MODULAR SYMBOLS AND ARITHMETIC, I

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1. Introduction

The story I want to describe in these talks goes (very roughly) as follows. Let F be a global field and $n \geq 2$. The philosophy is that

> "The geometry of GL_n/F (near a boundary component) is related to the arithmetic of GL_{n-1}/F."

Today I'm going to say a little bit about the direction geometry $\rightsquigarrow$ arithmetic for $F = \mathbf{Q}$ and $n = 2$. More precisely, I want to discuss a map

$$H_1(X_1(N), \mathbf{Z}) \to K_2(\mathbf{Z}[\mu_N]).$$

2. Modular curves

The cusps are denoted

$$C_1(N) := X_1(N) - Y_1(N) = \Gamma_1(N)\backslash\mathbf{P}^1(\mathbf{Q}).$$

We consider the group

$$M' := H_1(X_1(N), C_1(N); \mathbf{Z}),$$

which is generated by modular symbols $\{\alpha \to \beta\}$ for (representatives of cusps) $\alpha, \beta \in \mathbf{P}^1(\mathbf{Q})$. This is the class of the geodesic between α and β in the upper half plane.

Manin gave a presentation of M':

Theorem 2.1 (Manin). *M' is presented by symbols $[u : v]'$, for $u, v \in \mathbf{Z}/n\mathbf{Z}$ with $\gcd(u, v) = 1$, defined by*

$$[u : v]' = \gamma\{0 \to \infty\} = \{\frac{b}{d} \to \frac{a}{c}\}$$

for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Thus $(u, v) = (c, d) \mod N\mathbf{Z}^2$.

The relations are

$$[u : v]' = [-u : -v]' = -[-v : u]' = [u : u + v]' + [u + v : v]'$$

for all u, v as above.

I want to modify this for our purposes, because I want to consider only some cusps. Let

$$C_1^0(N) := \{x \in C_1(N) \mid x \not\mapsto 0 \in X_0(N)\}$$

be the set of "non-zero cusps". We want to consider relative cohomology with respect to these cusps. The presentation is actually basically the same; we just have to restrict to u, v non-zero.

I want to further restrict to the + part for complex conjugation, so define

$$S^0 := H_1(X_1(N), C_1^0(N); \mathbf{Z}')^+$$

where $\mathbf{Z}' = \mathbf{Z}[1/2]$. Write

$$[u:v] := w_N \left(\frac{[u:v]' + [-u:v]'}{2} \right)$$

where $w_N \leftrightarrow \begin{pmatrix} & -1 \\ N & \end{pmatrix}$ is the Atkin-Lehner involution. In the notation from before,

$$w_N[u:v]' = \{-\frac{d}{bN} \mapsto -\frac{c}{aN}\}.$$

Then S^0 has a presentation by $[u:v]$ for $u, v \neq 0$, with relations

$$[u:v] = [-u:v] = -[v:u] = [u:u+v] + [u+v:v].$$

Now $S^0 \supset S := H_1(X_1(N); \mathbf{Z}')^+$. In fact S is the object that we're really interested in, but it's easier to work with S^0 because S doesn't have such a nice presentation. The quotient S/S^0 has to do with the cusps: we have a short exact sequence

$$0 \to S \to S^0 \to \widetilde{H}_0(C_1^0(N); \mathbf{Z}') \to 0. \tag{2.1}$$

These homology groups have the action of Hecke algebras. Call $\mathfrak{h}^0$ the Hecke algebra acting on S^0, and $\mathfrak{h}$ the quotient acting on S. These consist of Hecke operators and diamond operators $j \in (\mathbf{Z}/N\mathbf{Z})^\times$, where -1 acts trivially by definition.

3. The Eisenstein ideal

Given a Dirichlet character χ of modulus N, we can define an Eisenstein series of weight 2 by

$$E_{2,\chi} = \frac{L(\chi,-1)}{2} + \sum_{n=1}^{\infty} \sum_{d|n} d\chi(d) q^n.$$

We form an ideal out of (the coefficients of) this Eisenstein series. Note that

$$\langle j \rangle E_{2,\chi} = \chi(j) E_{2,\chi}.$$

Definition 3.1. The *Eisenstein ideal* $I \subset \mathfrak{h}^0$ is generated by $T_\ell - 1 - \ell\langle \ell \rangle$ for ℓ prime $\nmid N$, and $U_\ell - 1$ for ℓ prime dividing N.

