COHOMOLOGY OF ARITHMETIC GROUPS AND AUTOMORPHIC FORMS: AN INTRODUCTION, I

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Contents

We will treat only the case of cocompact arithmetic groups, where the theory was discovered by Matsushima. (A sequel lecture series by Joachim Schwermer will cover more general situations.)

1. The basic description

1.1. **The data.** Let G be a semisimple, noncompact real Lie group. Suppose for simplicity that G is connected.

- Let $\mathfrak{g}_0$ be the real Lie algebra of G, and $\mathfrak{g} := \mathfrak{g}_0 \otimes \mathbf{C}$.
- Choose a maximal compact subgroup $K \subset G$. (This choice is necessary to talk about automorphic forms.)
- Let $\Gamma \subset G$ be a discrete subgroup. *We assume that* $\Gamma \backslash G$ *is compact.* We also assume for simplicity that Γ is torsion-free, or equivalently that Γ acts without fixed points on X.

1.2. **Locally symmetric spaces.** Let $X = G/K$ (a symmetric space). We won't recall the theory of symmetric spaces, but we remind you that X is simply-connected, and even contractible. Then $\Gamma \backslash X$ is a compact manifold, and $\Gamma = \pi_1(\Gamma \backslash X)$. Moreover the contractibility of X implies:

Fact 1.1. *The quotient* $\Gamma \backslash X$ *is a* $K(\Gamma, 1)$*-space.*

This implies that we can compute the group cohomology of Γ in terms of cohomology on the space $\Gamma \backslash X$. More precisely, if M is any Γ-module we denote by $\mathcal{M}$ the corresponding local system on $\Gamma \backslash X$ (explicitly, $\mathcal{M} = X \times_\Gamma M$).

Corollary 1.2. *We have* $H^k_{\mathrm{group}}(\Gamma, M) = H^k(\Gamma \backslash X, \mathcal{M})$.

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(For a reference, see [Br94].)

Here we are interested in very specific M. We will take M to be a finite-dimensional representation of Γ over $\mathbb{R}$ or $\mathbf{C}$. (In fact, M will even be the restriction to Γ of a G-representation.)

Remark 1.3. There are other examples which are quite natural and interesting, e.g.

$$\underbrace{\mathrm{SL}_2(\mathbf{Z}[i])}_{\Gamma} \subset \underbrace{\mathrm{SL}(2,\mathbf{C})}_{G}.$$

In this case the quotient is noncompact. Even though we will focus on the cocompact theory, the most interesting and natural examples are not cocompact, and we will discuss them anyway.

For M, we can take Γ acting on $(\mathbf{Z}[i]/\mathfrak{p})^2$ for $\mathfrak{p}$ a prime ideal of $\mathbf{Z}[i]$, or Γ acting on $\mathbf{Z}[i]^2 \subset \mathbf{C}^2$.

In fact for our purposes we can restrict to $M = \mathbf{C}$, the trivial representation. The interesting features of the general theory are already present in this situation.

2. The C^∞-description

We are now going to discuss Lie algebra cohomology. I want to emphasize at the outset that this is easy; it's just a reformulation of the de Rham complex.

We want to compute $H^*(\Gamma\backslash X, \mathbf{C})$. We will use the de Rham complex. The game is to rephrase it in group-theoretic terms.

2.1. **Lie algebra cohomology of symmetric spaces.** Write $X = G/K$. We want to describe the space of differential forms on X.

Proposition 2.1. *We have a canonical isomorphism*

$$\Omega^i(X) \cong \mathrm{Hom}_K(\Lambda^i\mathfrak{p}, C^\infty(G)).$$

Here the action of K on $C^\infty(G)$ is by right translation.

We need some more notation. Recall that we have a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. Therefore we have at the level of complexifications $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k}$. Note that $\mathfrak{k}$ acts on $\mathfrak{p}$ via the adjoint representation.

