COHOMOLOGY OF ARITHMETIC GROUPS AND EISENSTEIN SERIES: AN INTRODUCTION, II

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Contents

1. Review

We begin by reviewing the discussion from the previous talk.

1.1. Setup.

We recall some notation from last time.

Let G be a connected semisimple algebraic group over a number field k. Let $\mathcal{O}_k$ be the ring of integers of k. Choose an embedding $\rho\colon G \hookrightarrow \mathrm{GL}_n$. Then we can define

$$G(\mathcal{O}_k) := G(k) \cap \mathrm{GL}_n(\mathcal{O}_k).$$

Denote by V_∞ the set of archimedean places of k. Let $\Gamma \subset G(k)$ be a torsion-free arithmetic subgroup, and

$$G_\infty := (\mathrm{Res}_{k/\mathbf{Q}} G)(\mathbf{R}) = \prod_{v \in V_\infty} G_v$$

inside which we pick a maximal compact $K_\infty := \prod_{v \in V_\infty} K_v$.

Let $X = \prod_{v \in V_\infty} K_\infty \backslash G_\infty$. Then Γ acts on X, giving the quotient X/Γ the structure of a non-compact Riemannian manifold with finite volume.

For (η, E) a finite-dimensional representation of G, we have a corresponding local system on X/Γ, and we denote by $H^i_{\mathrm{dR}}(X/\Gamma, E)$ the de Rham cohomology. As we saw in Clozel's lectures, this admits an interpretation in terms of $(\mathfrak{g}, K)$-cohomology:

$$H^i_{\mathrm{dR}}(X/\Gamma, E) = H^*(\mathfrak{g}, K_\infty; C^\infty(G_\infty/\Gamma) \otimes E).$$

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1.2. **Boundary cohomology.** We introduced the Borel-Serre compactification

$$\overline{X}/\Gamma = \coprod_{P \in \mathcal{P}/\Gamma} e'(P).$$

Here $\mathcal{P}$ is the set of rational parabolics. Note that the term $e'(G)$ is X/Γ.

The key aspect of the Borel-Serre compactification is that the map $X/\Gamma \to \overline{X}/\Gamma$ is a homotopy equivalence. In particular, the map $H^*(\overline{X}/\Gamma, E) \to H^*(X/\Gamma, E)$ is an isomorphism.

Consider the restriction maps (for P a $\mathbf{Q}$-parabolic of G)

$$\gamma_P^* \colon H^*(\overline{X}/\Gamma, E) \to H^*(e'(P), E).$$

We explained last time that this restriction map can be realized in the space X/Γ. Namely, by reduction theory there is an neighborhood $V_{P,t}$ of $e'(P)$ such that $(X/\Gamma) \cap V_{P,t}$ is diffeomorphic to $A_{P,t} \times e'(P)$. Here $P(\mathbf{R}) = A_P \cdot {}^0P \cdot N_P$ and

$$A_{P,t} = \{a \in A_P \mid \alpha(a) \geq t \text{ for all } \alpha \in \Delta(P, A_P)\}.$$

An important point is that there exists $t_0 > 0$ such that Γ-equivalence and $\Gamma \cap P$-equivalence coincide in $V_{P,t}$, for $t > t_0$. As a consequence of this picture, $\gamma_P^*([\varphi])$ is represented by the restriction $[\varphi]|_{e'(P)}$.

Next we discussed the cohomology of $H^*(e'(P), E)$. We explained how topological arguments can be used to show that the image of γ_P^* is "large". The question is which classes lift to $\overline{X}/\Gamma$. We have a fibration $e'(P) \to Z_M/\Gamma_M$ where Z_M is the symmetric space of the Levi M_P of P. This yields an E_2-degenerate spectral sequence, hence a decomposition

$$H^*(e'(P), E) = \bigoplus_{w \in W^P} H^*(Z_M/\Gamma_M, \mathcal{F}_{\mu_w})$$

where W^P is the set of minimal coset representatives of $W_P \backslash W$, and F_{μ_w} is the representation of M with highest weight

$$\mu_w = w(\Lambda + \rho) - \rho|_{\mathfrak{b}_{\mathbf{C}}}$$

if (η, E) has highest weight Λ, where $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}_P$ is a splitting of a Cartan subalgebra of $\mathfrak{g}$ into a Cartan of ${}^0\mathfrak{m}_P$ and the Lie algebra of A_P.

Inside $H^*(e'(P), E)$ we have the cuspidal cohomology $H^*_{\text{cusp}}(e'(P), E)$.