Let $M = (M' \otimes \mathbf{Z}')^+$. It might be better to think of I as an ideal in the Hecke algebra $\mathfrak{H}$ acting on M. Then $\mathfrak{H}/I \xrightarrow{\sim} \mathbf{Z}[\Delta]$ where $\Delta = (\mathbf{Z}/N\mathbf{Z})^\times/\pm 1$, by the map $T_\ell \mapsto 1 + \ell\langle \ell \rangle$ and $U_\ell \mapsto 1$.

We're interested in congruences between cusp forms and Eisenstein series, and for that it is useful to look at I as being in $\mathfrak{h}$. The component of $\mathfrak{h}/I$ corresponding to a character χ has order essentially given by the constant coefficient of $E_{2,\chi}$.

Theorem 3.2. *The group $\mathfrak{h}/I$ is finite, with order divisible by the odd part of $|\prod_\chi L(\chi,-1)|$.*

Remark 3.3. In fact this divisibility is an equality.

4. Cyclotomic fields

Let's define objects corresponding to S and S_0. Let

$$Y = (K_2(\mathbf{Z}[\mu_N]) \otimes_{\mathbf{Z}} \mathbf{Z}')^+.$$

This sits inside

$$Y^0 = (K_2(\mathbf{Z}[\mu_N], 1/N) \otimes_{\mathbf{Z}} \mathbf{Z}')^+.$$

Comparing to (2.1), we have a short exact sequencec

$$0 \to Y \to Y^0 \to \bigoplus_{\mathfrak{p} \mid N} (\mathbf{Z}[\mu_N]^+/\mathfrak{p})^\times \otimes \mathbf{Z}' \to 0.$$

We also consider K_1, which is identified with "N-units":

$$K_1(\mathbf{Z}[\mu_N, 1/N]) \cong \mathbf{Z}[\mu_N, 1/N]^\times.$$

There is a "Steinberg symbol"

$$\{,\} \colon \mathbf{Z}[\mu_N, 1/N]^\times \times \mathbf{Z}[\mu_N, 1/N]^\times \to (K_2(\mathbf{Z}[\mu_N], 1/N) \otimes_{\mathbf{Z}} \mathbf{Z}')^+.$$

Inside the N-units we have the *cyclotomic N-units*

$$C_N := \langle \{1 - \zeta_N^i \mid 1 \leq i \leq N-1\}, \{\zeta_N, -1\} \rangle.$$

Then $C_N \subset \mathbf{Z}[\mu_N, 1/N]^\times$, with index having odd part equal to the plus part $h^+_{\mathbf{Q}(\mu_N)}$ of the class number (which by definition is the class number of the maximal totally real subfield $\mathbf{Q}(\mu_N)^+$).

Remark 4.1. Let me say a bit about how these K-groups (specifically K_2) relates to class groups. For $p \mid M$, there is an isomorphism (due to Tate)

$$K_2(\mathbf{Z}[1/M, \mu_N]) \otimes \mathbf{Z}_p \xrightarrow{\sim} H^2(\mathbf{Z}[1/M, \mu_N]; \mathbf{Z}_p(2)).$$

This second expression is the same as Galois cohomology of the maximal extension unramified outside M and infinite places. If $p^r \mid N$, for $r \geq 1$, then $\mathbf{Z}[1/M, \mu_N]$ has cohomological dimension 2 and we have

$$H^2_{\text{ét}}(\mathbf{Z}[1/M, \mu_N]; \mathbf{Z}_p(2))/p^r \cong H^2_{\text{ét}}(\mathbf{Z}[1/M, \mu_N]; \mathbf{Z}/p^r(2)).$$

This is quite close to the class group, more precisely "M-class group"

$$\mathrm{Cl}_{\mathbf{Q}(\mu_N), M} := \mathrm{Cl}_{\mathbf{Q}(\mu_N)} / \{[\mathfrak{p}] \colon \mathfrak{p} \mid M\}.$$

There is a short exact sequence

$$0 \to \mathrm{Cl}_{\mathbf{Q}(\mu_N), M} \otimes_{\mathbf{Z}} \mathbf{Z}_p \to H^2_{\text{ét}}(\mathbf{Z}[1/M, \mu_N]; \mathbf{Z}_p(1)) \to \underbrace{\bigoplus_{\mathfrak{p} \mid M} \mathbf{Z}_p \to \mathbf{Z}_p}_{\text{Brauer group}} \to 0.$$

Note that while K_2 is a finite group, the Galois cohomology is not finite if there are at least 2 places dividing M.

