TRANSFER OPERATORS FOR RELATIVE FUNCTORIALITY, II

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Contents

1. Recap

We are going to briefly review what we discussed last time.

1.1. **Notation.** Let's begin by reminding you about some notation from last time.

- We let k be a global field, $\mathbb{A}$ its ring of adeles, F a local field.
- We denote by X a variety over F. We will abuse notation by writing $X = X(F)$. We assume that X is smooth over F, and we denote by $\mathcal{S}(X)$ the space of Schwartz measures on X.
- When G acts on X, we assume that X is quasi-affine (i.e. open in affine) and $X//G := \operatorname{Spec} k[X]^G$. We let $\mathcal{S}(X/G)$ be the image of the pushforward map from $\mathcal{S}(X)$ to measures on X/G.
- The notation G/G always means G modulo G-conjugacy.

1.2. **Relative functoriality.** We have two spherical varieties (X_1, G_1) and (X_2, G_2), and we wanted to study some sort of transfer between them. There are L-groups ${}^L X_1$ and ${}^L X_2$ attached to the spherical varieties.

There is a notion of X-*distinguished representations.* Locally this means that the representation appears in $L^2(X)$, while globally it has to do with the relative trace formula.

When we have a map ${}^L X_1 \to {}^L X_2$, the expectation is that there should be a map from packets of X_1-distinguished representations to packets of X_2-distinguished representations. We want to realize this as a local transfer operator

$$\mathcal{T} : \mathcal{S}(X_2 \times X_2/G_2) \to \mathcal{S}(X_1 \times X_1/G_1)$$

such that if J_π is a relative character for X_1, then $\mathcal{T}^* J_\pi$ is a character for X_2. Recall that a relative character is a generalization of the notion of character: it's a functional that factors as

$$J_\pi : \mathcal{S}(X \times X) \to \pi \widehat{\otimes} \widetilde{\pi} \to \mathbf{C}.$$

In the group case there's a canonical normalization for π. In general there's a question on how to normalize. We do it by the local Plancherel measure, which rests on the local relative Langlands correspondence, plus some global comparison whose nature is unclear in general.

2. The two main examples

Now we want to specialize to the case where ${}^L X_1 = {}^L X_2$. Let's recall the basic examples that we want to consider.

Example 2.1. $X_2 = (N, \psi)\backslash \mathrm{SL}_2$ and $X_1 = \mathrm{SL}_2$. (This was studied in Rudnick's thesis.)

Example 2.2. $X_2 = (N, \psi)\backslash \mathrm{PGL}_2$ and $X_1 = \mathbf{G}_m\backslash \mathrm{PGL}_2$.

In these cases we want not only the local transfer map, but we'd also like to be able to put them together into an Euler product

$$\mathcal{T} \colon \mathcal{S}(X_2 \times X_2/G_2(\mathbb{A})) \xrightarrow{\sim} \mathcal{S}(X_1 \times X_1/G_1(\mathbb{A}))$$

preserving the global functionals given by the RTF. Here

$$\mathrm{RTF}_{X\times X/G}(f) = \sum_{\xi \in X \times X//G(k)} f(\xi).$$

So what we are saying is that the following diagram should commute:

$$\begin{tikzcd} \mathcal{S}(X_2 \times X_2/G_2(\mathbb{A})) \arrow[rr, "\mathcal{T}"] \arrow[dr, "\mathrm{RTF}_{X_2\times X_2/G_2}"'] & & \mathcal{S}(X_1 \times X_1/G_1(\mathbb{A})) \arrow[dl, "\mathrm{RTF}_{X_1\times X_1/G_1}"] \\ & \mathbf{C} & \end{tikzcd}$$

As we discussed last time, these are mutually incompatible at present. Indeed, when you spectrally decompose you should get L-functions of X_1 and X_2, which don't match.

Example 2.3. In our normalization the Whittaker space has L-function 1 always. So for Example 2.1 we have $L_{X_2} = 1$, while $L_{X_1} = L(\mathrm{Ad}, 1)$.