Definition 1.1. We say that $\varphi \in H^*_{\text{cusp}}(e'(P), E)$ is *a class of type* (π, w), with $w \in W^P$, if $[\varphi]$ comes from $H^*({}^0\mathfrak{m}, K_M, V_\pi \otimes F_{\mu_w})$ with $V_\pi \subset L^2({}^0M/\Gamma_M)$. This already imposes a condition on the central character. (We are implicitly using here that the weights μ_w are distinct, so that the type is well-defined)

2. Lifting differential forms from the boundary

Recall the isomorphism induced by the geodesic action:

$$X/P \cap \Gamma \xrightarrow{\sim} e'(P) \times A_P.$$

Let

$$\varphi \in \Omega^*(e'(P), E) = \Omega^*(e(P), E)^{\Gamma \cap P}.$$

We can pull back φ to $e'(P) \times A_P$ via the projection, and thus identify it with an element of $\Omega^*(X/P \cap \Gamma, E) = \Omega^*(X, E)^{\Gamma \cap P}$.

Definition 2.1. For $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, let $\varphi_\lambda := \varphi \cdot a^{\lambda+\rho} \in \Omega^*(X, E)^{\Gamma \cap P}$.

Since we want to produce cohomology classes, we want to know if the differential forms produced by this construction are closed.

Fact 2.2. *Let $[\varphi] \in H^*_{\text{cusp}}(e'(P), E)$ be of type (π, W) where W has highest weight Λ. Then φ_λ is a closed form if and only if*

$$\lambda = -w(\Lambda + \rho)|_{\mathfrak{a}}.$$

For $\omega \in \Omega^*(X/\Gamma, E)$, we denote by ω^0 the form in $\Omega^*(X, E)^\Gamma$ obtained by pullback. We can write ω^0 in terms of Maurer-Cartan forms.

Choose an orthonormal basis X_i for $\mathfrak{g}$, compatible with the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, and view the X_i as right-invariant vector fields on G (hence also for G/Γ). Let ω_i be the dual basis of 1-forms on G. For $J = \{i_1, \dots, i_q\} \subset \{1, \dots, \dim \mathfrak{g}\}$ an increasing sequence, we write

$$\omega^J := \omega^{i_1} \wedge \ldots \wedge \omega^{i_q}.$$

Let $\omega \in \Omega^i(X, E)^\Gamma$. We can write

$$\omega = \sum f_J \omega^J, \; f_J \in C^\infty(G/\Gamma) \otimes E.$$

Proposition 2.3. *Let $\omega \in \Omega^*(X, E)^\Gamma$ be closed. Write*

$$\omega^0 = \sum f_J \omega^J.$$

Suppose that the functions f_J are automorphic forms. Then

$$[\omega]|_{e'(P)} = [\omega_P]|_{e'(P)}$$

where ω_P is the constant Fourier coefficient of ω, defined by taking the constant terms of the functions:

$$\omega_P = \sum (f_J)_P \cdot \omega^J|_{e'(P)}.$$

Thus we have the description

$$\gamma_P^*([\omega]) = [\omega_P]|_{e'(P)}.$$

3. Eisenstein series

3.1. **Construction of Eisenstein series.** We now present a construction that attempts to lift forms from boundary cohomology.

Let $P \subsetneq G$ be a parabolic $\mathbf{Q}$-subgroup. We let $\mathfrak{a} := \mathfrak{a}_P$ be the Lie algebra of A_P, and $\mathfrak{a}^*$ be its the dual. We define

$$(\mathfrak{a}^*)^+ = \{\lambda \in \mathfrak{a}^* \mid (\lambda, \alpha) > 0 \text{ for all } \alpha \in \Delta(P, A)\}$$

and

$$(\mathfrak{a}^*_{\mathbf{C}})^+ = \{\lambda \in \mathfrak{a}^*_{\mathbf{C}} \mid \text{Re } \lambda \in \rho + (\mathfrak{a}^*)^+\}.$$

The subset $(\mathfrak{a}^*_{\mathbf{C}})^+$ corresponds to "real part > 1" under the right normalizations.

Let $f \in C^\infty(G/A_P N_P)^{\Gamma_P}$ be K-finite, such that for each $g \in G$, the function

$$m \mapsto f(gm)$$

is a square-integrable automorphic form on ${}^0M_P$ with respect to Γ_M. To f and $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, we can attach a function f_λ on G defined by: for $x \in G$,

$$f_\lambda(x) = f(x) a(x)^{\lambda + \rho}.$$

Here $a(x)$ is the part of the Cartan decomposition

$$G = A_P {}^0M_P N_P K.$$

Then we can form the Eisenstein series

$$E(f,\lambda)(x) = \sum_{\gamma \in \Gamma/\Gamma\cap P} f_\lambda(x\gamma), \qquad x \in G.$$

This is C^∞ and uniformly convergent on compact subsets in $G \times (\mathfrak{a}_{\mathbf{C}}^*)^+$. As a function of x, it is K-finite, and even an automorphic form with respect to Γ.

