TRANSFER OPERATORS BETWEEN RELATIVE TRACE FORMULAS IN RANK ONE, I

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Contents

1. Transfer operators

The goal of the talk is to study transfer operators. Later we'll recall what they are, but let's first mention the motivation.

1.1. **Motivation.** There are two main motivations.

(1) Relate periods of automorphic L-functions to special values of L-functions.
(2) To realize Langlands' "Beyond Endoscopy" proposal for realizing functoriality in terms of a comparison of stable trace formulas

We will focus on the first goal today, since it's easier to explain.

1.2. **Notation.** Let k be a global field, F a local field, $\mathbb{A}$ the ring of adeles over k, and $[G] = G(k)\backslash G(\mathbb{A})$. We will usually abbreviate $X = X(F)$.

2. Periods of L-functions

2.1. **Hecke periods.** The simplest example of a period of an automorphic form is the following. Let φ be a cusp form, and consider the integral

$$\int_0^\infty \varphi(y) y^s d^* y.$$

Adelically, this can be interpreted as follows. Consider $H := \mathbf{G}_m \subset \mathrm{PGL}_2$ and view φ as an element of an automorphic representation $\pi \subset C^\infty([G])$. We consider the period integral

$$\mathcal{P}_H(\varphi) = \int_{[\pi]} \varphi(h)\, dh.$$

In this special case, Hecke proved that if φ is a normalized newform then

$$\mathcal{P}_H(\varphi) = L(\pi, 1/2).$$

Here "normalized" means that the first Fourier coefficient is 1. Really this coefficient is another period, namely a Whittaker period: if

$$N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{\psi} \mathbf{C}^*$$

then the first Fourier coefficient is

$$\int_{[N]} \varphi(n)\psi(n)\,dn.$$

So we can reformulate Hecke's computation by saying that the quantity $L(\pi, 1/2)$ comes from comparing two different periods:

$$\int_{[N]} \varphi(n)\psi(n)\,dn \quad \text{and} \quad \int_{[\mathbf{G}_m]} \varphi.$$

2.2. **Gross-Prasad periods.** More generally, let $G = \mathrm{SO}_n \times \mathrm{SO}_{n+1}$ and $H = \mathrm{SO}_n^{\mathrm{diag}} \hookrightarrow G$. For $\pi = \pi_1 \otimes \pi_2 = \otimes_v' \pi_v$, and the corresponding factorization $\varphi = \otimes_v' \varphi_v$, the Ichino-Ikeda conjecture predicts that, fixing a normalized $H \times H$-invariant form

$$\pi \otimes \overline{\pi} \xrightarrow{H \times H} \mathbf{C}$$

we have

$$\left| \int_{[H]} \varphi \right|^2 = \underbrace{2^{-\beta}}_{\text{explicit}} \prod_v \int_{H(k_v)} \langle \pi_v(h)\varphi_v, \varphi_v \rangle \, dh.$$

For a.e. v, φ_v is unramified and our normalization is such that $\langle \varphi_v, \varphi_v \rangle = 1$ and the local factor is

$$\int_{H(k_v)} \langle \pi_v(h)\varphi_v, \varphi_v \rangle \, dh = \frac{L(\pi_{1,v} \times \pi_{2,v}, 1/2)}{L(\pi_v, \mathrm{Ad}, 1)}.$$

In the special case $n = 2$, $H = \mathrm{SO}_2 = \mathbf{G}_m$ and this follows from Hecke's formula (if we allow a character), except for issues of normalization. In Hecke's setup we normalized by the first Fourier coefficient, and here we are normalizing by local pairings. The compatibility between the two normalizations is expressed by a formula for the Petersson inner product, which is known for GL_n but conjectural in general:

$$\left| \int_{[N]} \varphi(n)\psi(n)\,dn \right|^2 = \prod_v \int_{N(k_v)} \langle \pi(n)\varphi_v, \varphi_v \rangle \psi(n)\,dn$$

where for a.e. v, the local factor is $\frac{1}{L(\pi_v, \mathrm{Ad}, 1)}$. This should be difficult in general. For $\widetilde{\mathrm{Sp}}_{2n}$ it is known by very difficult results of Lapid-Mao.

For $n = 2$, if we replace $\mathrm{SO}(2) = T$ by the non-split torus, then the formula is essentially equivalent to Waldspurger's formula.