Example 2.4. For Example 2.2 we have $L_{X_2} = 1$ and $L_{X_1} = L(\mathrm{Std}, 1/2)^2$.

We need to correct this by enlarging our space of test measures. For $X_2 = (N, \psi)\backslash G$, we want to enlarge the space of measures to $\mathcal{S}^-_{L_{X_1}}((N,\psi)\backslash G/(N,\psi))$. In fact we only get $\mathcal{T}$ to be an isomorphism after this enlargement. We then need a fundamental lemma telling us that

$$\sum_{\xi} f(\xi) = \sum_{\xi} \mathcal{T} f(\xi).$$

I want to remind you what this enlarged space of test functions means. Roughly,

$$\mathcal{S}^-_{L_{X_1}}((N,\psi)\backslash G/(N,\psi)) = h_{L_{X_1}} * \mathcal{S}((N,\psi)\backslash G/(N,\psi))$$

where $h_{L_{X_1}}$ is a series valued in the Hecke algebra, which is designed to produce the L-function.

Example 2.5. In Example 2.1, we have

$$X_2 \times X_2/G_2 = N\backslash \mathrm{SL}_2/N = \mathbb{A}^1 \supset \{\begin{pmatrix} & -\zeta^{-1} \\ \zeta & \end{pmatrix}\} = \mathbf{G}_m$$

The orbital integrals for KTF are measures on $\mathbb{A}^1$. Away from 0 we have something smooth ("Kloosterman germs"). If we were working with standard test functions, we would have rapid decay. But the nonstandard ones have a certain growth: the measure at infinity is $\sim c \cdot |\zeta|^{-s} \cdot d^* \zeta$, where s is such that the L-function obtained is $L(\mathrm{Ad}, 1+s)$.

Example 2.6. In Example 2.2,

$$X_2 \times X_2 / G_2 = N \backslash \mathrm{PGL}_2 / N = \mathbb{A}^1 \supset \begin{pmatrix} & 1 \\ \xi & \end{pmatrix}$$

Again, the measures on this quotient can be thought of as orbital integrals for KTF, and their growth at ∞ is $\sim |\xi|^{-s}(C_1 + C_2 \log |\xi|)$ if the relevant L-function is $L(\mathrm{Std}, 1/2+s)^2$.

This seems fishy globally. What matters is to have a good local transformation that satisfies Poisson summation. We'll explain this, but the point that we want to emphasize now is that we want good operators $\mathcal{T}$.

3. Examples of transfer operators

We're going to describe the local transfer operator in two special cases. Corresponding to Example 2.1, we want to define

$$\mathcal{T} : \mathcal{S}^-_{\mathcal{L}_{X_1}}(N, \psi \backslash \mathrm{SL}_2 / N, \psi) \to \mathcal{S}(\mathrm{SL}_2 / \mathrm{SL}_2).$$

Corresponding to Example 2.2, we want to define

$$\mathcal{T} : \mathcal{S}^-_{\mathcal{L}_{X_1}}(N, \psi \backslash \mathrm{PGL}_2 / N, \psi) \to \mathcal{S}(\mathbf{G}_m \backslash \mathrm{PGL}_2 / \mathbf{G}_m).$$

Here is the theorem that underlies Rudnick's thesis (although not stated or proved there).

Theorem 3.1. *The following (local) transformation*

$$\mathcal{T} : \mathcal{S}^-_{L_{X_1}}(N, \psi \backslash G / N, \psi) \xrightarrow{\sim} \mathcal{S}(X_1 \times X_1 / G_1)$$

is an isomorphism.