The Steinberg symbol in K-theory maps to the cup product in Galois cohomology: $\{\alpha, \beta\} \mapsto \alpha \smile \beta$. The Steinberg symbol is bilinear, antisymmetric, and satisfies the defining property of Milnor K-theory:

$$\{x, 1-x\} = 0 \text{ if } x, 1-x \in \mathbf{Z}[\mu_N, 1/N]^\times.$$

We can look for these relations in cyclotomic units. For instance,

$$\zeta_N^a + (1-\zeta_N^a) = 1,$$
$$\frac{1-\zeta_N^a}{1-\zeta_N^{a+b}} + \zeta_N^a \frac{1-\zeta_N^b}{1-\zeta_N^{a+b}} = 1,$$
$$\frac{1+\zeta_N^a}{1-\zeta_N^{a+b}} - \zeta_N^a \frac{1+\zeta_N^b}{1-\zeta_N^{a+b}} = 1.$$

5. The conjecture

Theorem 5.1 (Busuioc, S)**.** *There exists a map*

$$\Pi^0 \otimes \mathbf{Z}_p \colon S^0 \to Y^0$$

sending

$$[u : v] \mapsto \{1-\zeta_N^u, 1-\zeta_N^v\}.$$

Moreover, $\Pi^0((T_\ell - 1 - \ell\langle\ell\rangle)x) = 0$ *for all* $x \in S^0$*, if* $\ell \in \{2,3\}$*.*

We comment on the proof. We used the relations on the Steinberg symbol to try to cut down the size of Milnor K_2. We projected onto an eigenspace for the $(\mathbf{Z}/p\mathbf{Z})^\times$-action, and we found that each non-zero eigenspace (is nontrivial if p divides a certain Bernoulli number) is one-dimensional.

Conjecture 5.2 (S)**.** *The map*

$$\varpi \colon S/IS \to Y$$

is an isomorphism.

6. Argument of Fukaya-Kato

We will present a proof by Fukaya-Kato that $\varpi \otimes \mathbf{Z}_p$ is Eisenstein, i.e. that it factors through the Eisenstein quotient. We assume that $N \geq 4$.

We can extend $X_1(N)$ to a scheme over $\mathbf{Z}[1/N]$. This carries a universal family $\mathcal{E}$. For a point $(E, C) \in X_1(N)$ consisting of an elliptic curve E and $C \in E[N]$, the fiber in $\mathcal{E}$ is E. The unit group of $\mathcal{E}$ contains theta functions: if $(c, 6) = 1$ then there exists ${}_c\theta \in \mathcal{O}(\mathcal{E})^\times$ with divisor $c^2[0] - E[c]$. It is preserved under the norm map induced by multiplication by a, if $(a, c) = 1$.

We also have sections $\iota_a \colon X_1(N) \to \mathcal{E}$ sending $(E, C) \mapsto aC$ for $(a, 6N) = 1$. For $(a, c) = 1$ we can pull back the theta function to get what is called a *Siegel unit* on $X_1(N)$:

$${}_c g_a := \iota_a^*({}_c\theta).$$

We can also define a Siegel unit $g_a \in \mathcal{O}(Y_1(N))^\times \otimes_{\mathbf{Z}} \mathbf{Q}$ which is independent of c, if we allow denominators. These satisfy

$${}_c g_a = \frac{g_a^{c^2}}{g_{ac}}.$$

In fact, $g_a^{12N} \in \mathcal{O}(Y_1(N))^\times/\text{torsion}$. There is a map

$$\mathcal{O}(Y_1(N))^\times \to K_1(Y_1(N)).$$

Then we can take the Steinberg symbols to get elements of $K_2(Y_1(N))$. The elements $\{g_u, g_v\} \in K_2(Y_1(N)) \otimes_{\mathbf{Z}} \mathbf{Q}$ are called "Beilinson-Kato elements".

