PERIOD MAPPINGS ARE DEFINABLE IN THE O-MINIMAL STRUCTURE $R_{\text{an,exp}}$

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1. Hodge structures

The theme of this talk is about introducing algebraic structure to the theory of variations of Hodge structure, beyond the case of Shimura varieties.

Definition 1.1. A *polarized Hodge structure of weight* n is the following data:

(1) A lattice $L \approx \mathbf{Z}^n$,
(2) A bilinear form $Q\colon L \times L \to \mathbf{Z}$ such that $(-1)^n Q(v,w) = Q(w,v)$,
(3) A decomposition

$$L \otimes \mathbf{C} \cong \bigoplus_{p+q=n} H^{p,q}$$

such that
(a) $\overline{H^{p,q}} = H^{q,p}$, and
(b) $i^{p-q}Q(v,\overline{v})$ is positive on $H^{p,q}$, and $Q(H^{p,q}, H^{p',q'}) = 0$ if $p+p' \neq n$.

Example 1.2. The prototypical example is the following. Let X be a smooth, projective, complex variety and $L = H^n(X,\mathbf{Z})_{\text{prim}}/\text{torsion}$. By Hodge theory, we have a decomposition

$$L \otimes_{\mathbf{Z}} \mathbf{C} \cong H^n_{\text{prim}}(X;\mathbf{C}) = \bigoplus_{p+q=n} H^{p,q}_{\text{prim}}(X).$$

Finally, the bilinear form comes from the cup product.

$$Q(\gamma_1,\gamma_2) = \gamma_1 \smile \gamma_2 \smile [w]^{\dim x - n} \in H^{2\dim X}(X;\mathbf{Z}) \cong \mathbf{Z}.$$

It's better to work with the structure of a filtration instead of a splitting. Define

$$\mathcal{F}^p = \bigoplus_{p' \geq p} H^{p',q'}.$$

Then we have a filtration

$$L \otimes_{\mathbf{Z}} \mathbf{C} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \dots$$

and we can recover $H^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{F}^q}$. The reason it's better to work with filtrations is that we want to consider variations of Hodge structures in families. The complex conjugation shows that the direct sum decomposition cannot vary holomorphically in families - if $H^{p,q}$ varies holomorphically then $H^{q,p}$ varies antiholomorphically - while the filtration can (and does) vary holomorphically.

Definition 1.3. We say that $v \in L$ is *Hodge* if $v \in H^{n/2,n/2}$.

Define $\widehat{\Omega} = \{(L \xrightarrow{\sim} \mathbf{Z}^n, Q, \mathcal{F})\}$. This is some sort of symplectic Grassmanian. There's an open subset Ω parametrizing polarized Hodge structures with a basis.

Example 1.4. We have $\Omega = \mathbf{H} \subset \widehat{\Omega} = \mathbf{P}^1$.

Let $G = \mathrm{Aut}(L, Q)$. We have an action of $G(\mathbf{C})$ on $\widehat{\Omega}$, such that $G(\mathbf{R})$ preserves Ω. Then $G(\mathbf{Z})\backslash\Omega$ is the moduli space of polarized Hodge structures. One issue with this space is that it is not in general an algebraic variety (even though it is a complex manifold).

Remark 1.5. Shimura varieties can be placed in this context, if we allow a little more generality. They fall under the context of "Mumford Tate domains", which are obtained by insisting that certain tensors are Hodge.

Definition 1.6. A *variation of Hodge structures* (VHS) on S is $\mathscr{L} \to S$ be an integral local system and $\mathcal{F}^\bullet \subset \mathscr{L} \otimes \mathcal{O}_S$ be a filtration satisfying *Griffiths transversality*, which means that $d\mathcal{F}^p \subset \mathcal{F}^{p-1}$,

In other words, VHS are maps $\phi \colon S \to G(\mathbf{Z})\backslash\Omega$ satisfying Griffiths transversality.

The idea behind what we'll do is to say that even though these period maps ϕ are not algebraic, they can be described by a family of special functions. This is captured by the language of *o-minimality.*

2. O-minimality

Definition 2.1. An *o-minimal structure* is a collection of sets $S = (S_n)_{n \in \mathbb{N}}$ where each $S_n \subset 2^{\mathbf{R}^n}$ satisfying the following properties:

(1) For all n, S_n is a boolean algebra (i.e. closed under intersection, complements, and unions).
(2) We have $S_m \times S_n \subset S_{m+n}$.
(3) For any coordinate projection $\pi \colon \mathbf{R}^{n+1} \to \mathbf{R}^n$, we have $\pi(S_{n+1}) \subset S_n$.
(4) $\{(x, x) \colon x \in \mathbf{R}^n\} \in S_2$.
(5) S_1 consists of finite unions of points and intervals.[1]

Remark 2.2. In this theory the set $\mathbf{Z} \subset \mathbf{R}$ is the"enemy" – one is trying to get sets with finitely many connected components. Although it seems like we only enforced this kind of structure on S_1, the projection map enforces structure on higher-dimensional S_n.

Example 2.3 (Tarski-Seidenberg)**.** $\mathbf{R}_{\mathrm{sa}}$ is the o-minimal structure of semi-algebraic sets. This is the o-minimal structure generated by the graphs of addition and multiplication.

Example 2.4. $\mathbf{R}_{\exp} = \mathbf{R}_{sa} \cup \{(x, e^x), x \in \mathbf{R}\}$.

Example 2.5. The example of interest to arithmetic is the following. $\mathbf{R}_{\mathrm{an},\exp}$ is the o-minimal structure generated by sub-analytic functions and $R_{\exp}$. This means that for any subanalytic $f \colon \mathbb{B}_m \to \mathbf{R}$, i.e. converging on a bigger ball, you add the graph $\{(x, f(x)), x \in \mathbb{B}_m\}$.

Why care about o-minimality? The point is that you get some of the same consequences as for algebraic functions. For example, any set defined in an o-minimal structure has finitely many components.

Theorem 2.6 (Definable Chow, Peterzil-Starchenko)**.** *Let V be a quasiprojective algebraic variety. Take $W \subset V$ to be a closed complex-analytic subvariety which is definable in an o-minimal structure. Then W is algebraic.*

[1] This last condition is the content of the adjective "*o*-minimal"; the previous conditions form the content of the "structure".

Here "definable in an o-minimal structure" means that we allow definable transition functions, and the o-minimal structure is always $\mathbf{R}_{\mathrm{an,exp}}$.

Consider $\pi\colon \Omega \to G(\mathbf{Z})\backslash\Omega$. This isn't definable because there is infinite monodromy. Let $\mathcal{F}$ be a fundamental domain for π. By the isomorphism

$$\pi\colon \mathcal{F} \xrightarrow{\sim} G(\mathbf{Z})\backslash\Omega$$

we can transport a definable structure on $\mathcal{F}$ to one on $G(\mathbf{Z})\backslash\Omega$.

Choose $\mathcal{F}$ to essentially be a Siegel set. Use reduction theory (KAN decomposition) and $\mathbf{R}_{\mathrm{an,exp}}$ to put an o-minimal structure on $\mathcal{F}$.

Theorem 2.7 (B-T)**.** *Let S be an algebraic variety. Then the period map $\phi\colon S \to G(\mathbf{Z})\backslash\Omega$ is definable.*

Remark 2.8. The theorem is really about (punctured) polydisks; it is extended to varieties S by covering with polydisks.

Concretely, what does this say? Let $v_1, \dots, v_m$ be the basis of L in the definition of Ω. Then $\mathcal{F}$ can be described as the region where

(1) $\prod_{i=1}^m |v_i| \ll 1$ (Hodge norm).
(2) $|\langle v_i, v_j\rangle| \ll |v_i|$.

The theorem is that under a period map, you can pick a basis to satisfy (1) and (2). This means that there is a diagram

$$\begin{array}{ccc} (\Delta^\times)^\Gamma & \xrightarrow{\phi} & G(\mathbf{Z})\backslash\Omega \\ \uparrow & & \uparrow \\ V^r \subset \mathbf{H}^r & \dashrightarrow & \Omega \end{array}$$

with (1) and (2) satisfied on V^r. These results had a lot of classical input. The part (1) is essentially due to Kashiwara and Cattani-Kaplan-Schmid. The part (2) is due to Schmid in the 1-dimensional case.

3. Applications

One piece of evidence for the Hodge conjecture is the following.

Let S be an algebraic variety. Let $(\mathcal{F}^\bullet \subset \mathscr{L}) \to S$ be a variation of Hodge structure. Let v be a local section of $\mathscr{L}$. Consider the locus $(*)$ where v is Hodge. According to the Hodge Conjecture this is the locus where some algebraic cycle exists, hence should be algebraic.

Theorem 3.1 (Cattani-Deligne-Kaplan)**.** *The locus*

$$(*) = \{s \in S \colon \exists v \in \mathscr{L}_s, Q(v,v) \le K, v \ \textit{Hodge}\}$$

is algebraic

Proof. You can show that the analogue of $(*)$ in Ω, which we'll call W, is given by

$$W = \bigcup_{g \in G(\mathbf{Z})} gH(\mathbf{R}) \cdot p$$

where $H = \mathrm{Stab}(v) \subset G$. By standard Siegel-set theory, $G(\mathbf{Z})\backslash W$ is definable (i.e. the intersection with Siegel sets is reasonable).

Consider $\phi\colon S \to G(\mathbf{Z})\backslash\Omega$, the locus $(*) = \phi^{-1}(W)$ is closed, complex analytic and definable, hence algebraic by Theorem 2.6. □