(1) *For Example 2.1:*

$$\mathcal{T} = \mathcal{F}_1 := \text{ convolution by } \psi(\zeta)|\zeta| d^\times \zeta \text{ on } F^\times \subset F.$$

(2) *For Example 2.2:* $\mathcal{T} = \mathcal{F}_{1/2} \circ \mathcal{F}_{1/2}$, *where*

$$\mathcal{F}_{1/2} := \text{ convolution by } \psi(\xi)|\xi|^{1/2} d^\times \xi \text{ on } F^\times \subset F.$$

Moreover $\mathcal{T}$ *satisfies a fundamental lemma for the Hecke algebra: for* $f_2 \in \mathcal{S}^-_{L_{X_1}}(N, \psi \backslash G/N, \psi)$ *the "Dirichlet series of the L-function", and* $f_1^0 \in \mathcal{S}(X_1 \times X_1 / G_1)$ *the standard test function (i.e. the pushforward of* $\mathbb{I}_{G(\mathcal{O})} dg$*),*

$$\mathcal{T}(h * f_2^0) = h * f_1^0$$

for every Hecke function h.

Remark 3.2. In the fundamental lemma, the meaning of $h * f_1^0$ is as the pushforward of $h * \mathbb{I}_{G(\mathcal{O})} dg$.

Remark 3.3. Where does this construction come from? Recall that the secret goal is that our transfer operator satisfies

$$\mathcal{T}^* J_\pi^{X_1} = J_\pi^{X_2}.$$

The L-value appears in the nonstandard test function that we evaluate on.

In the first case, the guess can be reverse engineered from the statement that you want on characters. Namely, you want $\mathcal{T}^* \Theta_\pi$ to be equal to a relative character $J_\pi^{X_2}$ for the

$X_2 \times X_2/G_2$. In this case you can write explicitly what J_π should be, by an Ichino-Ikeda formula. We should have

$$J_\pi^{X_2}(g) = \int_N \Theta_\pi(ng)\psi(n)\,dn.$$

This is dual to the Ichino-Ikeda period

$$\widetilde{\pi} \otimes \pi \ni \widetilde{v} \otimes v \mapsto \int_N \langle \pi(n)v, \widetilde{v}\rangle \psi(n) dn.$$

Ichino-Ikeda tells us that this is the period related to the square of the L-function.

Notice the similarity between the nature of the transforms and the nature of the L-function. There is more than just the fact that the square of the measure matches the square of the L-function. In the parametrization of Example 2.5, ζ is the coroot and in the parametrization of Example 2.6, ξ is half the coroot. Later we'll try to fit this into a conceptual pattern.

4. Global applications

We'll now show how you can use this to derive a global comparison. We'll discuss Example 2.2, which is the only case where I've actually done this, so $G = \mathrm{PGL}_2$.

4.1. **Analytic continuation of RTF.** Let $f \in \mathcal{S}^-(N, \psi \backslash G/N, \psi(\mathbb{A}))$. We want to write down the following equality:

$$\text{“}\sum_{\xi \in k} f(\xi) = \sum_{\xi \in k} (\mathcal{T} f)(\xi)\text{”} \tag{4.1}$$

In this case these are really evaluations because the stabilizers are trivial. The measures are orbital integrals for the map

$$\begin{tikzcd} \mathrm{PGL}_2 \arrow[d] & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \arrow[d] \\ \mathbb{A}^1 = \mathbf{G}_m \backslash \mathrm{PGL}_2 / \mathbf{G}_m & \frac{ad}{\det} \end{tikzcd}$$

which is not smooth at $0, 1$, so we need to be a little careful at singular points. But the bigger problem is that the sum on the left hand side of (4.1) doesn't converge, because we inserted the L-function.

Our operator $\mathcal{T}$ is morally a Fourier transform. So in principle it should satisfy Poisson summation, which should be the comparison of RTF. But there are problems at points which are not regular for the group action. Almost all the the points are regular, but globally "almost everwhere" has adelic measure 0. More precisely, if $Y \subset \mathbb{A}^1$ is a proper open subset, $Y(\mathbb{A}) \subset \mathbb{A}^1$ has measure 0.

How do we solve the problem of divergence? We use the s parameter to deform to a region where we have convergence, and then deduce a result at $s = 0$ by analytic continuation.