There is an analytic continuation as a meromorphic function to all of $\mathfrak{a}_{\mathbf{C}}^*$. It is holomorphic on the imaginary plane $i\mathfrak{a}^*$.

3.2. **Representation-theoretic interpretation.** The preceding construction has the following representation-theoretic interpretation. For a representation of the Levi

$$(\sigma, H_\sigma) \hookrightarrow L^0_{\mathrm{disc}}({}^0M_P/\Gamma_M)$$

we have a map

$$\mathrm{Ind}_P^G[(\sigma, H_\sigma)\otimes\lambda] \to A(G,\Gamma) \subset C^\infty(G/\Gamma).$$

4. Eisenstein differential forms

We can play the same game with differential forms. Given $[\varphi] \in H^*(e'(P), E)$, we can lift it to $\varphi_\lambda \in \Omega^*(X/\Gamma\cap P, E)$ as discussed in §2, and then form the Eisenstein series

$$E(\varphi,\lambda) = \sum_{\gamma\in\Gamma/\Gamma\cap P} \varphi_\lambda \circ \gamma \in \Omega^*(X,E)^\Gamma$$

for a fixed $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, if $E(\varphi,\lambda)$ is holomorphic there.

The region of absolute convergence for $E(\varphi,\lambda)$, viewed as a function of λ with φ fixed, is $\mathrm{Re}\,\lambda \in \rho_P \in (\mathfrak{a}^*)^+$. Of course there is an analytic continuation. Since we want to construct cohomology classes, we are interested in:

(1) Determining for which λ_0, the form $E(\varphi,\lambda)$ evaluated at $\lambda=\lambda_0$ (subject to the condition that $E(\varphi,\lambda)$ is holomorphic at $\lambda=\lambda_0$) gives rise to a *closed* form on X/Γ (and hence a cohomology class).
(2) Determining when the cohomology class $[E(\varphi,\lambda_0)]$ is non-trivial.

Theorem 4.1. *Let (η, E) have highest weight Λ. Let $[\varphi] \in H^*_{\mathrm{cusp}}(e'(P), E)$ be of type (π, w), where π is cuspidal and $w \in W^P$. We can represent $[\varphi]$ by a harmonic differential form (this is a theorem, not an assumption). Suppose that $E(\varphi,\lambda)$, viewed as a function of $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, is holomorphic at λ_0.*

Then $E(\varphi,\lambda_0) \in \Omega^*(X/\Gamma, E)$ is a closed, harmonic form and $[E(\varphi,\lambda)] \neq 0$.

A cohomology that arises from this theorem is called a *regular* Eisenstein class. (What if it's not holomorphic? Then we have a residue. This happens if E is the trivial representation, for instance.)

Theorem 4.2. *Let G be a connected semisimple $\mathbf{Q}$-group, which is $\mathbf{Q}$-split. Let P_0 be a minimal $\mathbf{Q}$-parabolic subgroup of G.*

Suppose that (η, E) is a representation of G with regular highest weight. Then as R runs over the Γ-conjugacy classes of minimal $\mathbf{Q}$-parabolic subgroups of G, the classes $[E(\varphi,\lambda_0)] \neq 0$ attached to nontrivial classes $[\varphi] \in H^*(e'(R), E)$ of type (π, w_G), where w_G is the longest element in the Weyl group of G, generate a subspace $\mathrm{Eis}_{\{P_0\}} \subset H^*(\overline{X}/\Gamma, E)$ such that

$$\mathrm{Eis}_{\{P_0\}} \xrightarrow{\sim} \mathrm{Im}\,\gamma^*_{\{P_0\}/\Gamma}.$$

Remark 4.3. Note that $\ell(w_G) = \dim N_P$, which is the dimension of $e'(P_0)$. So the theorem is constructing classes in the top dimension, which is the same the virtual cohomological dimension of Γ, which is $\dim X - \mathrm{rank}_{\mathbf{Q}} G$.

The point is to show that there are no residues. Due to the fact that the highest weight of E is regular, the point of evaluation is moved away from the boundary of the region of absolute convergence of the Eisenstein series.

We will explain some strategies to show that $E(\varphi, \lambda)$ is a non-trivial cohomology class. We say that P_1 and P_2 are "associated", $P_1 \sim P_2$, if L_{P_1} and L_{P_2} are G-conjugate. Let $W(A_1, A_2)$ be the set of isomorphisms induced by G-conjugation. This is a finite set. For example,

$$W(A_1) := W(A_1, A_1) = N_G(A_1)/Z_G(A_1)$$

is the usual Weyl group. The set $W(A_1, A_2)$ is a torsor for $W(A_1)$: for $s \in W(A_1, A_2)$ you have $W(A_1, A_2) = sW(A_1)$.

