GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE

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CONTENTS

1. Introduction (Oct 30)

This will be a course on joint work with Laurent Fargues.

1.1. Setup.

Let E be an non-archimedean local field, i.e. $E = \mathbf{F}_q((t))$ or a finite extension of $\mathbf{Q}_p$. We introduce some notation:

- $\mathbf{F}_q$ will be the residue field of E.
- $\pi \in \mathcal{O}_E$ will be a uniformizer.
- G/E is a reductive group, e.g. $G = \mathrm{GL}_n$, Sp_{2n}. (Part of the appeal of this theory is that it works completely uniformly for any reductive group.)

We are interested in the representation theory of the locally profinite group $G(E)$. We will briefly recall some facts about this.

1.2. Representations of p-adic groups.

Let Γ be a locally profinite group.

Definition 1.1. A *smooth representation*[1] of Γ over a field L is an L-vector space V plus a map $\Gamma \to \mathrm{GL}(V)$ such that for all $v \in V$, $\mathrm{Stab}(v) \subseteq \Gamma$ is open.

Example 1.2. Let $K \subset G(E)$ be an open compact subgroup. As K is profinite, it has many finite quotients. Suppose we have such a finite quotient $K \twoheadrightarrow \overline{K}$, and a representation $\rho\colon \overline{K} \to \mathrm{GL}(V_0)$. Then we can form the compact induction

$$\text{c-Ind}_K^\Gamma(\rho) = \{f\colon \Gamma \to V_0 \mid f \text{ compact support}, f(gk) = f(g)k \forall g \in \Gamma, k \in K\}.$$

These are compact projective generators of the category of smooth representations, if L has characteristic zero.

Example 1.3 (Compact induction)**.** Let $K = \mathrm{GL}_n(\mathcal{O}_E) \subset \Gamma = \mathrm{GL}_n(E)$, $\overline{K} = \mathrm{GL}_n(\mathbf{F}_q)$, ρ a cuspidal representation of $\mathrm{GL}_n(\mathbf{F}_q)$. Then $\text{c-Ind}_K^\Gamma \rho$ is a "supercuspidal" representation, irreducible after fixing the central character.

Example 1.4 (Parabolic inducton)**.** If $P \subset G$ is a parabolic subgroup with Levi L, and (V_0, ρ_L) is a smooth representation of $L(E)$, then

$$\mathrm{Ind}_{P(E)}^{G(E)} V_0 = \{f\colon G(E) \to V_0 \mid f(\gamma p) = f(\gamma)p \forall \gamma \in G(E), p \in P(E)\}$$

is a smooth representation of $G(E)$. A supercuspidal representation is one which does not appear in such a representation. In some vague sense, all irreducible smooth representations of $G(E)$ are built via parabolic induction from supercuspidal representations of Levi subgroups.

Example 1.5 (Automorphic representations)**.** If $\mathbb{E}$ is a global field and $\mathbb{G}$ is a reductive group over E such that (G, E) arises as a localization of $(\mathbb{G}, \mathbb{E})$, then the space of automorphic forms $\mathcal{A}(\mathbb{G}(\mathbb{E})\backslash\mathbb{G}(\mathbf{A}_\mathbb{E}), \mathbf{C})$ is a smooth representation of $G(E)$.

In some sense, the study of the category of smooth representations of $G(E)$ is a local analogue of the study of the space of automorphic forms.

[1] Perhaps "locally constant" would be better terminology...

1.3. **Local Langlands correspondence.** We now state a somewhat imprecise form of the local Langlands conjecture. For simplicity, assume G is split.

Conjecture 1.6 (Langlands)**.** *Consider representations over $L = \mathbf{C}$. There is a "natural" map*

$$\mathrm{Irrep}(G(E))/\sim \to \mathrm{Hom}(W_E, \widehat{G}(\mathbf{C}))/\widehat{G}(\mathbf{C}) - \mathrm{conj}.$$

where $\widehat{G}$ is the Langlands dual group, W_E is the Weil group[2] *of E which is surjective with finite fibers (called L-packets).*

Remark 1.7. There are refined versions of the conjecture which describe the fibers precisely. There are also recent improvements which conjecturally describe the whole category $\mathrm{Rep}(G(E))$ in terms of coherent sheaves on the Artin stack $\mathrm{Hom}(W_E, \widehat{G}(\mathbf{C}))/\widehat{G}(\mathbf{C})$.

Here are some questions I had on first impression.

(1) Why does this make any sense – how does W_E relate to $\mathrm{Rep}(G(E))$?
(2) Where does $\widehat{G}$ come from?

To discuss (2) further, let's first recall what $\widehat{G}$ is. Recall that split reductive groups G over any field are classified by root data $(X, \Phi, X^*, \Phi^\vee)$. A funny observation is that $(X^*, \Phi^\vee, X, \Phi)$ is also a root datum, which defines $\widehat{G}$.

Example 1.8 (Some dual groups)**.** $\widehat{\mathrm{GL}}_n = \mathrm{GL}_n$, $\widehat{\mathrm{SL}}_n = \mathrm{PGL}_n$, $\widehat{\mathrm{Sp}}_{2n} = \mathrm{SO}_{2n+1}$, $\widehat{\mathrm{SO}}_{2n} = \mathrm{SO}_{2n}$.

1.4. **Examples of the LLC.**

1.4.1. $G = \mathbf{G}_m$. For $G = \mathbf{G}_m$, $G(E) = E^\times$ is abelian, so $\mathrm{Irrep}(E^\times) = \{\chi \colon E^\times \to \mathbf{C}^\times\}$ is the set of characters. The dual group is $\widehat{G} = \mathbf{G}_m$, and $\mathrm{Hom}(W_E, \widehat{G}(\mathbf{C})) = \mathrm{Hom}(W_E, \mathbf{C}^\times) = \mathrm{Hom}(W_E^{\mathrm{ab}}, \mathbf{C}^\times)$. Indeed, local class field theory identifies $E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}}$.

1.4.2. $G = \mathrm{GL}_n$. For $G = \mathrm{GL}_n$ the dual group is $\widehat{G} = \mathrm{GL}_n$.

Theorem 1.9 (Laumon-Rapoport-Stuhler, Harris-Taylor, Henniart)**.** *There is a "natural" bijection between supercuspidal representations of $\mathrm{GL}_n(E)$ and irreducible n-dimensional representations $W_E \to \mathrm{GL}_n(\mathbf{C})$.*

This does not quite give a natural bijection between $\mathrm{Irrep}(\mathrm{GL}_n(E))/\sim$ and $\mathrm{Hom}(W_E, \widehat{G}(\mathbf{C}))$, but one can say exactly what happens using Bernstein-Zelevinsky.

Example 1.10 (Automorphic induction)**.** Unlike for $n = 1$, in practice it's easier to write down things on the Galois side. For E'/E a degree n extension, and $\chi' \colon W_{E'} \to W_{E'}^{\mathrm{ab}} \cong (E')^\times \to \mathbf{C}^\times$ a generic character, $\varphi := \mathrm{Ind}_{W_{E'}}^{W_E}(\chi')$ will be an irreducible n-dimensional representation of W_E. This should correspond to a supercuspidal representation π_φ of $\mathrm{GL}_n(E)$. It is quite difficult to construct π_φ directly.

Remark 1.11. Many explicit examples of the LLC are known by work of Bushnell, Kutzko, Kaletha, etc. It is extremely complicated!

1.5. **Goal of the course.**

[2] This is the pre-image of $\mathbf{Z} \subset \widehat{\mathbf{Z}}$ under the surjection $\mathrm{Gal}(\overline{E}/E) \to \widehat{\mathbf{Z}}$ corresponding to the maximal unramified extension of E.

1.5.1. The first goal is to give a construction of the map $\pi \to \varphi_\pi$ from irreducible representations to L-parameters, that works uniformly for any reductive group G, and is purely local.

Remark 1.12 (Related work)**.** In the function field case, this has essentially been done by Genestier-Lafforgue.

For GL_n, the proof is via global methods. This has then been extended to classical groups by Jim Arthur, using twisted endoscopy.

1.5.2. The second goal is to "explain" where W_E and $\widehat{G}$ come from.

1.5.3. The third goal is to formulate a more structured form of the local Langlands correspondence as an equivalence of categories, and (essentially) construct a functor in one direction. This makes the statement much more precise: our particular functor is an equivalence.

Remark 1.13 (Coefficients)**.** The methods will work over any coefficient ring Λ where $p \in \Lambda^\times$.

1.6. **The rough idea.** We want to develop the geometric Langlands program over the "Fargues-Fontaine curve", using the geometry of "perfectoid space/diamonds".

We will not be able to review all these topics in detail. References include:

- Berkeley lectures on p-adic geometry.
- Lecture notes for Montreal workshop.

1.7. **The big picture.** We want to contemplate the space "Spec E". There are a couple different ways to look at a scheme.

- One is via its étale site, which in the case of Spec E is controlled by $\pi_1^{\text{ét}}(\text{Spec } E) = \text{Gal}(\overline{E}/E)$.
- Another is via coherent sheaves. In this case that just means E-vector spaces. More generally one could consider the groupoid of G-torsors
$$[\text{pt}/G](\text{Spec } E) = \coprod_{\alpha \in H^1(E,G)} [\text{pt}/G_\alpha(E)]$$
where $G_\alpha(E)$ is the inner form corresponding to the torsor α. In particular, $[\text{pt}/G(E)]$ sides as an open and closed substack, and $\text{Rep}(G(E)) = \text{Shv}([\text{pt}/G(E)])$, which is embedded in a natural way in $\text{Shv}([\text{pt}/G](\text{Spec } E))$. We will find it profitable to study this larger category.[3]

Now we are going to modify the picture a bit. We want to change $\text{Gal}(\overline{E}/E)$ into W_E. For a scheme $X/\mathbf{F}_q$, we replace X by $X_{\overline{\mathbf{F}}_q}$ with its Frob_q-action. Then $\pi_1(X_{\overline{\mathbf{F}}_q}/\text{Frob}_q) = \pi_1(X_{\overline{\mathbf{F}}_q}) \rtimes \mathbf{Z}$ is the "Weil form" of the fundamental group.

In our setting, this suggests replacing

- Spec $\mathbf{F}_q((t))$ by "Spec $\overline{\mathbf{F}}_q((t))/\text{Frob}_q$".
- Spec E by Spec $\breve{E}/\text{Frob}_q$ where $\breve{E}$ is the completion of the maximal unramified extension of E.

Now let's consider the picture for "Spec $\breve{E}/\text{Frob}_q$" instead:

- The étale site is controlled by $\pi_1(\text{Spec } \breve{E}/\text{Frob}_q) = W_E$.

[3]This idea has also been emphasized by Vogan, Bernstein.

- On the coherent side, the vector bundles are by descent the category of *isocrystals*
$$\mathrm{Isoc}_E = \{\breve{E} - \text{vector spaces} + \text{Frob-semilinear } \phi \colon V \xrightarrow{\sim} V\}.$$
This is now richer than just E-vector spaces. By the Dieudonné-Manin theorem, it decomposes into
$$\bigoplus_{\lambda \in \mathbf{Q}} \mathrm{Isoc}_E^{\lambda}$$
where $\mathrm{Isoc}_E^{\lambda}$ consists of isocrystals which are pure of slope λ. The category $\mathrm{Isoc}_E^{\lambda}$ can be identified with D_λ-modules, where D_λ/E is the central division algebra of invariant $[\lambda] \in \mathbf{Q}/\mathbf{Z}$.

More generally, we can consider the category of G-torsors in Isoc_E. This was classified by Kottwitz, with motivation coming from the theory of Shimura varieties. He showed that the groupoid of G-isocrystals is
$$\coprod_{b \in B(E,G)} [\mathrm{pt}/G_b(E)]$$
where G_b is an inner form of a Levi subgroup of G. There is an injection $H^1(E,G) \hookrightarrow B(E,G)$. Kottwitz and Kaletha already realized from a purely representation-theoretic perspective that it was profitable to consider all the G_b simultaneously.

1.8. **Stacks of isocrystals.** We want something more geometric than $\coprod_{b\in B(E,G)}[\mathrm{pt}/G_b(E)]$. We want to promote this to a "stack of G-isocrystals", and there are several ways of going about this.

1.8.1. Isoc_E is an E-linear category, so for any E-algebra A we can consider G-torsors in $\mathrm{Isoc}_E \otimes_E A$. This leads to an Artin stack over E, which is
$$\coprod_{b \in B(E,G)} [\mathrm{pt}/G_b].$$
We emphasize that here G_b is an algebraic group, and this point of view is instrumental in seeing that G_b can be promoted to an algebraic group. But from our perspective this is not the right thing to do, as we are after smooth representations of p-adic groups rather than algebraic representations of algebraic groups.

1.8.2. A better idea is to replace $\overline{\mathbf{F}}_q$ by any (perfect) $\overline{\mathbf{F}}_q$-algebra R, i.e. replace
$$\mathrm{Spec}\, \overline{\mathbf{F}}_q((t))/\mathrm{Frob}_q \rightsquigarrow \mathrm{Spec}\, R((t))/\mathrm{Frob}_q\,.$$
$$\mathrm{Spec}\, \breve{E}/\mathrm{Frob}_q \rightsquigarrow \mathrm{Spec}\,(W(R) \otimes_{W(\overline{\mathbf{F}}_q)} E)/\mathrm{Frob}_q\,.$$
We can define a stack on perfect $\overline{\mathbf{F}}_q$-algebras
$$R \mapsto \{G\text{-torsors on Spec } R((t))/\mathrm{Frob}_q \text{ or Spec}\,(W(R) \otimes_{W(\overline{\mathbf{F}}_q)} E)/\mathrm{Frob}_q\}.$$

It turns out that this is a well-behaved stack which we call $G - \mathrm{Isoc}$.[4]

[4] $G - \mathrm{Isoc}$ is equivalent to an object "$LG/_\sigma LG$" which has been considered by Gaitsgory, Genestier-Lafforgue, Zhu. They define a semi-orthogonal decomposition of $D(G - \mathrm{Isoc}, \overline{\mathbf{Q}}_\ell)$ into pieces $D(G - \mathrm{Isoc}^b, \overline{\mathbf{Q}}_\ell) \cong D(G_b(E), \overline{\mathbf{Q}}_\ell)$.

Theorem 1.14 (Rapoport-Richartz, Caraiani-Scholze, Ivanov, Anschütz)**.** $G - \mathrm{Isoc}$ *is a stack for the v/arc topology.*[5]

For any $b \in B(E,G)$*, there is a locally closed substack* $G - \mathrm{Isoc}^b \subset G - \mathrm{Isoc}$ *where the isocrystal is isomorphic to* b*, and* $G - \mathrm{Isoc}^b \cong [\mathrm{pt}/G_b(E)]$.[6]

1.8.3. We will still not quite consider this stack. We will consider a more analytic version, which is arguably better behaved, using perfectoid rings instead of perfect algebras.

To motivate this, we mention that the relation to the Langlands dual group $\widehat{G}$ is through *Hecke operators.*

We think of $R((t))$ as a small punctured disk. We want to consider correspondences on the space of G-torsors.

We consider the space parametrizing $\mathcal{E}_1, \mathcal{E}_2$ two G-torsors over $(\mathrm{Spec}\ R((t))/\mathrm{Frob}_q)$, and an isomorphism $\mathcal{E}_1 \cong \mathcal{E}_2$ away from some divisor $D \subset \mathrm{Spec}\ R((t))$. This requires a section $\mathrm{Spec}\ R \to \mathrm{Spec}\ R((t))$, which does not exist if R is a discrete ring. So we will need to take R to be a Banach algebra. This motivates the entrance of perfectoid geometry.

To any perfectoid affinoid algebra (R, R^+) over $\overline{\mathbf{F}}_q$, we can associate the *Fargues-Fontaine curve* $\mathbb{D}^*_{\mathrm{Spa}(R,R^+)}/\mathrm{Frob}_q$ (resp. a similar object in mixed characteristic). We can then consider the moduli space of G-torsors on the Fargues-Fontaine curve.

The advantage of the analytic situation is that the punctured disk is much more geometric. So we can really consider Hecke operators

$$\begin{array}{ccc} & \mathrm{Hecke}_G & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Spa}\, \breve{E}/\phi) \end{array}$$

Here $\mathrm{Spa}(\breve{E}/\phi)$ is the space parametrizing sections of the Fargues-Fontaine curve.

Hecke_G is infinite-dimensional, but it has finite-dimensional strata Hecke_G^{μ}. Let $T_\mu := \mathrm{pr}_{2!}\,\mathrm{pr}_1^* \colon D(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \to D(\mathrm{Bun}_G \times \mathrm{Spa}\, \breve{E}/\phi; \overline{\mathbf{Q}}_\ell) = D(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)^{W_E}$.

The *Geometric Satake equivalence* says that the operators T_μ are enumerated by $\mathrm{Rep}(\widehat{G})$.

Recall that $D(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)$ is some aggregate of $\mathrm{Rep}(G_b(E); \mathbf{Q}_\ell)$ for all b. It turns out that this categorical structure is *precisely* what is needed to define L-parameters attached to $\mathrm{Rep}(G_b(E); \mathbf{Q}_\ell)$.

[5] This is even stronger than being a stack in the fpqc topology!

[6] This forms a stratification, but the strata are not open and closed (the gluing data is complicated).

2. The Fargues-Fontaine Curve, I (Nov 2)

2.1. **Local fields.** Fix a nonarchimedean local field E, with residue field $\mathbf{F}_q$ of characteristic p. Let $\pi \subset \mathcal{O}_E$ be a uniformizer.

Example 2.1. Concretely, either $E \cong \mathbf{F}_q((t))$ or E is a finite extension of $\mathbf{Q}_p$.

The goal is to "make Spec E geometric".

- The Zariski site of Spec E is a point, which is not too geometric – we'll want to enhance this.
- The étale site of E is $\{\text{finite separable } E\text{-algebras}\}^{\text{op}}$. In other words, this is $B\,\text{Gal}(\overline{E}/E)$, which can be seen as the category of finite sets with continuous $\text{Gal}(\overline{E}/E)$-action. This is more interesting, but still not very geometric.

We have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_E & \longrightarrow & \text{Gal}(\overline{E}/E) & \longrightarrow & \text{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) & \longrightarrow & 1 \\ & & & & & & \downarrow \wr & & \\ & & & & & & \widehat{\mathbf{Z}} = \langle \text{Frob}_q \rangle & & \end{array}$$

and we can further decompose the inertia subgroup I_E as

$$0 \to P_E \to I_E \to \prod_{\ell \neq p} \mathbf{Z}_\ell \to 0.$$

2.1.1. *Local Tate duality.* If E has characteristic 0, then for all torsion $\text{Gal}(\overline{E}/E)$-representations M the pairing

$$H^i_{\text{ét}}(\text{Spec } E, M) \otimes H^{2-i}_{\text{ét}}(\text{Spec } E, M^*(1)) \to H^2_{\text{ét}}(\text{Spec } E, \mathbf{Q}/\mathbf{Z}(1)) = \mathbf{Q}/\mathbf{Z}.$$

Here $M^* = \text{Hom}(M, \mathbf{Q}/\mathbf{Z})$ and $\mathbf{Q}/\mathbf{Z}(1)$ is the Tate twist, i.e. $\bigcup_n \mu_n$. If E has characteristic p, then we restrict to prime-to-p M and the prime-to-p torsion in $\mathbf{Q}/\mathbf{Z}$.

This looks like Poincaré duality on a 2-manifold. (But the Tate twist means that its orientation sheaf is non-trivial.) We want to turn Spec E into something closer to a compact Riemann surface.

2.2. **The equal characteristic Fargues-Fontaine curve.** Let $E = \mathbf{F}_q((t))$. Let $\breve{E} = \overline{\mathbf{F}}_q((t))$. Intuitively, Spec $\overline{\mathbf{F}}_q((t))$ is a "formal punctured open unit disc over $\overline{\mathbf{F}}_q$". But of course it only has one point. We will "make more space" by passing to an extension.

Pick $C/\overline{\mathbf{F}}_q$ a complete algebraically closed non-archimedean field, e.g. $C = \overline{\mathbf{F}_q((u))}^{\wedge}$. Consider the adic spectrum $\text{Spa}\, C \times_{\text{Spa}\, \mathbf{F}_q} \text{Spa}\, \mathbf{F}_q((t))$. This is the "punctured open unit disk over C",

$$\mathbb{D}^*_C = \{x \mid 0 < |x| < 1\}.$$

It now has many points.

Base changing to C passes to a "geometric" situation; as in the first lectures, we'll want to remember the Frobenius ϕ_C.

Definition 2.2. The *Fargues-Fontaine curve*[7] (for E, C) is $X_{C,E} := \mathbb{D}^*_C/\phi_C^{\mathbf{Z}}$, viewed as an adic space over E.

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[7]This depends on the choice of C.

Remark 2.3. The action of $\phi_C^{\mathbf{Z}}$ on $\mathbb{D}_C^*$ is free. So the quotient is very well-behaved. However, we note that although $\mathbb{D}_C^*$ is a finite type object over C, since the action is non-trivial on the base C, it no longer lives over C; it is instead an infinite type object over E.

In what sense is this a geometrization of Spec E? The following properties are due to Fargues-Fontaine.

- $H^0(X_{C,E}, \mathcal{O}_{X_{C,E}}) = E$.
- The category of finite étale covers of $X_{C,E}$ is $(\text{Spec } E)_{\text{ét}}$.
- $H^i_{\text{ét}}(X_{C,E}, M) \cong H^i_{\text{ét}}(\text{Spec } E, M)$.

In particular, local Tate duality becomes global duality over $X_{C,E}$.

2.3. **Some reminders on adic spaces.** Roughly, adic spaces are variants of schemes associated to certain topological rings (e.g. Banach algebras). What they have in common with schemes are:

- There are specializations between points.
- There are very general: there are no finiteness assumptions in the foundations (unlike for rigid analytic geometry).

Definition 2.4. Let A be a topological ring.

(1) A is *adic* if there is some ideal $I \subseteq A$ such that $\{I^n \mid n \geq 0\}$ is a neighborhood basis of 0. Such an ideal I is called an *ideal of definition*. (This is not unique, but for any two I, J there exists n such that $I^n \subset J$ and $J^n \subset I$.)

(2) A is *Huber*[8] if there exists an open subring $A_0 \subset A$ that is an adic ring with the finitely generated ideal of definition. Such an $A_0 \subseteq A$ is called a *ring of definition*.

Remark 2.5. Any such A admits a completion $\widehat{A}$, which contains the I-adic completion $\widehat{A}_0 \subset \widehat{A}$ as an open subring.

The most important type of adic ring for us is:

Definition 2.6. A is a *Tate* ring if it contains a topologically nilpotent unit $\varpi \in A$. Such a ϖ is called a *pseudouniformizer*.

Example 2.7. Any nonarchimedean field is Tate. In $\mathbf{F}_p((t))$, we can take $\varpi = t$. In $\mathbf{Q}_p$ we can take $\varpi = p$. More generally, any Huber ring over a nonarchimedean local field is Tate.

Remark 2.8. If K is any nonarchimedean local field and $\varpi \in K$ a pseudouniformizer, and A/K is a complete Huber ring, then A has a natural structure as a Banach algebra over K, with the "unit ball" being $\{f \in A \colon |f_0| \leq 1\} = A_0$. Any ring of definition $A_0 \subset A$ has the ϖ-adic topology.

A norm on A can be constructed as follows. Declare $||\varpi||$ to be some arbitrary real number in $(0, 1)$. Then define

$$||a|| = \inf_{\{n \mid \varpi^n a \in A_0\}} 2^n.$$

There is an equivalence of categories between Banach algebras over K with continuous maps and Tate-Huber rings over K. So we could have formulated things in terms of Banach algebras instead of Tate rings. But there are at least a couple advantages of the Tate ring approach:

- We don't need to fix a nonarchimedean ground field K.

[8]This was called "f-adic" in Huber's original papers.

- The norm itself is extraneous structure.

Definition 2.9. The *valuation spectrum* of a Huber ring A is

$$\mathrm{Cont}(A) := \{\text{continuous valuations } |\cdot|: A \to \Gamma \cup \{0\}\}/\sim$$

where Γ is a totally ordered abelian group (e.g. $\mathbf{R}_{>0}$). We equip $\mathrm{Cont}(A)$ with the topology generated by opens $\{|f| \leq |g| \neq 0\} \subset \mathrm{Cont}(A)$ for any $f, g \in A$.

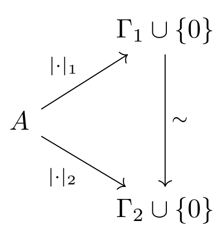
Explicitly, "continuous valuation" means:

- $|ab| = |a| \cdot |b|$, $0\gamma = \gamma 0 = 0$ and $0 \leq \gamma$ for all γ.
- $|a + b| \leq \max(|a|, |b|)$.
- $|0| = 0$ and $|1| = 1$.
- For all $\gamma \in \Gamma$, $\{a \mid |a| < \gamma\} \subset A$ is open.

Two continuous valuations $|\cdot|_1, |\cdot|_2$ are *equivalent* if the binary relations are the same:

$$|a|_1 \geq |b|_1 \iff |a|_2 \geq |b|_2.$$

This is equivalent to: if Γ_i are chosen minimal then there exists an isomorphism $\Gamma_1 \xrightarrow{\sim} \Gamma_2$ such that the diagram below commutes.



Example 2.10. We say that a valuation has "rank 1" if Γ can be embedded in $\mathbf{R}_{>0}$. An example of a Γ that comes up naturally and is *not* rank 1 is $\mathbf{R}_{>0} \times \gamma^{\mathbf{Z}}$ where $r > \gamma > 1$ for all $r \in \mathbf{R}_{>1}$.

Example 2.11. The strictness in the definition only matters for $\Gamma = \{1\}$.

Definition 2.12. A *Huber pair* is a pair (A, A^+) where A is a Huber ring, $A^+ \subset A$ is an open integrally closed subring of power-bounded elements. We write $A^\circ \subset A$ for the subring of power-bounded elements.

We define $\mathrm{Spa}(A, A^+) = \{|\cdot|: |A^+| \leq 1\} \subset \mathrm{Cont}(A)$.

We define $\mathrm{Spa}\, A = \mathrm{Spa}(A, A^\circ)$.

We can endow $\mathrm{Spa}\, A$ with a presheaf $\mathcal{O}_{\mathrm{Spa}\, A} \supset \mathcal{O}^+_{\mathrm{Spa}\, A}$ of Huber rings on the basis of rational subsets. Rational subsets are of the form $U(\frac{f_1, \dots, f_n}{g}) := \{|f_i| \leq |g| \neq 0\}$ where $(f_1, \dots, f_n, g)$ generate an open ideal. So

$$\mathcal{O}_{\mathrm{Spa}\, A}\left(U\left(\frac{f_1, \dots, f_n}{g}\right)\right) = A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle$$

"allows all convergent series in f_i/g". The role of the $+$ subring is to encode which elements are ≤ 1, so it includes the f_i/g. The subring $\mathcal{O}^+_{\mathrm{Spa}\, A}\left(U\left(\frac{f_1, \dots, f_n}{g}\right)\right)$ is the minimal subring that contains A^+ and all the f_i/g.

Theorem 2.13 (Huber, ...)**.** *In all practical cases $\mathcal{O}_{\mathrm{Spa}(A, A^+)}$ is a sheaf, but not always.*

Remark 2.14. Recently, it was realized (Bambozzi-Kremnizer, Clausen-S) that this non-sheafiness can be corrected by allowing the structure sheaf to be *derived*. The subtlety is in how to combine the derivedness with the topology, which we resolved with Clausen using the theory of condensed mathematics. This is not relevant for the present course.

Definition 2.15. An *adic space* is a triple $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ where

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of complete topological rings,
- $\mathcal{O}_X^+$ is a subsheaf of $\mathcal{O}_X$

that is locally of the form $(\operatorname{Spa}(A, A^+), \mathcal{O}_A, \mathcal{O}_A^+)$, plus the correct analog of "locally ringed".

Remark 2.16. This can be alternately formulated by giving the valuations on stalks of $\mathcal{O}_X$ instead of $\mathcal{O}_X^+$. The subsheaf $\mathcal{O}_X^+ \subset \mathcal{O}_X$ can be recovered as elements whose valuations are ≤ 1 at all points.

Remark 2.17. Why do we demand that $\mathcal{O}_X$ is a sheaf of *complete* rings? One advantage is that only by working with complete things can we formulate the nice universal property of the functions $A\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle$ on a rational subset.

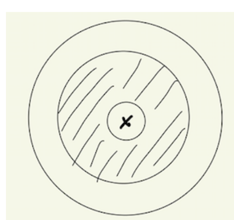
Example 2.18. Let C be a non-archimedean field. We explain the adic space structure on $\mathbb{D}_C^* =$ "$\{x \mid 0 < |x| < 1\}$. We have a *Tate algebra*

$$C\langle T\rangle = \{\sum a_n T^n \mid a_n \in C, a_n \to 0\}.$$

So $\operatorname{Spa} C\langle T\rangle =: \mathbb{B}_C =$ "$\{x \mid 0 \leq |x| \leq 1\}$", the "closed unit disc" over C.

We have $\mathbb{B}_C^* = \mathbb{B}_C \setminus \{0\} = \bigcup_{\epsilon > 0} \mathbb{A}(\epsilon, 1)$ where informally it is the annulus $\{x \mid \epsilon \leq |x| \leq 1\}$ for $\epsilon \in |C|$. More formally, $\mathbb{A}(\epsilon, 1) = \operatorname{Spa} A_\epsilon$ where

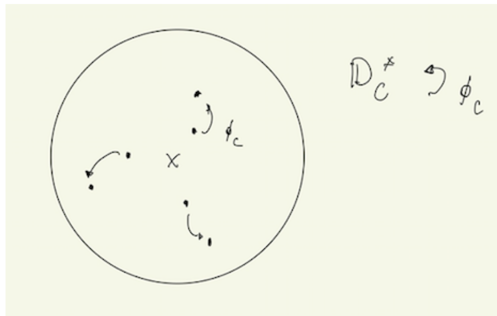
$$A_\epsilon = \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \colon a_n \in C, |a_n| \xrightarrow{n \to \infty} 0, \epsilon^n |a_n| \xrightarrow{n \to -\infty} 0 \right\}.$$



Similarly, $\mathbb{D}_C^* = \bigcup_{\epsilon > 0, r < 1} \mathbb{A}(\epsilon, r)$ where $\mathbb{A}(\epsilon, r) = \{\epsilon \leq |T| \leq r\}$.