Proof of Proposition 2.1. Assume we are given

$$c\colon \Lambda^i\mathfrak{p} \to C^\infty(G).$$

We need to construct from c a (smooth) differential form $\omega \in \Omega^i(X)$. If you think about it, there is only one way to do this. Let $x \in X$; we can write $x = g \cdot o$ for some $g \in G$, where $o \in X$ is the basepoint corresponding to the choice of K. We want to define

$$\omega(v_1, \dots, v_i) \in \mathbf{C}, \quad v_1, \dots, v_i \in T_x(X).$$

We can write $v_\alpha = g w_\alpha$ for $w_\alpha \in T_o(X) \cong \mathfrak{p}$. The only reasonable thing is to set

$$\omega(v_1, \dots, v_i) = c(w_1, \dots, w_i)(g).$$

One has to check certain properties:

- This is a smooth form.
- This is well-defined (independent of the choice of g). This ultimately comes from the K-equivariance of c.

The inverse map is given by restriction to $T_o(X) \cong \mathfrak{p}$, showing that this is an isomorphism. □

Given $\Gamma \subset G$, the same argument yields

$$\Omega^i(\Gamma\backslash X) \cong \mathrm{Hom}_K(\Lambda^i \mathfrak{p}, C^\infty(\Gamma\backslash G)).$$

To describe the de Rham complex, we still need to describe the differential

$$d\colon \Omega^i(\Gamma\backslash X) \to \Omega^{i+1}(\Gamma\backslash X).$$

We have

$$d\omega(v_0,\dots,v_i) = \sum_{\alpha=0}^{i} (-1)^\alpha v_\alpha \cdot \omega(v_0,\dots,\widehat{v}_\alpha,\dots v_i) \tag{2.1.1}$$

$$+ \sum_{\alpha<\beta} (-1)^{|\alpha|+|\beta|} \omega([v_\alpha, v_\beta], v_0,\dots,\widehat{v}_\alpha,\dots,\widehat{v}_\beta,\dots,v_i) \tag{2.1.2}$$

Here $v_\alpha \cdot \omega(v_0,\dots,\widehat{v}_\alpha,\dots v_i)$ is the differentiation of a function.

We will now make a specific choice that simplifies the differential. For the v_i we use vectors in $g \cdot T_o(X)$, i.e. a left-invariant vector field. With this choice we have

$$[v_\alpha, v_\beta] \in g \cdot \mathfrak{k}$$

by the following basic fact about the Cartan decomposition: $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{k}$. But ω kills $\mathfrak{k}$ because it is dual to $\mathfrak{g}/\mathfrak{k}$. Therefore, the second line in (2.1.1) disappears.

$$d\omega(v_0,\dots,v_i) = \sum_{\alpha=0}^{i} (-1)^\alpha v_\alpha \cdot \omega(v_0,\dots,\widehat{v}_\alpha,\dots v_i) \tag{2.1.3}$$

We have to understand the remaining term within the dictionary furnished by Proposition 2.1. This comes down to understanding the action of v_α, which is the derivative by a left-invariant vector field. Under the identification $\Omega^i(\Gamma\backslash X) = \mathrm{Hom}(\Lambda^i\mathfrak{p}, C^\infty(\Gamma\backslash G))$ we see that v_α acts on $h\colon \Lambda^i \to C^\infty$ through its action on $C^\infty(\Gamma\backslash G)$, which is

$$v_\alpha \cdot f = \frac{d}{dt}|_{t=0} f(g e^{t v_\alpha}).$$

2.2. $(\mathfrak{g},K)$**-cohomology.** We give a brief reminder on representation theory. Let H be an irreducible Hilbert space representation for G. Then we can take the space of K-finite vectors in H, call it V. Then V is an irreducible representation of $\mathfrak{g}$. Furthermore, the restriction of V to K has finite multiplicities for its irreducibles. Then V is a $(\mathfrak{g},K)$-*module*. This means that V is a complex $\mathfrak{g}$-module, with a locally finite (continuous) K-action that is compatible with the $\mathfrak{g}$-action in a way that is left as an exercise to write out.