2.3. **Spherical varieties.** More generally, given a (homogeneous) *spherical variety* $X = H \backslash G$ with some multiplicity one property, one expects "the same" factorization of

$$\left| \int_{[H]} \right|^2$$

as an Euler product with local factors almost everywhere of the form

$$\frac{L_X}{L(\pi, \mathrm{Ad}, 1)}$$

where L_X is an L-value determined by a recipe that will be discussed later, which involves a local Plancherel formula.

What is stunning is that there is a uniform conjectural relationship between periods and L-functions, but no general proof for these sorts of results. The general method of proof at this point goes under the name "good luck", which means that in some cases we are lucky enough to prove the relationships in a direct way. For example,

(1) The simplest case is the method of "unfolding" (which works for $\mathbf{G}_m \subset \mathrm{PGL}_2$). This originates with Hecke, who directly compared the q-expansion with the periods.
$$\int_{[\mathbf{G}_m]} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} d^* a = \int_{\mathbf{G}_m(\mathbb{A})} \int_{[N]} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \psi(x)\, dx d^* a$$
The unfolding method directly compares the two integrals.
$$\begin{tikzcd} & C^\infty(\mathbf{G}_m \backslash G(\mathbb{A})) \\ \pi \arrow[ur, "\int_{[\mathbf{G}_m]}"] \arrow[dr, "\int_{[N]} \psi"'] & \\ & C^\infty(N, \psi \backslash G(\mathbb{A})) \arrow[uu, "\int_{\mathbb{A}^*}"'] \end{tikzcd}$$

(2) Unfolding doesn't work for the nonsplit torus. An alternative strategy is to use the Theta correspondence / Howe duality, which relates (G, H)-periods to (G', H')-periods. You want to apply this to (G', H') that you understand better. One way to (G', H') is to use a dual pair.

(3) The third method of proof is via the *relative trace formula*, developed by Jacquet and collaborators. For example, $H = U_n^{\mathrm{diag}} \subset G = U_n \times U_{n+1}$. We will review the relative trace formula shortly, but let me just say for now that it gives an expression of
$$\sum_{\pi \in \widehat{G}^{\mathrm{Aut}}} \left| \int_{[H]} \right|^2$$
in terms of orbital integrals for $H \backslash G / H$.

You can naturally identify "most" of the orbits in $H \backslash G / H$ with orbits for a different quotient $H' \backslash G' / H''$, where $G' = \mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E}$ for some quadratic E/k, and $H' = \mathrm{GL}_n(E)^{\mathrm{diag}}$ and $H'' = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$. The benefit of this is that we understand these periods better. For example, the period for $H' \backslash G'$ is the Rankin-Selberg period. There has been a lot of progress using this strategy by Jacquet-Rallis, W. Zhang, and Zydor-Chardouard. This is what we call the "traditional/endoscopic-type comparison of RTFs". We will explain a "non-traditional" use of RTF.

3. The relative trace formula

The RTF is a global/automorphic analogue of the Plancherel formula for a space of the form $X = H \backslash G$. What is the Plancherel formula? Recall $X = X(F)$ for a local field F. The Plancherel formula studies the decomposition of an inner product spectrally: for $\Phi_1 \otimes \Phi_2 \in \mathcal{S}(X \times X)$, where $\mathcal{S}$ is a space of test functions (soon to be taken to be Schwartz measures instead), consider the functional
$$\Phi_1 \otimes \Phi_2 \in \mathcal{S}(X \times X) \xrightarrow{G^{\mathrm{diag}}} \int_X \Phi_1 \Phi_2 dx \in \mathbf{C}.$$
The Plancherel formula is the spectral decomposition of this inner product:
$$\langle \Phi_1, \Phi_2 \rangle = \int_{\widehat{G}} \mathcal{J}_\pi^{\mathrm{Pl}} (\Phi_1 \otimes \Phi_2)\, \mu_X(\pi).$$

Here $\mathcal{J}_\pi^{\mathrm{Pl}}$ is a local "relative character", i.e. a bilinear form that factors through the representation π:

$$S(X \times X) \to \pi \otimes \widetilde{\pi} \xrightarrow{\langle,\rangle} \mathbf{C}.$$

The RTF is a global analogue. We take $\Phi_1 \in \mathcal{S}(X_1(\mathbb{A}))$ and $\Phi_2 \in \mathcal{S}(X_2(\mathbb{A}))$. We will restrict out attention to $X_1 = X_2 = X$ which is quasi-affine, of the form $G\backslash H$. Then we "make them automorphic" by forming Θ series:

$$\sum \Phi_1(g) \coloneqq \sum_{\gamma \in X(k)} \Phi_1(\gamma g) \in C^\infty([G]).$$

Similarly for Φ_2. Now we can take the inner product

$$\int_{[G]} (\sum \Phi_1(g))(\sum \Phi_1(g)) \in \mathbf{C}.$$

This is the distribution

$$\mathrm{RTF}_{X \times X/G}(\Phi_1 \otimes \Phi_2).$$

This notation is because this is a diagonally G-invariant distribution on $X \times X(\mathbb{A})$. The RTF is an identity between two ways to write this distribution.