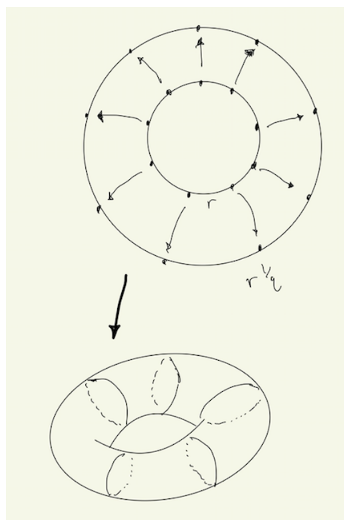
Warning 2.19. $\mathbb{D}_C^* \subset \mathbb{B}_C$ is open, but it is not equal to $\{0 < |T| < 1\}$. There is one point $x \in \mathbb{B}_C = \operatorname{Spa}\langle T\rangle$ such that $r < |T(x)| < 1$ for all $r \in |C|$ for $r < 1$, i.e., x is infinitesimally less than 1. The rising union of the annuli does not contain such a point. The subset $\{0 < |T| < 1\}$ is not open, as strict inequalities < 1 define a *closed* subset.

In fact there is a natural map $\mathrm{rad}\colon \mathbb{B}_C \to [0,1]$ and $\mathrm{rad}\colon \mathbb{D}^*_C \to (0,1)$ sending $x \mapsto |T(\widetilde{x})|$. The Frobenius ϕ_C moves $\mathbb{D}^*_C$ outwards from the origin, i.e. $\mathrm{rad} \circ \phi_C = \mathrm{rad}^{1/q}$.



This implies that the action of ϕ_C on $\mathbb{D}^*_C$ is free and properly discontinuous.

Hence we can define $X_{C,E} = \mathbb{D}^*_C/\phi_C^{\mathbf{Z}}$. This can be seen as $\mathbb{A}(r, r^{1/q})$ modulo the identification of the boundary annuli via $\phi\colon \mathbb{A}(r,r) \cong \mathbb{A}(r^{1/q}, r^{1/q})$.



Intuitively, the quotient looks like a bit like a complex torus.

3. The Fargues-Fontaine Curve, II (Nov 5)

3.1. **Recap.** Last time we considered the Fargues-Fontaine curve for an equal characteristic local field $E = \mathbf{F}_q((t))$.

Letting $C/\mathbf{F}_q$ be a complete algebraically closed nonarchimedean field, we considered

$$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} C = \mathbb{D}_C^*,$$

the punctured open unit disk over C. This had an action of ϕ_C, and we defined $X_{C,E} := \mathbb{D}_C^*/\phi_C^{\mathbf{Z}}$.

3.2. **Classical points.** There is an equivalence of categories

$$\{\text{rigid-analytic varieties}/C\} \cong \{\text{adic spaces “locally of finite type”}/\operatorname{Spa} C\}$$

The equivalence is defined by $X(C) \leftarrow\!\shortmid X$, where $X(C) \subset |X|$ are the “classical points”. Locally $X = \operatorname{Spa} A$ where $A = C\langle T_1, \ldots, T_n\rangle/I$, and

$$X(C) = \{(x_1, \ldots, x_n) \in C^n \colon |x_i| \leq 1, f(x_1, \ldots, x_n) = 0\} =: |\operatorname{Sp} A|.$$

Tate defined a Grothendieck topology $(\operatorname{Sp} A)_{\text{rig}}$ consisting of “quasicompact admissible opens”, which is equivalent to the topology on $\operatorname{Spa} A$ given by quasicompact opens. In these terms, Tate's “admissible covers” of $(\operatorname{Sp} A)_{\text{rig}}$ are equated with covers of $\operatorname{Spa} A$.

In particular, the quasicompact open subsets of $\operatorname{Spa} A$ are completely determined by their classical points.

For $\mathbb{D}_C^*$, the classical points are $\{x \in C \colon 0 < |x| < 1\}$. The Frobenius ϕ_C acts on the set of classical points by sending $x \mapsto x^{1/q}$. (The reason why it is not $x \mapsto x^q$ is because ϕ_C is acting through its action on C.)

For any connected affinoid $\operatorname{Spa} A \subset \mathbb{D}_C^*$, A is a principal ideal domain. That reflects the 1-dimensionality of $\mathbb{D}_C^*$. One checks this by hand; the maximal ideal corresponding to x is generated by $T - x$.

By descent, we can also define the classical points of $X_{C,E}$. They are

$$X_{C,E}^{\text{cl}} = \{x \in C \colon 0 < |x| < 1\}/\phi_C.$$

Again, any connected affinoid subset of $X_{C,E}$ is the adic spectrum of a principal ideal domain.

3.3. **The mixed characteristic Fargues-Fontaine curve.** Now we turn our attention to the case where E be a finite extension of $\mathbf{Q}_p$. Let $C/\mathbf{F}_q$ be as before (a complete algebraically closed nonarchimedean field).

What is “$\operatorname{Spa} E \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} C$”? Of course the immediate issue is that there is no map $\operatorname{Spa} E \to \operatorname{Spa} \mathbf{F}_q$ since E has characteristic 0.

The idea is that in characteristic p, we deformed any $\mathbf{F}_q$-algebra R to $\mathbf{F}_q[[t]]$ by taking $R[[t]]$. Now we want to deform R to mixed characteristic.

Note that if R is a *perfect* $\mathbf{F}_q$-algebra, there is a unique (up to unique isomorphism) lift $\widetilde{R}/\mathcal{O}_E$ that is flat, π-adically complete, and has $\widetilde{R}/\pi \xrightarrow{\sim} R$. One construction is via the (p-typical, ramified) *Witt vectors*

$$\widetilde{R} = W_{\mathcal{O}_E}(R) := W(R) \otimes_{W(\mathbf{F}_q)} \mathcal{O}_E.$$

The idea behind the construction of Witt vectors is the following. Any element of R has p-power roots, by assumption that R is perfect. Since raising to the pth power improves congruences, one can write down distinguished lifts by exponentiating arbitrary lifts of p-power roots. More precisely, there is a Teichmüller map

$$[\cdot]\colon R \to \widetilde{R} = W_{\mathcal{O}_E}(R)$$

sending $x \mapsto \lim_{n\to\infty} \widetilde{x}_n^{p^n}$ where $\widetilde{x}_n \in \widetilde{R}$ is any lift of x^{1/p^n}. This is multiplicative, but of course not additive.

Any element of $\widetilde{R}$ admits a unique expression of the form

$$\sum_{n\geq 0} [r_n]\pi^n, \quad r_n \in R.$$

Intuitively, elements of $\widetilde{R}$ look like "power series over R in the variable π".

The analog of $\operatorname{Spa} \mathbf{F}_q[[t]] \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} \mathcal{O}_C = \operatorname{Spa} \mathcal{O}_C[[t]]$ in the mixed characteristic setting should then be

$$\text{“}\operatorname{Spa} \mathcal{O}_E \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} \mathcal{O}_C\text{”} := \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C).$$

In equal characteristic we really worked with $\operatorname{Spa} \mathbf{F}_q((t)) \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} C$. This is obtained by asking $t \neq 0$ and $\varpi \neq 0$. So the analogue of $\operatorname{Spa} \mathbf{F}_q((t)) \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} C$ in mixed characteristic is

$$\text{“}\operatorname{Spa} E \times_{\operatorname{Spa} \mathbf{F}_q} \operatorname{Spa} C\text{”} := Y_{C,E} := \{\pi \neq 0, [\varpi] \neq 0\} \subset \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C).$$

This still carries an action of ϕ_C, which is free and totally discontinuous. (There is no longer a Frobenius action coming from E.)

Definition 3.1. The *Fargues-Fontaine curve* is $X_{C,E} := Y_{C,E}/\phi_C^{\mathbf{Z}}$.

Remark 3.2. The geometry of $X_{C,E}$ is in some sense extremely similar to that in equal characteristic, as we shall see below.

One difference is that $Y_{C,E}$ is much less explicit. In particular, $Y_{C,E}$ lives over $\operatorname{Spa} E$, hence has no structure map to $\operatorname{Spa} C$.

Theorem 3.3 (Fargues-Fontaine, Kedlaya)**.**

(1) There is a notion of classical points $Y_{C,E}^{\mathrm{cl}} \subset Y_{C,E}$ *such that for any connected affinoid open subset* $\operatorname{Spa} A \subset Y_{C,E}$*,* A *is a principal ideal domain, and* $\operatorname{Spm}(A) \xrightarrow{\sim} \operatorname{Spa} A \cap Y_{C,E}^{\mathrm{cl}}$.

(2) For any classical point $y \in Y_{C,E}^{\mathrm{cl}}$*, there is some* $x \in C$ *with* $0 < |x| < 1$ *such that* $y = V(\pi - [x])$*. (But beware that* x *is not unique.)*[9]

(3) For any classical point $y \in Y_{C,E}^{\mathrm{cl}}$*, the complete residue field*[10] *at* y *(which by part (1) is the complete residue field of* A*, where* $\operatorname{Spa} A$ *is an affinoid neighborhood of* y*) is a complete algebraically closed field* $C(y)$ *with a distinguished isomorphism*

$$\underbrace{C(y)^\flat}_{\text{“tilt of } C(y)\text{”}} \xrightarrow{\sim} C.$$

This induces a bijection $Y_{C,E}^{\mathrm{cl}} \xrightarrow{\sim} \{\text{untilts } C^\#/E \text{ of } C\}$.[11]

[9] If C were not algebraically closed, we could not choose the elements to be of this simple form $\pi - [x]$.

[10] If (A, A^+) is a Huber pair and $x \in \operatorname{Spa}(A, A^+)$, then we get a valuation $|\cdot|_x\colon A \to \Gamma_x \cup \{0\}$. The kernel $\mathfrak{p}_x := \{f \in A\colon |f|_x = 0\} \subset A$ is a prime ideal. Then the *completed residue field at* x is the completion of $\kappa_x := \operatorname{Frac}(A/\mathfrak{p}_x)$ with respect to x. In our present case neither the passage to completion nor fraction field is necessary.

[11] By definition an "untilt" of C includes the datum of an isomorphism of its tilt with C.

Remark 3.4. Recall that in equal characteristic the classical points of $Y_{C,E}$ were explicitly in bijection with $\{x \in C : 0 < |x| < 1\}$. So (2) is a bit different from its equal characteristic analog. The analog of (3) holds in equal characteristic, but is vacuous because tilting is the identity functor there.

3.4. **Tilting.** For a complete algebraically closed non-archimedean field K such that $|p|_K < 1$, one can define a complete algebraically closed non-archimedean (characteristic p) field $K^\flat$. As a multiplicative topological monoid, $K^\flat \cong \varprojlim_{x \mapsto x^p} K$.

Projection to the first coordinate in this inverse limit induces multiplicative (but obviously not additive) maps $K^\flat \to K$ and $\mathcal{O}_{K^\flat} \to \mathcal{O}_K$, denoted $x \mapsto x^\#$.

To see the ring structure, we note that

$$\mathcal{O}_{K^\flat} := \varprojlim_{x \mapsto x^p} \mathcal{O}_K \xrightarrow{\sim}{}^{12} \varprojlim_{x \mapsto x^p} \mathcal{O}_K/p.$$

Now the RHS has an evident ring structure.

3.5. **Proof of Theorem 3.3.**

3.5.1. *Step 1.* We first construct an injective map

$$\{C^\#/E \text{ untilt of } C\} \to |Y_{C,E}|.$$

Say $C^\#$ is an untilt of C. The projection

$$\mathcal{O}_C \cong \varprojlim_{x \mapsto x^q} \mathcal{O}_{C^\#} \xrightarrow{\mathrm{pr}_1} \mathcal{O}_{C^\#}$$

sending $x \mapsto x^\#$ induces a map[13] $\theta \colon W(\mathcal{O}_C) \to \mathcal{O}_{C^\#}$ sending

$$\sum_{n \geq 0} [x_n] \pi^n \mapsto \sum_{n \geq 0} x_n^\# \pi^n.$$

So we get $\operatorname{Spa} \mathcal{O}_{C^\#} \to \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C)$. The surjectivity of θ implies that this is injective. As with schemes, $\operatorname{Spa} \mathcal{O}_{C^\#}$ has two points – special and generic – and this map induces $\operatorname{Spa} C^\# \to Y_{C,E} \subset \operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C)$. The image is a point $y \in Y_{C,E}$ whose completed residue field at y is $C^\#$.

That gives the desired map

$$\{C^\#/E \text{ untilt of } C\} \hookrightarrow |Y_{C,E}|.$$

To get injectivity, one traces through to see that the untilt can be reconstructed from θ.

Define $Y^{\mathrm{cl}}_{C,E}$ to be the image – then we have already proved part (3) of the theorem.

3.5.2. *Tilting for $Y_{C,E}$.* Let $E_\infty = E(\pi^{1/p^\infty})^\wedge := (\bigcup_n E(\pi^{1/p^n}))^\wedge$. This is a "perfectoid field", meaning that $x \mapsto x^p$ is surjective on $\mathcal{O}_{E_\infty}/p$.

Then the tilt is $E_\infty^\flat \cong \mathbf{F}_q((t^{1/p^\infty}))$. The identification is by setting $t := (\pi, \pi^{1/p}, \ldots) \in \varprojlim_{x \mapsto x^p} E_\infty = E_\infty^\flat$.

Lemma 3.5. *The perfectoid space $(Y_{C,E} \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty)^\flat$ is canonically isomorphic to $\mathbb{D}_C^* \times_{\operatorname{Spa} \mathbf{F}_q((t))} \operatorname{Spa} \mathbf{F}_q((t^{1/p^\infty}))$. Moreover, classical points biject under this correspondence.*

[12] The fact that this is an isomorphism boils down to the same elementary considerations used to construct the Teichmüller map.

[13] We will call this "Fontaine's map."

Note that since perfection doesn't affect the underlying topological space, we have

$$|\mathbb{D}_C^*| \xleftarrow{\sim} |\mathbb{D}_C^* \times_{\operatorname{Spa} \mathbf{F}_q((t))} \operatorname{Spa} \mathbf{F}_q((t^{1/p^\infty}))|.$$

Also tilting is a homeomorphism, so

$$|\mathbb{D}_C^* \times_{\operatorname{Spa} \mathbf{F}_q((t))} \operatorname{Spa} \mathbf{F}_q((t^{1/p^\infty}))| \cong |Y_{C,E} \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty|.$$

This certainly still admits a map to $|Y_{C,E}|$, which is easily checked to be a surjection, although it is not a homeomorphism.

$$\begin{array}{ccc} |Y_{C,E} \times_{\operatorname{Spa} E} \operatorname{Spa} E_\infty| & = & |\mathbb{D}_C^* \times_{\operatorname{Spa} \mathbf{F}_q((t))} \operatorname{Spa} \mathbf{F}_q((t^{1/p^\infty}))| \\ \downarrow\!\!\downarrow & & \downarrow \wr \\ |Y_{C,E}| & & |\mathbb{D}_C^*| \end{array}$$

Composing gives $\{x \in C \colon 0 < |x| < 1\} = \mathbb{D}_C^{*,\mathrm{cl}} \twoheadrightarrow Y_{C,E}^{\mathrm{cl}}$, and chasing through the definitions easily shows that it sends $x \mapsto V(\pi - [x])$.

3.5.3. *Aside on perfectoid spaces.*

Definition 3.6. A *perfectoid Tate ring* is a complete Tate ring A (meaning there exists topologically nilpotent unit $\varpi \in A$, and ring of definition A_0) such that

- there exists ϖ satisfying $\varpi^p \mid p$ in A°, and
- A° is ϖ-adic (equivalently, A° is a ring of definition), and
- $x \mapsto x^p$ is surjective on A°/p.

A *perfectoid space* is an adic space X covered by $\operatorname{Spa}(A, A^+)$ with A a perfectoid Tate ring.

Example 3.7. Perfectoid fields include $E_\infty, C, \mathbf{F}_q((t^{1/p^\infty}))$.

An example of a perfectoid ring which is not a field is $A = C\langle T^{1/p^\infty}\rangle$.

Example 3.8. If $A/\mathbf{F}_p$ is a Tate ring in characteristic p, then A is perfectoid if and only if A is perfect.

Tilting extends to perfectoid rings.[14] If A is perfectoid, the tilt $A^\flat = \varprojlim_{x \mapsto x^p} A$ (with suitable addition). Note that there is a multiplicative monoid map $A^\flat \xrightarrow{\mathrm{pr}_1} A$ denoted $a \mapsto a^\#$.

Example 3.9. $E_\infty\langle T^{1/p^\infty}\rangle^\flat = E_\infty^\flat\langle T^{1/p^\infty}\rangle$.

Tilting also extends to perfectoid spaces $X \mapsto X^\flat$, by gluing $\operatorname{Spa}(A, A^+) \mapsto \operatorname{Spa}(A^\flat, A^{\flat+})$.

Theorem 3.10 (Tilting equivalence)**.**

(1) There is a homeomorphism $|X| \xrightarrow{\sim} |X^\flat|$ sending $x \mapsto x^\flat$, the valuation $|f(x^\flat)| := |f^\#(x)|$. Slogan: "Tilting preserves topological spaces".

(2) Given a perfectoid space X, tilting induces an equivalence of categories

$$\{\textit{perfectoid spaces } Y/X\} \xrightarrow{\sim} \{\textit{perfectoid spaces } Y'/X^\flat\}$$

In particular, untilting is canonical after fixing a base perfectoid space.

(3) If $X = \operatorname{Spa}(A, A^+)$ and $X^\flat = \operatorname{Spa}(A^\flat, A^{\flat+})$ then the Zariski closed subsets (meaning the vanishing locus of some ideal) of X and $X^\flat$ correspond under tilting:

$$\{Z \subseteq |X|\} \leftrightarrow \{Z^\flat \subseteq |X^\flat|\}.$$

[14] For us "perfectoid ring" means "perfectoid Tate ring". There is a different notion called "integral perfectoid ring", which won't play a role in this course.

Example 3.11. Challenge: $X = \operatorname{Spa} C^{\#}\langle T^{1/p^\infty}\rangle \supseteq Z = V(T-1)$. Show that $Z^\flat$ is Zariski closed.

3.5.4. *Back to the proof.* To prove Lemma 3.5, compute by hand that[15]

$$(W_{\mathcal{O}_E}(\mathcal{O}_C)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty})^\flat \cong \mathcal{O}_C[[t^{1/p^\infty}]].$$

Basically we have to identify the LHS mod p with the RHS mod t. For this we note that

$$(W_{\mathcal{O}_E}(\mathcal{O}_C)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty})^\flat/p \cong \mathcal{O}_C[[p^{1/p^\infty}]]^\wedge/p$$

which is isomorphic to $\mathcal{O}_C[[t^{1/p^\infty}]]/t$.

Since the classical points are defined through tilting, it is not hard to see that classical points of $W_{\mathcal{O}_E}(\mathcal{O}_C)\widehat{\otimes}_{\mathcal{O}_E}\mathcal{O}_{E_\infty}$ and $\mathcal{O}_C[[t^{1/p^\infty}]]$ correspond, and their ideals have very explicit generators. That establishes (2) of Theorem 3.3

It remains to establish (1), the principality. Basically what we have to prove is that for any function on A, it can only vanish at classical points. This is proved by tilting again. In the tilted picture it is clear because it is clear what the Zariski closed subsets are. That shows $\operatorname{Spm}(A) \hookrightarrow \operatorname{Spa} A \cap Y_{C,E}^{\mathrm{cl}}$. But we also got the surjectivity already in (2). Since the points have principal ideals, it is easy to conclude that every ideal is principal.

[15]This is an example of an integral perfectoid ring, for which tilting also behaves well, although it is not Tate.

4. The Fargues-Fontaine Curve, III (Nov 9)

4.1. **Recap.** Let E be a local field, $\mathcal{O}_E \ni \pi, \mathbf{F}_q$ be as before. Fixed a complete nonarchimedean algebraically closed field $C/\mathbf{F}_q$.

We have defined the Fargues-Fontaine curve $X_{C,E} = Y_{C,E}/\phi_C^{\mathbf{Z}}$, as an adic space over E. (This will be sometimes abbreviated as X_C or even X, as E will always be fixed while C will vary.) Here we had defined $Y_{C,E}$ as the open subset of $\operatorname{Spa} W_{\mathcal{O}_E}(\mathcal{O}_C)$ where $\pi \neq 0$ and $[\varpi] \neq 0$, for $\varpi \in C$ a pseudouniformizer.

A consequence of Theorem 3.3 from last time is that

- The classical points $X_{C,E}^{\mathrm{cl}} \subset |X_{C,E}|$ correspond to untilts C^*/E of C up to $\phi_C^{\mathbf{Z}}$.
- Any connected affinoid open subset $\operatorname{Spa} A \subset X_{C,E}$ has the property that A is a PID, and $\operatorname{Spm}(A) = X_{C,E}^{\mathrm{cl}} \cap |\operatorname{Spa} A| \subset |X_{C,E}|$.
- For any classical point $y \in Y_{C,E}^{\mathrm{cl}}$, there exists $t \in C$ with $0 < |t| < 1$ such that $y = V(\pi - [t])$. (In mixed characteristic t was not necessarily unique.)

4.2. **Isocrystals.** The most important theorem about the Fargues-Fontaine curve is the classification of vector bundles on it. This will first require some knowledge of isocrystals.

Recall that $\breve{E}$ is the completion of the maximal unramified extension of E. Explicitly, $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q)[1/\pi]$.

Recall that an *isocrystal* is a pair (V, ϕ) where V is a finite-dimensional $\breve{E}$-vector space, and $\phi_V \colon V \xrightarrow{\sim} V$ is a $\phi_{\breve{E}}$-linear automorphism. Isocrystals form an E-linear $\otimes$-category Isoc_E.

Example 4.1.

(1) The unit in Isoc_E is $(\breve{E}, \phi_{\breve{E}})$.

(2) The 1-dimensional objects are $(\breve{E}, b\phi_{\breve{E}})$ for some $b \in \breve{E}^\times$. In fact, any such isocrystal is isomorphic to $(\breve{E}, \pi^n \phi_{\breve{E}})$ for a unique $n \in \mathbf{Z}$. We say that n is the *slope* of $(\breve{E}, b\phi_{\breve{E}})$. This is because a change of basis replaces b by $a^{-1}b\phi(a)$, i.e. changes b by "ϕ-conjugation". In this 1-dimensional case we can write this as $a^{-1}\phi(a)b$. Now, it is easy to prove that anything in $\mathcal{O}_{\breve{E}}^\times$ can be expressed as $a^{-1}\phi(a)$.

Also, we have

$$\operatorname{Hom}((\breve{E}, \pi^n \phi_{\breve{E}}), (\breve{E}, \pi^m \phi_{\breve{E}})) = \begin{cases} E & m = n \\ 0 & m \neq n \end{cases} \tag{4.2.1}$$

Indeed, this Hom space is the same as $\breve{E}^{\phi = \pi^{m-n}}$. Since ϕ preserves the valuation, this can only be non-zero if $m - n = 0$.

For any $\lambda \in \mathbf{Q}$, write $\lambda = s/r$ with $s, r \in Z$ coprime and $r > 0$. Let

$$(V_\lambda, \phi_{V_\lambda}) = \left(\breve{E}^r, \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ \pi^s & & & 0 \end{pmatrix} \phi_{\breve{E}} \right).$$

Theorem 4.2 (Dieudonné-Manin)**.** $\mathrm{Isoc}_E = \bigoplus_{\lambda \in \mathbf{Q}} \mathrm{Isoc}_E^\lambda$*, where* Isoc_E^λ *is the subcategory of isocrystals which are "isoclinic of slope* λ*". The category* Isoc_E^λ *is equivalent to* V_λ *tensored with the category of finite-dimensional* E*-vector spaces.*

Proof sketch. For any non-zero $V = (V, \phi_V) \in \mathrm{Isoc}_E$, let $\mu(V) = \frac{\deg V}{\operatorname{rank} V}$, where

- $\operatorname{rank}(V)$ is the dimension of the underlying $\breve{E}$-vector space, and
- $\deg(V)$ is the slope of $\det V = \wedge^{\operatorname{rank} V}(V)$, an isocrystal of rank 1.

The rank and degree behave well in short exact sequences, and give rise to a "Harder-Narasimhan formalism of slopes". In fact it is stronger than usual because of (4.2.1). (Usually one only gets vanishing of Homs in one direction, i.e. a "semi-orthogonal decomposition".)

We can define (V, ϕ_V) to be *semistable* if for all subobjects $0 \subsetneq (V', \phi_{V'}) \subsetneq (V, \phi_V)$ we have $\mu(V') \leq \mu(V)$, and *stable* if the inequality is strict. By formal considerations, any object $(V, \phi_V) \in \operatorname{Isoc}_E$ has a unique "Harder-Narasimhan filtration", which is a decreasing separated exhaustive filtration (in Isoc_E)

$$V^{\geq \lambda} \subset V$$

indexed by $\lambda \ni \mathbf{Q}$, such that $V^\lambda := V^{\geq \lambda} / \bigcup_{\lambda' > \lambda} V^{\geq \lambda'}$ is semistable of slope λ.

Now, in this special case, the slope function $\mu'(V) := -\mu(V) = \frac{-\deg V}{\operatorname{rank} V}$ *also* gives a Harder-Narasimhan formalism, ultimately because of (4.2.1). These two filtrations canonically split each other. So

$$\operatorname{Isoc}_E \cong \bigoplus_{\lambda \in \mathbf{Q}} \operatorname{Isoc}_E^\lambda$$

where $\operatorname{Isoc}_E^\lambda$ is the full subcategory of isocrystals which are semistable of slope λ.

It remains to classify $\operatorname{Isoc}_E^\lambda$. First, consider $\lambda = 0$. We want to show that Isoc_E^0 is equivalent to the category of finite-dimensional E-vector spaces, via $W \mapsto (W \otimes_E \breve{E}, \operatorname{Id} \otimes \phi_{\breve{E}})$. It is easy to show that this is fully faithful, so we need to show that everything comes from it. This comes down to showing that there are enough ϕ_V-invariants: letting $W = V^{\phi_V = \operatorname{Id}}$, one has $W \otimes_E \breve{E} \xrightarrow{\sim} V$.

For the classification of vector bundles on the Fargues-Fontaine curve, the argument proceeds similarly up until this point, and then it diverges. In the case of isocrystals, it is relatively easy to finish. The idea is to show that V contains a ϕ_V-stable *lattice* $L \subset V$, i.e. a finite free $\mathcal{O}_{\breve{E}}$-module generating V over $\breve{E}$. This can be done by taking any lattice, and forming the intersection of its pre-images under iterates of ϕ_V. The issue is to show that this is a lattice, which one does by arguing that otherwise it would break the semistability.

Now, in the integral situation one can show that $L^{\phi_L = \operatorname{Id}}$ is finite free over $\mathcal{O}_E$ of the correct rank, by Artin-Schreier theory. (First study it mod π, and then build up inductively.)

For general λ, use that if $(V, \phi_V) \in \operatorname{Isoc}_E^\lambda$ then the internal $\underline{\operatorname{Hom}}((V_\lambda, \phi_{V_\lambda}), (V, \phi_V)) \in \operatorname{Isoc}_E^0$, and thus reduce to $\lambda = 0$.

□

Remark 4.3. One can show that $\operatorname{End}(V_\lambda) = D_\lambda$, the central division algebra of invariant λ. This completes the description of Isoc_E.

4.3. **Vector bundles on the Fargues-Fontaine curve.** Note that $\overline{\mathbf{F}}_q \hookrightarrow C$. So $Y_{C,E} \to \operatorname{Spa} \breve{E}$, equivariant for $\phi_C \curvearrowright Y_{C,E}$ and $\phi_{\breve{E}} \curvearrowright \operatorname{Spa}(\breve{E})$. This induces a pullback functor

$$\operatorname{Isoc}_{\breve{E}} \to \{\phi_C - \text{equivariant vector bundles}/Y_{C,E}\} \overset{\text{descent}}{=} \operatorname{VB}(X_{C,E}).$$

We denote this by $V \mapsto \mathcal{E}(V)$. Let $\mathcal{O}_{X_{C,E}}(\lambda) := \mathcal{E}(V_{-\lambda})$. The normalization is chosen so that $\mathcal{O}_{X_{C,E}}(1)$ is "ample".

Theorem 4.4 (Fargues-Fontaine for p-adic E, Hartl-Pink '04 for $E = \mathbf{F}_q((t))$, Kedlaya '04)**.** *Any vector bundle $\mathcal{E}$ on $X_{C,E}$ is isomorphic to a direct sum of $\mathcal{O}(\lambda)$'s. Equivalently,* *the functor*

$$\mathrm{Isoc}_E \to \mathrm{VB}(X_{C,E}) \tag{4.3.1}$$

induces a bijection on isomorphism classes.

More precisely,

(1) *Any* $\mathcal{E}$ *admits a Harder-Narasimhan filtration* $\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$ *such that each*

$$\mathcal{E}^{\lambda} := \mathcal{E}^{\geq \lambda} / \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$$

is semistable of slope λ.

(2) $\mathrm{Isoc}_E^{\lambda} \xrightarrow{\sim} \mathrm{VB}(X_{C,E})^{\lambda}$ *(the subcategory of vector bundles semistable of slope* λ*).*

(3) *The HN filtration splits (but not uniquely). Hence the functor* (4.3.1) *is exact.*

Part (1) of the theorem resembles the theory for all smooth projective curves. Part (2) is similar to $\mathbf{P}^1$ but not other curves. Part (3) is similar to curves of genus $g = 0, 1$ but not higher genus.

Remark 4.5. The functor (4.3.1) is nowhere near to being an equivalence of categories. There are many more maps in $\mathrm{VB}(X_{C,E})$, e.g.

$$H^0(X_{C,E}, \mathcal{O}(n)) = \begin{cases} \infty\text{-dim'l } E\text{-vector space} & n > 0 \\ E & n = 0 \\ 0 & n < 0. \end{cases}$$

We have

$$H^1(X_{C,E}, \mathcal{O}(n)) = \begin{cases} 0 & n > 0 \\ 0 & n = 0 \\ \infty\text{-dim'l } E\text{-vector space} & n < 0. \end{cases}$$

We will revisit the question of "what structure" these cohomology groups have later, when we discuss Banach-Colmez spaces.

Example 4.6. Let $C^{\#}/E$ be some untilt of C. This gives a point $i\colon \operatorname{Spa} C^{\#} \hookrightarrow X_{C,E}$. Then we get

$$0 \to I \to \mathcal{O}_{X_{C,E}} \to i_* C^{\#} \to 0. \tag{4.3.2}$$

It turns out that $I \approx \mathcal{O}_{X_{C,E}}(-1)$. By Remark 4.5 we have $H^1(X_{C,E}, \mathcal{O}(-1)) = C^{\#}/E$. This is an example of a "Banach-Colmez Space" – it is built from finite-dimensional $C^{\#}$-vector spaces and finite-dimensional E-vector spaces.

Example 4.7. Similarly, twisting (4.3.2) by $\mathcal{O}(1)$ gives a short exact sequence

$$0 \to \mathcal{O}_{X_{C,E}} \to \mathcal{O}_{X_{C,E}}(1) \to i_* C^{\#} \to 0.$$

Taking cohomology then gives an extension

$$0 \to E \to H^0(X_{C,E}, \mathcal{O}(1)) \to C^{\#} \to 0.$$

This should be considered as a "non-split" extension, although this is tricky to make precise.