Definition 4.4. Suppose $\omega = E(\varphi, \lambda) \in \Omega^*(X/\Gamma, E)$. Write

$$\omega = \sum f_J \omega^J, \qquad \text{where } f_J \in C^\infty(G/(\Gamma \cap P)A_P N_P) \otimes E.$$

Then we define the *constant term along* Q:

$$E(\varphi, \lambda)_Q = \sum_J E(f_J, \lambda)_Q \omega^J.$$

Recall that the *parabolic rank* $\mathrm{prk}(P)$ of P is the dimension of A_P. We discuss the behavior of the constant term, which depends qualitatively on the parabolic rank:

(1) If $\mathrm{prk}(Q) > \mathrm{prk}(P)$, then $E(\varphi, \lambda)_Q = 0$.
(2) If $\mathrm{prk}(Q) = \mathrm{prk}(P)$ and $P \not\sim Q$, then $E(\varphi, \lambda)_Q = 0$.
(3) If $\mathrm{prk}(Q) < \mathrm{prk}(P)$, then $E(\varphi, \lambda)_Q$ is negligible with respect to Q, meaning orthogonal to cusp forms. However, it can still be a residue of an Eisenstein series from a subparabolic of M_Q.

The most interesting case is $\mathrm{prk}(P) = \mathrm{prk}(Q)$ and $Q \sim P$. Changing notation, suppose we have two associated parabolics $P_1 \sim P_2$. For an Eisenstein series $E(f, \lambda)$ coming from P_1, we have

$$E(f, \lambda)_{P_2}(x) = \sum_{s \in W(A_1, A_2)} \underbrace{c(s, f)_{s\lambda}(x)}_{\in C^\infty(G/(\Gamma \cap P_2)N_2)}$$

where $c(s, f)_\lambda$ are certain intertwining operators.

In terms of representation theory, the intertwiner is an operator

$$\mathrm{Ind}_{P_1}^G(\pi_1 \otimes \lambda) \to \mathrm{Ind}_{P_2}^G({}^s\pi \otimes s\lambda).$$

The analytic behavior of $E(f, \lambda)$ is closely related to the analytic behavior of $c(s, \lambda)f$, for all $s \in W(A_1, A_2)$, because of the formula

$$E(\varphi, \lambda)_{P_2} = \sum_{s \in W(A_1, A_2)} (c(s, \lambda)(\varphi))_{s\lambda}$$

where $c(s, \lambda)$ can be viewed as operators between $(\mathfrak{g}, K)$-modules (for M) of induced representations.

How do you show that Eisenstein classes are nontrivial? You consider $E(\varphi, \lambda)_{P_1}$. There is a term corresponding to $s = \mathrm{Id}$, and it is nontrivial. There are other summands, and the question is if there is cancellation. Looking at the weights with respect to the split component, one can often see that there is no cancellation. If there are two terms with the same weight, then one can rule out cancellation by looking at the infinitesimal character.

There was a question about whether $E(\varphi, \lambda)$ is holomorphic at $-w(\Lambda+\rho)|_{\mathfrak{a}}$. Suppose $E = \mathbf{C}$, and P is a maximal parabolic. Then $\dim A_P = 1$, so $\mathfrak{a}^*_{\mathbf{C}} \xrightarrow{\sim} \mathbf{C}^*$ and $\lambda_s \leftrightarrow s$. In the region of absolute convergence for $E(\varphi, \lambda_s)$, this is a point on the boundary, for the longest Weyl element w_P. As we know from the adelic context, the holomorphicity of $c(s, \lambda)$ is governed by normalizing factors. The evaluation points of $E(\varphi, \lambda)$ you are interested in are always integral or half-integral.

Example 4.5. Let P be the maximal parabolic, with $|w(A)| = 2$ corresponding to

$$\mathrm{GL}_v \times \mathrm{Sp}_{v'} \subset \mathrm{Sp}_{v+v'}\,.$$

The normalizing factor is

$$\frac{L(s, \tau \times \sigma) L(2s, \tau, \wedge^2)}{L(s+1, \epsilon \times \sigma) L(2s+1, \wedge^2)}$$

Remark 4.6. One can also approach $H^*(X/\Gamma, \mathbf{C})$ more geometrically. If $H \subset G$, then you can arrange after passing to a finite index subgroup of Γ that $X_H/\Gamma_H \hookrightarrow X_G/\Gamma$ is a totally geodesic submanifold. You can show that $[X_H/\Gamma_H] \neq 0$. You can try to relate this to cohomology coming automorphic representations. Sometimes you are lucky, and by looking at certain degrees you can relate this to the automorphic stuff; see [Sch].

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