3.1. **Spectral side.** The RTF has a spectral side, which is an integral over the space of automorphic representations:

$$\int_{\widehat{G}^{\mathrm{Aut}}} \mu(\pi)$$

of the period integrals

$$\left| \int_{[H]} \right|^2 \text{ on } \pi.$$

I don't want to say much more about the spectral side, since it's not a focus of this talk.

Example 3.1. When $X = H$ and $G = H \times H$, this is the trace formula. In this case $\sum \Phi_i$ is the kernel function K_{Φ_i}, so

$$\langle K_{\Phi_1}, K_{\Phi_2} \rangle_{[H \times H]} = \langle \Phi_1 *, \Phi_2 * \rangle_{\mathrm{HS}(L^2([H]))} = \mathrm{Tr}(\Phi_1^\vee * \Phi_2 *)_{L^2([H])}.$$

3.2. **Geometric side.** We discuss the geometric side of the stable RTF. The "stable" refers to orbits that coincide with their stable orbits; it could be everything.

We recall some of our standard notation.

- $\mathcal{S}(X(F))$ denotes Schwartz measures. This is a Schwartz function times a smooth measure of polynomial growth. Here Schwartz function means C_c^∞ if F is non-archimedean, and rapidly decreasing smooth functions if F is archimedean.
- $X//G = \mathrm{Spec}\ k[X]^G$. If X is affine and G is reductive, then $X//G(\overline{k})$ is in bijection with closed $G(\overline{k})$-orbits on $X(\overline{k})$.
- Given a map $p \colon X \to X//G$, we have a pushforward measure

  $$p_! \colon \mathcal{S}(X) \to \mathrm{Meas}(X//G)$$

  with $p_! \mathcal{S}(X) = \mathcal{S}(X/G)$. Think of these as "stable orbital integrals".
- The notation G/G means G modulo the conjugation action.

Example 3.2. For $G = \mathrm{SL}_2$, consider G/G. Then $G//G = \mathbb{A}^1$, and the map $G \to \mathbb{A}^1$ is the trace map. This is smooth away from ± 2. So elements of $\mathcal{S}(G/G)$ are smooth away from ± 2, and in neighborhoods of ± 2 look like

$$c_1 + c_2 \sqrt{|D|}$$

where $D = \mathrm{tr}^2 - 4$, c_1 is a smooth measure (meaning a smooth function times $d(\mathrm{tr})$) and c_2 is complicated: when restricted to any $[D] \in F^\times/(F^\times)^2$ is a smooth measure. So we could say that $c_2(t)$ is some function times dt, where the function depends on whether D is a square, $F(\sqrt{D})$ is unramified, or $F(\sqrt{D})$ is ramified, and is smooth within each case.

The next example is simpler.

Example 3.3. Consider $\mathbf{G}_m\backslash \mathrm{PGL}_2/\mathbf{G}_m$. (Remark: if $X = H\backslash G$ then $X \times X/G^{\mathrm{diag}} = H\backslash G/H$. The RTF is a distribution on $X \times X/G^{\mathrm{diag}} = H\backslash G/H$.)

In this case one again has $(\mathbf{G}_m\backslash \mathrm{PGL}_2)//\mathbf{G}_m \xrightarrow{\sim} \mathbb{A}^1$ via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \xi = \frac{ad}{\det}.$$

Then $\mathcal{S}(\mathbf{G}_m\backslash \mathrm{PGL}_2/\mathbf{G}_m)$ consists of measures on $\mathbb{A}^1$ which are smooth away from $0, 1$ and are $c_1 + c_2 \log|\xi|$ at 0 and $c_1 + c_2 \log|\xi - 1|$ at 1, where c_1 and c_2 are smooth measures.

Example 3.4. Consider

$$(N, \psi)\backslash(\mathrm{SL}_2 \text{ or } \mathrm{PGL}_2)/(N, \psi).$$

How can we think of these as measures? First consider the case of SL_2. Then $(N\backslash \mathrm{SL}_2)//N = \mathbb{A}^1$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c.$$

We are taking a twisted integral, so we need to choose a basepoint on each orbit. So we need to choose a section of the quotient map over $\mathbb{A}^1 - \{0\}$. We take the section

$$\begin{pmatrix} & -\zeta^{-1} \\ \zeta & \end{pmatrix} \leftarrow \zeta$$

We have

$$\mathcal{S}(N, \psi\backslash G^*/N, \psi) \subset \mathrm{Meas}(\mathbb{A}^1).$$

The things look like smooth measures away from 0, but near 0 they look like "Kloosterman integrals". (For non-archimedean places they really involve Kloosterman sums.)

3.3. **The "traditional/endoscopic comparison of RTFs".** Consider $\mathcal{X} = X_1 \times X_2/G$ and $\mathcal{Y} = Y_1 \times Y_2/G'$. For example, we could take $\mathcal{X} = U_n\backslash U_n \times U_{n+1}/U_n$ and $\mathcal{Y} = \mathrm{GL}_{n,E}\backslash \mathrm{GL}_{n,E} \times \mathrm{GL}_{n+1,E} // \mathrm{GL}_n \times \mathrm{GL}_{n+1}$.

In the traditional comparison we can identify the coarse quotient spaces, in this case canonically. Therefore the spaces $\mathcal{S}(\mathcal{X}))$ and $\mathcal{S}(\mathcal{Y}))$ are viewed of spaces of measures on the same space. The endoscopic paradigm is to show that these are the same spaces of measures, up to scalar transfer factors. This is what we call "matching", but we need more than just this "matching" statement. We need a fundamental lemma, which says that

$$p_!(h * \mathbb{I}_{X\times X(\mathcal{O})}) \sim p'_!(h * \mathbb{I}_{Y\times Y(\mathcal{O})})$$

Example 3.5. For $\mathrm{SL}_2/\mathrm{SL}_2$, we need to take "$\kappa$-orbital integrals" instead. These live on $\mathbb{A}^1$, and we can compare this $\mathcal{S}(T)$, using the identification $\mathbb{A}^1 \xleftarrow{\sim} T//W$.

The problem with this approach is that we need to find two quotient spaces that match. It is clear that this sort of matching cannot hold for the examples in Example 3.2, 3.3, 3.4, since the singularities of the function spaces are different. In Example 3.3 there were logarithmic singularities at 0 and 1, while in Example 3.4 there were Kloosterman integrals.

Our main theorem is the following, although for the sake of exposition we state it slightly inaccurately at first.

Theorem 3.6. *There are (local) explicit "transfer operators"*

(1) $\mathcal{T}\colon\mathcal{S}(N,\psi\backslash\mathrm{PGL}_2/N,\psi)\xrightarrow{\sim}\mathcal{S}(\mathbf{G}_m\backslash\mathrm{PGL}_2/\mathbf{G}_m)$ *and*
(2) $\mathcal{T}\colon\mathcal{S}(N,\psi\backslash\mathrm{SL}_2/N,\psi)\xrightarrow{\sim}\mathcal{S}(\mathrm{SL}_2/\mathrm{SL}_2)$ *and*

which

- *are linear isomorphisms,*
- *satisfy the fundamental lemma for the Hecke algebra.*

By "explicit" we mean that we can give formulas. In case (2), let t be the trace coordinate on RHS, and ζ,ζ^{-1} the coordinates on the LHS. Then the formula for (2) is

$$\mathcal{T}f=\int f(\zeta a^{-1})\underbrace{\psi(a)d^*a}_{D_1}=D_1*f$$

where $D_s=\psi(a)|a|^s\,da$ on $\mathbf{G}_m$. The formula for (1) is

$$\mathcal{T}f=D_{1/2}*D_{1/2}*f.$$

4. Nontraditional comparison of RTFs

Let X be a homogeneous, affine, rank one spherical variety. So X is of the form $H\backslash G$ with H reductive. We assume that G,H are split. The "rank one" means that the spectrum "looks like" the spectrum of SL_2 or PGL_2.

To explain what this means, we need to recall that to X we can attach an L-group ${}^LX$, which governs its local and global spectrum. The L-group comes with a map ${}^LX\to{}^LG$. Locally, we have a decomposition

$$L^2(X)=\int_\varphi\mathcal{H}_\varphi\,d\varphi$$

where φ runs over tempered Langlands parameter into ${}^LX$.

Example 4.1. We have

$$L^2(N,\psi\backslash\mathrm{PGL}_2)=\int_{\widehat{G}}\pi\cdot\mu(\pi)$$

where $\mu(\pi)$ is the same Plancherel measure as for $L^2(G=\mathrm{PGL}_2)$.

Example 4.2. We have $L^2(\mathbf{G}_m\backslash\mathrm{PGL}_2)\cong L^2(N,\psi\backslash\mathrm{PGL}_2)$.

Similarly, globally one would expect only the "global Langlands parameters factoring through ${}^LX$" to appear, under

$${}^LX\to{}^LG$$

(although strictly speaking this is not quite right, because one has to account for an Arthur SL_2 as well). The relative trace formula should be the distribution

$$\mathrm{RTF}_{X\times X/G}=\int_{\mathcal{L}_k\to{}^LX}\left(\left|\int_{[H]}\right|^2-\text{periods}\right)$$

Example 4.3. As an example of the above principle (spectral decomposition of RTF in terms of periods), for $\mathbf{G}_m\backslash\mathrm{PGL}_2/\mathbf{G}_m$, $f\in\mathcal{S}(\mathrm{PGL}_2(\mathbb{A}))$ you have

$$\begin{aligned}\mathrm{RTF}(f)&=\sum_{\pi\in\widehat{G}^{\mathrm{Aut}}}\sum_{\varphi\in\mathrm{ON}(\pi)}\int_{[\mathbf{G}_m]}(f*\varphi)(h)\,dh\cdot\left(\int_{[\mathbf{G}_m]}\varphi\right)\\&=\sum_{\pi\in\widehat{G}^{\mathrm{Aut}}}\langle\int_{[\mathbf{G}_m]},f*\int_{[\mathbf{G}_m]}\rangle_\pi.\end{aligned}$$

For the examples in the main theorem, the spaces are different but the spectrum is the same. In particular the L-groups ${}^L X$ coincide.

What are the periods? They look like

$$\frac{L_X}{L(\mathrm{Ad}_{{}^L X}, 1)}.$$

The adjoint representation is for the group ${}^L X$. On the Whittaker (Kuznetsov) side, you just get $\frac{1}{L(\mathrm{Ad},1)}$, i.e. $L_X = 1$. Otherwise you get $\frac{L(\mathrm{Ad},1)}{L(\mathrm{Ad},1)}$, i.e. $L_X = L(\mathrm{Ad})$, corresponding to the fact that you don't see L-functions in the classical formula.

So to compare the function spaces, we need to enlarge the test functions to something that we call "$\mathcal{S}^-_{L_X}$".

Philosophically, we view RTF as a functional

$$\mathcal{S}(X \times X/G(\mathbb{A})) \xrightarrow{\mathrm{RTF}} \mathbf{C}.$$

This factors through

$$\begin{tikzcd} \mathcal{S}(X \times X/G(\mathbb{A})) \arrow[rr, "\mathrm{RTF}"] \arrow[dr] & & \mathbf{C} \\ & \mathcal{S}(X \times X/G(\mathbb{A}))_{\mathrm{Aut}} \arrow[ur] & \end{tikzcd}$$

The Langlands program can be viewed as saying that RTF is a sheaf on the space of Langlands parameters in ${}^L X$. However, it is actually something a little more complicated, which one might call the sheaf on the space of Langlands parameters is also equipped with trivializations away from the zeros and poles of the L-functions.

The usual KTF can be thought of as follows. For $\Phi, \Phi' \in \mathcal{S}(N, \psi \backslash G(\mathbb{A}))$ we form Poincaré series

$$\sum \Phi, \sum \Phi'.$$

Then you consider

$$\langle \sum \Phi, \sum \Phi' \rangle_{[G]} = \sum_\pi \langle \sum \Phi, \sum \Phi' \rangle_\pi$$

there are coefficients corresponding to the test functions. You can make a good choice Φ_m, Φ_n where the sum becomes just

$$\sum_\pi a_m(\pi) \overline{a_n(\pi)}. \tag{4.1}$$

We don't want this, we want to "insert" an L-function. So we write the Dirichlet series for the L-function

$$L_X = \sum_{n \ge 1} c_n a_n(\pi)$$

Replace Φ by $\sum_{n\ge1} c_n \Phi_n$. Take $\Phi' = \Phi_1$ for simplicity. Then (**??**) becomes

$$\sum_\pi \underbrace{a_1(\pi)\overline{a_1(\pi)}}_{|\int_{[N]} \psi|^2 = 1/L(\mathrm{Ad},1)} \cdot L_X$$

The point is that the test function $\sum c_n \Phi_n$ is not of compact support, so its orbital integrals are no longer of rapid decay; they will have some asymptotic at ∞. Given L_X, which is

$$L_X = \begin{cases} L(\mathrm{Std}, s_1) L(\mathrm{Std}, s_2) & G^* = \mathrm{PGL}_2 \\ L(\mathrm{Ad}, s_0) & G^* = \mathrm{SL}_2 \end{cases}$$

I can define $\mathcal{S}^-_{L_X}(N,\psi\backslash G^*/N,\psi)$ by specifying a behavior at ∞. For $G^* = \mathrm{SL}_2$, it looks like $C \cdot |\zeta|^{1-s_0} d^*\zeta$.

Representation theoretically, we are adding the trivial representation.

$$0 \to \mathcal{S}(N,\psi\backslash G)_{(N,\psi)} \to \mathcal{S}^-_{L_X}(N,\psi\backslash G)_{(N,\psi)} \to I(\delta^{1/2})_{(N,\psi)} \to 0.$$

Theorem 4.4. *Let X be a rank one spherical variety, with ${}^LX = \begin{cases} \mathrm{SL}_2 \\ \mathrm{PGL}_2 \end{cases}$, and let $G^* = \begin{cases} \mathrm{PGL}_2 \\ \mathrm{SL}_2 \end{cases}$, so $L_X = \begin{cases} L(Std,s_1)L(Std,s_2) \\ L(\mathrm{Ad},s_0) \end{cases}$. Then we have the enlarged KTF space $\mathcal{S}^-_{L_X}(N,\psi\backslash G^*/N,\psi)$. There is an explicit transfer operator $\mathcal{T}$ which affords a matching*

$$\mathcal{T}\colon \mathcal{S}^-_{L_X}(N,\psi\backslash G^*/N,\psi) \xrightarrow{\sim} \mathcal{S}(X\times X/G).$$

The formula has the following shape. We have

$$X\times X \to (X\times X)//G \cong \mathbb{A}^1$$

which is smooth away from 2 points. Fix coordinates as in $\begin{cases} \text{Example } refex:2 \\ \text{Example } ex:1 \end{cases}$ for the two cases. Then

$$\mathcal{T}f = \begin{cases} |\cdot|^{\max(s_1,s_2)-1/2} D_{s_1} * D_{s_2} * f \\ |\cdot|^{s-1} D_s * f \end{cases}$$

Here $*$ is convolution on F^*, and have to be understood as Fourier transform of distributions.

The fundamental lemma is known for $\mathrm{SL}_2 / \mathrm{SL}_2$ and $\mathrm{GL}_n \backslash \mathrm{PGL}_{n+1} / \mathrm{GL}_n$. There is ongoing work of D. Johnstone and R. Krishna to generalize this.

In higher rank we would like to have a comparison

$$\mathcal{S}(X\times X/G) \to \mathcal{S}^-_{L_X}(N,\psi\backslash G^*/N,\psi).$$

where ${}^LG^* = {}^LX$.

We suspect that $\mathcal{T}$ should be a convolution dictated by the weights of L_X. The quotient $N\backslash G^*//N$ is a toric variety in the limit, and the shape of the convolution seems to be dictated by the weights of L_X, i.e. the coweights of $T_{G^*} \hookrightarrow N\backslash G^*//N$. However, we also know that we need to insert some correction factors that we don't yet understand.

Remark 4.5. Friedrich Knop has shown that although the X's are very different, their cotangent bundles T^*X are very similar. These have the structure of a multiplicity-free Hamiltonian manifold $T^*X \to \mathfrak{g}^*$. The proof uses this structure.

Another situation where symplectic structure arises is for a symplectic vector space W, and $G = G_1\times G_2 \hookrightarrow \mathrm{Sp}(W)$ a dual pair. Then you have a moment map $W \to \mathfrak{g}^*\times\mathfrak{g}_2^*$ exhibiting W as a multiplicity-free Hamiltonian manifold, and there is a similar Euler factorization of periods

$$\int_{[G]} \Theta_\Phi(g_1,g_2)\varphi_1(g_1)\varphi_2(g_2).$$

References

[SV] Sakellaridis, Yiannis and Venkatesh, Akshay. *Periods and harmonic analysis on spherical varieties.*