The next goal is to sketch a proof of the classification theorem. The original proofs by Hartl-Pink and Kedlaya rely on very difficult computations. The proof of Fargues-Fontaine is a significant simplification. However, it does use non-trivial input from p-divisible groups. We will give a new proof that involves no computations, but uses heavily the theory of p-adic geometry (perfectoid spaces, diamonds, and v-descent).

We begin with some reductions, which are the same in all known proofs.

4.4. **Classification of line bundles.** For this, the first step is to show that "$\mathcal{O}(1)$ is ample".

Theorem 4.8 (Kedlaya-Liu '15)**.** *For any vector bundle $\mathcal{E}$ on $X_{C,E}$, and all $n \gg 0$, the bundle $\mathcal{E}(n) := \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ is globally generated and $H^1(X_{C,E}, \mathcal{E}(n)) = 0$.*

Remark 4.9. We always have $H^i(X_{C,E}, \mathcal{E}) = 0$ for $i \geq 2$, for dimension reasons.

We omit the proof; it is a fairly direct computation.

Corollary 4.10 (GAGA)**.** *For $P := \bigoplus_{n \geq 0} H^0(X_{C,E}, \mathcal{O}(n))$, and $X_{C,E}^{\mathrm{alg}} := \mathrm{Proj}\,(P)$, then there is a natural map of locally ringed topological spaces*

$$X_{C,E} \xrightarrow{f} X_{C,E}^{\mathrm{alg}}$$

such that $f^* \colon \mathrm{VB}(X_{C,E}^{\mathrm{alg}}) \xrightarrow{\sim} \mathrm{VB}(X_{C,E})$ is an equivalence, and preserves cohomology.

In what sense is this a Corollary? It follows from a formal argument, which applies very abstractly for any locally ringed space satisfying the conclusion of the Theorem.

Remark 4.11. The *scheme-theoretic Fargues-Fontaine curve* $X_{C,E}^{\mathrm{alg}}$ is a regular scheme, Noetherian of Krull dimension 1, locally the spectrum of a PID. This was the incarnation originally studied by Fargues-Fontaine.

Corollary 4.12. *Any line bundle $\mathcal{L} \in \mathrm{Pic}(X_{C,E})$ is isomorphic to $\mathcal{O}(D)$ for some divisor $D \in \bigoplus_{x \in X_{C,E}^{\mathrm{cl}}} \mathbf{Z}$.*

Proposition 4.13. *For any $x \in X_{C,E}^{\mathrm{cl}}$, $\mathcal{O}(x) := I_x^{-1}$ is isomorphic to $\mathcal{O}_{X_{C,E}}(1)$.*

We will give a proof of this next time, using Lubin-Tate theory.

Corollary 4.14. *We have $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$, and $\mathrm{Pic}(X_{C,E}) \cong \mathbf{Z}$. We normalization the identification so that $\mathcal{O}(1) \mapsto 1$.*

One can now define $\deg \mathcal{E}$ to be the image of $\det \mathcal{E} = \wedge^{\mathrm{rank}\,\mathcal{E}}(\mathcal{E}) \in \mathrm{Pic}(X_{C,E}) \cong \mathbf{Z}$. This gives a Harder-Narasimhan formalism, and then a Harder-Narasimhan filtration. That gives part (1) of Theorem 4.4.

It remains to classify bundles that are semistable of slope 0. It is easy to see that the category of finite-dimensional E-vector spaces embeds fully faithfully in $\mathrm{VB}(X_{C,E})^0$, by sending $W \mapsto W \otimes_E \mathcal{O}_{X_{C,E}}$.

The **key point** is that if $\mathcal{E}/X_{C,E}$ is non-zero and semistable of slope 0, then $H^0(X_{C,E}, \mathcal{E}) \neq 0$. Once we know this, we get a map from the trivial bundle, and the cokernel is also semistable of slope 0, so we will win by induction.

Idea: we will allow ourselves to (a priori) enlarge C. Consider the functor $C'/C \mapsto H^0(X_{C',E}, \mathcal{E}_{X_{C',E}})$. This is some functor on the extensions $\{C'/C\}$. The hope is that this functor is representable by a geometric object, whose C'-valued points are $H^0(X_{C',E}, \mathcal{E}_{X_{C',E}})$. That motivates us to consider even more general test objects than C'. However, the construction of $X_{C,E}$ really depends on C being a perfect ring. So what we do is extend the functor to all *perfectoid* C-algebras. This gives a sheaf on the category of (affinoid) perfectoid spaces.

Example 4.15. Fix $C^\#/E$ an untilt of C. Let R be a perfectoid C-algebra. By the tilting equivalence, there is a unique perfectoid untilt $R^\#/C^\#$. We will define $X_{R,E}$ fitting in the diagram below.

$$\begin{array}{ccc} \operatorname{Spa} R^{\#} & \xhookrightarrow{i_{R^{\#}}} & X_{R,E} \\ \downarrow & & \downarrow \\ \operatorname{Spa} C^{\#} & \xhookrightarrow{i_{C^{\#}}} & X_{C,E} \end{array} \qquad \begin{array}{c} 0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow i_{R^{\#}*}\mathcal{O}_{R^{\#}} \longrightarrow 0 \\ \\ 0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow i_{C^{\#}*}\mathcal{O}_{C^{\#}} \longrightarrow 0 \end{array}$$

The exact sequence shows that $H^1(X_{R,E}, \mathcal{O}(-1)) = R^{\#}/E$. So in equal characteristic $R^{\#} = R$, the functor $H^1(X_{C,?}, \mathcal{O}(-1))$ should be $\mathbf{A}^1_C/E$, where $E \subset \mathbf{A}^1_C$ is embedded as a closed subset by the choice of untilt. This is the quotient of $\mathbf{A}^1_C$ by a *pro-étale equivalence relation*, which is pathological compared to what we usually consider in algebraic geometry.

This will be the general picture: $H^0(X_{?,E}, \mathcal{E})$ and $H^1(X_{?,E}, \mathcal{E})$ are quotients of perfectoid spaces under pro-etale quivalence relations. Such objects are called *diamonds*. They are formally analogous to Artin's algebraic spaces, which are equivalence of schemes by étale equivalence relations. We will show using abstract properties of diamonds that $H^0(X_{?,E}, \mathcal{O}(1))$ cannot be trivial. That guarantees sections over some large extension, which we can descend.

Example 4.16. For $n > 0$ in equal characteristic, $H^0(X_{C,E}, \mathcal{O}(n)) \cong \mathfrak{m}^n_{\mathcal{O}_C}$, which is represented by $\mathbb{D}^n_C$.

5. The relative Fargues-Fontaine Curve (Nov 13)

5.1. **Recap of last time.** Recall that our aim is to classify vector bundles on "the" Fargues-Fontaine curve $X_{C,E}$. Last time we started giving the classification of line bundles, and the classification of vector bundles semistable of slope 0. This relies on putting a "geometric structure" on $H^0(X_{C,E}, \mathcal{E})$ for $\mathcal{E} \in \mathrm{VB}(X_{C,E})$. Similarly for $H^1(X_{C,E}, \mathcal{E})$. These are called "Banach-Colmez spaces".

An intermediate aim, which will be essential for everything to come, is to explain these "geometric structures", which are called *diamonds*, and to define the *relative* Fargues-Fontaine curve $X_{S,E}$ for a perfectoid space S. This is where we are headed.

Reference: [S17].

5.2. **The relative Fargues-Fontaine curve.** Recall that a perfectoid algebra over $\mathbf{F}_p$ is just a perfect Tate algebra R over $\mathbf{F}_p$. So it has a topologically nipotent unit ϖ. (In characteristic p, this just means R is perfect.)

A perfectoid space over $\mathbf{F}_p$ is an adic space $X/\mathbf{F}_p$ covered by opens $U = \mathrm{Spa}(R, R^+) \subset X$ such that R is perfectoid.

If $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid, we can mimic the construction of the Fargues-Fontaine curve, replacing $\mathcal{O}_C$ by R^+. Precisely, we start by forming $\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$. Then we look at the open subspace $Y_{(R,R^+),E} := \{[\varpi] \neq 0, \pi \neq 0\} \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$. (Recall that $\pi \in \mathcal{O}_E$ is a uniformizer.) This is an adic space over E.

We may occasionally also consider $\mathcal{Y}_{(R,R^+),E} := \{[\varpi] \neq 0\}$, which is an adic space over $\mathcal{O}_E$. So we have inclusions

$$\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \supset \mathcal{Y}_{(R,R^+),E} \supset Y_{(R,R^+),E}.$$

The spaces $\mathcal{Y}_{(R,R^+),E}, Y_{(R,R^+),E}$ are *analytic* adic spaces, meaning they are locally the adic spectrum of a Tate ring.

There is a radius function

$$\mathrm{rad}\colon \mathcal{Y}_{(R,R^+),E} \to [0, \infty)$$

sending

$$y \mapsto \frac{\log |\varpi(\widetilde{y})|}{\log |\pi(\widetilde{y})|}$$

where $\widetilde{y}$ is the maximal rank 1 generalization[16] of y, and it sends $Y_{(R,R^+),E} \to (0,\infty)$. There is an endomorphism ϕ_{R^+} acting freely and totally discontinuously on $Y_{(R,R^+),E}$, as one sees by multiplying the radius by q:

$$\mathrm{rad} \circ \phi_{R^+} = q \circ \mathrm{rad}\,.$$

Hence we may take the quotient of $Y_{(R,R^+),E}$ by $\phi_R^{\mathbf{Z}}$.

Definition 5.1. We define $X_{(R,R^+),E} = Y_{(R,R^+),E}/\phi_{R_+}^{\mathbf{Z}}$. It is an adic space over E, called the "relative Fargues-Fontaine curve".

Example 5.2 (Equal characteristic)**.** $E = \mathbf{F}_q((t))$. Then $W_{\mathcal{O}_E}(R^+) = R^+[[t]]$. So

$$\mathcal{Y}_{(R,R^+),E} = \mathrm{Spa}(R, R^+) \times_{\mathrm{Spa}\, \mathbf{F}_q} \mathrm{Spa}\, \mathbf{F}_q[[t]] = \mathbb{D}_{\mathrm{Spa}(R,R^+)}$$

the open unit disc. (Note that it is not quasicompact, illustrating that fibered products of affinoids are not affinoid.)

The subspace $Y_{(R,R^+),E} = \mathbb{D}^*_{\mathrm{Spa}(R,R^+)}$ is the punctured open unit disc.

[16]"Rank 1" means that its value group can be embedded in $\mathbf{R}$. We choose such an embedding in order to take the logarithm; it is of course not unique, but the ratio is well-defined.

Note that in this case, $Y_{(R,R^+),E}$ and $\mathcal{Y}_{(R,R^+),E}$ have a structure map over $\mathrm{Spa}(R,R^+)$, but not after quotienting by $\phi_{R_+}^{\mathbf{Z}}$. So $X_{S,E}$ **does not map to** S, even in equal characteristic.

Now we will glue this construction. We claim that for general perfectoid spaces, the diagram $\mathcal{Y}_{S,E} \supset Y_{S,E} \twoheadrightarrow X_{S,E} = Y_{S,E}/\phi_S^{\mathbf{Z}}$ can be glued from the affinoid case.

This is easy to see in positive characteristic, $E = \mathbf{F}_q((t))$, where the whole diagram glues:

$$\begin{array}{ccc}
\mathcal{Y}_{S,E} & = & S \times_{\mathrm{Spa}\,\mathbf{F}_q} \mathrm{Spa}\,\mathbf{F}_q[[t]] \\
\uparrow & & \uparrow \\
Y_{S,E} & = & S \times_{\mathrm{Spa}\,\mathbf{F}_q} \mathrm{Spa}\,\mathbf{F}_q((t)) \\
\downarrow & & \downarrow \\
X_{S,E} & = & Y_{S,E}/\phi_S^{\mathbf{Z}}
\end{array}$$

Note that Φ_S is the identity on the topological space $|S|$. So the map $|Y_{S,E}| \to |S|$ descends to a map of topological spaces $|X_{S,E}| \to |S|$. We would like to argue similarly in the p-adic case.

Remark 5.3. In general, there's a "diamond equation" $\mathcal{Y}_{S,E}^{\diamond} = S \times (\mathrm{Spa}\,\mathcal{O}_E)^{\diamond}$, and the similar diagram of diamonds glues:

$$\begin{array}{ccc}
\mathcal{Y}_{S,E}^{\diamond} & = & S \times (\mathrm{Spa}\,\mathcal{O}_E)^{\diamond} \\
\uparrow & & \uparrow \\
Y_{S,E}^{\diamond} & = & S \times (\mathrm{Spa}\,E)^{\diamond} \\
\downarrow & & \downarrow \\
X_{S,E}^{\diamond} & = & Y_{S,E}^{\diamond}/\phi_S^{\mathbf{Z}}
\end{array}$$

5.3. **Diamonds.** The idea is that diamonds are quotients of perfectoid spaces (of characteristic p) by pro-étale equivalence relations.

There will be a functor from analytic adic spaces over $\mathrm{Spa}\,\mathbf{Z}_p$ to diamonds, denoted $X \mapsto X^{\diamond}$, which you should think of as a generalization of tilting. Recall that tilting took perfectoid spaces of characteristic 0 to perfectoid spaces of characteristic p. In some sense it was like "forgetting the structure morphism over $\mathbf{Q}_p$".

The idea to construct $X^{\diamond}$ is to take a pro-étale cover $\widetilde{X} \to X$ where $\widetilde{X}$ is perfectoid, and then define $X = \widetilde{X}/R$ where $R \subseteq \widetilde{X} \times \widetilde{X}$ is perfectoid and a pro-étale equivalence relation. Then we define $X^{\diamond} = \widetilde{X}^{\diamond}/R^{\diamond}$.

The moral is that after tilting, $\mathcal{Y}_{S,E}$ will decompose into a product as in Remark 5.3. This is congruous with our intuition of what the Fargues-Fontaine curve "should" be (cf. the equal characteristic case).

5.4. **Pro-étale maps of perfectoid spaces.**

Definition 5.4. Let $f\colon Y \to X$ be a map of perfectoid spaces (possibly of mixed characteristic). We say f is *finite étale* if for any open affinoid perfectoid[17] $U = \mathrm{Spa}(R,R^+) \subset X$ (equivalently for a cover by such), the pre-image $V = f^{-1}(U) = \mathrm{Spa}(S,S^+) \subseteq Y$ is affinoid perfectoid, with S is a finite étale R-algebra and $S^+ \subset S$ is the integral closure of R^+.

[17] It is not known if any affinoid subset of a perfectoid space is represented by a perfectoid ring (the definition only stipulates a cover by such).

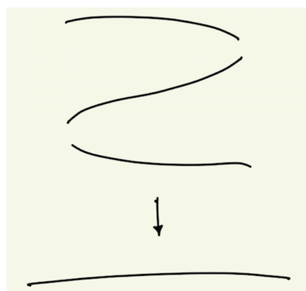
For this to be well-behaved, we are implicitly using that any finite étale algebra over a perfectoid algebra is automatically perfectoid; this is a version of Faltings' almost purity theorem. (This is obvious in characteristic p.) So $\mathrm{Spa}(R, R^+)_{\text{fét}} \cong (\mathrm{Spec}\ R)_{\text{fét}}$.

Definition 5.5. Let $f \colon Y \to X$ be a map of perfectoid spaces (possibly of mixed characteristic). We say f is *étale* if it is locally on X and Y of the form

$$\begin{array}{ccc} Y & \xhookrightarrow{\text{open imm.}} & Y' & \\ & & \downarrow{\scriptstyle \text{finite étale}} & \\ & & U & \xhookrightarrow{\text{open imm.}} & X \end{array}$$

It is non-trivial but true that compositions of étale maps are étale.

Warning 5.6. Note that the analogous characterization is false for schemes. Consider a ramified map of curves, then puncture at the ramification points.



Any Zariski open subset of the domain knows the neighborhood of the ramification points, but not so for analytic open subsets.

Definition 5.7. Let $f \colon Y \to X$ be a map of perfectoid spaces (possibly of mixed characteristic). We say f is *pro-étale* if it is locally on X and Y *affinoid pro-étale*, meaning that $Y = \mathrm{Spa}(S, S^+)$ is a cofiltered limit of the form $\varprojlim_i \mathrm{Spa}(S_i, S_i^+) \to X = \mathrm{Spa}(R, R^+)$ with

$$f_i \colon (\mathrm{Spa}, S_i, S_i^+) \to \mathrm{Spa}(R, R^+) \quad \text{is étale for each } i.$$

We restrict to affinoids because they have all (connected) limits; in particular $S^+ = (\varinjlim S_i^+)^{\wedge}_{\varpi}$ with $S = S^+[1/\varpi]$.

Example 5.8. Suppose $p \neq 2$. Let $X = \mathrm{Spa}(C\langle T^{1/p^\infty}\rangle)$ and $Y = \mathrm{Spa}(C\langle T^{1/2p^\infty}\rangle)$. The map $Y \to X$ looks like it's ramified at the origin. However, we will produce a pro-étale cover $\widetilde{X} \to X$ such that the base change of $Y \to X$ is affinoid pro-étale. This shows that although $Y \to X$ is not pro-étale, it is pro-étale locally in the pro-étale topology, which is weird, and very different from the usual étale topology for schemes.

$$\begin{array}{ccc} \widetilde{Y} & \xrightarrow{\text{pro-étale}} & Y \\ {\scriptstyle \text{affinoid pro-étale}}\downarrow & & \downarrow \\ \widetilde{X} & \xrightarrow{\text{pro-étale}} & X \end{array}$$

Let $U_n = \{|T| \leq 1/p^n\}$ and $U_{n,n+1} = \left\{ \frac{1}{p^{n+1}} \leq |T| \leq \frac{1}{p^n} \right\} \subset U_n$. These are all rational subsets, so affinoid perfectoid. For each n, let $X_n = U_{0,1} \sqcup U_{1,2} \sqcup \ldots U_{n-1,n} \sqcup U_n$. Each $X_n \to X$ is an étale cover.

Now consider $\widetilde{X} = \varprojlim_n X_n \to X$. Note that $\pi_0 \widetilde{X} = \mathbf{N} \cup \{\infty\}$. The fiber of $\widetilde{X}$ over $n \in \mathbf{N}$ is $U_{n,n+1}$, and the fiber over ∞ is $\operatorname{Spa} C$, the origin. So $Y \times_X \widetilde{X} \to Y$ is affinoid pro-étale.

Example 5.9. Any Zariski closed immersion is affinoid pro-étale. Let $X = \operatorname{Spa}(R, R^+) \supset V(f) = Z = \operatorname{Spa}(S, S^+)$. Let $S = R/\overline{(f^{1/p^\infty})} \supseteq S^+$ be the integral closure of R^+. Let $U_n = \{|f| \leq 1/p^n\} \subset X$ be a rational open subset. Consider $U_\infty := \bigcap_n U_n = \varprojlim_n U_n \subset X$. On U_∞, $|f| = 0$ everywhere, so $f = 0$ (perfectoid spaces are reduced, even uniform). So $U_\infty = V(f)$.

Said slightly differently, on U_∞ the function f becomes more and more divisible by ϖ, and then in the completion it is 0.

Note that the analogous construction in algebraic geometry would instead produce the henselization.

Example 5.10. $f\colon \operatorname{Spa} C\langle T^{1/p^\infty}\rangle \to C$ is not pro-étale, for dimension reasons. (The fibers of a pro-étale map are profinite sets.)

Theorem 5.11. *A map $f\colon Y \to X$ of affinoid perfectoid spaces is pro-étale locally on X affinoid pro-étale if and only if:*

- *For all geometric (rank 1) points $\operatorname{Spa}(C, \mathcal{O}_C) \to X$, the fiber product*

$$Y \times_X \operatorname{Spa}(C, \mathcal{O}_C) \to \operatorname{Spa}(C, \mathcal{O}_C)$$

  is affinoid pro-étale, equivalently isomorphic to $\underline{S} \times \operatorname{Spa}(C, \mathcal{O}_C) = \varprojlim_i (S_i \times \operatorname{Spa}(C, \mathcal{O}_C))$ for a profinite set $S = \varprojlim_i S_i$.

Definition 5.12. Such maps are called *quasi-pro-étale*. The point is that this can be checked locally in the pro-étale topology.

5.5. **The v-topology.**

Definition 5.13. A map $f\colon Y \to X$ is a *v-cover* if for any affinoid $U \subset X$ there exists a quasicompact open subset $V \subset Y$ such that $|V| \to |U|$ is surjective. We say f is a *pro-étale cover* if it is a v-cover and f is quasi-pro-étale.

Theorem 5.14. *(1) The presheaves $X \mapsto \mathcal{O}_X(X)$ and $X \mapsto \mathcal{O}_X^+(X)$ are sheaves for the v-topology on perfectoid spaces.*

(2) For any perfectoid space X, $\operatorname{Hom}(-, X)$ is a sheaf for the v-topology.

(3) $X \mapsto \operatorname{VB}(X)$ is a v-stack.

(4) If $X = \operatorname{Spa}(R, R^+)$ is affinoid, then

$$H^i_v(X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

$$H^i_v(X, \mathcal{O}_X^+) \overset{a}{=} 0 \text{ for } i > 0.$$

Here $\overset{a}{=} 0$ means "almost 0", which means killed by ϖ^{1/p^n} for all n.

Remark 5.15. The v-topology is an analog of the fpqc topology, which usually involves a flatness condition plus a topological condition. Note that "everything is flat" in the perfectoid world, so the only condition that remains is the topological condition.

Part (2) suggests that the v-topology is essentially the canonical topology. On affinoid perfectoid spaces, it is literally true that the v-topology is the canonical topology (the finest topology for which representable objects are sheaves).

Proof sketch. First you prove that $X \mapsto \mathcal{O}_X^+(X)$ is a sheaf for the étale topology, and that $H^i_{\text{ét}}(X, \mathcal{O}_X^+) \overset{a}{=} 0$ for $i > 0$ when X is affinoid. For this, you split into the cases of open and finite étale covers, by a combinatorial argument of de Jon and van der Put. For the open covers it is basically "classical". For the finite étale case, this basically amounts to Faltings' almost purity theorem.

Then one gets similar assertions for $\mathcal{O}_X^+/\varpi$. This extends to affinoid pro-étale things by filtered colimits (passing to $\mathcal{O}_X^+/\varpi$ behaves well with respect to colimits, since it is discrete). So $\mathcal{O}_X^+/\varpi$ has good properties as a pro-étale sheaf. Then you bootstrap to $\mathcal{O}_X^+ = \varprojlim_n \mathcal{O}_X^+/\varpi^n$ as well, and then to $\mathcal{O}_X = \mathcal{O}_X^+[1/\varpi]$.

Then, one uses that pro-étale locally v-covers are faithfully flat on $\mathcal{O}^+/\varpi$-level. This seems surprising since we didn't assume any flatness. But the point is that pro-étale locally, you break up your space so much until you get basically just a collection of points, where flatness is automatic. So "pro-étale locally everything is flat". Then you reduce to the usual theory of faithfully flat descent.

□

6. Diamonds and the relative Fargues-Fontaine curve (Nov 16)

6.1. **Setup.** We fix a non-archimedean local field E with residue field $\mathbf{F}_q$. Now $S/\mathbf{F}_q$ is a perfectoid space (replacing $\operatorname{Spa} C$ from before). The aim is to introduce the relative Fargues-Fontaine curve $X_{S,E} = Y_{S,E}/\phi_S^{\mathbf{Z}}$. Intuitively, $Y_{S,E} =$ "$S \times \operatorname{Spa} E$". What is literally true is that at the level of diamonds, $Y_{S,E}^{\diamond} = S \times (\operatorname{Spa} E)^{\diamond}$.

There is a functor $X \mapsto X^{\diamond}$ extending the tilting functor on perfectoid spaces:

$$\begin{array}{ccc} \{\text{analytic adic spaces}/\mathbf{Z}_p\} & \longrightarrow & \{\text{diamonds}\} \\ \uparrow & & \uparrow \\ \{\text{perfectoid spaces}\} & \xrightarrow{\text{tilt}} & \{\text{perfectoid spaces}/\mathbf{F}_p\} \end{array}$$

6.2. **Pro-étale local structure of perfectoid spaces.** Last time we defined pro-étale morphisms of perfectoid spaces.

Definition 6.1. A perfectoid space X is *(strictly) totally disconnected* if it is qcqs[18] and every

- étale cover splits (for strictly totally disconnected)
- open cover splits (for totally disconnected).

Proposition 6.2. *A perfectoid space X is (strictly) totally disconnected if and only if all fibers of* $X \to \underbrace{\pi_0(X)}_{\textit{profinite}}$ *are of the form* $\operatorname{Spa}(K, K^+)$ *where K is a perfectoid field*[19] *and $K^+ \subset \mathcal{O}_K$ is an open valuation subring (resp. K is algebraically closed in the strictly totally disconnected case).*

Remark 6.3. The underlying topological space of a qcqs adic space is always a *spectral space*, which has a profinite set of connected components.

The open-ness of the ring of definition K^+ implies that $K^+ \supset \mathfrak{m}_{\mathcal{O}_K}$. The subring $K^+/\mathfrak{m}_{\mathcal{O}_K} \subset \mathcal{O}_K/\mathfrak{m}_K = k$ is a valuation ring (exercise). This induces a bijection

$$|\operatorname{Spa}(K, K^+)| \cong |\operatorname{Spec}(K^+/\mathfrak{m}_{\mathcal{O}_K})|,$$

where $(K^+/\mathfrak{m}_{\mathcal{O}_K})$ is a valuation ring. This is a totally ordered chain of points. The generic point is $\operatorname{Spa}(K, \mathcal{O}_K) \hookrightarrow \operatorname{Spa}(K, K^+)$. It is always what we referred to as the "rank 1 generalization" of $\operatorname{Spa}(K, K^+)$.

How to prove this? There's a general structure theorem for spectral topological spaces. First, there is a profinite set of connected components. Second, each connected component must be local, since if it had two closed points then you could find two opens which cover and each missing a closed point, and such an open cover could not split. Finally, one knows that the only local analytic adic spaces are of the form $\operatorname{Spa}(K, K^+)$.

Corollary 6.4. *Assume $X = \operatorname{Spa}(R, R^+)$ is totally disconnected. Let $f \colon Y = \operatorname{Spa}(S, S^+) \to X$ be any affinoid adic space over X. Then, $R^+/\varpi \to S^+/\varpi$ is flat for any pseudo-uniformizer $\varpi \in R^+$, and faithfully flat if $|f|$ is surjective.*

This says that "everything is flat" over totally disconnected spaces.

[18] In fact, the definition will force X to be affinoid.

[19] A perfectoid field is a complete non-archimedean non-discretely valued field such that Φ is surjective on $\mathcal{O}_K/p$.

Proof. We can check flatness locally, so on connected components. Then $(R, R^+) = (K, K^+)$ where K^+ is a valuation ring. Note that $S^+ \subseteq S = S^+[1/\varpi]$, so S^+ is ϖ-torsion free. Hence S^+ is torsion free over K^+, and therefore flat over K^+. By base change, we then get that S^+/ϖ is flat over K^+/ϖ.

For the faithful flatness, we use that $|\operatorname{Spa}(K, K^+)| \cong |\operatorname{Spec}(K^+/\varpi)|$. We can also apply the same reasoning to Y, by making a further covering of it by totally disconnected spaces. □

Remark 6.5. The localization onto connected components uses compatibility with filtered colimits. So the ϖ-adic completion prevents from deducing that S^+ is flat over R^+.

This allows us to deduce v-descent results (maps $f\colon Y \to X$ such that X, Y are quasi-compact and $|f|$ surjective) from pro-étale descent and faithfully flat descent.

Definition 6.6. A *diamond* is a pro-étale sheaf on Perf, the category of perfectoid spaces over $\mathbf{F}_p$, that can be written in the form $Y = X/R$ where

- X is a perfectoid space, and
- $R \subset X \times X$ is a pro-étale equivalence relation represented by a perfectoid space such that $s, t\colon R \to X$ are pro-étale.

Here we use the Yoneda embedding $\mathrm{Perf} \hookrightarrow \{\text{pro-étale sheaves on Perf}\}$ sending $X \mapsto \operatorname{Hom}(-, X)$. So we are implicitly using the fact, sketched last time, that representable objects are pro-étale sheaves.

Definition 6.7. A map $f\colon Y \to X$ of pro-étale sheaves on Perf is *quasi-pro-étale* if for all strictly totally disconnected perfectoid spaces X' and maps $X' \to X$, the fiber product $f'\colon Y' = Y \times_X X' \to X'$ is representable in perfectoid spaces, and pro-étale. (The definition we stated last time, Definition 5.12, was slightly different-looking, but it is equivalent because any Y admits a pro-étale cover by a strictly totally disconnected space.).

Some facts:

- The category of diamonds has all fiber products, cofiltered inverse limits (and even all non-empty limits), but no final object. Indeed, the final object would be $\operatorname{Spa} \mathbf{F}_p$, but this is not a perfectoid space because it does not have a topologically nilpotent unit.[20] Moreover, it cannot be made into a perfectoid space through pro-étale covers, as "adjoining a variable" increases the dimension and is therefore not pro-étale.
- If $f\colon Y \to X$ is a quasi-pro-étale map, then Y is a diamond if X is a diamond, and the converse holds if f is surjective as a map of pro-étale sheaves.
- Y is a diamond if and only if there exists a surjective quasi-pro-étale map $X \to Y$ with X a perfectoid space.
- One can introduce an underlying topological space $|Y| = |X|/|R|$. It turns out that this is independent of the presentation.

Example 6.8. Fix a geometric base point $S = \operatorname{Spa}(C, \mathcal{O}_C)$. There is a fully faithful embedding $\mathrm{ProFinSet} \hookrightarrow \mathrm{Perf}_S$, which can be constructed by taking $T = \varprojlim_i T_i$ to

$$\underline{T} \times \operatorname{Spa}(C, \mathcal{O}_C) := \varprojlim_i (T_i \times \operatorname{Spa}(C, \mathcal{O}_C)).$$

Alternatively, this can be written as $\operatorname{Spa}(\operatorname{Cont}(T, C), \operatorname{Cont}(T, \mathcal{O}_C))$. (In this way profinite sets are identified with the affinoid pro-étale spaces over S.)

[20] The reason that we demand topologically units is because we need to have enough points on the curve to where there will be Hecke stacks of modifications.

Recall that any compact Hausdorff space T can be written as a quotient $\widetilde{T}/R$ where $\widetilde{T}$ is a profinite set, i.e. totally disconnected compact Hausdorff space, and $R \subset \widetilde{T} \times \widetilde{T}$ is a closed equivalence relation. For example, $\widetilde{T}$ can be taken to be the Stone-Cech compactification of the underlying set of T as a discrete space.