Definition 2.2. Let V be a $(\mathfrak{g},K)$-module. Then we define $H^*(\mathfrak{g},K;V)$ to be the cohomology of the complex $C^*(\mathfrak{g},K;V)$ given by

- $C^i(\mathfrak{g},K;V) = \mathrm{Hom}_K(\Lambda^i\mathfrak{p}, V)$,
- with the differential

$$d\omega(v_0,\dots,v_i) = \sum_{\alpha=0}^{i} (-1)^{|\alpha|} v_\alpha \omega(v_0,\dots,\widehat{v}_\alpha,\dots,v_i).$$

We can summarize the preceding discussion with the following formula of Matsushima:

Theorem 2.3 (Matsushima)**.** *We have*

$$H^*(\Gamma\backslash X, \mathbf{C}) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G))$$

Remark 2.4. One should perhaps write $H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G)_K)$, the K-finite part. But the complex $\mathrm{Hom}_K(\Lambda^i\mathfrak{p}, V)$ automatically lands in the K-finite part of $C^\infty(\Gamma\backslash X)$, so it doesn't really make a difference.

3. Hodge theory and representation theory

3.1. **Input from Hodge theory.** Let $\Omega^i = \Omega^i(\Gamma\backslash X)$. By Proposition 2.1 we can view this as the space of global sections of a bundle on $\Gamma\backslash X$, which is obtained by descending the vector bundle $G \times_K \Lambda^i\mathfrak{p}^*$ from $X = G/K$.

We have a natural scalar product on the Lie algebra, namely on $\mathfrak{g}$ we have the Killing form, which is positive definite on $\mathfrak{p}$. We use it to get Hermitian structures on all the Ω^i. Then Hodge theory applies. In particular we get adjoint operators

$$\begin{aligned} d &\colon \Omega^i \to \Omega^{i+1} \\ d^* &\colon \Omega^{i+1} \to \Omega^i \end{aligned}$$

We can consider the corresponding Laplacian

$$\Delta_i := dd^* + d^*d \colon \Omega^i(\Gamma\backslash X) \to \Omega^i(\Gamma\backslash X).$$

Theorem 3.1. *By Hodge theory, we have:*

(1) The space of harmonic forms $\mathcal{H}^i(\Gamma\backslash X) \subset \Omega^i(\Gamma\backslash X)$ is finite-dimensional.
(2) There is a canonical isomorphism $H^i(\Gamma\backslash X) \cong \mathcal{H}^i(\Gamma\backslash X)$.
(3) Δ_i has positive eigenvalues λ_n, tending to ∞ and each with finite multiplicity.

3.2. **Input from representation theory.** Remember that we assume $\Gamma\backslash X$ is compact. If $\varphi \in C_c^\infty(G)$, then φ acts on $L^2(\Gamma\backslash G)$ by convolution on the right. It is easy to check that this operator is given by a smooth kernel on $(\Gamma\backslash G)^2$.

Thus $R(\varphi)$ is a compact operator for all φ, implying that

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus}_\pi H_\pi,$$

a *discrete* sum of irreducibles. Moreover, if π_0 is an irreducible unitary representation of G, then the mutiplicity $m(\pi_0)$ of π_0 in this decomposition is finite.

We make some obvious remarks about smooth vectors. Recall that if H is a Hilbert space representation of G, irreducible or not, then there is a notion of *smooth vectors* $H^\infty \subset H$.

Example 3.2. In particular we have $L^2(\Gamma\backslash G)^\infty = C^\infty(\Gamma\backslash G)$.

Then $C^\infty(\Gamma\backslash G) \subset \prod_\pi H_\pi^\infty$. Taking K-types, we get

$$\mathrm{Hom}_K(\Lambda^i\mathfrak{p}, C^\infty(\Gamma\backslash G)) \subset \prod_\pi \mathrm{Hom}_K(\Lambda^i\mathfrak{p}, V_\pi)$$

where V_π is the space of K-finite vectors in H_π^∞.

3.3. **The Casimir element.** Recall the decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Pick an orthonormal basis (z_β) for $\mathfrak{k}_0$ and (x_α) for $\mathfrak{p}_0$. Consider the element

$$C = \sum x_\alpha^2 - \sum z_\beta^2 \in U\mathfrak{g}.$$

This is called the *Casimir element.* This is an element of $\mathfrak{z} := Z(U\mathfrak{g})$. So by Schur's Lemma, it must act on any irreducible $U\mathfrak{g}$-module by scalars.

Lemma 3.3 (Kuga)**.** *Let (π, V) be a unitary $(\mathfrak{g}, K)$-module, so in particular we can define $(C^k(\mathfrak{g}, K; V), d, d^*)$. If $\omega \in C^k(\mathfrak{g}, K; V)$ then we have*

$$\Delta_k \omega = -C\omega.$$

Remark 3.4. To clarify, a *unitary* $(\mathfrak{g}, K)$-module is one such that for $X \in \mathfrak{g}_0$ we have

$$\langle Xu, v\rangle + \langle u, Xv\rangle = 0.$$

Proof. For $\omega \in C^k$, write

$$\omega = \sum_I \omega_I \beta^I$$

where $I \subset \{1, \dots, q\}$ where $q = \dim \mathfrak{p}$ with $|I| = k$. Then $\beta^I = x_{i_1}^* \wedge \dots \wedge x_{i_k}^*$ where x_i^* is the dual basis to x_i, and $\omega_I \in V$. We can extend the hermitian structure to $\Lambda^* \mathfrak{p}$ so that the β^I are an orthonormal basis.

We're just going to do the computations by brute force. Let $\omega \in C^k$ and $|I| = k+1$. Then, as we found earlier (2.1.3)

$$(d\omega)_I = \sum_{\alpha=1}^{k+1} (-1)^{\alpha-1} x_{i_\alpha} \omega_{I-\{i_\alpha\}}. \tag{3.3.1}$$

Next we need to compute $d^*\omega$. Let $\theta \in C^{k+1}$ and $J \subset \{1, \dots, q\}$ with $|J| = k$. We have

$$(d^*\theta)_J = \sum_{\substack{|I|=k+1 \\ I = J \sqcup \{j\}}} (-1)^{\iota_I(j)} x_j \theta_I = \sum_{j \notin J} (-1)^{\iota_I(j)} x_j \theta_{I := J \sqcup j} \tag{3.3.2}$$

Here $(-1)^{\iota I(j)}$ is the order rank of j, i.e. if you write $j = i_\alpha$ then it's α.

A trivial observation, which ends up being how things come out, is that if $J = \{j_1, \dots, j_k\}$ and $j \in J$ then

$$\iota_I(j) = \sum_{j_\alpha \le 1} 1.$$

Next we compute $(dd^*\omega)_J$. By (3.3.1) and (3.3.2) it is

$$(d^*d\omega)_J = -\sum_{j \notin J} x_j^2 \omega_J + \sum_{j \notin J} \sum_{i_\alpha \neq j} (-1)^{\iota_I(j)} (-1)^{\alpha-1} x_j x_{i_\alpha} \omega_{J+j-i_\alpha}$$

Here $|J| = k$ and $\alpha = 1, \dots, k$.

Next we compute $(d^*d\omega)_J$. Write $|J| = k$.

$$(d^*d\omega)_J = s \sum_\beta (-1)^{\beta-1} x_{j_\beta} \sum_{|J|=k} (-1)^{\iota_{J'}(j)} x_j \omega_{J'}$$

where $J' = J - j_\beta \sqcup j$. The summand $(-1)^{\beta-1} x_{j_\beta} \sum_{|J|=k} (-1)^{\iota_{J'}(j)} x_j \omega_{J'}$ is

$$-\sum_{j \in J} x_j^2 \omega_J, \quad j = j_\beta$$

and if $j \neq j_\beta$ is

$$\sum_{\beta=1}^{k} (-1)^{\beta-1} \sum_{J'=J-j_\beta \sqcup j} (-1)^{\iota_{J'}(j)} x_{j_\beta} x_j \omega_{J'}.$$

You can see that the two "main terms" are correct. You have to check that the "remainder terms" cancel. Basically the point is that they concern the same J', and the coefficients are opposite: the cancellation is

$$(-1)^{\beta-1}(-1)^{\iota_{J'}(j)} = -(-1)^{\iota_I(j)}(-1)^{\alpha-1} = 0.$$

This means that one gets terms like $X_{j_\beta}X_j - X_jX_{j_\beta}$, which lie in $\mathfrak{k}$ and hence vanish by the $\mathfrak{k}$-equivariance.

□

3.4. **Matsushima's formula.** Recall that we had written

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} H_\pi$$

with

$$C^k(\mathfrak{g}, K; V) \subset \prod \mathrm{Hom}_K(\Lambda^k \mathfrak{p}, V_\pi).$$

By Kuga's Lemma we have $\Delta = -C$ on $C^k(\mathfrak{g}, K; V)$. Since the Casimir $C \in \mathfrak{z}$, it acts by a scalar on each V_π, and moreover there are only a finite number of representations V_π where $C|_{V_\pi} =: C_\pi = 0$. This implies that the space of harmonic forms $\mathcal{H}^k$ is identified with $\bigoplus_{C_\pi=0} \mathrm{Hom}_K(\Lambda^k\mathfrak{p}, V_\pi)$, where C_π is the constant by which C acts on H_π. (The sum is finite by general properties of elliptic operators.)

Theorem 3.5 (Matsushima)**.** *We have*

$$\mathcal{H}^k(\Gamma\backslash X, \mathbf{C}) = \bigoplus_{C_\pi=0} \mathrm{Hom}_K(\Lambda^k\mathfrak{p}, V_\pi).$$

This is basically the abstract part of Matsushima' formula, without any representation theory.

4. Complements and variants

4.1. **Restrictions on Γ.** We demanded that Γ have no torsion. However, this doesn't matter since we're working over $\mathbf{C}$, so we can take Γ to be any discrete cocompact subgroup. The point here is a theorem of Selberg which says that there is a finite index subgroup $\Gamma' \triangleleft \Gamma$ such that Γ' is torsion-free (no fixed points), so over $\mathbf{C}$

$$H^k(\Gamma, V) = H^0(\Gamma/\Gamma', H^k(\Gamma', V)).$$

4.2. **Coefficient systems.** Let E be a representation of G. Then E induces a local system $\mathcal{E}$ on $\Gamma\backslash X$, and

$$H^k(\Gamma, E) = \bigoplus_\pi H^k(\mathfrak{g}, K; V_\pi \otimes E).$$

There is again a Hodge theory, where you transform the Kuga Laplacian by one from the local system. We need a K-invariant hermitian form on E.

4.3. **Action of $\mathfrak{z}$.** Let $\mathfrak{z} = Z(U\mathfrak{g})$. We go back to the trivial local system for simplicity. This cuts out the π with $C_\pi = 0$.

Assume V is an irreducible $(\mathfrak{g}, K)$-module (not necessarily unitary). Then it has an infinitesimal $\omega_V : \mathfrak{z} \to \mathbf{C}$ by Schur's Lemma.

Theorem 4.1 (Wigner)**.** *If $\omega_V \neq \omega_{\mathbf{C}}$ (the infinitesimal character of the trivial representation), then $H^*(\mathfrak{g}, K; V) = 0$.*

The point is that you can also define $\mathrm{Ext}_{\mathfrak{g},K}(V, W)$, and interpret $(\mathfrak{g}, K)$-cohomology using it. Then you use Yoneda's description of Ext in terms of extensions: in an exact sequence with incompatible central characters, it must split.

Corollary 4.2. *Let $\mathcal{A}(\Gamma\backslash G, K)$ be the space of automorphic forms, which in our cocompact case are just functions in $C^\infty(\Gamma\backslash G)$ which are K-finite and $\mathfrak{z}$-finite. Then*

$$H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G)) = H^*(\mathfrak{g}, K; \mathcal{A}(\Gamma\backslash G, K)).$$

4.4. **Restrictions on G.** In general the groups G which come up in algebraic geometry will be reductive, not merely semisimple. Any reductive G has a Langlands decomposition

$$G = {}^0GA_G$$

where A_G is a central split torus, connected for the real topology. Then we look at $\Gamma A_g\backslash G$, and make the same construction. Here ${}^0G$ is not necessarily connected, but there is a maximal compact subgroup K meeting all connected components, and we note that it is really crucial to consider $\mathrm{Hom}_K(\Lambda^i\mathfrak{p}, V)$ for the full group K, as we have been doing all along. Then everything is true, as the key point was ${}^0G/K$ being contractible, which is true because it's still a symmetric space.

References

[Br94] Brown, Kenneth S. Cohomology of groups. Corrected reprint of the 1982 original. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994. x+306 pp. ISBN: 0-387-90688-6