So the idea is to replace $\mathcal{S}^-_{L_X(1/2)}$ by $\mathcal{S}^-_{L_X(1/2+s)}$, take an analytic (in s) section $f_s \in \mathcal{S}^-_{L_X(1/2+s)}$, and apply a modified transform $\mathcal{T}_s$, getting something of rapid decay. Now for s in the region of convergence, we should be able to prove an identity

$$\sum_{\xi \in k} f_s(\xi) = \sum_{\xi \in k} \mathcal{T}_s f_s(\xi) \quad \text{for Re } s \gg 0.$$

The right hand side makes sense for all s, so you can meromorphically continue the left hand side. Then of course the identity is a tautology: at $s = 0$, we get

$$\mathrm{AnalyticContinuation}_{s=0}(\mathrm{KTF}(f_s)) = \mathrm{RTF}_{\mathbf{G}_m\backslash G/\mathbf{G}_m}(\mathcal{T}_f). \tag{4.2}$$

Think of these sides as being

$$\mathrm{AnalyticContinuation}_{s=0}\int_{\widehat{G}^{\mathrm{Aut}}} J_\pi(f_s)\,d\pi = \int_{\widehat{G}^{\mathrm{Aut}}} I_\pi(\mathcal{T}f)\,d\pi \tag{4.3}$$

The left side is a sum of global relative character for the Kuznetsov formula, and the right side is a sum of global relative characters for the torus.

4.2. **Spectral decomposition.** In particular, (2.1) is an identity of averages. We now want to extract each L-packet separately.

Let's recall how we do this in endoscopy. We fix $f = f_0$ unramified outside S, and then act on it by Hecke operators $\mathcal{H}^S = \otimes_{v\notin S}\mathcal{H}(G_v, K_v)$ supported away from S. By the Satake isomorphism we have

$$\mathcal{H}^S \cong \prod_{v\notin S}{}' \mathbf{C}[\widehat{T}//W].$$

If (4.3) were just a finite sum, then by using the Hecke algebra you could completely separate out the different π. But since you have an infinite sum, you want to view the Hecke algebra as functions on the space of Satake parameters. Then the map

$$h \mapsto \text{either side applied of (4.3) applied to } (h * f_0)$$

can be viewed as a measure on the space of Satake parameters. We know that this is supported on unitary representations. So we want to view KTF and RTF as a measure on the space of unitary parameters, which is a compact subset of $\prod_v \widehat{T}//W(\mathbf{C})$. If you knew that they really were measures (as in the case with endoscopy), then by denseness of the polynomials (Stone-Weierstrass) in the continuous functions on the space of parameters, the two sides represent equal measures.

We've used Hecke eigenvalues outside a finite number of places, so you get an identity L-packet by L-packet. (Of course for GL_2 there are no L-packets.)

In the case at hand, the problem is that from extending polynomials to continuous functions is a big problem, because KTF is defined by analytic continuation. To know that it really defines a measure, we need *a priori* estimates. The goal is to obtain estimates for the LHS as a functional on $\|\widehat{h}\|$. In order to do this without any hard analytic number theory, we're going to implement the functional equation at the level of the KTF. We'll know that it's a measure for large s, and we'll deduce by the functional equation that it is a measure for small s. Then we'll apply the Phragmen-Lindelöf principle to get the middle.

4.3. **Hankel operators.** What does it mean to implement the functional equation on KTF? It should be yet another transfer operator that takes $s \mapsto -s$. That is, we want a new operator H, which following B. C. Ngô we'll call a *Hankel operator*, which takes

$$H\colon \mathcal{S}^-_{L(1/2+s)}(N,\psi\backslash G/N,\pi) \to \mathcal{S}^-_{L(1/2-s)}(N,\psi\backslash G/N,\psi)$$

which corresponds to the functional equation of L-functions, and enjoys the following properties:

(1) Locally, we have a Fundamental Lemma for the Hecke algebra.

(2) Globally, we have

$$\mathrm{KTF}(f_s) = \mathrm{AnalyticContinuation}_{-s}(\mathrm{KTF}(\mathcal{H}_s f_s)) \text{ for } \mathrm{Re}\, s \gg 0.$$

This will be again be a kind of Poisson summation formula, for $\mathcal{H}$.

The Hankel operator is an incarnation of the γ factor. What does it do to relative characters? Secretly

$$\mathcal{H}_s^* J_\pi = \gamma(\ldots) J_\pi$$

i.e. $\mathcal{H}$ acts on the relative characters by gamma factors. (This is a difference with the transfer operators $\mathcal{T}$.)

4.4. **Hankel operators for the standard representation.** Hankel operators exist. I constructed one for $\mathcal{H}_{L(\mathrm{Std},1/2)^2}$. This turns out to obtained by applying $\mathcal{H}_{L(\mathrm{Std},1/2)}$ twice.

Let me tell you the Hankel transform for the automorphic L-function $L(\mathrm{Std},1/2)$ of GL_n. This is due to Jacquet, although I did not know about it at the time. Here it is more convenient to work with half-densities. So we seek

$$\mathcal{H}_{L(\mathrm{Std},1/2)} \colon \mathcal{D}^-_{L(\mathrm{Std},1/2)}(N,\psi\backslash G/N,\psi) \to \mathcal{D}^-_{L(\mathrm{Std},1/2)}(N,\psi\backslash G/N,\psi)$$

(As in Godement-Jacquet theory, the s parameter is in the power of the determinant. That explains the dualization.)

What are these spaces? They are pushforwards of half-densities on the space of $n \times n$ matrices, i.e. pushforwards from $\mathcal{D}^-(\mathrm{Mat}_n)$. (Since we are dealing with half-densities, you can think of the pushforward as halfway between pushforward of measures and orbital integrals.)

Upstairs the Hankel is just Fourier transform, and by the equivariance it descends.

$$\begin{tikzcd}
\mathcal{D}^-(\mathrm{Mat}_n) \arrow[r, "\text{Fourier transform}"] \arrow[d] & \mathcal{D}^-(\mathrm{Mat}_n) \arrow[d] \\
\mathcal{D}^-_{L(\mathrm{Std},1/2)}(N,\psi\backslash G/N,\psi) \arrow[r, "\mathcal{H}_{\mathrm{Std}}"] & \mathcal{D}^-_{L(\mathrm{Std},1/2)}(N,\psi\backslash G/N,\psi)
\end{tikzcd}$$

Theorem 4.1 (Jacquet)**.** *Choose representatives*

$$\left\{\begin{pmatrix} & & & a_n \\ & & \iddots & \\ & a_2 & & \\ a_1 & & & \end{pmatrix}\right\} \leftrightarrow N\backslash G/N$$

and identify the corresponding subset of the double coset space with the universal Cartan

$$\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \ddots & * \\ & & & a_n \end{pmatrix}/N$$

Then

$$\mathcal{H}_{L(\mathrm{Std},1/2)} = \mathcal{F}_{-\epsilon_1^\vee,1/2} \circ (\psi(e^{-\alpha_1})) \circ \mathcal{F}_{-\epsilon_2^\vee,1/2} \circ (\psi(e^{-\alpha_2})) \circ \ldots \circ \mathcal{F}_{-\epsilon_n^\vee,1/2}$$

where $\mathcal{F}_{-\epsilon_1^\vee,1/2}$ *is convolution with* $\epsilon^\vee(\psi(x)|x|^2 d^\times x)$*, and the* $\epsilon_i^\vee$ *are the weights of the standard representation. Concretely, the operator* $\psi(e^{-\alpha_1})$ *is multiplication by the function* $\psi(a_2/a_1)$*.*

In principal this still satisfies Poisson summation.

Example 4.2. Recall from Theorem 3.1 that in the situation of Example 2.1, the transfer operator

$$\mathcal{T}: \mathcal{S}^-(N,\psi\backslash \mathrm{SL}_2 / N,\psi) \xrightarrow{\sim} \mathcal{S}(\mathrm{SL}_2 / \mathrm{SL}_2).$$

was given by $\mathcal{T} = \mathcal{F}_{\alpha^\vee,1}$.

Example 4.3. Recall from Theorem 3.1 that in the situation of Example 2.2, the transfer operator

$$\mathcal{T}: \mathcal{S}^-(N,\psi\backslash \mathrm{PGL}_2 / N,\psi) \xrightarrow{\sim} \mathcal{S}(\mathbf{G}_m\backslash \mathrm{PGL}_2 / \mathbf{G}_m).$$

was given by $\mathcal{T} = \mathcal{F}_{\alpha^\vee/2,1/2} \circ \mathcal{F}_{\alpha^\vee/2,1/2}$. Is there a conceptual explanation for the obvious parallel with the Hankel operator?

Just to illustrate how interesting these formulas are: if we had the formulas for all spherical varieties, we would get as a consequence the Gan-Gross-Prasad conjectures.

5. Boundary degenerations and asymptotics

Next I want to tell you how to guess these formulas. First we need a little crash course on local harmonic analysis. I want to compare transfer operators / Hankel transforms for the X_i's and their "boundary degenerations".

5.1. **Boundary degenerations of spherical varieties.** What is a boundary degeneration? It is a simpler version of your variety. Namely, if (X,G) is a spherical variety, we can define a boundary degeneration $X_\varnothing$ which has a G-action and also an action of a torus T_X.

Example 5.1. If $X = \mathrm{SL}_2$, then $X_\varnothing$ is the space of 2×2 matrices of rank 1.

Example 5.2. If $X = \mathrm{SO}_2 \backslash \mathrm{SO}_3$, the hyperboloid corresponding to $x^2+y^2-z^2 = \alpha \neq 0$, then $X_\varnothing$ is the cone $x^2+y^2-z^2 = 0$ minus the origin $\{(0,0,0)\}$.

In general, X lives in a family $\mathcal{X} \to \mathbb{A}^1$. The generic fiber is X, and the special fiber over 0 is $X_\varnothing$. In the first example $\mathcal{X}$ is 2×2 matrices ("Vinberg monoid" in general), then we take the open G-orbit on the special fiber as the boundary degeneration $X_\varnothing$. In the second example there is the obvious degeneration α.

5.2. **Asymptotics.** There is a universal "asymptotics" morphism

$$e_\varnothing^*: \mathcal{S}(X) \to \mathrm{Meas}^\infty(X_\varnothing).$$

More generally, this can be extended to $C^\infty(X) \to C^\infty(X)$.

Example 5.3. For $X = \mathrm{SL}_2$, with the action of $\mathrm{SL}_2\times \mathrm{SL}_2$, we can think of

$$X_\varnothing = T^{\mathrm{diag}}\backslash (N\backslash \mathrm{SL}_2 \times N\backslash \mathrm{SL}_2).$$

An example of an element in $C^\infty(\mathrm{SL}_2)$ is a matrix coefficient, which comes from the map $\tau\otimes\widetilde{\tau} \to C^\infty(\mathrm{SL}_2)$. By Frobenius reciprocity this is equivalent to a T-invariant pairing $\tau_{N^-}\otimes \widetilde{\tau}_N \to \mathbf{C}$, where τ_{N^-} denotes the Jacquet module with respect to N^-. From this get a function on $X_\varnothing$, which is the result of applying the map $e_\varnothing^*$.

Fact 5.4. *There is a space $\mathcal{S}^+(X_\varnothing)^{W_X} \subset \mathrm{Meas}^\infty(X_\varnothing)$, such that the image of $\mathcal{S}(X)$ under $e_\varnothing^*$ is $\mathcal{S}^+(X_\varnothing)^{W_X}$.*

Let us say something about the mysterious superscipsts + and −? If $X_\varnothing = N\backslash G$, then a + superscript denotes the functions can extend to the cusp, while a − superscript denotes the function that can be extended to the funnel.

5.3. **Scattering operators.** How does the Weyl group W_X act on $\mathcal{S}^+(X_\varnothing)$? By something we call the *scattering operator*. For each $w \in W_X$, we have a scattering operator $\mathscr{S}_w \in \mathrm{Aut}(\mathcal{S}^+(X_\varnothing))$.

Example 5.5. This is familiar globally: if $X = G(k)\backslash G(\mathbb{A})$, then the asymptotics map is just the constant term. In this case, the scattering operators are just the standard intertwining operators. The image being invariant under the scattering operators reducing to the fact that integrating the constant term against a character coincides with integrating the original function against an Eisenstein series, so the content of this statement is the functional equation of Eisenstein series.

Remark 5.6. The scattering operators $\mathscr{S}_w$ depend on X, not just on $X_\varnothing$. So you should think of the boundary degenerations as coming equipped with the scattering operators. The analogy is to a wave going out to infinity.

To describe them, we describe them on (T_X, χ)-coinvariants. The coinvariants $\mathcal{S}^+(X_\varnothing)_{T_X,\chi}$ is a principal series representation. Thus

$$\mathscr{S}_w : I(\chi) \to I({}^w\chi).$$

for $\chi \in \widehat{T}_X$. This has to be a meromorphic multiple of the usual (unnormalized) M_w. Which multiple?

We don't know in general, but I'll describe the answer in our main examples.

Example 5.7. If $X = (N,\psi)\backslash \mathrm{SL}_2$ or $X = (N,\psi)\backslash \mathrm{PGL}_2$, we have $X_\varnothing = N\backslash G$ and

$$\mathscr{S}_w = \prod_{\alpha>0} \gamma(\chi \circ e^{\alpha^\vee}, 1, \psi)^{-1} M_w.$$

Example 5.8. If $X = \mathrm{SL}_2 = H$ then $X_\varnothing = T^{\mathrm{diag}}\backslash(N^-\backslash \mathrm{SL}_2 \times N\backslash \mathrm{SL}_2)$, which has an $H \times H$ action, and $I(\chi) = I^H(\chi) \otimes I^H(\chi^{-1})$. Then

$$\mathscr{S}_w = M_w \otimes M_{w^{-1}}^{-1} = c(\chi) \cdot M_w \otimes M_w$$

where $c(\chi)$ is the Plancherel measure, and can be given explicitly by

$$c(\chi) = \prod_{\alpha>0} \gamma(\chi \circ e^{-\alpha}, 1, \psi)\gamma(\chi \circ e^{\alpha}, 1, \psi).$$

Example 5.9. If $X = \mathbf{G}_m\backslash \mathrm{PGL}_2$, then we have

$$\mathscr{S}_w = \gamma(\chi \circ e^{-\alpha^\vee/2}, 1/2, \psi)^{-2} \cdot \gamma(\chi \circ e^{\alpha}, 1, \psi)^{-1} \cdot M_w.$$

We have a diagram of asymptotics maps

$$\begin{array}{ccc} \mathcal{S}(N,\psi\backslash G/N,\psi) & \xrightarrow{e_\varnothing^* \otimes e_\varnothing^*} & \mathcal{S}^+(N\backslash G/N) \\ \Big\downarrow \mathcal{T} & & \Big\downarrow \mathcal{T}_\varnothing \\ \mathcal{S}(\mathrm{SL}_2/\mathrm{SL}_2 \text{ or } \mathbf{G}_m\backslash \mathrm{PGL}_2/\mathbf{G}_m) & \xrightarrow{e_\varnothing^*} & \mathcal{S}^+(X_\varnothing \times X_\varnothing/G) \end{array}$$

Morally $\mathcal{T}$ is a degeneration/deformation of $\mathcal{T}_\varnothing$. For this to preserve Plancherel normalizations of relative characters, we need $\mathcal{T}_\varnothing$ to be the Fourier convolution with underlying γ-factors.

References

[SV] Sakellaridis, Yiannis and Venkatesh, Akshay. *Periods and harmonic analysis on spherical varieties.*