This gives rise to a fully faithful embedding from {compact Hausdorff spaces} into the category of diamonds over S, taking $T \mapsto (X \in \mathrm{Perf}/S \mapsto \mathrm{Cont}(|X|, T))$. This is represented by $\underline{\widetilde{T}} \times \mathrm{Spa}(C, \mathcal{O}_C)/\underline{R} \times \mathrm{Spa}(C, \mathcal{O}_C)$.

For our purposes, these compact Hausdorff spaces are a nuisance that we want to get rid of, although they are important in condensed math.

Definition 6.9. A diamond $Y = X/R$ is *spatial* if it is qcqs (meaning that we can choose X, R to be qcqs) and $|Y|$ is *spectral*, which means of the following equivalent conditions:

- $|Y|$ is homeomorphic to an inverse limit of finite T_0 spaces,
- $|Y|$ is homeomorphic to Spec A for some ring A,
- $|Y|$ has a good basis of quasicompact open subsets,

and if furthermore $|X| \to |Y|$ is spectral (i.e., the pre-image of a quasicompact open is a quasicompact open).

We say Y is *locally spatial* if it has an open cover by spatial $U \subset Y$. This implies that $|Y|$ is locally spectral.

In practice, all relevant diamonds are locally spatial. Y is spatial if and only if Y is locally spatial and $|Y|$ is qcqs.

Remark 6.10. The category of locally spatial diamonds has all fiber products and all cofiltered inverse limits with qcqs transition maps.

Remark 6.11. For algebraic spaces, qcqs automatically implies the spectrality. That's because étale maps are open, so étale equivalence relations cannot change the topology much, in contrast to pro-étale maps.

6.3. **Structure of locally spatial diamonds.** Let Y be a locally spatial diamond. Then it has an underlying locally spectral space $|Y|$.

For each $y \in |Y|$ we have a localization $Y_y \subset Y$ of Y at y, which is $\varprojlim_{U \ni y} U$. It has a presentation $Y_y = \mathrm{Spa}(C, C^+)/\underline{G}$ where C is a complete algebraically closed nonarchimedean field, $\mathfrak{m}_{\mathcal{O}_C} \subset C^+ \subset \mathcal{O}_C$ is a valuation subring, and G is a profinite group acting continuously and faithfully on C. This action of G to C is formally similar to a Galois action.

6.4. **Diamond functor.** We will now construct the functor

$$\{\text{analytic adic spaces}/\mathbf{Z}_p\} \longrightarrow \{\text{diamonds}\}.$$

Proposition 6.12. *For an analytic adic space $X/\mathbf{Z}_p$, the functor*

$$X^\diamond \colon S \in \mathrm{Perf} \mapsto \{S^\#/\mathbf{Z}_p \text{ untilt of } S \text{ plus map } S^\# \to X\}$$

defines a locally spatial diamond. Moreover, there are canonical equivalences $|X| \cong |X^\diamond|$ and $X_{ét} \cong X^\diamond_{ét}$.

If X is perfectoid then $X^\diamond \cong X^\flat$.

Slogan: "$X^\diamond$ remembers topological information about X, but forgets the structure map to Spa $\mathbf{Z}_p$."

Proof sketch. If X is perfectoid, then the tilting equivalence induces

$$\{S^\#, S^\# \to X\} \xrightarrow{\sim} \{S \to X^\flat\}.$$

This shows that for perfectoid X, $X^\diamond$ is indeed represented by $X^\flat$. The isomorphisms $|X| \cong |X^\diamond|$ and $X_{\text{ét}} \cong X^\diamond_{\text{ét}}$ are also part of the tilting equivalence for perfectoid spaces.

In general, we use that any X admits a pro-étale surjection from a perfectoid space $\widetilde{X}$. Sketch of a construction: locally $X = \operatorname{Spa}(A, A^+)$. Suppose $A/\mathbf{Q}_p$ for simplicity. Then adjoining x^{1/p^∞} is pro-étale whenever $x \in A^\times$, for example $x \in 1 + pA^+$. This defines an affinoid pro-étale perfectoid cover.

□

Remark 6.13. By a proposition of Kedlaya-Liu, the restriction of $X \mapsto X^\diamond$ to semi-normal rigid-analytic spaces over $\mathbf{Q}_p$, as a functor

$$\{\text{seminormal rigid-analytic spaces}/\mathbf{Q}_p\} \longrightarrow \{\text{diamonds}/(\operatorname{Spa}\mathbf{Q}_p)^\diamond\}$$

is fully faithful. Note that $(\operatorname{Spa}\mathbf{Q}_p)^\diamond(S) = \{S^\#/\mathbf{Q}_p \text{ untilt of } S\}$ parametrizes untilts of S.

6.5. **Relative Fargues-Fontaine curve.** Now let's go back to the Fargues-Fontaine curve. For an affinoid perfectoid space $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}_{/\mathbf{F}_q}$, we defined

$$\begin{array}{ccc} \mathcal{Y}_{S,E} & = & \operatorname{Spa} W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\} \\ \uparrow & & \uparrow \\ Y_{S,E} & = & \{\pi \neq 0\} \end{array}$$

Theorem 6.14. $Y^\diamond_{S,E} = S \times (\operatorname{Spa} E)^\diamond$. *In other words, given a perfectoid $T/\mathbf{F}_q$, an untilt $T^\#/Y_{S,E}$ is the same as an untilt $T^\#/E$ plus a map $T \to S$.*

Proof sketch. Given an untilt $T^\#/E$, we need to see that maps $T \to S$ over $\mathbf{F}_q$ are equivalent to maps $T^\# \to Y_{S,E}$ over E.

Let $T = \operatorname{Spa}(A, A^+)$. Then maps $T^\# \to Y_{S,E} \subset \operatorname{Spa} W_{\mathcal{O}_E}(R^+)$ are given by maps $W_{\mathcal{O}_E}(R^+) \to A^+$ such that $[\varpi], \pi$ map to units of A. The condition on π is automatic, since $T^\#$ lives over E. As discussed before, a universal property of the Witt vectors on perfect rings gives an adjunction between maps $W_{\mathcal{O}_E}(R^+) \to A^+$ and $R^+ \to (A^+)^\flat = \varprojlim_{x \mapsto x^p} A^+/p$.

That $[\varpi]$ maps to a unit is equivalent to ϖ mapping to a unit of $A^\flat$.

These are the same as map $T = \operatorname{Spa}(A^\flat, A^{\flat+}) \to S = \operatorname{Spa}(R, R^+)$.

Where does the adjunction come from? By rigidity of perfect rings (they deform uniquely), maps $W_{\mathcal{O}_E}(R^+) \to A^+$ are the same as $R^+ \to A^+/\pi$ (any such map lifts uniquely), which factors uniquely through the inverse limit perfection because R^+ is perfect. □

In particular, there is a canonical map $|Y_{S,E}| \cong |Y^\diamond_{S,E}| \cong |S \times (\operatorname{Spa} E)^\diamond| \to |S|$.

Proposition 6.15. *For $S' \subset S$ an open affinoid subset, $Y_{S',E} \hookrightarrow Y_{S,E}$ is an open immersion with $|Y_{S',E}| = |Y_{S,E}| \times_S |S'|$.*

Proof. This all follows after passing to diamonds, using the diamond equation. □

The proposition then allows to glue $Y_{S,E}$ for general perfectoid spaces $S/\mathbf{F}_q$, such that $Y^\diamond_{S,E} \cong S \times (\operatorname{Spa} E)^\diamond$. Slogan: "$Y_{S,E}$ is *the* analytic adic space over E with $Y^\diamond_{S,E} = (S \times (\operatorname{Spa} E)^\diamond)$ over $(\operatorname{Spa} E)^\diamond$."

Definition 6.16. We define $X_{S,E} = Y_{S,E}/\phi_S^{\mathbf{Z}}$, the "relative Fargues-Fontaine curve". So $X_{S,E}^\diamond = (S/\phi_S^{\mathbf{Z}}) \times (\operatorname{Spa} E)^\diamond$.

Proposition 6.17. *All diamonds are v-sheaves.*

7. Untilts, $\mathcal{O}(1)$, and Lubin-Tate Theory (Nov 21)

7.1. **Untilts.** We fix the usual notation: E a non-archimedean local field, $\mathcal{O}_E \ni \pi$, $\mathbf{F}_q \subset \overline{\mathbf{F}}_q$, $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q)[1/\pi]$ the completion of the maximal unramified extension of E.

For $S \in \mathrm{Perf}_{\mathbf{F}_q}$ a perfectoid spaces, we defined $Y_{S,E} \twoheadrightarrow X_{S,E} = Y_{S,E}/\phi_S^{\mathbf{Z}}$. At the level of diamonds, we have

$$Y_{S,E}^{\diamond} = S \times (\mathrm{Spa}\, E)^{\diamond}.$$

We want to formulate the statement that "untilts = degree 1 Cartier divisors on $Y_{S,E}$".

Locally any S is of the form $\mathrm{Spa}(R, R^+)$ and an untilt will be of the form $S^{\#} = \mathrm{Spa}(R^{\#}, R^{\#+})$. There is a canonical surjection

$$\theta \colon W_{\mathcal{O}_E}(R^+) \twoheadrightarrow R^{\#+}.$$

(This already came up before – by rigidity of perfect rings, such a map is adjoint to $R^+ \to R^{\#+}/\pi$. Any such map factors uniquely through the inverse limit perfection of $R^{\#+}/\pi$, which comes with an identification with R^+.) By a general structure result on integral perfectoid rings (equivalently, "perfect prisms"), we will have $\ker(\theta) = (\xi)$ for ξ a non zero-divisor in $W_{\mathcal{O}_E}(R^+)$.

As in last time, this induces $S^{\#} \hookrightarrow \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{\pi = 0, [\varpi] = 0\}$ with $S^{\#}$ realized as a Cartier divisor $V(\xi) \subset Y_{S,E}$.

7.2. **Closed Cartier divisors.**

Definition 7.1. Let X be a uniform analytic[21] adic space (e.g., X perfectoid). A *closed Cartier divisor* on X is an ideal sheaf $I \subset \mathcal{O}_X$, locally free of rank 1, such that for all affinoid $U \subset X$, the map $I(U) \to \mathcal{O}_X(U)$ has closed image.

In other words, locally I is generated by one element, which is a non zero-divisor generating a *closed* ideal. This closedness condition is necessary to make the quotient well-behaved: $(V(I), \mathcal{O}_X/I)$ will then define an adic space.

Warning 7.2. The condition cannot be checked on an open cover of affinoids – it must really be checked for *every* affinoid $U \subset X$. So to check that the (ξ) which came up in §7.1 is a closed Cartier divisor, there is a bit more work to do.

Proposition 7.3. *(1) $V(\xi) = S^{\#} \hookrightarrow Y_{S,E}$ is a closed Cartier divisor.*
(2) The composite $S^{\#} \hookrightarrow Y_{S,E} \to X_{S,E}$ is a closed Cartier divisor.

Definition 7.4 ("Moduli space of degree 1 Cartier divisors")**.** Let Div^1_Y, Div^1_X be the functors $\mathrm{Perf}_{\overline{\mathbf{F}}_q} \to \mathrm{Sets}$ taking S to the set of closed Cartier divisors on $Y_{S,E}$ (resp. $X_{S,E}$) that locally on S arise as $S^{\#} \hookrightarrow Y_{S,E}$ (resp. $S^{\#} \hookrightarrow X_{S,E}$) for untilts $S^{\#}/E$.

Remark 7.5. The functor could have been defined on $\mathrm{Perf}_{\mathbf{F}_q}$. However later we'll see that we want to work geometrically for various reasons.

Proposition 7.6. *We have the following identification of diamonds.*

(1) $\mathrm{Div}^1_Y = (\mathrm{Spa}\, \breve{E})^{\diamond}$.
(2) $\mathrm{Div}^1_X = \mathrm{Div}^1_Y / \phi_E^{\mathbf{Z}} = (\mathrm{Spa}\, \breve{E})^{\diamond}/\phi_E^{\mathbf{Z}}$.

To explain the ϕ_E appearing in (2), note that $(\mathrm{Spa}\, \breve{E})^{\diamond} = (\mathrm{Spa}\, E)^{\diamond} \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$, and $(\mathrm{Spa}\, E)^{\diamond}$ is a functor on $S \in \mathrm{Perf}_{\mathbf{F}_q}$ hence has a Frobenius coming from the Frobenius on the test category. By the same token, *any* diamond has a Frobenius.

[21] Analytic means that X is locally $\mathrm{Spa}(R, R^+)$ for a Tate ring R. Uniform means that the spectral norm on R is a norm, or equivalently that $R^0 \subset R$ is bounded. Hence $R^+ \subset R$ is bounded.

Proof. (1) By definition, $(\operatorname{Spa}\breve{E})^\diamond \twoheadrightarrow \operatorname{Div}^1_Y$ is surjective, as any Cartier divisor parametrized by Div^1_Y locally comes from an untilt $S^\#/E$. Since we are considering objects over $\overline{\mathbf{F}}_p$, the E-structure on $S^\#$ automatically upgrades to a $\breve{E}$-structure.

But conversely, a closed Cartier divisor on $Y_{S,E}$ determines $Z \subset Y_{S,E}$, which locally on S is an untilt of S. This glues to a global untilt $S^\#$ of S. Hence $(\operatorname{Spa}\breve{E})^\diamond \xrightarrow{\sim} \operatorname{Div}^1_Y$.

(2) Take the quotient by Frobenius in part (1). The map is still surjective; we just have to match up which things are identified; this is left as an exercise. □

Warning 7.7. Note that $X^\diamond_{S,E} = (S/\phi_S^{\mathbf{Z}}) \times (\operatorname{Spa} E)^\diamond$ but $\operatorname{Div}^1_X = (\operatorname{Spa}\breve{E})^\diamond/\phi_E^{\mathbf{Z}}$. The quotients are being taken for different Frobenii. Fargues calls Div^1_X the "mirror curve". It is "only" a diamond (i.e., not coming from an adic space).

Note that the topological space of Div^1_X is a point. It is not quasi-separated or locally spatial (since it has no interesting subsets, whereas spatial includes quasi-separated.)

Slogan: "the moduli space of points on the curve is *not* the same as the curve".

7.3. $\mathcal{O}(1)$ **and untilts.** Recall that $\mathcal{O}_{X_{S,E}}(1)$ is the line bundle on $X_{S,E}$ corresponding to the isocrystal $(\breve{E}, \pi^{-1}\sigma)$. In particular, by descent we have

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y, \mathcal{O}_{Y_{S,E}})^{\phi_S=\pi}.$$

The goal is to see that if $S^\#$ is any untilt$/E$ of S, then $I_{S^\#} \subset \mathcal{O}_{X_{S,E}}$ is (after pro-étale localization on S) isomorphic to $\mathcal{O}_{X_{S,E}}(-1)$.[22]

Remark 7.8. In this sense $S^\# \hookrightarrow X_{S,E}$ is a divisor of "degree 1". Later we will prove that for S a geometric point, $\operatorname{Pic}(X_{S,E}) \cong \mathbf{Z}$, and this will give a precise notion of degree.

To prove the goal, we want to construct maps $\mathcal{O}(-1) \overset{?}{\cong} I_{S^\#} \hookrightarrow \mathcal{O}$. By twisting, this is equivalent to maps $\mathcal{O} \to \mathcal{O}(1)$, i.e. construct elements $H^0(X_{S,E}, \mathcal{O}(1))$ vanshing along $S^\#$. We will give a "formula" for $H^0(X_{S,E}, \mathcal{O}(1))$ in terms of Lubin-Tate formal groups.

7.4. **Lubin-Tate theory.** Recall that a *Lubin-Tate formal group* is a 1-dimensional formal group $G/\mathcal{O}_{\breve{E}}$, with an action of $\mathcal{O}_E$ such that the two induced actions on $\operatorname{Lie}(G)$ agree.

A 1-dimensional formal group law has $[\pi]_G(X) = \pi X + O(X^2)$, and the first non-zero term of $[\pi]_G(X) \pmod{\pi}$ is $a_{q^h} X^{q^h}$ for some $h = 1, 2, \ldots, \infty$. This h is called the *height* of G.

Theorem 7.9 (Lubin-Tate)**.** *Up to (very non-unique) isomorphism, there is a unique Lubin-Tate formal group of any fixed height h.*

Example 7.10. We are interested in Lubin-Tate formal groups of height 1. Suppose $E = \mathbf{Q}_p$. Then we can take $G = \widehat{\mathbf{G}}_m$ to be the formal multiplicative group $\operatorname{Spf}\breve{\mathbf{Z}}_p[[X]]$, and $X +_G Y = (1+X)(1+Y) - 1$.

For any Lubin-Tate formal group G, over the generic fiber we have an isomorphism $\log_G \colon G \times_{\mathcal{O}_{\breve{E}}} \breve{E} \xrightarrow{\sim} \widehat{\mathbf{G}}_{a,\breve{E}}$ with the additive group. (All 1-dimensional formal groups in characteristic 0 are isomorphic to the formal additive group, by the logarithm.) Then the group law on G can be written as

$$X +_G Y = \exp_G(\log_G(X) + \log_G(Y)).$$

[22] These isomorphisms will not glue in general.

Remark 7.11. We can choose the isomorphism $G \cong \operatorname{Spf} \mathcal{O}_{\breve{E}}[[X]]$ so that its logarithm takes the shape

$$\log_G(X) = X + \frac{1}{\pi}X^q + \frac{1}{\pi^2}X^{q^2} + \dots$$

This gives a way to construct G: transport the addition on $\mathbf{G}_a$ to G via the logarithm, and check that the coefficients are integral.

7.5. **Connection to local class field theory.** For any $n \geq 1$, let $G[\pi^n] \subset G$ be the kernel of multiplication by π^n on G. So $G[\pi^n] \cong \operatorname{Spf} \mathcal{O}_{\breve{E}}[[X]]/([\pi^n]_G(X))$.

Example 7.12. For $E = \mathbf{Q}_p$, we have $G \cong \widehat{\mathbf{G}}_m$ as discussed earlier, so $G[\pi^n] \cong \mu_{p^n}$.

We have $G[\pi^n] \times_{\mathcal{O}_{\breve{E}}} \breve{E} = \sqcup_{i=0}^n \operatorname{Spec} \breve{E}_i$ where $\breve{E}_0 = \breve{E}$, and $\breve{E}_0 \subset \breve{E}_1 \subset \ldots \subset \breve{E}_n \subset \ldots$ is a tower of extensions with $\breve{E}_n/\breve{E}_0$ obtained by adjoining a primitive π^n-torsion point.

Theorem 7.13. *The maximal abelian-over-E extension of $\breve{E}$ is $\bigcup_n \breve{E}_n$, and we have a canonical isomorphism*

$$\operatorname{Gal}(\breve{E}_n/\breve{E}) \cong (\mathcal{O}_E/\pi^n)^\times.$$

Also, $\breve{E}_\infty :=$ completion of $\bigcup_n \breve{E}_n$ is a perfectoid field.

7.6. **Universal cover of G.**

Definition 7.14. We define $\widetilde{G} := \varprojlim_{[\pi]_G} G$ (limit in the category of formal schemes).

Proposition 7.15. *There is an isomorphism $\widetilde{G} \cong \operatorname{Spf} \mathcal{O}_{\breve{E}}[[\widetilde{X}^{1/p^\infty}]]$.*

Proof. Because $\widetilde{G}$ is p-adically complete and flat over $\mathcal{O}_{\breve{E}}$, by rigidity of such objects it's enough to show that $\widetilde{G} \times_{\mathcal{O}_{\breve{E}}} \overline{\mathbf{F}}_q \cong \operatorname{Spf} \overline{\mathbf{F}}_q[[\widetilde{X}^{1/p^\infty}]]$. But modulo π, the formal group law was arranged to by $X \mapsto X^q$, so this is clear. □

We have projection maps $f_n \colon \widetilde{G} \to G$ given at the level of rings by $\mathcal{O}_{\breve{E}}[[\widetilde{X}^{1/p^\infty}]] \leftarrow \mathcal{O}_{\breve{E}}[[X_n]]$. One checks that

$$\widetilde{X} = \lim_{n\to\infty} X_n^{q^n}.$$

On the generic fibers, we have a map $\widetilde{G}^{\mathrm{ad}} \times_{\mathcal{O}_{\breve{E}}} \breve{E} \to G^{\mathrm{ad}} \times_{\mathcal{O}_{\breve{E}}} \breve{E} \xrightarrow{\log_G} \mathbf{G}_a$. At the level of adic spaces, $G^{\mathrm{ad}} \times_{\mathcal{O}_{\breve{E}}} \breve{E}$ is an open unit disk and $\widetilde{G}^{\mathrm{ad}} \times_{\mathcal{O}_{\breve{E}}} \breve{E}$ is an infinite cover. The composite function is given by (exercise)

$$\sum_{i\in\mathbf{Z}} \pi^i \widetilde{X}^{q^{-i}} \in \mathcal{O}(\widetilde{G}^{\mathrm{ad}} \times_{\mathcal{O}_E} \breve{E}).$$

7.7. **Relation to global sections.** Let $S = \operatorname{Spa}(R, R^+) \in \operatorname{Perf}_{\overline{\mathbf{F}}_q}$. We have the Fargues-Fontaine curve $X_{S,E}$, and the line bundle $\mathcal{O}_{X_{S,E}}(1)$. Let $S^\# = \operatorname{Spa}(R^\#, R^{\#+})$ be an untilt of S. Then the logarithm defines $\widetilde{G}(S^\#) = \widetilde{G}(R^{\#+}) \xrightarrow{\sim} R^{\circ\circ}$, which has a map to $H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})$ sending $X \mapsto \sum_{i\in\mathbf{Z}} \pi^i [X^{q^{-i}}]$.

Proposition 7.16. *This induces an isomorphism*

$$\widetilde{G}^{\mathrm{ad}}(S^\#) \xrightarrow{\sim} H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})^{\phi_S = \pi}.$$

Under this isomorphism, the evaluation $H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}}) \to R^\#$ at $S^\# \subseteq Y_{S,E}$ corresponds to the logarithm map

$$\widetilde{G}^{\mathrm{ad}}(S^\#) = \widetilde{G}(R^{\#+}) \xrightarrow{\log_{\widetilde{G}}} R^\#$$

In particular,

$$H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) \cong \mathrm{Hom}(S, \mathrm{Spa}\, \overline{\mathbf{F}}_q[[\widetilde{X}^{1/p^\infty}]])$$

Remark 7.17. If $n \leq [E : \mathbf{Q}_p]$ (resp. all n if E has characteristic p) then we can compute $H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(n)) \cong \mathrm{Hom}(S, \mathrm{Spa}\, \overline{\mathbf{F}}_q[[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]])$. Slogan: "the functor $S \mapsto H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(n))$ is represented[23] by $\mathrm{Spa}\, \overline{\mathbf{F}}_q[[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}]]$ as an n-dimensional perfectoid open unit disc."

However, if $n > [E : \mathbf{Q}_p]$ then this functor is not representable. These functors are called Banach-Colmez spaces, and they will be interesting examples of diamonds.

Proof. The commutation with $\log_{\widetilde{G}}$ follows from inspection of the formulas. It is clear that one gets a map to $H^0(X_{S,E}, \mathcal{O}_{X_{S,E}}(1)) = H^0(Y_{S,E}, \mathcal{O})^{\phi=\pi}$, as from inspection of the formulas the Frobenius evidently multiplies by π.

The equal characteristic case follows from a direct computation. If $E = \mathbf{F}_q((t))$, then $Y_{S,E}$ is a punctured open unit disc so $H^0(Y_{S,E}, \mathcal{O})$ are certain explicit rings of power series $\sum_{n \in \mathbf{Z}} t^n r_n$, with $r_n \in R$, subject to convergence on the punctured open unit disk. The condition that ϕ_S is multiplication by $\pi = t$ is equivalent to $r_{n+1}^q = r_n$. So everything is determined by $r_0 = r \in R$, and this is subject to the condition $\sum_{n \in \mathbf{Z}} \pi^n r^{1/q^n}$ converges. It is easy to check that this happens if and only if $r \in R^{\circ\circ}$ is topologically nilpotent.

If $E/\mathbf{Q}_p$, then this can be deduced from a result in Dieudonné theory in [SW]. □

Warning 7.18. If E is p-adic, then one *cannot* describe $H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})$ as certain sums

$$\sum_{n \in \mathbf{Z}} \pi^n [r_n], \quad r_n \in R.$$

More precisely, it is known that there cannot be such unique power series expansions for all elements of $H^0(Y_{S,E}, \mathcal{O}_{Y_{S,E}})$. Otherwise, one could give a simple description of $H^0(X_{S,E}, \mathcal{O}_{Y_{S,E}}(n))$ as in the function field case.

In particular, we had a short exact sequence of étale sheaves

$$0 \to \bigcup_n G^{\mathrm{ad}}_{\breve{E}}[\pi^n] \to G^{\mathrm{ad}}_{\breve{E}} \xrightarrow{\log_G} \mathbf{G}_{a,\breve{E}} \to 0.$$

Passing to universal covers, we get

$$0 \to V_\pi(G^{\mathrm{ad}}_{\breve{E}}) \to \widetilde{G}^{\mathrm{ad}}_{\breve{E}} \xrightarrow{\log_G} \mathbf{G}_{a,\breve{E}} \to 0$$

where V_π is the rational π-adic Tate module. As G had height 1, we have $V_\pi(G^{\mathrm{ad}}_{\breve{E}}) \cong E$ on geometric points. As an adic space, we have

$$V_\pi(G^{\mathrm{ad}}_{\breve{E}}) \setminus \{0\} = \bigcup_{n \in \mathbf{Z}} \mathrm{Spa}\, \breve{E}_\infty \tag{7.7.1}$$

as an element of $V_\pi(G^{\mathrm{ad}}_{\breve{E}})$ involves a compatible choice of π^n-roots for all n. In particular, given an untilt $S^\#/\breve{E}_\infty$, we get a distinguished section $s \in \widetilde{G}^{\mathrm{ad}}(S^\#)$ corresponding to the chosen compatible π^n-torsion points, which map to 0 under the logarithm. Thus, under the isomorphism of the Proposition, we get a map

$$\mathcal{O}_{X_{S,E}} \to \mathcal{O}_{X_{S,E}}(1).$$

[23] The "representing" object is not quite a perfectoid space.

corresponding to s. Further composing with the evaluation map $\mathcal{O}_{X_{S,E}}(1) \to \mathcal{O}_{S^\#}$, the composite is 0 because s came from $V_\pi \subset \ker(\log_G)$. So the above map factors through $\mathcal{O}_{X_{S,E}} \to I_{S^\#}(1)$.

Proposition 7.19. *This map $\mathcal{O}_{X,S} \to I_{S^\#}(1)$ is an isomorphism.*

This is formal if you think about it in the right way. The key is to understand the identification (7.7.1). You reduce to the universal case $S^\# = \operatorname{Spa} \breve{E}_\infty$. In this case $Y_{S,E}$ gets canonically identified with $\widetilde{G}^{\mathrm{ad}}_{\breve{E}} \setminus \{0\}$, as both are $\operatorname{Spa} \mathcal{O}_{\breve{E}}[[\widetilde{X}^{1/p^\infty}]] \setminus \{\pi = 0 \text{ or } \widetilde{x} = 0\}$. Then you have to analyze the vanishing locus of the section $\mathcal{O}_{X_{S,E}} \to \mathcal{O}_{X_{S,E}}(1)$, and see that it has order 1 at all the relevant points. You can then go to $Y_{S,E}$ and check it there.

For a general untilt, which only lives over $\breve{E}$, we get an identification $I_{S^\#} \cong \mathcal{O}(-1)$ after pro-étale cover $S^\# \times_{\breve{E}} \breve{E}_\infty \to S^\#$.

This also completes the classification of line bundles: all the ideal sheaves of untilts are isomorphic.

Remark 7.20. More generally, we can consider any π-divisible $\mathcal{O}_E$-module $G/\overline{\mathbf{F}}_q$. It has a Dieudonné module $(V/\breve{E}, \phi_V)$. We then have $\widetilde{G}(S^\#) \cong \widetilde{G}(S) \cong H^0(X_{S,E}, \mathcal{E}(V))$.

8. Banach-Colmez spaces and the classification of vector bundles (Nov 23)

8.1. **Recap of last time.** Fix E, π. For $S \in \mathrm{Perf}_{\mathbf{F}_q}$, we will abbreviate $Y_S := Y_{S,E}$ and $X_S := X_{S,E}$.

8.1.1. *Divisors on Y.* Last time we were talking about "sections of $Y_{S,E} \dashrightarrow S$". We showed that the following sets are canonically in bijection.

- Sections of $Y_{S,E}^{\diamond} \to S$.
- Maps $S \to (\operatorname{Spa} E)^{\diamond}$.
- Untilts $S^{\#}/E$ of S.
- Degree 1 closed Cartier divisors $D \subset Y_{S,E}$. (In terms of the previous bullet point, $D \cong S^{\#}$.)

The moduli problem Div_Y^1 classifies these sets.

8.1.2. *Divisors on X.* There was also a variant for X_S. The following sets are canonically in bijection.

- Maps $S \to (\operatorname{Spa} E)^{\diamond}/\varphi_S^{\mathbf{Z}}$.
- Degree 1 closed Cartier divisors $D \subset X_S$.

The moduli problem Div_X^1 classifies these sets.[24]

We often work over $\mathrm{Perf}_{\overline{\mathbf{F}}_p}$ instead of $\mathrm{Perf}_{\mathbf{F}_p}$. Then we get $\mathrm{Div}_Y^1 \cong (\operatorname{Spa} \breve{E})^{\diamond}$, $\mathrm{Div}_X^1 \cong (\operatorname{Spa} \breve{E})^{\diamond}/\varphi_S^{\mathbf{Z}}$. Another presentation of this will be useful. Take $C = \widehat{\overline{E}}$. Then

$$(\operatorname{Spa} \breve{E})^{\diamond}/\varphi_S^{\mathbf{Z}} = (\operatorname{Spa} C)^{\diamond}/\underbrace{(I_E \rtimes \varphi_S^{\mathbf{Z}})}_{W_E},$$

where W_E is the Weil group of E. Hence $\pi_1(\mathrm{Div}_X^1) \cong W_E$.

8.1.3. *The fundamental group.* There will be a map $\mathrm{Div}_X^1 \to \mathrm{Pic}_X^1$, where Pic_X^1 is the moduli space of line bundles of degree 1, sending $D \mapsto \mathcal{O}(D)$. The bundle $\mathcal{O}(1)$ defines a particular $\mathrm{pt} \to \mathrm{Pic}_X^1$. The discussion of last time computed the fiber product in terms of Lubin-Tate groups:

$$\begin{array}{ccc} (\operatorname{Spa} \breve{E}_\infty)^{\diamond} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathrm{Div}_X^1 & \xrightarrow{\mathrm{AJ}^1} & \mathrm{Pic}_X^1 \end{array} \tag{8.1.1}$$

Let $G/\mathcal{O}_{\breve{E}}$ be the Lubin-Tate group with $\log_G(X) = X + \frac{1}{\pi}X^q + \frac{1}{\pi^2}X^{q^2} + \ldots$. Then we defined $\breve{E}_\infty = \breve{E}(G[\pi^\infty])^\wedge$.

We showed that the following sets are canonically in bijection.[25]

- Maps $S \to (\operatorname{Spa} \breve{E}_\infty)^{\diamond} = \operatorname{Spa} \breve{E}_\infty^\flat$. (Explicitly, $\breve{E}_\infty^\flat \cong \overline{\mathbf{F}}_q((X^{1/p^\infty}))$.
- Degree 1 closed Cartier divisors $D \subset X_S$ plus an isomorphism $\mathcal{O}(D) \cong \mathcal{O}(1)$.

The vertical maps in (8.1.1) are $\underline{E}^\times$-torsors.

The Abel-Jacobi map $\mathrm{AJ}^1 \colon \mathrm{Div}_X^1 \to \mathrm{Pic}_X^1 = [\mathrm{pt}/\underline{E}^\times]$ gives a map $W_E = \pi_1(\mathrm{Div}_X^1) \to \pi_1(\mathrm{Pic}_X^1) = E^\times$, which is the Artin-Reciprocity map of local class field theory.

Fargues in [F17] builds on this to give a new proof of local class field theory, imitating Deligne's argument for geometric class field theory. (Any 1-dimensional character of W_E

[24] One could also describe this in terms of "untilts up to Frobenius", but that requires sheafification.

[25] The bijections depend on the choice of π, however.

induces a rank 1 local system on $\mathrm{Div}^d_X = (\mathrm{Div}^1_X)^d/\Sigma_d$, by descending the dth exterior power. The key point is to show that the fibers of AJ^d are simply connected for $d \gg 0$, which allows to descend to Pic^d_X.)

8.2. **Banach-Colmez spaces.** Reference: [LB18].

Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$, and $\mathcal{E}$ a vector bundle (i.e., a locally free $\mathcal{O}_{X_S}$-module of finite rank) on X_S.

Results of Kedlaya-Liu show that the notion of vector bundles on adic spaces is well-behaved.

Theorem 8.1 (Kedlaya-Liu)**.** *If $X = \mathrm{Spa}(A, A^+)$ is an affinoid analytic adic space (so $\mathcal{O}_X$ is a sheaf), then we have an equivalence of categories*

$$\mathrm{VB}(X) \xleftarrow{\sim} \{\textit{finite projective } A\textit{-modules}\}/\sim$$

sending $M \otimes_A \mathcal{O}_X \leftarrow M$.

Furthermore, we have $H^i(X, \mathcal{E}) = 0$ for $i > 0$ and $\mathcal{E} \in \mathrm{VB}(X)$, and also for $H^i_{\acute{e}t}$ if $\mathcal{O}_X$ is an étale sheaf[26].

Proposition 8.2. *If S is affinoid, then $H^i(X_S, \mathcal{E}) = 0$ for $i \geq 2$ and $H^i(Y_S, \mathcal{E}) = 0$ for $i \geq 1$.*

Proof sketch. Pick ϖ a pseudo-uniformizer. We have a radius function $\mathrm{rad}\colon Y_S \to (0, \infty)$ comparing $|[\varpi]|$ and $|\pi|$. In our normalization, ϕ_S multiplies the radius by q.

For an interval $I = [a, b]$ with $a, b \in \mathbf{Q}$, we have a rational subset $Y_{S,I} \subset Y_S \subset \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+)$. This can be described as

$$\{|[\varpi]|^b \leq |\pi| \leq |[\varpi]|^a \neq 0\} = Y_{S,I} \subset \mathrm{rad}^{-1}(I).$$

We remark that the inclusion is an equality on rank 1 points, but $Y_{S,I}$ is open while $\mathrm{rad}^{-1}(I)$ is closed (since I is a closed interval). So $Y_{S,I}$ is affinoid and analytic. Then X_S can also be presented as

$$Y_{S,[1,q]}/(Y_{S,[1,1]} \xrightarrow{\sim}_{\varphi} Y_{S,[q,q]}). \tag{8.2.1}$$

This cover allows to build a Cech complex computing the cohomology of X_S:

$$R\Gamma(X_S, \mathcal{E}) \cong \left[\mathcal{E}(Y_{S,[1,q]}) \xrightarrow{\varphi-1} \mathcal{E}(Y_{S,[q,q]})\right] \tag{8.2.2}$$

This gives vanishing in degree ≥ 2.

For Y_S, we write $Y_S = \bigcup_I Y_{S,I}$ and the transition maps $\mathcal{O}(Y_{S,I'}) \to \mathcal{O}(Y_{S,I})$ have dense image. Then

$$R\Gamma(Y_S, \mathcal{E}) = R \varprojlim_I \mathcal{E}(Y_{S,I})$$

and the $R \varprojlim^1$ vanishes by Mittag-Leffler. Slogan: "Y_S is Stein". □

Remark 8.3. We could also have deduced the vanishing for X from the vanishing for Y by using group cohomology. However, the proof also gave a recipe to compute the cohomology of X in terms of an *affinoid*, which is useful in practice.

[26]This will always be the situation for us. The usual criteria which show that $\mathcal{O}_X$ is a sheaf in the analytic topology will show that $\mathcal{O}_X$ is a sheaf in the étale topology.

Proposition 8.4. *The functor $T \in \mathrm{Perf}_{/S} \mapsto H^0(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf. In fact, $T \mapsto R\Gamma(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf of complexes in $D(\mathbf{Z})$.*

In particular, if $H^0(X_T, \mathcal{E}|_{X_T}) = 0$ for all $T \in \mathrm{Perf}_{/S}$, then $T \mapsto H^1(X_T, \mathcal{E}|_{X_T})$ is a v-sheaf.

Proof. It is enough to check this after $\widehat{\otimes}_E E_\infty$, because $E \hookrightarrow E_\infty$ splits as an inclusion of Banach E-vector spaces (e.g. by Hahn-Banach). Now the point is that E_∞ is perfectoid, and $X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ is perfectoid. So v-covers on S induce v-covers of this object. Now we use v-sheaf and acyclicity properties for general perfectoid spaces. □

Definition 8.5. (1) The functor $\mathcal{BC}(\mathcal{E})\colon \mathrm{Perf}_{/S} \to \mathrm{Sets}$ sending $T \mapsto H^0(X_T, \mathcal{E}|_{X_T})$ is the *Banach-Colmez space* associated to $\mathcal{E}$. (For a usual curve, this would be an affine space of dimension $h^0(X, \mathcal{E})$.

(2) If $\mathcal{BC}(\mathcal{E}) = 0$, then we define $\mathcal{BC}(\mathcal{E}[1])\colon T \to H^1(X_T, \mathcal{E}|_{X_T})$ to be the *negative Banach-Colmez space* associated to $\mathcal{E}$.

Proposition 8.6. *(1) $\mathcal{BC}(\mathcal{E})$, $\mathcal{BC}(\mathcal{E}[1])$ are locally spatial diamonds over S.*

(2) If $E = \mathbf{F}_q((t))$, $\mathcal{BC}(\mathcal{E})$ is even represented by a perfectoid space.

(3) If $\mathcal{E} = \mathcal{O}(\lambda)$ for $0 < \lambda \leq [E : \mathbf{Q}_p]$, writing $\lambda = \frac{r}{s}$ with $(s, r) = 1$ and $r, s > 0$, then $\mathcal{BC}(\mathcal{E}) \cong \widetilde{\mathbb{D}}^r_S$ is an r-dimensional perfectoid disk over S, and for S affinoid $H^1(X_S, \mathcal{E}) = 0$.

(4) $R\Gamma(X_S, \mathcal{O}_{X_S}) \cong R\Gamma_{\text{proét}}(S, \underline{E})$. In particular, $R\Gamma(X_C, \mathcal{O}_{X_C}) \cong E[0]$.

Sketch. Start with (3). This is similar to the identification last time, where we computed

$$\mathcal{BC}(\mathcal{O}(1)) \cong \widetilde{\mathbb{D}}_S.$$

Say $E = \mathbf{Q}_p$ for simplicity. Then $\mathcal{BC}(\mathcal{O}(\lambda)) \cong \widetilde{G}_S$ where $\widetilde{G}$ is the universal cover of the p-divisible group $G/\overline{\mathbf{F}}_q$ with Dieudonné module $D_{-\lambda}$.

The vanishing of H^1 can be proved by direct computation, using (8.2.2). The details are left as an exercise.

(4) Pro-étale locally on S, we use the exact sequence

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S}(1) \to \mathcal{O}_{S^\#} \to 0.$$

On global sections, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{O}) & \longrightarrow & H^0(\mathcal{O}(1)) & \xrightarrow{\log_{\widetilde{G}}} & R^\# & \longrightarrow & H^1(\mathcal{O}) & \longrightarrow & 0 \\ & & & & \| & & & & & & \\ & & & & \widetilde{G}(S^\#) & & & & & & \end{array}$$

Now use that $\log_{\widetilde{G}}$ is pro-étale locally surjective, with kernel identified with E.

(1) and (2): bootstrap from (3) using various exact sequences. Example, $\mathcal{BC}(\mathcal{O}(-1)[1]) \cong (\mathbf{A}^1_E)^\diamond/\underline{E}$ for $S/(\mathrm{Spa}\, E^{\mathrm{LT}}_\infty)^\diamond$ use

$$0 \to \mathcal{O}_{X_S}(-1) \to \mathcal{O}_{X_S} \to \mathcal{O}_{S^\#} \to 0$$

and you get

$$0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_{S^\#}) \to H^1(\mathcal{O}(-1)) \to H^1(\mathcal{O})$$

and use that $H^0(\mathcal{O}_{S^\#}) = (\mathbf{A}^1_E)^\diamond(S)$, $H^0(\mathcal{O}) = \underline{E}(S)$, and $H^1(\mathcal{O})$ is locally zero. This takes care of shifting by 1. The case of $\mathcal{O}(1/n)$ can be handled directly. □

8.3. **Classification of vector bundles.** Back to $S = \operatorname{Spa} C$ a geometric point.

Theorem 8.7. *We have a bijection*

$$\mathrm{Isoc}_E / \sim \leftrightarrow \mathrm{VB}(X_C)/\sim,$$

explicitly, any $\mathcal{E} \in \mathrm{VB}(X_C)$ is isomorphic to

$$\bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}_{X_C}(\lambda)^{n_\lambda} \text{ for unique } n_\lambda \in \mathbf{Z}.$$

8.3.1. *Step 1.* First establish that $\mathcal{O}_{X_C}(1)$ is ample. This means that for any $\mathcal{E}$ and all $n \gg 0$, $\mathcal{E}(n)$ is globally generated and $H^1(X_C, \mathcal{E}(n)) = 0$. (This works for any affinoid S, for any affinoid S there is an algebraic version of the Fargues-Fontaine curve.)

This is a theorem of Kedlaya-Liu, which is proved by hands-on computation using (8.2.2) and explicit estimates. The task is to find enough Frobenius invariants, up to Frobenius twist, and the point is that twisting enough makes certain series converge.

8.3.2. *Step 2.* Prove that $\mathrm{Pic}(X_C) \cong \mathbf{Z}$, via $\mathcal{O}_{X_C}(n) \leftarrow n$. Step 1 implies that any $\mathcal{L} \in \mathrm{Pic}(X_C)$ is generically trivial on the schematic curve. So $\mathcal{L} \cong \mathcal{O}(D)$ for some divisor D on the schematic curve. But all closed points on the schematic curve correspond to untilts, whose divisors are isomorphic to $\mathcal{O}(1)$. Hence $\mathcal{O}(D) \cong \mathcal{O}(\deg D)$, with deg defined naively as the sum of the coefficients. This shows that $\mathbf{Z} \twoheadrightarrow \mathrm{Pic}(X_C)$. As $H^0(\mathcal{O}(-n)) = 0$ for $n > 0$, there are no non-trivial relations, so this surjection is an isomorphism.

8.3.3. *Step 3.* Build a Harder-Narasimhan formalism. The functions $\mathrm{rank}, \deg \colon \mathrm{VB}(X_C) \to \mathbf{Z}$ give a slope function $\mu := \frac{\deg}{\mathrm{rank}}$. Then for formal reasons we get a Harder-Narasimhan filtration.

Using that $H^1(X_C, \mathcal{O}_{X_C}(\lambda)) = 0$ for $\lambda \geq 0$ by Proposition 8.6, we get that the Harder-Narasimhan filtration splits, and so we reduce to the case of semi-stable $\mathcal{E}$. A small further argument reduces to the case where $\mathcal{E}$ is semi-stable of slope 0.

8.3.4. *Step 4.* The goal is to show that any $\mathcal{E}$ which is semi-stable of slope 0, is actually trivial.

We claim it is enough to show this after possibly enlarging C. This is by v-descent: if it is true over C'/C (using that $\operatorname{Spa} C' \to \operatorname{Spa} C$ is a v-cover), the torsor of isomorphisms $\mathcal{E} \cong \mathcal{O}^n$ is a $\mathrm{GL}_n(E)_v$-torsor over $\operatorname{Spa} C$. Any such v-torsor is split, by a theorem[27] in [S17] that any such v-torsor comes from a pro-étale torsor, and any pro-étale torsor over $\operatorname{Spa} C$ is split. Then you can find a splitting over the base by v-descent.

Also, we can assume by induction that the Theorem is true in smaller rank. Consider the minimal $d \geq 0$ (up to enlarging C) such that there exists an injection

$$0 \to \mathcal{O}_{X_C}(-d) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0.$$

If $d = 0$ then we are done by induction, using that $H^1(X_C, \mathcal{O}_{X_C}) = 0$. If $d \geq 2$, it is rather simple contradiction. The key case is $d = 1$. We get

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0.$$

So $\overline{\mathcal{E}}$ has rank $n-1$, degree 1 and slopes ≥ 0. By induction, we must have $\overline{\mathcal{E}} \cong \mathcal{O}_{X_C}^i \oplus \mathcal{O}_{X_C}(\frac{1}{n-1-i})$. The key case is where $\overline{\mathcal{E}} = \mathcal{O}(\frac{1}{n-1})$. This reduces to the following lemma.

[27] Looks be [S17, Lemma 10.13].

Lemma 8.8. *Let $\mathcal{E}$ be an extension*

$$0 \to \mathcal{O}_{X_C}(-1) \to \mathcal{E} \to \mathcal{O}_{X_C}(1/n) \to 0.$$

Then after possibly enlarging C, we get $H^0(X_C, \mathcal{E}) \neq 0$.

Remark 8.9. The reduction to this lemma goes back to [HP04], and is the same in all known proofs of the classification.

The proof of the Lemma by Hartl-Pink (which was the equal characteristic situation) was a difficult computation. Kedlaya-Liu generalized this to mixed characteristic using more difficult computations. Fargues-Fontaine gave a neat geometric argument in terms of the period maps on the Lubin-Tate and Drinfeld moduli spaces of p-divisible groups. Here is a new proof.

Proof. Assume the contrary. Then for all $S \in \mathrm{Perf}_{/C}$, we have an injection

$$H^0(X_S, \mathcal{O}_{X_S}(1/n)) \hookrightarrow H^1(X_S, \mathcal{O}_{X_S}(-1)),$$

i.e. $\mathcal{BC}(\mathcal{O}(1/n)) \hookrightarrow \mathcal{BC}(\mathcal{O}(-1)[1])$. Geometrically, this is

$$\widetilde{\mathbb{D}}_C \hookrightarrow (\mathbf{A}^1_{C^\#})^\diamond/\underline{E}.$$

We claim that this is also necessarily surjective. (This is an example where it's advantageous to think of diamonds as actual geometric objects, rather than "just" functors.) The image could not be contained in the classical points, as the classical points in the target are totally disconnected, while $\widetilde{\mathbb{D}}_C$ is connected. So the image contains some non-classical point. By the understanding of the affine line, the image contains a non-empty open subset after possibly enlarging C (a general property of non-classical points: after extending scalars, their fibers contains an open subset[28]). By translation, it contains a non-empty open neighborhood of 0. Then the map is surjective because both sides are E-vector spaces (so any quasi-compact open subset can be scaled by a power of π to land in a small neigborhood of 0), and is then an isomorphism.

But the map cannot be an isomorphism because $(\mathbf{A}^1_{C^\#})^\diamond/\underline{E}$ is easily shown to be non-representable (it has no global functions). □

[28]Example: on $\mathbf{A}^1$, a non-classical point x is "the generic point $B(x,r)$". After base change to the residue field $C(x)$, there is a tautological point $\widetilde{x}$, we claim that $B(\widetilde{x}, < r)$ is contained in the pre-image of $\{x\} \subset |(\mathbf{A}^1_C)^{\mathrm{ad}}|$. You can think of the Gauss point of $B(x,r)$ as the complement of all balls of smaller radii around classical points. After the base change, there are new classical points whose balls have not been deleted.

9. Families of vector bundles (Nov 27)

9.1. **Example: Hodge-Tate period map.** We wave seen that for the Fargues-Fontaine curve over C, isocrystals give rise to vector bundles and all vector bundles arise in this way. We want to begin with an example that illustrates how the two notions *differ* in families.

Consider the moduli space of elliptic curves $\mathcal{M}_{\text{ell}}/\mathbf{Z}$. It is a Deligne-Mumford stack, although it would become an affine scheme if we imposed a bit of auxiliary level structure; this distinction is immaterial for our discussion.

In characteristic p, we have two strata, parametrizing *ordinary* and *supersingular* elliptic curves:

$$\mathcal{M}_{\text{ell},\mathbf{F}_p} = \underbrace{\mathcal{M}^{\text{ord}}_{\text{ell},\mathbf{F}_p}}_{\text{open}} \cup \underbrace{\mathcal{M}^{\text{ss}}_{\text{ell},\mathbf{F}_p}}_{\text{closed (finite)}} .$$

One way to say what it means for $E/k = \overline{\mathbf{F}}_p$ to be ordinary is by looking at $H^1_{\text{crys}}(E/W(k))[1/p] \in \text{Isoc}_{\mathbf{Q}_p}$. It is an isocrystal, which is either

$$\underbrace{\left(\breve{\mathbf{Q}}_p^2, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\sigma\right)}_{\text{ordinary (slopes } 0,1)} \text{ or } \underbrace{\left(\breve{\mathbf{Q}}_p^2, \begin{pmatrix} & 1 \\ p & \end{pmatrix}\sigma\right)}_{\text{supersingular (slope } 1/2)}$$

So, over $\mathcal{M}_{\text{ell},\mathbf{F}_p}$ we have a family of isocrystals degenerating from ordinary to supersingular. (To make sense of "families of isocrystals", consider the perfection of $\mathcal{M}_{\text{ell},\mathbf{F}_p}$. Over a (perfect) affine open, take crystalline cohomology of the universal family.) However, this picture gets *reversed* when studying vector bundles on the FF curve.

Consider $\mathcal{M}_{\text{ell},\overline{\mathbf{F}}_p} \subset \mathcal{M}^{\wedge}_{\text{ell},\overline{\mathbf{Z}}_p^{\wedge}}$, where the latter is regarded as a formal scheme over $\overline{\mathbf{Z}}_p^{\wedge} = \mathcal{O}_{\mathbf{C}_p}$. There is the adic generic fiber $\mathcal{M}^{\text{ad}}_{\text{ell},\mathbf{C}_p}$. It has a specialization map

$$\text{sp}\colon |\mathcal{M}^{\text{ad}}_{\text{ell},\mathbf{C}_p}| \to |\mathcal{M}_{\text{ell},\overline{\mathbf{F}}_p}|$$

which is continuous, and induces a similar ordinary-supersingular stratification of $\mathcal{M}_{\text{ell},\mathbf{C}_p}$ by pullback.

On the other hand, there is a version of $\mathcal{M}_{\text{ell}}$ with infinite level structure

$$\mathcal{M}_{\text{ell},\mathbf{C}_p,\infty} \sim \varprojlim_m \mathcal{M}_{\text{ell},\mathbf{C}_p,p^m},$$

where $\mathcal{M}_{\text{ell},\mathbf{C}_p,p^m}$ parametrizes elliptic curves E along with level structure $E[p^m] \cong (\mathbf{Z}/p^m)^2$. Hence on the left we have a full trivialization $E[p^\infty] \cong (\mathbf{Q}_p/\mathbf{Z}_p)^2$. In [S15] we showed that $\mathcal{M}_{\text{ell},\mathbf{C}_p,p^\infty}$ exists as a perfectoid space.

A new phenomenon that only exists at infinite level is the *Hodge-Tate period map*

$$\pi_{\text{HT}}\colon \mathcal{M}_{\text{ell},\mathbf{C}_p,p^\infty} \to \mathbf{P}^1_{\mathbf{C}_p}.$$

To briefly recall, $E/\mathbf{C}_p$ has a Hodge-Tate filtration

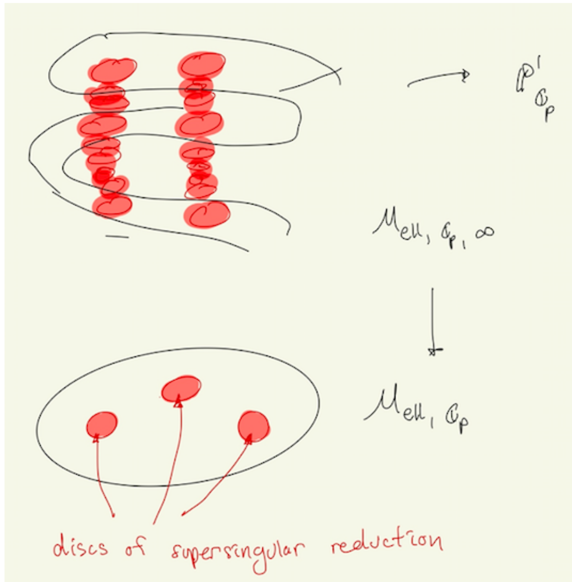
$$0 \to (\text{Lie}\, E)(1) \to T_p(E) \otimes_{\mathbf{Z}_p} \mathbf{C}_p \to (\text{Lie}\, E^*)^* \to 0$$

Then a full level structure gives an isomorphism $T_p(E) \otimes_{\mathbf{Z}_p} \mathbf{C}_p \cong \mathbf{C}_p^2$.

Proposition 9.1. *E has ordinary reduction if and only if the Hodge-Tate filtration is $\mathbf{Q}_p$-rational.*

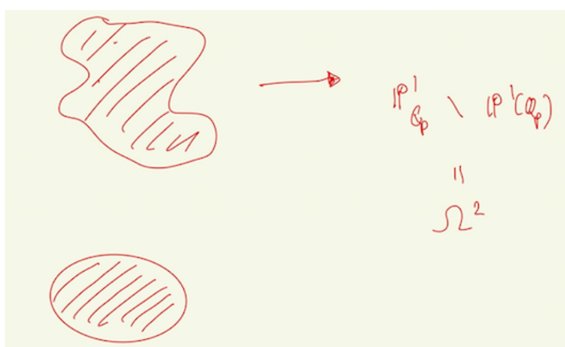
The point is that if E has ordinary reduction, then there is a natural filtration on $T_p(E)$ (given by the slope filtration, i.e. lifting of the connected-étale sequence mod p). Thus descends the Hodge-Tate filtration.

Somewhat paradoxically, the Hodge-Tate period map sends $\mathcal{M}^{\mathrm{ord}}_{\mathrm{ell},\mathbf{C}_p,p^\infty}$ to $\mathbf{P}^1(\mathbf{Q}_p)$, and it's mostly the supersingular locus where the action happens.



Essentially, $\mathcal{M}^{\mathrm{ord}}_{\mathrm{ell},\mathbf{C}_p,p^\infty} = \pi_{\mathrm{HT}}^{-1}(\mathbf{P}^1(\mathbf{Q}_p))$ is the pre-image of the *closed* subset $\mathbf{P}^1(\mathbf{Q}_p) \subset \mathbf{P}^1_{\mathbf{C}_p}$. The qualifier "essentially" is necessary because this is only really true on rank 1 points. (Otherwise, $\mathcal{M}^{\mathrm{ord}}_{\mathrm{ell},\mathbf{C}_p,p^\infty}$ would be open and closed, but $\mathcal{M}_{\mathrm{ell},\mathbf{C}_p,p^\infty}$ is connected.) There are certain rank 2 points which "just barely" specialize to supersingular points, and whose image also lies in $\mathbf{P}^1(\mathbf{Q}_p)$.

Let's zoom in on what happens on the pre-image of supersingular discs.



As you go to the boundary of the discs of supersingular reduction, the image under the Hodge-Tate period map goes to the boundary of Ω^2, which is $\mathbf{P}^1(\mathbf{Q}_p)$. In the adic space there actually is a rank 2 point on the boundary of the disk of supersingular reduction, and it maps into $\mathbf{P}^1(\mathbf{Q}_p)$.

In particular, this does not feel like a map of complex manifolds. Over Ω^2 the fibers are profinite sets, but over $\mathbf{P}^1(\mathbf{Q}_p)$ the fibers are 1-dimensional.

Remark 9.2. This picture illustrates the isomorphism between the Lubin-Tate tower at infinite level and the Drinfeld tower at infinite level.

9.2. **Families of vector bundles on the FF curve.** In terms of vector bundles on the Fargues-Fontaine curve, the $\mathbf{P}^1_{\mathbf{C}_p}$ in the preceding example parametrizes "modifications of trivial rank 2 vector bundles at an untilt". More precisely, we have a degree 1 divisor

$$\operatorname{Spa} \mathbf{C}_p \hookrightarrow X_{\mathbf{C}_p^\flat, \mathbf{Q}_p}.$$

Then choosing a varying line $\mathbf{C}_p^2 \twoheadrightarrow L$, we have a bundle $\mathcal{E}(L)$ over $X_{\mathbf{C}_p^\flat, \mathbf{Q}_p}$ determined by the short exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{E}(L) & \longrightarrow & \mathcal{O}^2_{X_{\mathbf{C}_p^\flat}} & \longrightarrow & L \to 0 \\ & & & \searrow & \nearrow & \\ & & & & \mathbf{C}_p^2 & \end{array}$$

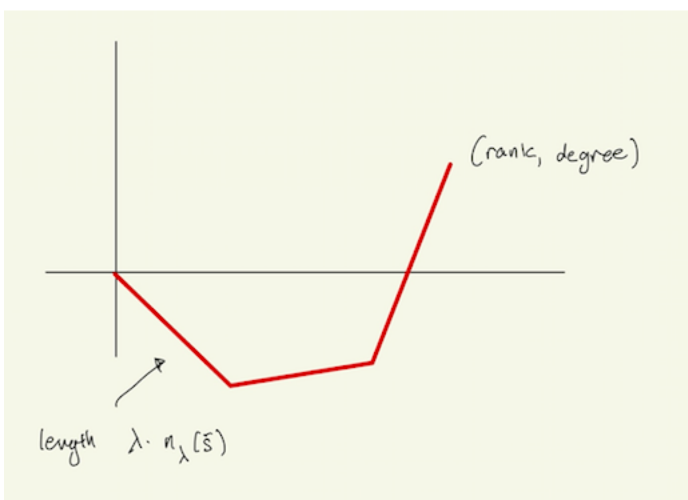
- Over $\Omega^2 \subset \mathbf{P}^1_{\mathbf{C}_p}$ we have $\mathcal{E}(L) = \mathcal{O}(-1/2)$ (which is semi-stable).
- Over $\mathbf{P}^1(\mathbf{Q}_p) \subset \mathbf{P}^1_{\mathbf{C}_p}$, we have $\mathcal{E}(L) \cong \mathcal{O} \oplus \mathcal{O}(-1)$.

Now, fix as before E a nonarchimedean local field of residue field $\mathbf{F}_q$, and $\pi \in \mathcal{O}_E$ a uniformizer.

Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$ be a perfectoid space. Then we have the Fargues-Fontaine curve $X_S = X_{S,E}$. Let $\mathcal{E}$ be a vector bundles on X_S. For each "geometric point"[29] $\overline{s} = \operatorname{Spa}(C, C^+) \to S$. We can consider $\mathcal{E}_{\overline{s}}/X_{\overline{s}}$.

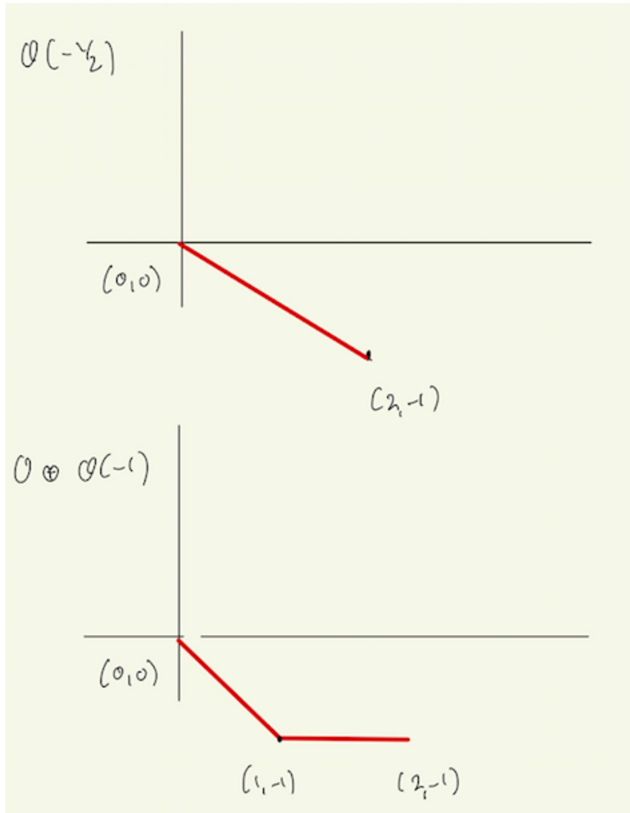
Note: $\mathrm{VB}(X_{\overline{s}}) \cong \mathrm{VB}(X_{\operatorname{Spa}(C, \mathcal{O}_C)})$ so we can forget about C^+ (the plus subring doesn't play a role in the theory of vector bundles, although it will be important in the theory of étale cohomology).

We have a classification of vector bundles $\mathcal{E}_{\overline{s}} \cong \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}_{X_{\overline{s}}}(\lambda)^{n_\lambda(\overline{s})}$. Then we can form a Newton polygon.



[29]In the analogy to algebraic geometry, this is more like a strict henselization at a geometric point. Maps of adic spaces are generalizing, so you cannot have a map whose image is a "single point".

Example 9.3.



How does the Newton polygon vary in families of vector bundles? We define the following ordering:

$P \geq P'$ iff P lies on or above P' with the same endpoints.

Theorem 9.4 (Kedlaya-Liu '15)**.**

(1) The function $\overline{s} \mapsto \mathrm{NP}(\mathcal{E}_{\overline{s}})$ *defines a map* $|S| \to \{$*Newton polygons*$\}$ *which is semi-continuous.*

(2) If the Newton polygon is constant, then there is a global Harder-Narasimhan filtration

$$\mathcal{E}^{\geq \lambda} \subseteq \mathcal{E}$$

by (saturated) vector sub-bundles, and each $\mathcal{E}^{\lambda} := \mathcal{E}^{\geq \lambda} / \bigcup_{\lambda' > \lambda} \mathcal{E}^{\geq \lambda'}$ *is everywhere semistable of slope* λ*. Pro-étale locally on* S*, there is an isomorphism*

$$\mathcal{E} \cong \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}_{X_S}(\lambda)^{n_\lambda}.$$

We will give a different (substantially shorter) proof, which again relies on the geometry of diamonds and v-descent. The key is that projectivized Banach-Colmez spaces are *proper*. To explain this, we need to go through some technical foundations.

9.3. **Separated and proper maps.** Go back to the setting of v-sheaves on

$$\mathrm{Perf} := \{\text{perfectoid spaces of characteristic } p\}.$$

We have the following valuative criteria.

Definition 9.5. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a map of small v-sheaves. We say:

(1) f is a *closed immersion* if for all strictly totally disconnected X and all maps $X \to \mathcal{G}$, the fiber product $\mathcal{F} \times_{\mathcal{G}} X$ is representable by a perfectoid space X', and $X' \to X$ is a (Zariski) closed immersion. (Equivalently, there is a generalizing closed subset $Z \subset |\mathcal{G}|$ such that $\mathcal{F} \subset \mathcal{G}$ is the subfunctor of all maps $X \to \mathcal{G}$ such that $|X| \to |\mathcal{G}|$ factors over Z.)
(2) f is *separated* if Δ_f is a closed immersion.
(3) f is *proper* if f is separated, quasi-compact and universally closed.

Proposition 9.6. *A $f\colon \mathcal{F} \to \mathcal{G}$ of small v-sheaves is separated (resp. proper) iff it is quasi-separated (resp. qcqs) and for all diagrams* $\mathrm{Spa}(R, R^\circ)$

$$\begin{array}{ccc} \mathrm{Spa}(R,R^\circ) & \longrightarrow & \mathcal{F} \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spa}(R,R^+) & \longrightarrow & \mathcal{G} \end{array}$$

then there exists at most one (resp. exactly one) dotted arrow for all affinoid perfectoid $\mathrm{Spa}(R, R^+)$*. In fact, it is enough to check this for $(R, R^+) = (C, C^+)$ where C is a complete algebraically closed field and C^+ is a valuation subring.*

In adic geometry there are many reasonable spaces which are not quasi-compact, e.g. open unit disks. Hence it's desirable to have a notion of a map being "proper without the quasicompactness".

Definition 9.7. A map of small v-sheaves $f\colon \mathcal{F} \to \mathcal{G}$ is *partially proper* if it is separated and there exists a unique filler

$$\begin{array}{ccc} \mathrm{Spa}(R,R^\circ) & \longrightarrow & \mathcal{F} \\ \downarrow & \overset{\exists!}{\nearrow} & \downarrow \\ \mathrm{Spa}(R,R^+) & \longrightarrow & \mathcal{G} \end{array}$$

Let us return to address a property that was implicitly used last time.

Proposition 9.8. *Let $f\colon \mathcal{F} \to \mathcal{G}$ be quasicompact. Then f is surjective as a map of v-sheaves if and only if $|f|\colon |\mathcal{F}| \to |\mathcal{G}|$ is surjective.*

Sketch. Reduce to representable $\mathcal{F} = X, \mathcal{G} = Y$. But then $X \to Y$ is a v-cover. ("v-covers are surjective maps as long as things are quasicompact".) □

9.4. **Projectivized Banach-Colmez spaces.**

Proposition 9.9. *Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$ and $\mathcal{E} \in \mathrm{VB}(X_S)$. Then $\mathcal{BC}(\mathcal{E})\colon T_{/S} \mapsto H^0(X_T, \mathcal{E}|_{X_T})$ is a locally spatial diamond, partially proper over S. The projectivized Banach-Colmez sapce $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E}^\times$ is a locally spatial diamond, proper over S.*

Proof sketch. By the ampleness of $\mathcal{O}(1)$, there exists a surjection $\mathcal{O}_{X_S}(-n)^N \twoheadrightarrow \mathcal{E}^\vee$ for $n, N \gg 0$. Dually, we get an embedding $\mathcal{E} \hookrightarrow \mathcal{O}_{X_S}(n)^N$. One shows that this gives a closed immersion $\mathcal{BC}(\mathcal{E}) \hookrightarrow \mathcal{BC}(\mathcal{O}_{X_S}(n)^N)$. Closed embeddings in locally spatial diamonds are still locally spatial. This reduces to $\mathcal{E} = \mathcal{O}_{X_S}(n)^N$. The valuative criterion part of the definition of partially proper is clear, as the theory of vector bundles does not depend on + subrings.

Assume that S is qcqs. The difficulty is proving the total space of $(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E}^\times$ is qcqs. (For schemes, the qcqs part is usually the easy part and the valuative criterion is hard. In p-adic geometry it is often the opposite situation.) We can analyze $\mathcal{BC}(\mathcal{O}(n))$ inductively,

$$0 \to \mathcal{O}_{X_S}(n-1) \to \mathcal{O}_{X_S}(n) \to \mathcal{O}_{S^\#} \to 0.$$

This gives

$$0 \to \mathcal{BC}(\mathcal{O}_{X_S}(n-1)^N) \to \mathcal{BC}(\mathcal{O}_{X_S}(n)^N) \to (\mathbf{A}^N_{S^\#})^\diamond \to 0.$$

Now the target is a quasi-separated locally spatial diamond. The fiber is also a locally spatial diamond by induction. The exact sequence is pro-étale locally on the target split, so the middle term is also locally spatial. This argument also gives the quasiseparatedness.

The hard part is that $)\mathcal{BC}(\mathcal{E})\setminus\{0\})/\underline{E}^\times$ is quasi-compact. Since $\mathcal{O}_E^\times$ is compact, the main point is that $(\mathcal{BC}(\mathcal{E})\setminus\{0\})/\pi^{\mathbf{Z}}$ is quasicompact. This is a purely topological statement, which follows from a general lemma about "contracting" automorphisms of locally spectral spaces, which says the following.

Suppose you have γ acting on T and $T_0 = T^\gamma$. Hypothesis:

- "T looks like an analytic adic space" (generalizations of a point form a totally ordered chain), and
- "for $n \to \infty$ the action of γ^n contracts towards T_0",
- "for $n \to -\infty$ the action of γ^n on $T \setminus T_0$ diverges"

Output: γ acts freely $T \setminus T_0$ and discontinuously, and $(T \setminus T_0)/\gamma^{\mathbf{Z}}$ is spectral, i.e. qcqs. The proof is some nasty point-set topology.

□

9.5. **Proof of Theorem 9.4.** Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$, $\mathcal{E} \in \mathrm{VB}(X_S)$.

(1) $\mathrm{NP}(\mathcal{E})\colon s \mapsto \mathrm{NP}(\mathcal{E}_s)$ is semi-continuous.

Elementary observation: $\mathrm{NP}(\mathcal{E}_{\overline{s}})$ is the convex hull of all points (i, d_i) for $i = 0, 1, \ldots, \operatorname{rank} \mathcal{E}_{\overline{s}}$ such that there exists a non-zero section of $(\wedge^i \mathcal{E}_{\overline{s}})(-d_i)$. This is an exercise using the classification theorem.

So you can detect the NP if you can detect the bundles which have a non-zero section. So it is enough to prove that the locus of all points where $\mathcal{E}$ has a non-zero section is closed in S. (You apply this to twists of the exterior powers, as well as the inverse of the determinant to get the endpoint.) But this locus is precisely the image of $|(\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E}^\times| \to S$. Since this map is proper, the image is closed.

(2) We want that if $\mathrm{NP}(\mathcal{E})$ is constant, then there exists global HN filtration and pro-étale locally we have

$$\mathcal{E} \cong \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{O}_{X_S}(\lambda)^{n_\lambda}. \tag{9.5.1}$$

For this we use v-descent again.

We claim that it is enough to see this v-locally. Indeed, once we have this we can define an HN filtration v-locally, and then by uniqueness it will descend (by v-decent of vector bundles on the Fargues-Fontaine curve, to be explained next time). Isomorphisms $\mathcal{E}^\lambda \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$ form a v-torsor under $\mathrm{GL}_{m_\lambda}(D_\lambda)$, which necessarily comes from a pro-étale torsor by [S17].

So we can after pro-étale localization find $\mathcal{E}^\lambda \cong \mathcal{O}_{X_S}(\lambda)^{n_\lambda}$. We split the HN filtration using that $H^1(X_S, \mathcal{O}_{X_S}(\lambda)) = 0$ for $\lambda > 0$, for S affinoid.

It remains to produce the splitting (9.5.1) v-locally. Let λ be the maximal slope of $\mathcal{E}$. We want to find a fiberwise non-zero map $\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$ after a v-cover. Given such, then we'll have a short exact sequence

$$0 \to \mathcal{O}_{X_S}(\lambda) \to \mathcal{E} \to \overline{\mathcal{E}} \to 0$$

and we win by induction. (Still using that it's split pro-étale locally.)

But now letting $\mathcal{E}' = \underline{\mathrm{Hom}}(\mathcal{O}_{X_S}(\lambda), \mathcal{E})$, we have

$$\mathcal{BC}(\mathcal{E}') \setminus \{0\} \twoheadrightarrow (\mathcal{BC}(\mathcal{E}') \setminus \{0\})/\underline{E}^\times \to S$$

and the latter map is proper and surjective on geometric points, because we already know the result at geometric points. The properness (hence quasicompactness) of $(\mathcal{BC}(\mathcal{E}') \setminus \{0\})/\underline{E}^\times \to S$ implies that is a v-cover by Proposition 9.8. The map $\mathcal{BC}(\mathcal{E}') \setminus \{0\} \twoheadrightarrow (\mathcal{BC}(\mathcal{E}') \setminus \{0\})/\underline{E}^\times$ is a v-cover by definition (it is a surjection of v-sheaves). So the composite is a v-cover. But on $\mathcal{BC}(\mathcal{E}') \setminus \{0\}$ we have tautologically a non-zero map $\mathcal{O}_{X_S}(\lambda) \to \mathcal{E}$ which is fiberwise non-zero. □

Example 9.10. What does $(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})/E^\times$ look like? Geometrically, $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}$ is a perfectoid punctured open unit disc. The action of π contracts towards the origin, gluing radii along their boundaries.

In fact, we have $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} = (\mathrm{Spa}\, \breve{E}_\infty^{\mathrm{LT}})^\diamond$ by Lubin-Tate theory, and so

$$(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\})/\underline{E}^\times \cong (\mathrm{Spa}\, \breve{E}_\infty^{\mathrm{LT}})^\diamond/\underline{E}^\times \cong (\mathrm{Spa}\, \breve{E})^\diamond/\varphi^{\mathbf{Z}} = \mathrm{Div}_X^1 .$$

Another way to see this is that $\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}$ parametrizes non-zero maps $\mathcal{O} \hookrightarrow \mathcal{O}(1)$. The cokernel is $\mathcal{O}_D$ for some degree 1 divisor D, which is what Div_X^1 parametrizes. The divisor D is invariant under changing the map by an invertible function, i.e. $\underline{E}^\times$. More generally,

$$(\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\})/\underline{E}^\times \cong \mathrm{Div}_X^d .$$

10. The stack of vector bundles on the curve (Nov 30)

10.1. **v-descent for vector bundles.** Let E be a non-archimedean local field with residue field $\mathbf{F}_q$, $\pi \in \mathcal{O}_E \subset E$ a uniformizer.

For $S \in \mathrm{Perf}_{\mathbf{F}_q}$, we have the relative Fargues-Fontaine curve $X_S = X_{S,E}$.

Definition 10.1. Fix $n \geq 1$. Let Bun_n be the moduli (pre)stack on $\mathrm{Perf}_{\mathbf{F}_q}$, taking

$$S \mapsto \{\text{rank } n \text{ vector bundles on } X_S\}.$$

Proposition 10.2 ([Berk, Lemma 17.1.8])**.**

(1) Bun_n *is a v-stack.*

(2) On $\mathrm{Perfd} := \{$*perfectoid spaces*$/\mathbf{Z}_p\}$*, we have v-descent for vector bundles.*

Proof. (2) We know analytic descent by Kedlaya-Liu. So we can reduce to the case of affinoids, and we have to prove: if $Y = \mathrm{Spa}(S, S^+) \to X = \mathrm{Spa}(R, R^+)$ is a v-cover (meaning here $|Y| \twoheadrightarrow |X|$) of affinoid perfectoid spaces, then letting $\mathrm{Proj}\,(R)$ be the category of finite projective R-modules, the functor

$$\mathrm{Proj}\,(R) \xrightarrow{\sim} \left\{ N \in \mathrm{Proj}(S) + \substack{\alpha\colon N\widehat{\otimes}_R S \cong S\widehat{\otimes}_R N \\ \text{isom of finite proj } S\widehat{\otimes}_R S\text{-modules} \\ \text{satisfying the cocycle condition over} \\ S\widehat{\otimes}_R S\widehat{\otimes}_R S} \right\}$$

is an equivalence of categories.

We already know fully faithfulness, as there is the right adjoint

$$\mathrm{eq}(N \rightrightarrows N \otimes_R S) \leftarrow\!\shortmid (N, \alpha).$$

The unit of the adjunction $M \mapsto \mathrm{eq}(M\widehat{\otimes}_R S \rightrightarrows M \otimes_R S\widehat{\otimes}_R S)$ is an isomorphism, as the structure sheaf is a v-sheaf (that gives the statement for $M = R$, and then tensor with M to get the statement in general).

Step 1: handle the case where $R = K$ is a perfectoid field. In this case, S being a v-cover is just equivalent to it being non-zero. We may assume that S (a K-Banach algebra) is topologically countably generated. That is because any K-Banach algebra is a filtered colimit of topologically countably generated subspaces, and the completed tensor products commute with *countably* filtered colimits, because the completion process only adds countable sums. The advantage of this is that a countably generated Banach space is free as a Banach space, so S is free as a K-Banach space. In particular, $(-)\widehat{\otimes}_K S$ is exact and conservative, as K is a direct summand of S.

Let (N, α) be a descent datum, $M = \mathrm{eq}(N \rightrightarrows N\widehat{\otimes}_K S)$. We want to show that

$$M\widehat{\otimes}_K S \xrightarrow{\sim} N.$$

But $M\widehat{\otimes}_K S = \mathrm{eq}(N\widehat{\otimes}_K S \rightrightarrows N\widehat{\otimes}_K S\widehat{\otimes}_K S)$, and this equalizer is always N because

$$0 \to N \to N\widehat{\otimes}_K S \to N\widehat{\otimes}_K S\widehat{\otimes}_K S \to \dots$$

is always exact, as it admits a contracting homotopy. (The point was these completed tensor products are tensor products in the category of K-Banach spaces; then it's the usual formal argument.)

Step 2: now we tackle general $R \to S$. Let $x \in X = \mathrm{Spa}(R, R^+)$ be any point, with completed residue field $K(x)$ (which is some perfectoid field). By base change, we get

$$K(x) \to S\widehat{\otimes}_R K(x)$$

and we can do the descent along this by Step 1. In particular, given any descent datum (N, α), we have that $(N\widehat{\otimes}_S(S\widehat{\otimes}_R K(x)))$ is finite free and admits a basis that is invariant under α. Thus for some small rational neighborhood $U \subset X$ of x, $N\widehat{\otimes}_S(S\widehat{\otimes}_R\mathcal{O}_X(U))$ is finite free, and admits a basis such that α is given by a matrix $\equiv 1 \pmod{\varpi}$ some $\varpi \in R^+$ a pseudouniformizer. That's because you can approximate a basis at x on a rational neighborhood and then by a standard limit argument it will still be a basis on some small neighborhood, and being $\equiv 1 \pmod{\varpi}$ is an open condition, which is satisfied at the point x and therefore extends to some open neighborhood.

By analytic descent we may assume $U = X$. So $(N, \alpha) = (S^n, \alpha \in \mathrm{GL}_n(S\widehat{\otimes}_R S))$. The reductions arranged that $\alpha \in \mathrm{GL}_n(S^+\widehat{\otimes}_{R^+}S^+)$ and $\alpha \equiv 1 \pmod{\varpi}$. We want to find a change of basis so that α becomes Id.

We prove the claim by successive approximation. Note that

$$\frac{\alpha - 1}{\varpi} \pmod{\varpi} \in M_n(S^+\widehat{\otimes}_{R^+}S^+/\varpi)$$

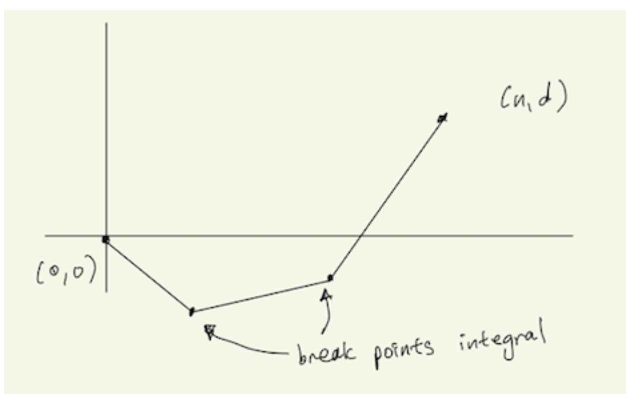
is an *additive* cocycle. But $H^1_v(X, \mathcal{O}^+/\varpi)$ is almost 0, i.e. killed by ϖ^ϵ for any $\epsilon > 0$. Hence we can change basis to ensure that $\alpha \equiv 1 \pmod{\varpi^{2-\epsilon}}$ for any $\epsilon > 0$. Then continue inductively.

(1) We want to show that Bun_n is a v-stack. For this we use that $X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ is perfectoid, where $E_\infty = E(\pi^{1/p^\infty})^\wedge$, and its formation takes v-covers to v-covers. So we can descend vector bundles on $X_S \times_{\mathrm{Spa}\, E} \mathrm{Spa}\, E_\infty$ by (2). Now, descend along E_∞/E by the argument from Case 1. □

Question: can one also descend perfect complexes? We expect the answer is positive.

Remark 10.3. In [BS17], similar v-descent result for vector bundles and perfect complexes on perfect schemes are proved.

10.2. **Structure of** Bun_n**.** By the classification of vector bundles, $|\mathrm{Bun}_n|$ has only countably many points, enumerated by Newton polygons of width n, whose break points have integral coordinates.



Let $B(\mathrm{GL}_n)$ be the set of such Newton polygons, a.k.a. $\{n\text{-dimensional isocrystals}\}/\sim$. So we get a bijection $|\mathrm{Bun}_n| = B(\mathrm{GL}_n)$. There is a natural topology on $|\mathrm{Bun}_n|$, where we say $U \subset |\mathrm{Bun}_n|$ is open if it corresponds to an open substack. Equivalently, if $X \twoheadrightarrow \mathrm{Bun}_n$ is a v-cover by a perfectoid space, and $Y \twoheadrightarrow X \times_{\mathrm{Bun}_n} X$, then $|\mathrm{Bun}_n| = |X|/|Y|$. What is the structure of this topology?

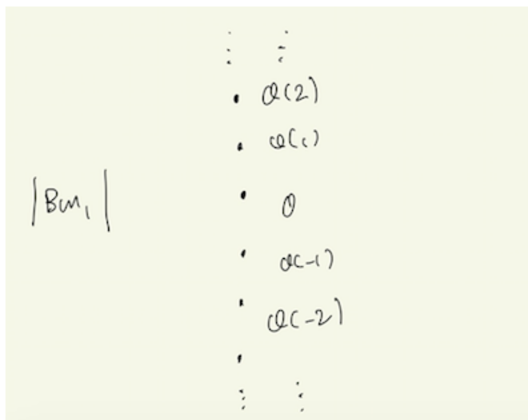
Introduce a partial order on $B(\mathrm{GL}_n)$ by majorization order: $P \geq P'$ if P lies on or above P' with the same endpoints. This induces a topology on $B(\mathrm{GL}_n)$, where $U \subset B(\mathrm{GL}_n)$ is open if for all $P \in U$, all $P' \geq P$ also lie in U.

Theorem 9.4 implies that $|\operatorname{Bun}_n| \to B(\mathrm{GL}_n)$ is continuous.

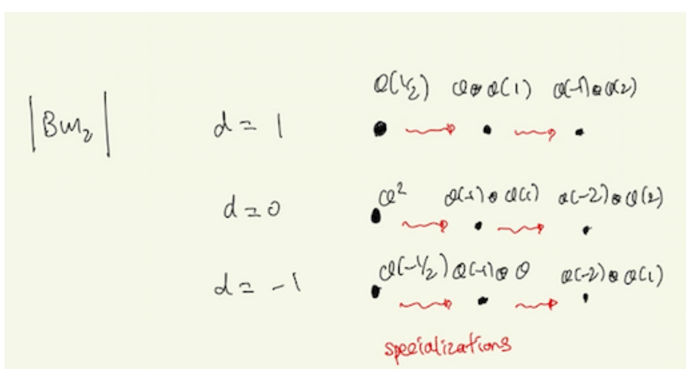
Theorem 10.4 ([BFHHLWY]). *The map $|\operatorname{Bun}_n| \to B(\mathrm{GL}_n)$ is a homeomorphism.*

Remark 10.5. For general G, this was recently announced by Viehmann, by completely different methods. The methods of [BFHHLWY] had previously been extended to some classical groups by Hamann.

Example 10.6. $|\operatorname{Bun}_1| = |\operatorname{Pic}| \cong \mathbf{Z}$ is discrete, with points $\mathcal{O}, \mathcal{O}(1), \ldots$

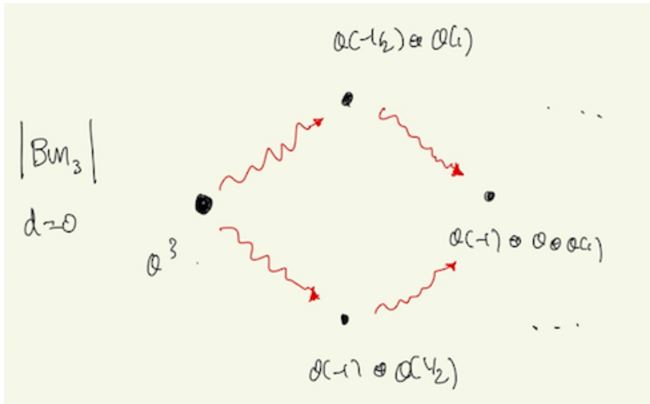


Example 10.7. Let's look at $|\operatorname{Bun}_2|$. The endpoint gives a connected component. In this case it measures the degree d. For each d, there is a semistable bundle.



The generic points have codimension 0, while the specializations have higher and higher codimensions. Tensoring with $\mathcal{O}(1)$ makes the picture 2-periodic in the vertical direction.

Example 10.8. We draw the beginnings of the $d = 0$ stratum for $|\operatorname{Bun}_3|$.



Remark 10.9. This looks a bit like Bun_n for $\mathbf{P}^1$.

Observation: $\pi_0(\operatorname{Bun}_n) \xrightarrow{\sim} \mathbf{Z}$ is given by degree, and each connected component has a *unique* semistable point $\mathcal{O}(d/n) = \mathcal{O}(a/b)^c$ if $\frac{d}{n} = \frac{ac}{bc}$ with $(a, b) = 1$.

10.3. **Description of strata.** For $b \in B(\mathrm{GL}_n)$, let $\operatorname{Bun}_G^b \subset \operatorname{Bun}_n$ be the corresponding locally closed stratum.

10.3.1. *Trivial stratum.* First let $b \cong \mathcal{O}^n$. It is a consequence of Theorem 9.4 that for $S \in \operatorname{Perf}_{\mathbf{F}_q}$, vector bundles on X_S that are semistable of slope 0 in each fiber are equivalent to $\underline{E}$-local systems via

$$\mathcal{E} \mapsto \mathcal{BC}(\mathcal{E})$$
$$\mathbb{L} \otimes_{\underline{E}} \mathcal{O}_{X_S} \leftarrow\!\shortmid \mathbb{L}$$

This follows from pro-étale descent and the fact that any such $\mathcal{E}$ is pro-étale locally trivial.

Corollary 10.10. *The stratum of* Bun_n *corresponding to* $\mathcal{O}^n$ *is* $[\mathrm{pt}/\underline{\mathrm{GL}_n(E)}]$*, the stack classifying* $\underline{E}$*-local systems.*

(For any topological space T, the functor $\underline{T}\colon S \to \operatorname{Cont}(|S|, T)$ is a v-sheaf.)

So we have an open immersion $j\colon [*/\mathrm{GL}_n(E)] \hookrightarrow \operatorname{Bun}_n$. In particular, representations of $\mathrm{GL}_n(E)$ are sheaves on $[*/\mathrm{GL}_n(E)]$, and we can use $j_!$ to embed this in the category of sheaves on Bun_n.

10.3.2. *Semistable bundles.* Next we analyze semistable points, $\mathcal{O}(d/n) := \mathcal{O}(a/b)^{\oplus c}$. Then $\underline{\operatorname{Aut}}(\mathcal{O}(d/n)) = \mathrm{GL}_e(D_{a/b})$ where $D_{a/b} = \operatorname{End}(\mathcal{O}(a/b))$ is the central division algebra over E of Hasse invariant a/b.

So we have $\operatorname{Bun}_n^b \cong [*/\mathrm{GL}_c(D_{a/b})]$. Note that $\mathrm{GL}_c(D_{a/b})$ is the E-valued points of an inner form of GL_n/E. All the inner forms of GL_n arise in this way. So $j_b\colon B(\mathrm{GL}_c(D_{a/b})) \to \operatorname{Bun}_n$ induces an embedding $\operatorname{Rep}(\mathrm{GL}_c(D_{a/b}))$ into the sheaves on Bun_n. So Bun_n sees the representations of $\mathrm{GL}_n(E)$ and all its inner forms.

Remark 10.11. If we want to get these profinite stabilizers, then we need to work with a pro-étale topology. Indeed, the meaning of $* \to */G$ being surjective is that a G-torsor can be trivialized in the relevant topology, and if G is profinite then this could only be possible in a profinite topology.

10.3.3. *Non-semistable points.* We always have $\mathrm{Bun}_n^b = [\mathrm{pt}/\underline{\mathrm{Aut}}(\mathcal{E}_b)]$ where $\mathcal{E}_b$ is the vector bundle corresponding to b, and $\underline{\mathrm{Aut}}(\mathcal{E}_b)$ is the v-sheaf $S \mapsto \mathrm{Aut}(\mathcal{E}_b|_{X_S})$.

Example 10.12. Let's look at Bun_2 for $b \cong \mathcal{O} \oplus \mathcal{O}(1)$. Then

$$\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) = \begin{pmatrix} \underline{E}^\times & \mathcal{BC}(\mathcal{O}(1)) \\ & \underline{E}^\times \end{pmatrix}$$

In particular, recall that $\mathcal{BC}(\mathcal{O}(1))$ is a perfectoid open unit disc. In particular, $\underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1))$ is positive-dimensional. It comes with a natural filtration

$$1 \to \underbrace{\mathcal{BC}(\mathcal{O}(1))}_{\text{1-dimensional, connected}} \to \underline{\mathrm{Aut}}(\mathcal{O} \oplus \mathcal{O}(1)) \to \underbrace{\underline{E}^\times \times \underline{E}^\times}_{\text{0-dimensional}} \to 1.$$

In general, write $\mathcal{E}_b = \bigoplus_{\lambda \in \mathbf{Q}} \underbrace{\mathcal{O}(\lambda)^{n_\lambda}}_{\mathcal{E}_b^\lambda}$ with the slopes λ arranged in increasing order. Then

$$\underline{\mathrm{Aut}}(\mathcal{E}_b) = \begin{pmatrix} \boxed{\underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_1})} & \mathcal{BC}(\underline{\mathrm{Hom}}(\mathcal{E}_b^{\lambda_1}, \mathcal{E}_b^{\lambda_2})) & \cdots & \\ & \boxed{\underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_2})} & \mathcal{BC}(\underline{\mathrm{Hom}}(\mathcal{E}_b^{\lambda_2}, \mathcal{E}_b^{\lambda_3})) & \\ & & \boxed{\underline{\mathrm{Aut}}(\mathcal{E}_b^{\lambda_3})} & \ddots \\ & & & \ddots \end{pmatrix}$$

So in general we have an extension

$$1 \to \underbrace{\text{“unipotent”}}_{\text{ext. of dim} > 0 \text{ Banach-Colmez spaces}} \to \underline{\mathrm{Aut}}(\mathcal{E}_b) \to \underbrace{\text{locally profinite group}}_{\mathrm{Aut}(V_b)} \to 1$$

where V_b is the isocrystal corresponding to b. Traditionally, one writes $J_b(E) = \mathrm{Aut}(V_b)$. It is the E-valued points of an inner form of a Levi subgroup of GL_n.

Remark 10.13. The unipotent groups cannot act on ℓ-adic sheaves, so

$$\{\ell\text{-adic sheaves on } \mathrm{Bun}_n^b\} \cong \{\ell\text{-adic representations of } J_b(E)\}.$$

Hence $\{\ell$-adic sheaves on $\mathrm{Bun}_n\}$ are built via a semi-orthogonal decomposition from pieces that look like $\{\ell$-adic representations of $J_b(E)\}$.

10.4. **How do strata interact?**

Example 10.14 ($n = 2$)**.** We can build a map $(\mathbf{P}_E^1)^\diamond \to \mathrm{Bun}_2$ as follows. A map $S \to (\mathbf{P}_E^1)^\diamond$ is equivalent to an untilt $S^\#/E$ plus a surjection $\mathcal{O}_{S^\#}^2 \twoheadrightarrow L$. From this data, we can build the vector bundle

$$\mathcal{E}(L) := \ker(\mathcal{O}_{X_S}^2 \underbrace{\twoheadrightarrow}_{\text{comes from } S^\# \hookrightarrow X_S} \mathcal{O}_{S^\#}^2 \twoheadrightarrow L)$$

This gives a rank 2 vector bundle on X_S, so a map $S \to \mathrm{Bun}_2$.

Proposition 10.15. *The image lands in* $\mathrm{Bun}_2^{\mathcal{O}(-1/2)} \cup \mathrm{Bun}_2^{\mathcal{O} \oplus \mathcal{O}(-1)}$*, with*

- $(\Omega^2)^\diamond := \mathbf{P}_E^1 \setminus \mathbf{P}^1(E)$ *landing in* $\mathrm{Bun}_2^{\mathcal{O}(-1/2)}$ *and*
- $\mathbf{P}^1(E)$ *landing in* $\mathrm{Bun}_2^{\mathcal{O} \oplus \mathcal{O}(-1)}$.

Proof. For S a geometric point, $\mathcal{E}(L)$ is necessarily of rank 2 and degree -1. Then $\mathcal{E}(L) \approx \mathcal{O}(-1/2)$ or $\mathcal{O}(-i-1) \oplus \mathcal{O}(i)$ for some $i \geq 0$. But we also have $\mathcal{E}(L) \hookrightarrow \mathcal{O}^2$, so only $i = 0$ is possible.

If $\mathcal{E}(L) \cong \mathcal{O}(-1) \oplus \mathcal{O}$, then we get

$$\mathcal{O} \hookrightarrow \mathcal{E}(L) \hookrightarrow \mathcal{O}^2 \twoheadrightarrow \mathcal{O}^2_{S^\#} \twoheadrightarrow L$$

and taking global sections gives non-zero maps $E \to E^2 \to C^2 \twoheadrightarrow L$. So there is a rational line killed by the quotient to L, so L must be E-rational, giving a point of $\mathbf{P}^1(E)$. Conversely, if the point lies in $\mathbf{P}^1(E)$ then up to the $\mathrm{GL}_2(E)$-action we may assume that it is $[0:1]$ and then we see explicitly that $\mathcal{E}(L) \cong \mathcal{O} \oplus \mathcal{O}(-1)$. □

Remark 10.16. The map $(\mathbf{P}^1_E)^\diamond \to \mathrm{Bun}_2$ is close to being "smooth". What is true is that $(\mathbf{P}^1_E)^\diamond/\underline{\mathrm{GL}_2(E)} \to \mathrm{Bun}_2$ is smooth, and so gives a chart.

11. G-bundles on the Fargues-Fontaine curve (Dec 4)

Fix an *arbitrary* (connected) reductive group G/E.

11.1. G-bundles.

We review the general notion of G-bundles (aka "G-torsors).

Proposition 11.1. *Let X be a scheme over E. The following are naturally equivalent:*

(1) ("Geometric G-torsors") Schemes $Y \to X$ with a G-action over X such that locally (étale/smooth/fppf/fpqc) on X, there is a G-equivariant isomorphism

$$Y \cong G \times X.$$

(2) ("Cohomological G-torsors") Sheaves $\mathcal{F}$ on $X_{ét}$ plus an action of G such that locally on $X_{ét}$, there is a G-equivariant isomorphism of étale sheaves $\mathcal{F} \cong G$.

(3) ("Tannaka G-torsor") Exact $\otimes$-functors $\mathrm{Rep}_E(G) \to \mathrm{VB}(X)$.

Example 11.2. If $G = \mathrm{GL}_n$, then GL_n-bundles are equivalent to rank n vector bundles on X. E.g., to get the description (1), consider the space of bases for the G-torsor.

Example 11.3. If $G = \mathrm{Sp}_{2n}$, then these notions of bundles are equivalent to rank $2n$ vector bundles E on X, plus a perfect alternating form on E.

Proof. For (1) $\to$ (2), take the sheaf of sections of $Y \to X$.

For (2) $\to$ (3), for $V \in \mathrm{Rep}_E(G)$ and $\mathcal{F}$ a cohomological G-torsor, then $V \times^G \mathcal{F}$ is an $\mathcal{O}_X$-module on $X_{\text{ét}}$, locally free of finite rank. Then you get a vector bundle on X by étale descent.

For (3) $\to$ (1), you consider $\mathcal{O}(G)$ with its $G \times G$-action by left and right translation. This lies in $\mathrm{Ind}(\mathrm{Rep}_E G)$; in fact it is an algebra object. Hence applying the exact $\otimes$-functor $F\colon \mathrm{Rep}_E(G) \to \mathrm{VB}(X)$ to $\mathcal{O}(G) \in \mathrm{Ind}(\mathrm{Rep}_E G)$ gives an object in $\mathrm{Alg}(\mathrm{Ind}\,\mathrm{VB}(X)) \subset \mathrm{Alg}(\mathrm{QCoh}(X))$ with a G-action. Then take $Y = \underline{\mathrm{Spec}}\, F(\mathcal{O}(G))$. □

Remark 11.4. A similar discussion applies to G-torsors on adic spaces. Note that the notion of quasicoherent sheaves on adic spaces is not well-behaved, but the theory of vector bundles is, and we only really use vector bundles here. For this one uses the scheme-theoretic curve: the total space of the vector bundle can be constructed *algebraically*, using the relative Spec, and then analytified.

In practice the most convenient description is that of exact $\otimes$-functors.

Corollary 11.5. *G-torsors on X are classified by $H^1(X_{ét}, G)$.*

11.2. G-isocrystals.

Recall that we defined the category of E-isocrystals

$$\mathrm{Isoc}_E = \left\{(V, \phi)\colon \substack{V \text{ fin. dim. } \breve{E}-\text{vec. space} \\ \phi\colon V \xrightarrow{\sim} V \ \phi_E\text{-linear}}\right\}.$$

Definition 11.6. A *G-isocrystal* is an exact $\otimes$-functor $\mathrm{Rep}_E(G) \to \mathrm{Isoc}_E$.

Proposition 11.7. *Any G-isocrystal is of the form*

$$V \mapsto (V \otimes_E \breve{E}, b\sigma)\colon \mathrm{Rep}_E(G) \to \mathrm{Isoc}_E$$

for some $b \in G(\breve{E})$. This induces a bijection between

$$G(\breve{E})/\sigma - \mathrm{conj} \cong \{G\text{-}isocrystals\}/\sim.$$

Remark 11.8. G-isocrystals are meant to be "G-torsors on Spec $\breve{E}/\sigma^{\mathbf{Z}}$".

Proof. It's enough to see that all G-torsors on Spec $\breve{E}$ are trivial. This follows from a Theorem of Steinberg, that $H^1_{\text{ét}}(\text{Spec } \breve{E}, G) = *$. (This in turn is based on the fact that $\breve{E}$ has cohomological dimension one.) □

Definition 11.9. We define $B(G) := G(\breve{E})/\sigma - \text{conj} \cong \{G\text{-isocrystals}\}/\sim$. Elements are denoted $b \in B(G)$, and often a choice of representative in $G(\breve{E})$ is implicit.

Example 11.10. For $G = \text{GL}_n$, $B(G)$ is the set of isomorphism classes of rank n isocrystals. This can be identified with the set of Newton polygons of width n.

11.3. **Kottwitz classification.** Kottwitz gives a combinatorial description of $B(G)$ for all G. There is a generalization of the Newton polygon, but one needs a little more information in general. The answer is, roughly, in terms of Newton polygons (satisfying a certain symmetry condition) plus a finite amount of extra data (which is not needed for GL_n).

11.3.1. *Newton point.* For any $(V, \phi) \in \text{Isoc}_E$, V is naturally $\mathbf{Q}$-graded by the slope decomposition, $V = \bigoplus_{\lambda \in \mathbf{Q}} V^\lambda$. Hence we get a map

$$\mathbb{D} \to \text{GL}_{\breve{E}}(V),$$

where $\mathbb{D}$ is the pro-torus with character group $X^*(\mathbb{D}) = \mathbf{Q}$ (so that $\text{Rep}_E(\mathbb{D})$ is equivalent to the category of $\mathbf{Q}$-graded E-vector spaces).

If $F\colon \text{Rep}_E(G) \to \text{Isoc}_E$ is an exact $\otimes$-functor, then compatible maps $\mathbb{D} \to \text{GL}_{\breve{E}}(F(V))$ for all $V \in \text{Rep}_E(G)$ are equivalent to, by the Tannakian formalism, a homomorphism $\mathbb{D} \to G_{\breve{E}}$. Hence, for any isocrystal $b \leftrightarrow F$ with underlying functor $V \mapsto V \otimes_E \breve{E}$, we get a well-defined conjugacy class of maps $\nu(b)\colon \mathbb{D} \to G_{\breve{E}}$. This can be factored over a torus. Let $X = X_*(T)$ for some $T \subset B \subset G_{\overline{E}}$. Although it is not clear from the presentation, $X_*(T)$ is canonically independent of the choice of T, hence inherits an action of $\Gamma := \text{Gal}(\overline{E}/E)$. If $X^+ \subset X$ is the subset of dominant cocharacters, then there is a unique representative $\nu(b) \in G_{\breve{E}} \in (X^+_{\mathbf{Q}})^\Gamma$. This is the *Newton point* of b, and the map $B(G) \to (X^+_{\mathbf{Q}})^\Gamma$ is the *Newton map*.

Example 11.11. For $G = \text{GL}_n$ with the usual B, T, we have $X = X_*(T) = \mathbf{Z}^n$ and $X^+ = \{(m_1 \geq m_2 \geq \ldots \geq m_n)\}$. The action of Γ is trivial as G is split. We have

$$X^+_{\mathbf{Q}} = \{(\lambda_1 \geq \ldots \geq \lambda_n)\colon \lambda_i \in \mathbf{Q}\}.$$

For a rank n isocrystal, this is just recording the slopes.

Example 11.12. For GL_n, the Newton map

$$\nu\colon B(G) \to (X^+_{\mathbf{Q}})^\Gamma$$

is injective, but this fails for general G.

Example 11.13. Let $G = T$ be a torus. This is equivalent to the datum of $X = X_*(T_{\overline{E}})$ together with its Γ-action. So $B(T) = T(\breve{E})/\sigma - \text{conj} \cong T(\breve{E})/(\sigma - 1)$.

Proposition 11.14. *There is an isomorphism $B(T) \cong X_*(T)_\Gamma$ functorial for maps of tori over E. Under this isomorphism, $\nu\colon B(T) \to (X^+_{\mathbf{Q}})^\Gamma = X^\Gamma_{\mathbf{Q}}$ is given by averaging over Γ:*

$$\text{“}\frac{1}{|\Gamma|}\sum_{\gamma \in \Gamma} \gamma\colon X_*(T)_\Gamma \to X_*(T)^\Gamma_{\mathbf{Q}}.\text{”} \tag{11.3.1}$$

Note that this map is not injective if $X_*(T)_\Gamma$ has torsion.

Sketch. First treat the case $T = \mathbf{G}_m$. Then $B(T) = \breve{E}^\times/(\sigma - 1)$, which has a surjection $\twoheadrightarrow \mathbf{Z}$ sending $b \mapsto \nu(b)$. This is easily seen to be an isomorphism; in fact it is a restatement of the classification of rank 1 isocrystals.

Next treat $T = \operatorname{Res}_{E'/E} \mathbf{G}_m$ for a finite separable extension E'/E. Then we have

$$B(E, T) \cong B(E', \mathbf{G}_m) \cong \mathbf{Z}$$

by a variant of Shapiro's Lemma: $B(E, \operatorname{Res}_{E'/E} G) \cong B(E', G)$. We have $X_*(T) \cong \operatorname{Ind}_{\Gamma_{E'}}^{\Gamma_E}(\mathbf{Z})$ and $X_*(T)_{\Gamma_E} \cong \mathbf{Z}_{\Gamma_{E'}} \cong \mathbf{Z}$, also by Shapiro's Lemma. One checks that the identifications are compatible with (11.3.1) (in fact, (11.3.1) could have been defined as the map which makes these compatible).

Finally, resolve a general torus by induced tori. Any torus T admits a surjection

$$\prod_{i=1}^{n} \operatorname{Res}_{E_i/E} \mathbf{G}_m \twoheadrightarrow T.$$

This reduces to the previous case. However, we note that this is the step at which torsion can appear. □

Now we return to general G. We can define the "Borovoi fundamental group"

$$\Gamma \curvearrowright \pi_1(G) = \pi_1(G_{\overline{E}}) = X_*(T)/\text{coroot lattice}.$$

The point is that for $G/\mathbf{C}$, this would recover the usual *topological* π_1 (not something profinite!).

Proposition 11.15. *There is a unique functorial extension*

$$\kappa \colon B(G) \to \pi_1(G)_\Gamma$$

extending the map $B(T) \xrightarrow{\sim} X_*(T)_\Gamma = \pi_1(T)_\Gamma$ *for tori.*

Sketch. Consider G such that G_{der} is simply connected. Then we have a short exact sequence

$$1 \to G_{\mathrm{der}} \to G \to \underbrace{D}_{\text{torus}} \to 1$$

with $\pi_1(G) \xrightarrow{\sim} \pi_1(D)$ so κ is defined by projecting to D.

For general G, there exists a z-extension $G' \twoheadrightarrow G$ (with central kernel) such that G'_{der} is simply connected. Then $B(G') \twoheadrightarrow B(G)$, and we try to define

$$\begin{array}{ccc} B(G') & \twoheadrightarrow & B(G) \\ \downarrow & & \downarrow \exists \\ \pi_1(G')_\Gamma & \longrightarrow & \pi_1(G)_\Gamma \end{array}$$

□

Example 11.16. $E = \mathbf{F}_q((t))$. Then we think of $G(\overline{\mathbf{F}}_q((t)))$ as being the "algebraic loops in G". There is a map $G(\overline{\mathbf{F}}_q((t))) \to \pi_1(G)_\Gamma$ which could be thought of as remembering the class as a "topological loop". It descends to the Kottwitz map.

$$\begin{array}{ccc} G(\overline{\mathbf{F}}_q((t))) & & \\ \downarrow & \searrow & \\ B(G) & \xrightarrow{\kappa} & \pi_1(G)_\Gamma \end{array}$$

Theorem 11.17 (Kottwitz)**.** *For all G, the map*

$$(\nu, \kappa)\colon B(G) \twoheadrightarrow (X_{\mathbf{Q}}^{+})^{\Gamma} \times \pi_1(G)_{\Gamma}$$

is injective.

This allows to define a partial order on $B(G)$: $b \le b'$ if $\nu(b) \prec \nu(b')$ in the dominance order and $\kappa(b) = \kappa(b')$. Minimal elements in this order are called *basic.*

Proposition 11.18. *The Kottwitz map κ induces $B(G)_{\mathrm{basic}} \xrightarrow{\sim} \pi_1(G)_{\Gamma}$.*

Non-basic elements can be understood in terms of Levi subgroups (at least if G is quasi-split).

Proposition 11.19. *An element $b \in B(G)$ is basic if and only if $\nu(b)$ is central.*

For any $b \in B(G)$, we can look at the σ-centralizer of b, which is the same thing as the automorphisms of the corresponding $\otimes$-functor. This defines a connected reductive group G_b over E. If b is basic, then G_b is an inner form of G. More generally, if G is quasi-split then it is an inner form of a Levi subgroup (the centralizer of $\nu(b)$) of G. This is usually denoted J_b, although for $b = 1$ we prefer to write G instead of J_1.

11.4. **G-bundles on the Fargues-Fontaine curve.** Fix the usual notation: $E \supset \mathcal{O}_E \ni \pi$ with residue field $\mathbf{F}_q$. Choose $\overline{\mathbf{F}}_q$, giving $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q)[1/\pi]$.

Definition 11.20. Let $S \in \mathrm{Perf}_{\mathbf{F}_q}$. A *$G$-torsor on X_S* is an exact $\otimes$-functor

$$\mathcal{E}\colon \mathrm{Rep}_E(G) \to \mathrm{VB}(X_S).$$

Definition 11.21. We define the *stack of G-bundles on the Fargues-Fontaine curve* Bun_G to be the v-stack on $\mathrm{Perf}_{\overline{\mathbf{F}}_q}$ sending

$$S \mapsto \{G\text{-bundles on } X_S\}.$$

(The fact that this is a v-stack follows from v-descent for vector bundles.)

Warning 11.22. There is no such thing as "the" Fargues-Fontaine curve, as the construction of $X_{C,E}$ depends on some input field C. But the "stack of G-bundles on the Fargues-Fontaine curve" is well-defined.

Theorem 11.23 (Fargues if $E/\mathbf{Q}_p$, Anschütz in general)**.** *If $S = \mathrm{Spa}(C, C^+)$ where C is a complete algebraically closed field, then the functor*

$$G - \mathrm{Isoc} \to \mathrm{Bun}_G(S)$$

sending a G-torsor on $\mathrm{Spa}\,\breve{E}/\sigma^{\mathbf{Z}}$ to its pullback to $Y_S/\phi^{\mathbf{Z}} = X_S$ induces a bijection on isomorphism classes

$$\mathrm{Bun}_G(S)/\sim \xrightarrow{\sim} B(G)$$

which is even a homeomorphism $|\mathrm{Bun}_G(S)| \cong B(G)$.

Sketch. Let $\mathcal{E}$ be a G-torsor on $X_C := X_S$. For any $V \in \mathrm{Rep}_E(G)$, $\mathcal{E}(V)$ has a HN-filtration, so we get a functor $\mathrm{Rep}_E(G) \to \mathbf{Q}-\mathrm{FilVB}(X_C)^{\mathrm{HN}}$, the category of $\mathbf{Q}$-filtered vector bundles on X_C such that all $\mathcal{E}^{\lambda}$ are semistable of slope λ.

This is still a $\otimes$-functor, which one checks by the classification of vector bundles. (The statement that the HN filtration on a tensor product is the tensor product of the HN filtrations is hard in general, but easy in this case given the classification theorem.) We need to get the exactness. This is trivial over $E/\mathbf{Q}_p$, as then $\mathrm{Rep}_E(G)$ is semisimple, so it's just a question of additivity. In positive characteristic it is subtler because $\mathrm{Rep}_E(G)$

is complicated; you use certain "geometric reductivity" properties proved by Haboush as a substitute for complete reducibility.

Now taking the associated gives a projection to $\mathbf{Q} - \mathrm{GrVB}(X_C)^{\mathrm{HN}}$. This is actually just equivalent to Isoc_E, because the subcategory of semistable bundles is just equivalent to the category of isocrystals. This gives a candidate G-isocrystal, and it remains to split the filtration. For this you use that $H^1(X_C, \mathcal{O}(\lambda)) = 0$ for $\lambda > 0$.

□

Corollary 11.24. *An isocrystal $b \in B(G)$ is basic if and only if $\mathcal{E}_b \in \mathrm{Bun}_G(X_C)$ is semistable in the sense of Atiyah-Bott.*

Theorem 11.25. *The map $|\mathrm{Bun}_G| \to B(G)$ is continuous, i.e.*

$$-\nu\colon |\mathrm{Bun}_G| \to (X_{\mathbf{Q}}^{+})^{\Gamma} \text{ is semi-continuous}$$

and

$$-\kappa\colon |\mathrm{Bun}_G| \to \pi_1(G)_{\Gamma} \text{ is locally constant.}$$

In fact,

$$\kappa\colon \pi_0(\mathrm{Bun}_G) \xrightarrow{\sim} \pi_1(G)_{\Gamma}.$$

The proof will be given next time.

Remark 11.26. This is an analogue of a Theorem of Rapoport-Richartz [RR96] for families of G-isocrystals (although the specialization arrows are reversed). They were not able to prove the statement about κ being locally constant, in full generality. This Theorem can be used to complete their results. The point is that it gives a nice geometric description of Kottwitz' map κ, which a priori is defined by a rather complicated process.

12. The $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannian (Dec 7)

12.1. **Recap.** Fix the usual notation: $E \supset \mathcal{O}_E \ni \pi$ with residue field $\mathbf{F}_q$. Choose $\overline{\mathbf{F}}_q$, giving $\breve{E} = W_{\mathcal{O}_E}(\overline{\mathbf{F}}_q)[1/\pi]$.

Let G/E be a reductive group. Last time we discussed the notion of "isocrystals with G-structure". The isomorphism classes of such were denoted $B(G)$. This is a countable set, which can be presented concretely as $G(\breve{E})/\sigma-\text{conj}$, i.e. $b \in G(\breve{E})$ modulo $b \sim g^{-1}b\sigma(g)$.

We defined two maps

$$\nu\colon B(G) \to (X_{\mathbf{Q}}^+)^\Gamma \text{ for } \Gamma = \mathrm{Gal}(\overline{E}/E)$$

and

$$\kappa\colon B(G) \to \pi_1(G)_\Gamma.$$

The main point was that $(\nu,\kappa)\colon B(G) \to (X_{\mathbf{Q}}^+)^\Gamma \times \pi_1(G)_\Gamma$ is injective, so this classifies $B(G)$ in terms of "combinatorial" data.

12.2. **Structure of** Bun_G**.**

Definition 12.1. Bun_G is the v-stack on $\mathrm{Perf}_{\overline{\mathbf{F}}_q}$, taking S to the groupoid of G-bundles on X_S.

Note that there is a functor $G-\mathrm{Isoc} \to \mathrm{Bun}_G(S)$ for any S, which can be thought of as pullback for

$$X_S = Y_S/\phi_S^{\mathbf{Z}} \to \mathrm{Spa}\,\breve{E}/\sigma^{\mathbf{Z}}.$$

Remark 12.2. In fact, the G-isocrystals are exactly the "universal G-torsors on X_S", by the following theorem.

Theorem 12.3 (Anschütz)**.** *The functor* $G-\mathrm{Isoc} \to \varprojlim_S \mathrm{Bun}_G(S)$ *is an equivalence.*

Theorem 12.4 (Fargues if $E/\mathbf{Q}_p$, Anschütz in general)**.** *If* $S = \mathrm{Spa}(C, C^+)$ *where* C *is a complete algebraically closed field, then the functor*

$$G-\mathrm{Isoc} \to \mathrm{Bun}_G(S)$$

sending a G*-torsor on* $\mathrm{Spa}\,\breve{E}/\sigma^{\mathbf{Z}}$ *to its pullback to* $Y_S/\phi^{\mathbf{Z}} = X_S$ *induces a bijection on isomorphism classes*

$$\mathrm{Bun}_G(S)/\sim \;\xrightarrow{\sim} B(G).$$

Last time we promised to give the proof of:

Theorem 12.5. *The map* $|\mathrm{Bun}_G| \to B(G)$ *is continuous, i.e.*

(1) $-\nu\colon |\mathrm{Bun}_G| \to (X_{\mathbf{Q}}^+)^\Gamma$ *is semi-continuous.*
(2) $-\kappa\colon |\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$ *is locally constant.*

In fact, κ *induces a bijection*

$$\kappa\colon \pi_0(\mathrm{Bun}_G) \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

Remark 12.6. A complete determination of $|\mathrm{Bun}_G|$ was obtained by Hansen for GL_n, by Hamann for some classical groups, and by Viehmann for general G.

Proof. (1) We know this for GL_n. A simple argument explained by Rapoport-Richartz [RR96] shows that you can reduce to GL_n by considering an embedding $G \hookrightarrow \mathrm{GL}_n$.

(2) This is harder; indeed, the construction of the Kottwitz map κ was complicated! The proof follows the same group-theoretic gymnastics involved in the construction.

Lemma 12.7. *Let $G' \twoheadrightarrow G$ be a map of groups that is an extension by a central torus (i.e. a z-extension). Then the induced map $\mathrm{Bun}_{G'} \to \mathrm{Bun}_G$ is a surjective map of v-stacks.*

We will see the proof later. Classically it comes down to the fact that the obstruction to lifting is an H^2, which therefore vanishes on a curve.

Assuming the Lemma for now, we will complete the proof of Theorem 12.5. We go through steps parallel to the construction of κ:

- If G is a product of induced tori, then $\pi_1(G)_\Gamma$ is torsion-free.

$$\begin{array}{ccccc} & & (X^+_{\mathbf{Q}})^\Gamma & & \\ & \overset{\nu}{\nearrow} & & \searrow & \\ B(G) & & & & (\pi_1(G)_{\mathbf{Q}})^\Gamma \\ & \underset{\kappa}{\searrow} & & \underset{\text{average}}{\nearrow} & \\ & & \pi_1(G)_\Gamma & & \end{array}$$

Since $\pi_1(G)_\Gamma$ is torsion-free, κ is determined by ν in this case. The only non-trivial order relations in $(X^+_{\mathbf{Q}})^\Gamma$ lie in the fibers over $(\pi_1(G)_{\mathbf{Q}})^\Gamma$, so κ is locally constant.

- For general torus T, we can find a surjection $\widetilde{T} \to T$ where $\widetilde{T}$ is a product of induced tori. By Lemma 12.7,

$$\mathrm{Bun}_{\widetilde{T}} \to \mathrm{Bun}_T$$

is a surjective map of v-stacks. In particular, $|\mathrm{Bun}_{\widetilde{T}}| \to |\mathrm{Bun}_T|$ is a quotient map[30]. So continuity of $\kappa_T \colon |\mathrm{Bun}_T| \to \pi_1(T)_\Gamma$ follows from continuity of $\kappa_{\widetilde{T}} \colon |\mathrm{Bun}_{\widetilde{T}}| \to \pi_1(\widetilde{T})_\Gamma$, which we just established.

$$\begin{array}{ccc} |\mathrm{Bun}_{\widetilde{T}}| & \twoheadrightarrow & |\mathrm{Bun}_T| \\ \downarrow{\scriptstyle \kappa_{\widetilde{T}}} & & \downarrow{\scriptstyle \kappa_T} \\ \pi_1(\widetilde{T})_\Gamma & \longrightarrow & \pi_1(T)_\Gamma \end{array}$$

- Consider G with G_{der} simply connected. Then we have a short exact sequence

$$1 \to G_{\mathrm{der}} \to G \to T \to 1$$

inducing $\pi_1(G) \xrightarrow{\sim} \pi_1(T)$. Hence we have a diagram

$$\begin{array}{ccc} |\mathrm{Bun}_G| & \longrightarrow & |\mathrm{Bun}_T| \\ \downarrow{\scriptstyle \kappa_G} & & \downarrow{\scriptstyle \kappa_T} \\ \pi_1(G)_\Gamma & \xrightarrow{\sim} & \pi_1(T)_\Gamma \end{array}$$

with κ_T continuous, and that shows that κ_g is continuous as well.

- For a general G, take $G' \to G$ a z-extension such that G'_{der} simply connnected, and apply Lemma 12.7.

□

[30]See [S17, Proposition 11.13]

Corollary 12.8 (Slight strengthening of [RR96]). *If S is a perfect scheme over $\overline{\mathbf{F}}_q$, and $\mathcal{G}$ is a G-isocrystal over S, then*

$$\kappa\colon |S| \to \pi_1(G)_\Gamma$$

is locally constant.

Proof. We may assume that $S = \operatorname{Spec} R$. For any perfectoid space $S'/\overline{\mathbf{F}}_q$ with a map $S' \to \operatorname{Spa}(R, R)$, we get a map $S' \to \operatorname{Bun}_G$ by the same procedure using $\mathcal{G}$. For this map we get a local constancy statement. (One needs to be careful though, as we discussed that the specialization relations are reversed for bundles versus isocrystals.) ☐

Remark 12.9. The missing technical ingredient for Rapoport-Richartz was some analogue of Lemma 12.7.

So now we are back to proving Lemma 12.7.

We will deduce this from a version of the Beauville-Laszlo uniformization. We will introduce the $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannian and a surjection of v-stacks $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+} \to \operatorname{Bun}_G$. We will see that if $G' \twoheadrightarrow G$ is a central isogeny, then $\mathrm{Gr}_{G'}^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$ is a surjection. Lemma 12.7 follows immediately from this.

The uniformization is a generalized version of the map $\mathbf{P}^1 \to \operatorname{Bun}_2$ from §9.2. That was obtained by starting with the trivial bundle, and modifying it at a point. The general case is of the same nature.

12.3. $\mathrm{B}_{\mathrm{dR}}^+$.

Let R be any perfectoid ring over $\mathcal{O}_E$. Let $\varpi \in R^\flat$ be a pseudo-uniformizer. Then we get a Fontaine map

$$\theta\colon W_{\mathcal{O}_E}(R^{\flat\circ}) \twoheadrightarrow R$$

with $\ker(\theta) = (\xi)$ where $\xi = \pi + [\varpi^\epsilon]a$ where $a \in W_{\mathcal{O}_E}(R^{\flat\circ})$.

So from θ we get a surjection

$$W_{\mathcal{O}_E}(R^{\flat\circ})[1/[\varpi]] \twoheadrightarrow R = R^\circ[1/\varpi^\#].$$

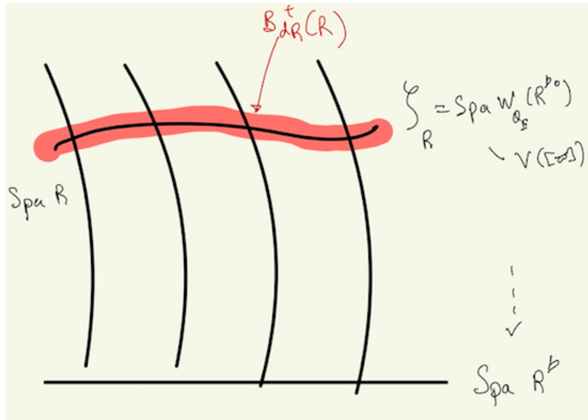
Definition 12.10. We define $\mathrm{B}_{\mathrm{dR}}^+(R)$ to be the ξ-adic completion of $W_{\mathcal{O}_E}(R^{\flat\circ})[1/[\varpi]]$.

What is the geometric meaning of this?

- This is a "1-parameter deformation" of R, and in particular we have

$$\mathrm{B}_{\mathrm{dR}}^+(R)/(\xi) \xrightarrow{\sim} R.$$

  In particular, $(\xi)^i/(\xi)^{i+1}$ is also isomorphic to R as an R-module.
- $\mathrm{B}_{\mathrm{dR}}^+(R) \twoheadrightarrow R$ is the universal pro-infinitesimal thickening in *solid* $\mathcal{O}_E$-algebras.
- If $R/\mathbf{F}_q$, then $\mathrm{B}_{\mathrm{dR}}^+(R) = W_{\mathcal{O}_E}(R)$.
- If $R = C/E$ is complete algebraically closed, then $\mathrm{B}_{\mathrm{dR}}^+(R)$ is isomorphic to $C[[\xi]]$ as an *abstract* E-algebra (but not as a topological E-algebra – in fact $\mathrm{B}_{\mathrm{dR}}^+(R)$ is not a topological ring, although it is a ring with a topology).
- The name is due to Fontaine, and it arises as the "ring of p-adic de Rham periods".



12.4. $\mathrm{B}^+_{\mathrm{dR}}$**-affine Grassmannian.**

Definition 12.11. The $\mathrm{B}^+_{\mathrm{dR}}$-*affine Grassmannian* $\mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}}$ is the étale sheafification of the functor on $\mathrm{Perf}/(\mathrm{Spa}\, E)^\diamond \cong \mathrm{Perf}/E$, taking $\mathrm{Spa}(R, R^+)/E$ to $G(\mathrm{B}_{\mathrm{dR}}(R))/G(\mathrm{B}^+_{\mathrm{dR}}(R))$, where $\mathrm{B}_{\mathrm{dR}}(R) = \mathrm{B}^+_{\mathrm{dR}}(R)[1/\xi]$.

Equivalently, this is classifying G-bundles on $\mathrm{Spa}\,\mathrm{B}^+_{\mathrm{dR}}(R)$ with a trivialization on $\mathrm{Spa}\,\mathrm{B}_{\mathrm{dR}}(R)$. The reason is that étale locally on R, a G-bundle on $\mathrm{Spa}\,\mathrm{B}^+_{\mathrm{dR}}(R)$ is trivial.

Remark 12.12. This is a p-adic version of the usual affine Grassmannian

$$R/\mathbf{C} \mapsto G(R((t))/G(R[[t]]).$$

Example 12.13. Note that for $E = \mathbf{F}_q((t))$, this is literally

$$R/\mathbf{F}_q((t)) \mapsto G(R((t-\zeta)))/G(R[[t-\zeta]]).$$

This is because if $E = \mathbf{F}_q((t))$, then $W_{\mathcal{O}_E}(R^\circ) = R^\circ[[t]]$, and $\xi = t - \zeta$. So this recovers the usual affine Grassmannian.

Proposition 12.14 (Cartan decomposition)**.** *We have*

$$\mathrm{Gr}_G^{\mathrm{B}^+_{\mathrm{dR}}}(C) = \bigcup_{\mu \in X^+} G(\mathrm{B}^+_{\mathrm{dR}}(C))[\mu(\xi)].$$

Example 12.15. For $G = \mathrm{GL}_n$, if $\mu = (a_1 \geq \ldots \geq a_n)$ then $\mu(\xi)$ is

$$\begin{pmatrix} \xi^{a_1} & & & \\ & \xi^{a_2} & & \\ & & \ddots & \\ & & & \xi^{a_n} \end{pmatrix}$$

Proof. We have an abstract isomorphism $\mathrm{B}^+_{\mathrm{dR}}(C) \cong C[[\xi]]$ as abstract rings. The question at hand is about the algebraic ring, since it can be reduced to a question about vector bundles, which only depends on the underlying algebraic ring. So it actually just follows from the usual Cartan decomposoition. □

Assume G is split for simplicity. (In general, this happens after finite étale base change on E, and $\mathrm{Gr}_{G_{E'}} = \mathrm{Gr}_G \times_E E'$, and so one can reduce to this case by descent.)

Definition 12.16. For $\mu \in X^+$, let

$$\mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+} \subset \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$$

be the "Schubert variety" subfunctor of all $S \to \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$ such that at all geometric points, it lies in $G(\mathrm{B}_{\mathrm{dR}}^+)$-orbit of $\mu'(\xi)$, where $\mu' \leq \mu$.

Remark 12.17. Since perfectoid spaces are automatically reduced, one can specify subsets by specifying conditions on geometric points. This is not a good idea for schemes.

Theorem 12.18. *$\mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+} \to (\mathrm{Spa}\, E)^\diamond$ is proper and $\mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+}$ is represented by a spatial diamond.*

$$\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+} = \bigcup_\mu \mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+}$$

with transition maps being closed immersions.

If μ is minusucle, then $\mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+} \cong (G/P_\mu)^\diamond$.

The proof is a little subtle. It uses a variant of Artin's recognition principle for algebraic spaces. In this case it is hard to find explicit pro-étale covers. Instead we find atlases in the v-topology, which are enough to show that it's spatial. Then there's a criterion for a spatial v-sheaf to be a diamond, in terms of geometric points. The geometric points are exhausted by strata, and on strata one can argue directly.

Corollary 12.19. *If $G' \to G$ is a z-extension, then $\mathrm{Gr}_{G'}^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$ is a v-cover.*

Proof. By making a field extension, we can assume that G is split. It is enough to prove that

$$\mathrm{Gr}_{G',\leq\mu'}^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Gr}_{G,\leq\mu}^{\mathrm{B}_{\mathrm{dR}}^+}$$

is a v-cover for all $\mu' \mapsto \mu$. We are using here that $(X')^+ \twoheadrightarrow X^+$ because the kernel of $G' \to G$ is a *connected* torus. These are spatial diamonds, in particular qcqs. Hence we can check surjectivity on geometric points. Then it is clear from Cartan decomposition and $G'(\mathrm{B}_{\mathrm{dR}}^+(C)) \twoheadrightarrow G(\mathrm{B}_{\mathrm{dR}}^+(C))$. □

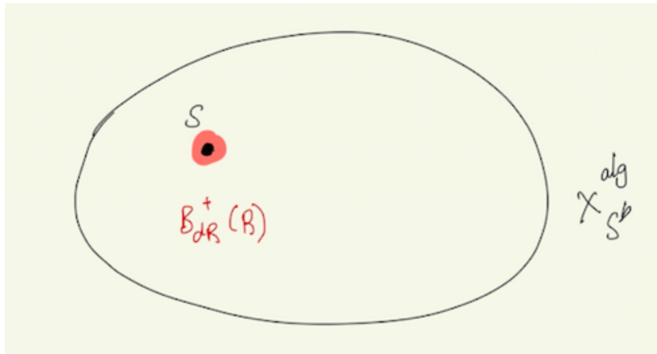
Remark 12.20. For $E/\mathbf{F}_q((t))$, all the $\mathrm{Gr}_{G,\leq\mu}$ are represented by projective varieties, but for $E/\mathbf{Q}_p$, we get actual diamonds not coming from rigid varieties.

12.5. Beauville-Laszlo uniformization.

Definition 12.21. Let G be a reductive group over E (not necessarily split). We have a map

$$\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Bun}_G$$

defined as follows: a point of the domain is $S = \mathrm{Spa}(R, R^+)/E$ and a G-torsor $\mathcal{E}_0$ over $\mathrm{B}_{\mathrm{dR}}^+(R)$, trivialized over $\mathrm{B}_{\mathrm{dR}}(R)$. We can glue the trivial G-torsor on $X_S^{\mathrm{alg}} \setminus \mathrm{Spec}\, R$ with $\mathcal{E}_0/\mathrm{B}_{\mathrm{dR}}^+(R)$ along the given identification over $\mathrm{B}_{\mathrm{dR}}(R)$.



We are using here:

Lemma 12.22 (Beauville-Laszlo)**.** *If X is a scheme over E and $Z \subset X$ is a Cartier divisor, with Z affine, then G-torsors over X are equivalent to the groupoid of: a G-torsor over $X \setminus Z$, a G-torsor over $X_Z^\wedge$, and an identification of the two over $(X_Z^\wedge \setminus Z)$.*

Remark 12.23. This is one place where it is really useful to have the algebraic version of the Fargues-Fontaine curve.

Example 12.24. For $G = \mathrm{GL}_2$, $\mu = (1,0)$, restricting to $\mathrm{Gr}_{\mathrm{GL}_2,\le\mu}^{\mathrm{B}_{\mathrm{dR}}^+} \cong (\mathbf{P}_E^1)^\diamond$ recovers the example from before.

Theorem 12.25. $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Bun}_G$ *is surjective map of v-stacks.*

Remark 12.26. This is an analogue of a result of Drinfeld-Simpson for usual curves. There it's only true if G is semisimple, but not for tori. In this case, it's really true for all reductive G. The key is that the Picard group of the punctured curve is trivial. (This is clearly necessary for the statement to be true.)

Proof. The key step is to check what happens on geometric points. This is due to Fargues / Anschütz, using the classification of G-bundles.

From this we bootstrap to the general case. For a general S, an S-point of Bun_G corresponds to $\mathcal{E}/X_S$. Assume S is strictly totally disconnected. At geometric points we can lift to $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$ by the previous step, to get modifications $\mathcal{E}'_s$ of $\mathcal{E}_s$. Pick a modification $\mathcal{E}'$ of $\mathcal{E}$ recovering $\mathcal{E}'_s$ at s. (Using Cartan decomposition, this is a question of lifting matrices.) It's enough to show that $\mathcal{E}'$ is trivial in a neighborhood of s. We claim that in general for $\mathcal{E}'/X_S$, the locus where $\mathcal{E}'|_{X_s}$ is trivial is open in S. This will be completed next time. $\square$

Remark 12.27 (Witt vector affine Grassmannian)**.** Let $\mathcal{G}/\mathcal{O}_E$. The *Witt vector affine Grassmannian* $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Witt}}$ is the functor on perfect $\mathbf{F}_q$-algebras, taking

$$R \mapsto G(W_{\mathcal{O}_E}(R)[1/\pi])/G(W_{\mathcal{O}_E}(R)).$$

This is representable by an ind(perfect scheme).

There is a degeneration of $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannian. We can define $\mathrm{Gr}_{\mathcal{G}}^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Spa}\,\mathcal{O}_E^\diamond$. This is an ind-(proper, relatively representable in spatial diamond) over $\mathrm{Spa}\,\mathcal{O}_E^\diamond$. This is a degeneration from $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$ to $(\mathrm{Gr}_{\mathcal{G}}^{\mathrm{Witt}})^\diamond$.

13. Bun_G (Dec 11)

13.1. **Loose end.** We need to complete an argument from last time. We were trying to prove:

Theorem 13.1. *The map* $|\mathrm{Bun}_G| \to B(G)$ *is bijective and continuous.*

What was missing to complete the proof is:

Theorem 13.2. *Let* $\mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$ *be the substack of all* G*-bundles* $\mathcal{E}/X_S$ *such that at all geometric points*[31] $\mathrm{Spa}(C, C^+) \to S$, $\mathcal{E}|_{X_{\mathrm{Spa}(C,C^+)}}$ *is trivial. Then* $\mathrm{Bun}_G^1 \subseteq \mathrm{Bun}_G$ *is an open substack and* $\mathrm{Bun}_G^1 \cong [\mathrm{pt}/\underline{G(E)}]$.

Remark 13.3. We already stated this for GL_n in §10.3.1. Note that this open-ness follows immediately from the statement of Theorem 13.1, as the trivial isocrystal evidently defines an open point in $B(G)$.

The strategy is to reduce to the case of GL_n via the Tannakian formalism. But this is not formal.

Proof. We know that the Newton map ν is semi-continuous, and Bun_G^1 is contained the locus where $\nu = 0$, which is open. So we can restrict our attention to G-bundles $\mathcal{E}$ where $\nu \equiv 0$ for every geometric point of S.

In this case, for all representations $\rho\colon G \to \mathrm{GL}(V)$, we can consider $\rho_* \mathcal{E} \in \mathrm{VB}(X_S)$. This is semi-stable of slope 0. As such, it is equivalent to (pro-étale) $\underline{E}$-local system on S (Corollary 10.10).

This is still complicated, so we make a pro-étale localization to pass to the case where S is strictly totally disconnected. Let $A = \mathrm{Cont}(|S|, E) = \mathrm{Cont}(\pi_0(S), E)$. Since S is strictly totally disconnected, the category of $\underline{E}$-local systems is equivalent to $\mathrm{Proj}\,(A)$ (the category of finitely projective A-modules) via the global sections functor.

In summary, $\mathcal{E}$ defines an exact $\otimes$-functor

$$\mathrm{Rep}_E(G) \to \{\mathrm{VB}(X_S) \text{ everywhere semistable of slope } 0\} \cong \mathrm{Proj}\,(A).$$

This is the same as the data of a G-torsor $\mathcal{F}$ on $\mathrm{Spec}\ A$, and the triviality of the G-torsor $\mathcal{E}$ is equivalent to the triviality of the G-torsor $\mathcal{F}$. So it is enough to see that if $\mathcal{F}$ is trivial at the point $\mathrm{Spec}\ E \hookrightarrow \mathrm{Spec}\ A$ corresponding to $s \in S$ (evaluation at s defines a map $A \to E$), then it is trivial after pullback to

$$\mathrm{Spec}\ \mathrm{Cont}(U, E) \subset \mathrm{Spec}\ \underbrace{\mathrm{Cont}(|S|, E)}_{A}$$

for some open neighborhood (and therefore also closed) neighborhood $U \ni s$. This follows from two facts:

(1) The local ring
$$\varinjlim_{U \ni s} \mathrm{Cont}(U, E)$$
is henselian along the kernel of evaluation at s, with residue field E.

(2) If (B, I) is a henselian pair, then $H^1_{\text{ét}}(\mathrm{Spec}\ B, G) \hookrightarrow H^1_{\text{ét}}(\mathrm{Spec}\ B/I, G)$, i.e. any G-torsor over B that splits over B/I already splits over B.

[31] By this we always mean that C is complete and algebraically closed and $C^+ \subset C$ is a valuation subring

To conclude the proof, one uses (2) to get a splitting over the henselian local ring, and then (1) to spread it out to a neighborhood. For (1), we note that A is a Tate algebra whose adic spectrum is $\pi_0(S)$. The colimit is then the local ring of this analytic adic space, and a general lemma says that the local rings of analytic adic spaces are *always* henselian. (This is analogous to the situation for complex analytic varieties, where local rings are henselian by the inverse function theorem.) □

13.2. **Digression on local Shimura varieties.** "Local Shimura Varieties" (cf. Rapoport-Viehmann) are a p-adic (hence "local") analogue of Shimura varieties, and are related to Shimura varieties via uniformization results. For example, while complex Shimura varieties are quotients of Hermitian symmetric domains by arithmetic group actions, certain loci in Shimura varieties can be written as quotients of p-adic symmetric spaces by arithmetic group actions. This phenomenon was first studied by Cerednik in the 1970s. In the 1980s Rapoport-Zink considered PEL-type Shimura varieties, in which case the relevant "local Shimura varieties" are moduli spaces of p-divisible groups, generalizing an idea of Drinfeld. These are now called "Rapoport-Zink spaces".

Now, we know that there general Shimura varieties do not admit moduli-theoretic descriptions. [RV14] conjectured the existence of local analogues of such objects, and it turns out that these can be constructed using the machinery we have set up (cf. [Berk]).

13.2.1. *Local Shimura data.* We give the abstract group-theoretic datum for defining local Shimura varieties. Usually one considers $E = \mathbf{Q}_p$, but our formalism allows more general E.

Definition 13.4. A *local Shimura datum* is a triple $(G, [b], \{\mu\})$ where

- G/E is a reductive group,
- $\{\mu\colon \mathbf{G}_m \to G_{\overline{E}}\}$ is a conjugacy class of *minuscule* cocharacters.
- $[b] \in B(G)$.

In order for the associated local Shimura variety to be non-empty, we need to ask that

$$[b] \in \underbrace{B(G,\mu)}_{\text{finite}} \subset B(G)$$

where $B(G,\mu)$ is a finite subset given by an explicit combinatorial criterion. We will give a geometric formulation for this criterion, later.

13.2.2. *Local Shimura varieties.* A *local Shimura variety* is a tower

$$(\mathcal{M}_{(G,b,\mu),K})_{K \text{ compact open } \subseteq G(E)}$$

of smooth rigid analytic varieties over $\breve{E}$ (plus a Weil descent datum, i.e. an isomorphism with the pullback via the Frobenius on $\breve{E}$), equipped with compatible étale period maps

$$\pi_K\colon \mathcal{M}_{(G,b,\mu),K} \to \mathcal{F}\ell_\mu/\breve{E}.$$

Here $\mathcal{F}\ell_\mu$ parametrizes parabolic subgroups of G with conjugacy class determined by μ. The nonempty geometric fibers of π_K are $\cong G(E)/K$.

Example 13.5 (Drinfeld case). [32] If $G = D^\times$ for a quaternion algebra D/E, we can take $\mu\colon \mathbf{G}_m \to G_{\overline{E}} \cong \mathrm{GL}_2$ to be

$$t \mapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix}$$

[32] Discussion is up to signs

and b to be the "basic" isocrystal, of slope $1/2$. (This is the unique element of $B(G,\mu)$.) Then $\mathcal{F}\ell_\mu = \mathbf{P}^1$ (as an adic space), and the period map π_K will be valued in the "Drinfeld upper half space" $\Omega^2 = \mathbf{P}^1 - \mathbf{P}^1(E)$. The $\mathcal{M}_K$ will be the *Drinfeld covers* of Drinfeld upper half space.

There is a similar example for GL_n for $n \geq 2$. You take G to be the units in the division algebra of invariant $1/n$ and b to be the basic element of slope $1/n$. The target would be $\mathbf{P}^{n-1}$ minus all the rational hyperplanes.

Example 13.6 (Lubin-Tate case)**.** [33] Let $G = \mathrm{GL}_n$ and $\mu\colon \mathbf{G}_m \to G$ to be

$$t \mapsto \begin{pmatrix} t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and b to be the basic element of slope $1/n$. In this case $\mathcal{F}\ell_\mu = \mathbf{P}^{n-1}/\breve{E}$, and the period map

$$(\mathcal{M}_K)_{K \subseteq \mathrm{GL}_n(E)} \to \mathcal{F}\ell_\mu$$

is the *Gross-Hopkins period map*. It is surjective étale with fibers $\mathrm{GL}_n(E)/K$. In particular, the adic space $\mathbf{P}^{n-1}$ admits non-trivial infinite degree étale covering spaces!

The spaces $\mathcal{M}_K$ parametrize deformations of 1-dimensional height n π-divisible $\mathcal{O}_E$-modules plus level structures. For $K = \mathrm{GL}_n(\mathcal{O}_E) \subset \mathrm{GL}_n(E)$, $\mathcal{M}_K$ is a disjoint union of $\mathbf{Z}$ copies of an $(n-1)$-dimensional open unit disk, hence isomorphic to

$$\coprod_{\mathbf{Z}} (\operatorname{Spa} W_E(\overline{\mathbf{F}}_q)[[u_1, \ldots, u_{n-1}]])_{\breve{E}}.$$

In particular, an $(n-1)$-dimensional open unit disk can be presented as an infinite degree étale covering of $\mathbf{P}^{n-1}$.

Remark 13.7. These examples are quite special. For example, the Lubin-Tate case is essentially the only case where the period map π_K is surjective. Also, they are rare instances where the image of the period map can be understood explicitly.

Now let us say something about the construction of local Shimura varieties. We want an open subset $\mathcal{F}\ell_\mu^{\mathrm{adm}} \subset \mathcal{F}\ell_\mu$ (the "admissible locus") plus a $\underline{G(E)}$-local system $\mathbb{L}$ on $\mathcal{F}\ell_\mu^{\mathrm{adm}}$. Then $\mathcal{M}_K$ can be defined to parametrize reductions of $\mathbb{L}$ to $\underline{K}$; equivalently, considering $\mathbb{L}$ as a $\underline{G(E)}$-torsor

$$\mathbb{L} \to \mathcal{F}\ell_\mu^{\mathrm{adm}}$$

we can define $\mathcal{M}_K = \mathbb{L}/\underline{K} \to \mathcal{F}\ell_\mu^{\mathrm{adm}}$. This will automatically be étale because $G(E)/K$ is discrete. As $\mathcal{F}\ell_\mu^{\mathrm{adm}}$ is open inside the flag variety, it naturally has the structure of a smooth rigid-analytic variety, so $\mathcal{M}_K$ is a smooth rigid-analytic variety.

So the content of the construction is in finding the right open subset, and the right $\underline{G(E)}$-local system. Recall that we have defined $\mathrm{Gr}_G := \mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+}$. Inside Gr_G we have the Schubert variety $\mathrm{Gr}_{G,\leq\mu}$ which because μ is **minuscule**, is isomorphic to $\mathcal{F}\ell_\mu^\diamond$. (If μ is not minuscule, we get a similar story replacing $\mathcal{F}\ell_\mu^\diamond$ by $\mathrm{Gr}_{G,\leq\mu}$.)

[33] Discussion is up to signs

We have constructed a uniformization map $\mathrm{Gr}_G^{\mathrm{B}_{\mathrm{dR}}^+} \to \mathrm{Bun}_G$ by modifying the trivial bundle. But we now want to consider a different map given $b \in B(G)$, by modifying $\mathcal{E}_b$. So we get a map

$$\mathcal{F}\ell_\mu^\diamond \to \mathrm{Bun}_G$$

Proposition 13.8 (Appendix by Rapoport to [S18]). [34] *The image of $\mathcal{F}\ell_\mu^\diamond \to \mathrm{Bun}_G$ meets Bun_G^1 if and only if $b \in B(G,\mu)$.*

So $B(G,\mu)$ are those isocrystals for which there exists a modification of type μ taking $\mathcal{E}_b$ to $\mathcal{E}_1$, or equivalently such that there exists a modification of type μ^{-1} taking $\mathcal{E}_1$ to $\mathcal{E}_b$, i.e. $\mathcal{E}_b$ lies in the image of the analogous map $\mathcal{F}\ell_{\mu^{-1}} \to \mathrm{Bun}_G$ obtained by modifying $\mathcal{E}_1$.

The open substack $\mathrm{Bun}_G^1 \hookrightarrow \mathrm{Bun}_G$ then induces an open subvariety $(\mathcal{F}\ell_\mu^{\mathrm{adm}}) \hookrightarrow \mathcal{F}\ell_\mu^\diamond$.

$$\begin{array}{ccc} \mathcal{F}\ell_\mu^\diamond & \longrightarrow & \mathrm{Bun}_G \\ \uparrow & & \uparrow \\ \mathcal{F}\ell_\mu^{\mathrm{adm}} & \longrightarrow & \mathrm{Bun}_G^1 \\ & & \downarrow \sim \\ & & [*/\underline{G(E)}] \end{array}$$

This gives the desired data of the open subvariety $\mathcal{F}\ell_\mu^{\mathrm{adm}}$ plus a $\underline{G(E)}$-torsor over it.

Corollary 13.9. *The infinite level local Shimura variety*

$$\varprojlim_K \mathcal{M}_K^\diamond$$

parametrizes modifications $\mathcal{E}_b \cong \mathcal{E}_1$ of type μ. More precisely, for all $S \in \mathrm{Perf}\,/(\mathrm{Spa}\,\breve{E})^\diamond$,

$$\varprojlim_K \mathcal{M}_K^\diamond(S) = \left\{ \textit{isom. } \mathcal{E}_b|_{X_S \setminus S^\#} \cong \mathcal{E}_1|_{X_S \setminus S^\#} \ \substack{\textit{modifications of type } \mu \\ \textit{(at geometric points)}} \right\}.$$

Example 13.10. In the situation of Example 13.6, the inverse limit of the Lubin-Tate tower

$$\varprojlim_{K \subset \mathrm{GL}_n(E)} \mathcal{M}_K^\diamond \cong \{\mathcal{O}_{X_S}^n \hookrightarrow \mathcal{O}_{X_S}(1/n) \text{ cokernel supp. at } S^\#\}.$$

Note that the cokernel is necessarily a line bundle on $S^\#$, for degree reasons.

One can also study the Drinfeld tower

$$\varprojlim_{K' \subset D_{1/n}^\times} \mathcal{M}_K^\diamond$$

and it turns out to have the same description at ∞ level, as modifications between $\mathcal{O}(1/n)$ and the trivial bundle (but in the opposite order).

Remark 13.11. This analysis generalizes to give a duality for all local Shimura varieties whenever b is basic. It relates (G,b,μ) and (G_b, b^{-1}, μ^{-1}).

Remark 13.12. For general μ, there is a Bialynicki-Birula map

$$\mathrm{Gr}_{G,\mu} \to \mathcal{F}\ell_\mu^\diamond.$$

[34] Everything here is up to signs

The fibers are iterated affine spaces (but not itself an affine space, because there are non-split extensions $(\mathbf{A}^1)^\diamond \to X \to (\mathbf{A}^1)^\diamond$ with X not represented by a rigid analytic variety). Let S be a rigid-analytic variety, smooth for simplicity, and consider a map $S^\diamond \to \mathcal{F}\ell_\mu^\diamond$. You can ask about lifts to $\mathrm{Gr}_{G,\mu}$.

$$\begin{array}{ccc} & & \mathrm{Gr}_{G,\mu} \\ & \nearrow & \downarrow \\ S^\diamond & \longrightarrow & \mathcal{F}\ell_\mu^\diamond \end{array}$$

You might guess that there are lots of lifts, but in fact this is not the case.

Theorem 13.13 ([S14][35])**.** *The map $\mathrm{Gr}_{G,\mu}(S^\diamond) \hookrightarrow \mathcal{F}\ell_\mu^\diamond(S^\diamond) = \mathcal{F}\ell_\mu(S)$ is injective, and the image consists of those maps that satisfy Griffiths transversality.*

This is already interesting when S is a point.

13.3. General points on Bun_G.

13.3.1. *Semistable points.*

Theorem 13.14. *The semistable locus $\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$ is open, and*

$$\mathrm{Bun}_G^{ss} = \coprod_{b \in B(G)_{\mathrm{basic}}} \underbrace{[\mathrm{pt}/\underline{G_b(E)}]}_{\mathrm{Bun}_G^b}.$$

where Bun_G^b is the locus where $\mathcal{E} \cong \mathcal{E}_b$ at geometric points.

Proof. The statement about open-ness follows from semicontinuity of ν. The decomposition in Bun_G^b follows from the local constancy of $\kappa \colon |\mathrm{Bun}_G^b| \to B(G)$ and that κ restricted to the basic locus is an isomorphism $B(G)_{\mathrm{basic}} \xrightarrow{\sim} \pi_1(G)_\Gamma$. So it only remains to show that $\mathrm{Bun}_G^b \cong [\mathrm{pt}/\underline{G_b(E)}]$.

But note that $\mathcal{E}_b$ is a G-torsor on X_S, and $\underline{\mathrm{Aut}}_{X_S}(\mathcal{E}_b) = G_b \times_E X_S$ for basic b. (In general the automorphism group would be an inner form because it is easily check to be locally isomorphic to G.) Indeed, G_b acts on $\mathcal{E}_b$ by automorphisms in the obvious way, and since it is an inner form for G it must fill out all the automorphisms. In particular, we get an equivalence

$$\{G-\text{torsors on } X_S\} \cong \{G_b - \text{torsors on } X_S\}$$

sending

$$\mathcal{E} \mapsto \underbrace{\underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{E}_b)}_{\underline{\mathrm{Aut}}(\mathcal{E}_b)-\text{torsor}}.$$

Hence basic b induce isomorphisms $\mathrm{Bun}_G \cong \mathrm{Bun}_{G_b}$, sending Bun_G^b isomorphically to $\mathrm{Bun}_{G_b}^1$.

$$\begin{array}{ccccc} \mathrm{Bun}_G & \xrightarrow{\sim} & \mathrm{Bun}_{G_b} & & \\ \uparrow & & \uparrow & & \\ \mathrm{Bun}_G^b & \xrightarrow{\sim} & \mathrm{Bun}_{G_b}^1 & = & [\mathrm{pt}/\underline{G_b(E)}] \end{array}$$

□

[35] This owes partly to discussions with Kedlaya, and was probably already known to Fontaine and Faltings.

13.4. **Non-semistable b.**

Theorem 13.15. $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$ *is locally closed, and*

$$\mathrm{Bun}_G^b \cong [\mathrm{pt}/\mathcal{G}_b]$$

where $\mathcal{G}_b$ *is a group v-sheaf fitting into a short exact sequence*

$$1 \to \mathcal{G}_b^0 \to \mathcal{G}_b \to \underline{G_b(E)} \to 1$$

with $\mathcal{G}_b^0$ *an extension of positive Banach-Colmez spaces, of dimension* $\langle 2\rho, \nu(b)\rangle$.

14. Étale cohomology of diamonds (Dec 14)

Our goal is to set the foundations to define $D(\mathrm{Bun}_G, \mathbf{Z}_\ell)$ for $\ell \neq p$. So we need a theory of ℓ-adic étale sheaves on general small v-stacks. This theory was developed in [S17], in the setting of torsion coefficients. Today we will summarize it.

14.1. Setup.

Recall that Perf is the category of perfectoid spaces of characteristic p. We endow this with the v-topology. Now, there is a technical issue that Perf is large. So it is useful to introduce the notion of a *small* v-sheaf, which is a small colimit of representable sheaves. Equivalently, a v-sheaf $\mathcal{F}$ is *small* if and only if there exists a surjection $X \to \mathcal{F}$ for some perfectoid space X.

Remark 14.1. Using such a presentation, one can see that a sub-v-sheaf of a small v-sheaf is automatically small.

Similarly, there is the notion of "small v-stack".

14.2. General idea.

Slogan: we will always use to descent to reduce to the case of strictly totally disconnected spaces.

A priori this is weird because these strictly totally disconnected spaces are rather ungeometric, and for example don't enjoy any kind of Poincaré duality. Nevertheless, it works.

14.3. Formalism.

Let Λ be a ring annihilated by some integer n prime to p. **Goal:** for any small v-stack, define a triangulated Λ-linear category $D_{\text{ét}}(X, \Lambda)$ and the following 6 functors:

(1) If $f\colon Y \to X$ is any map of small v-stacks, there is a pullback

$$f^*\colon D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$$

with a right adjoint

$$f_*\colon D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda).$$

(2) Any $D_{\text{ét}}(X, \Lambda)$ has a symmetric monoidal tensor product $- \overset{L}{\otimes}_\Lambda -$ and f^* preserves $- \overset{L}{\otimes}_\Lambda -$ (i.e. is symmetric monoidal). It has a partial right adjoint

$$\mathcal{RH}om_\Lambda(-,-)\colon D_{\text{ét}}(X, \Lambda)^{\mathrm{op}} \times D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(X, \Lambda)$$

such that

$$\mathcal{RH}om_{D_{\text{ét}}(X,\Lambda)}(A, \mathcal{RH}om_\Lambda(B, C)) = \mathcal{RH}om_{D_{\text{ét}}(X,\Lambda)}(A \overset{L}{\otimes}_\Lambda B, C)$$

(3) If $f\colon Y \to X$ is representable in locally spatial diamonds and compactifiable[36] and $\mathrm{dim.tr.g.}(f) < \infty$ (locally), there is a functor

$$Rf_!\colon D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$$

satisfying base change and a projection formula

$$Rf_!(A \overset{L}{\otimes}_\Lambda f^*B) \cong (Rf_!A) \overset{L}{\otimes}_\Lambda B$$

for $A \in D_{\text{ét}}(Y, \Lambda)$, $B \in D_{\text{ét}}(X, \Lambda)$. This admits a right adjoint

$$Rf^!\colon D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$$

[36] In analytic geometric there is always a *distinguished* compactification if any exists at all, so this is not difficult to check in practice.

Really we only need the three functors $f^*, - \overset{L}{\otimes}_\Lambda -, Rf_!$; the others then arise as right adjoints. Among these three functors, we need f^* to commute with $- \overset{L}{\otimes}_\Lambda -$, and satisfy base change against $Rf_!$ in the following sense: if we have a cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

such that $Rf_!$ is defined, then

$$g^* Rf_! \xrightarrow{\sim} R(f')_!(g')^* \text{ as functors } D_{\text{ét}}(Y,\Lambda) \to D_{\text{ét}}(X',\Lambda).$$

Actually, all the $D_{\text{ét}}(X,\Lambda)$ arise as the homotopy categories of stable Λ-linear ∞-categories $\mathcal{D}_{\text{ét}}(X,\Lambda)$, and all functors are defined at this level. (However, we do not claim to develop a full 6-functor formalism for $\mathcal{D}_{\text{ét}}(X,\Lambda)$. There are some technical problems in doing so.)

14.4. **Definition of the derived category.** The functor $X \mapsto \mathcal{D}_{\text{ét}}(X,\Lambda)$ will be a v-sheaf of ