_B(M,\mathbf{R})}{\mathrm{Real}(F^1 H_B(M))} \xleftarrow{\sim} H_B(M,\mathbf{R})^{(0,0)}$$

i.e. $H_B(M,\mathbf{R})^{(0,0)}$ is the fixed space for the Hodge $\mathbf{C}^*$. This is also fixed by the real Frobenius if it came from an extension over $\mathbf{R}$. We're just going to explain the map after tensoring up to $\mathbf{C}$, so we don't have to worry about real structures.

So this gives a map

$$\Lambda \to H_B(M,\mathbf{R})^{W_{\mathbf{R}}}.$$

This is Beilinson's regulator. Next we need to explain why (ignoring real structures)

$$H_B(M,\mathbf{C})^{W_{\mathbf{R}}} \cong \mathfrak{a}_G.$$

Again it comes down to the fact that you can "twist" π_∞ (a cohomological tempered representation of $G(\mathbf{R})$) by characters of $\mathfrak{a}_G^\vee$. We can see what happens to the L-parameter when we do this. The effect is that the L-parameter $\varphi\colon W_{\mathbf{R}} \to {}^L G$ gets multiplied by a character $W_{\mathbf{R}} \to Z(\varphi)$ (this is a simple fact about parabolic induction).

This gives a map

$$\mathfrak{a}_G^\vee \xrightarrow{\sim} (\widehat{\mathfrak{g}})^{\varphi(W_{\mathbf{R}})}.$$

Dualizing, we get

$$\mathfrak{a}_G \xleftarrow{\sim} (\widehat{\mathfrak{g}}^\vee)^{\varphi(W_{\mathbf{R}})}$$

and $(\widehat{\mathfrak{g}}^\vee)$ is the target of the Beilinson regulator, as desired.

References

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