Theorem 6.1 (Goncharov, Brunault)**.** *There is a well-defined map*

$$H_1(X_1(N), C_1^0(N); \mathbf{Z}) \to K_2(Y_1(N)) \otimes_{\mathbf{Z}} \mathbf{Q}$$

defined by

$$[u : v] \mapsto \{g_u, g_v\}.$$

This is nice, but there are two issues.

(1) The proof doesn't show that the map is Hecke-equivariant.
(2) The theorem requires tensoring with $\mathbf{Q}$. Since we want to end up with K_2 of a ring of integers, which is finite, tensoring with $\mathbf{Q}$ is really bad. One can show that denominators don't show up in a certain "primitive part" that we're interested in.

Roughly, Fukaya-Kato showed that there exists a Hecke-equivariant map

$$z^{\#} \colon H_1(X_1(N), C_1^0(N); \mathbf{Z}) \to K_2(Y_1(N)) \otimes_{\mathbf{Z}} \mathbf{Z}_p$$

sending $[u : v] \mapsto \{g_u, g_v\}$ by a computation of a p-adic regulator. This statement is not really true, and we'll explain the correction later.

Now we want to focus on ∞. There is a map

$$\operatorname{Spec} \mathbf{Z}[\mu_N, 1/N]^+((q)) \to Y_1(N)$$

so we can pull back

$$K_2(Y_1(N)) \to K_2(\mathbf{Z}[\mu_N, 1/N]((q))) \xleftarrow{\sim} K_2(\mathbf{Z}[\mu_N, 1/N][[q]]) \oplus K_1(\mathbf{Z}[\mu_N, 1/N]).$$

Then we can project to the first component, and specialize $q = 0$:

$$K_2(\mathbf{Z}[\mu_N, 1/N][[q]]) \oplus K_1(\mathbf{Z}[\mu_N, 1/N]) \xrightarrow{(q \mapsto 0, 0)} K_2(\mathbf{Z}(\mu_N, 1/N]).$$

The composition is a map $\infty \colon K_2(Y_1(N)) \to K_2(\mathbf{Z}(\mu_N, 1/N])$ which is called "specialization at ∞".

Now you can ask: what happens to a Siegel unit when you specialize to ∞?

$$g_u \sim q^{1/12N} \prod_{n=0}^{\infty} (1 - q^n \zeta_N^u) \prod_{n=1}^{\infty} (1 - q^n \zeta_N^{-u})$$

The process of forgetting K_1 kills factor $q^{1/12N}$. The processing of taking $q \to 0$ leaves us with $1 - \zeta_N^u$. So we conclude that

$$\infty(\{g_u, g_v\}) = \{1 - \zeta_N^u, 1 - \zeta_N^v\}.$$

The upshot is that we have found

$$\Pi^0 = \infty \circ z^{\#}.$$

Now, $z^{\#}$ is Hecke-equivariant. What happens at ∞? We compute that

$$\infty((T_\ell^* - 1 - \ell \langle \ell \rangle^*)x) = 0, \quad x \in K_2, \ell \nmid N$$

and

$$\infty((U_\ell^* - 1)\{g_u, g_v\}) = 0 \quad \forall u, v, \ell \mid N.$$

This shows that Π^0 factors through S^0/IS^0.

Now we have to go back and clear up some of the lies we told earlier. First of all, they do not quite construct a map $z^{\#}$ like we stated. Assume that $p \mid N$. To explain what they really do, let

$$\widetilde{\mathcal{T}} = \varprojlim H^1_{\text{ét}}(Y_1(Np^r)_{\overline{\mathbf{Q}}}; \mathbf{Z}_p(1))^{\text{ord}}$$

where the transition maps are corestriction (i.e. trace). Consider also

$$\mathcal{H}^0 := \varprojlim H_1(X_1(Np^r)_{\overline{\mathbf{Q}}}, C_1^0(Np^r); \mathbf{Z}_p)^{\mathrm{ord}}$$

They show that there is a map

$$z^{\#} \colon \mathcal{H}^0 \to H^1_{\text{ét}}(\mathbf{Z}[1/N], \widetilde{\mathcal{T}}(1)) \otimes \Lambda_\mu^{-1} \tag{6.1}$$

where Λ is an Iwasawa algebra of diamond operators, and μ is an explicit quantity which I don't want to describe.

There is a map

$$K_2(Y_1(N)) \otimes \mathbf{Z}_p \to H^2_{\text{ét}}(Y_1(N)/\mathbf{Z}[1/N]; \mathbf{Z}_p(2)) \to H^1(\mathbf{Z}[1/N]; H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}; \mathbf{Z}_p(2))).$$

Now take ordinary parts, the kernel disappears and we get

$$K_2(Y_1(N)) \otimes \mathbf{Z}_p \to H^2_{\text{ét}}(Y_1(N)/\mathbf{Z}[1/N]; \mathbf{Z}_p(2))^{\mathrm{ord}} \xrightarrow{\sim} H^1(\mathbf{Z}[1/N]; H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}; \mathbf{Z}_p(2)))^{\mathrm{ord}}.$$

We think of H^2 as a substitute for K_2.

Now you might ask: how do you show that there exists the map (6.1)? They realize $(1-U_p)z^{\#}$ as the corestriction of a map

$$z \colon \Lambda \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{H}^0 \to \varprojlim H^1_{\text{ét}}(\mathbf{Z}[1/N, \mu_{Np^r}], \widetilde{\mathcal{T}}(1))^{\mathrm{ord}} \otimes (\Lambda \widehat{\otimes} \Lambda)\lambda^{-1}.$$

There are two Λ's, one coming from the level (diamond operators) and one coming from the Iwasawa tower. This map is given explicitly by Siegel units. To check that it is well-defined, they compose with a p-adic regulator map to $\Lambda \widehat{\otimes}_{\mathbf{Z}_p} \mathcal{H}^0$. The composition is multiplication by a non zero-divisor (a p-adic L-function): it sends

$$[u : v] \mapsto x[u : v].$$

The punchline is that the composite is clearly Hecke-equivariant, and the regulator is injective and Hecke-equivariant.

After taking the corestriction back down to the original level, you see that the map is divisible by $1 - U_p$. After specializing at ∞ the inverse limit of H^2 is closely related to a characteristic ideal, and the denominator μ introduced misses it.

Remark 6.2. If $p \nmid N$ then it is not clear that the map is Hecke-equivariant, because we cannot project to the ordinary part.

7. Preview of next week

Next time we will essentially construct a map in the opposite direction:

$$\Upsilon \colon Y \otimes \mathbf{Z}_p \to S/IS \otimes \mathbf{Z}_p$$

which is inverse to ϖ. This is close to a map that appears in the Mazur-Wiles proof of the Main Conjecture of Iwasawa theory. It uses the Galois action on $H^1_{\text{ét}}(X_1(Np^r); \mathbf{Z}_p(1))/I$. You should think of Υ as being completely inexplicit, in contrast to ϖ which was very explicit.

8. Generalizations

For the remainder of the talk we discuss some cases where one might be able to construct generalizations of ϖ.

A couple of things are required. We need a nice description of units / elements in H^2. On the other hand, we need an explicit description of homology.

8.1. **Imaginary quadratic fields.** Let K be an imaginary quadratic field and $n = 2$, p split. Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal with $p \mid \mathfrak{n}$. We have the hyperbolic upper-half space $\mathbf{H}_3$. The congruence subgroup $\Gamma_1(\mathfrak{n}) \subset \mathrm{SL}_2(\mathcal{O}_K)$ acts on $\mathbf{H}_3$, and this actions extends to the extended upper-half space $\mathbf{H}_3 \cup \mathbf{P}^1(K)$. The quotient $X_1(\eta) = \Gamma_1(\eta)\backslash\mathbf{H}_3$ is a Bianchi space. We'd like to think of this as analogous to the quotient of the modular curve by complex conjugation, although of course it has no algebraic structure. So the group in question is $H_1(X_1(\mathfrak{n}); \mathbf{Z}_p)$.

Similarly, we can consider $H_1(X_1(\mathfrak{n}), C^0; \mathbf{Z}_p)$. This has modular symbols $\gamma\{0 \to \infty\}$. If K is Euclidean, Cremona showed that this group has an explicit presentation on symbols $[u : v]$ for non-zero $u, v \in \mathcal{O}_K$ with $\gcd(u, v) = 1$.

On the other hand, inside $K_2(\mathcal{O}_K[1/\mathfrak{n}])$ we have Steinberg symbols of elliptic units $\{\alpha_u, \alpha_v\}$, where α_u, α_v are elliptic $\mathfrak{n}$-units in $\mathcal{O}_{K(\mathfrak{n})}[1/n]^\times$ where $K(\mathfrak{n})$ is the Ray class field of conductor $\mathfrak{n}$ over K. You can think of elliptic units as the specializations of Siegel units at CM points.

We can speculate that there should exist a map

$$[u : v] \mapsto \{\alpha_u, \alpha_v\}$$

which is Hecke-equivariant, and moreover Eisenstein. Goncharov first studied this idea.

8.2. $n = 3$ **for Q.** In this case the space in question is $\mathrm{SL}_3(\mathbf{R})/\mathrm{SO}_3(\mathbf{R})$. This has an action of $\mathrm{SL}_3(\mathbf{R})$. Let $\Gamma_1^{(3)}(N)$ be the group of matrices in $\mathrm{SL}_3(\mathbf{Z})$ with bottom row congruent to $(1, 0, 0) \mod N$.

Let $X_1^{(3)}(N)$ be an appropriate compactification of the quotient. This is a 5-manifold, which again suffers from the problem of not being algebra. However, by work of Ash-Rudolph and Ash, we know that $H_2(X_1^{(3)}(N), \partial, \mathbf{Z})$ has a presentation on symbols $[u : v : w]$ for $u, v, w \in \mathbf{Z}/n\mathbf{Z}$ with $\gcd(u, v, w) = 1$.

What about a K-group? Recall the philosophy that the geometry of GL_n/F (near a boundary component) is related to the arithmetic of GL_{n-1}/F. So now we should be looking at GL_2, i.e. the modular curve, over $\mathbf{Z}[1/N]$. Now, $K_3(Y_1(N)/\mathbf{Z}[1/N]) \ni \{g_u, g_v, g_w\}$.

Question 8.1. Does there exist a Hecke-equivariant map

$$H_2(X_1^{(3)}(N), \partial, \mathbf{Z}) \to K_3(Y_1(N)/\mathbf{Z}[1/N])$$

sending $[u : v : w] \mapsto \{g_u, g_v, g_w\}$?

Remark 8.2. That there exists a map after tensoring with $\mathbf{Q}$ should be easy; the hard part would be to show Hecke-equivariance.

There is a map

$$K_3(X_1(N)) \to H^3(X_1(N)/\mathbf{Z}[1/N]; \mathbf{Z}_p(3)) \to H^2(\mathbf{Z}[1/N]; H^1_{\text{ét}}(X_1(N); \mathbf{Z}_p(3)).$$

So we see H^2 of interesting Galois representations. The target is closely related to the dual Selmer groups of cusp forms. So if we had the desired map, then we might be able to get information here.

8.3. **Function fields.** Let K be a global function field of characteristic $\ell \neq p$ and $n \geq 2$. We will focus on the case $n = 2$, $K = \mathbf{F}_q(t)$.

In this case the symmetric space is the Bruhat-Tits tree BT. It has an action of $\mathrm{PGL}_2(\mathcal{O}_K)$, which extends to a compactification by "ends". An end is an apartment in the tree, and the Bruhat-Tits tree has a partial compactification $\overline{BT}$ whose boundary is the ends. Then we can consider

$$H_1(\Gamma_1(\mathfrak{n})\backslash\overline{BT}), \text{ends}; \mathbf{Z}_p).$$

By work of Teitelbaum, this has a nice presentation on symbols $[u : v]$ for $u, v \in \mathcal{O}_K/\mathfrak{n}$ for $(u, v) = 1$ with essentially the same relations.

We also have an analogue of the notion of ray class field. There is an $\mathfrak{n}$-unit $\lambda_u \in K(\mathfrak{n})^\times$ which is an $\mathfrak{n}$-torsion point for a Drinfeld module called the *Carlitz module.*

We have the action of a Hecke algebra $\mathfrak{h}$ on $H_1(\Gamma_1(\mathfrak{n})\backslash\overline{BT}); \mathbf{Z}_p)$, which contains an Eisenstein ideal I.

Theorem 8.3 (Fukaya-Kato-S)**.** *There exists a map*

$$\varpi\colon H_1(X_1(\mathfrak{n}); \mathbf{Z}_p)/I \to K_2(\mathcal{O}_{K(\mathfrak{n})}) \otimes \mathbf{Z}_p$$

sending

$$[u : v] \mapsto \{\lambda_u, \lambda_v\}.$$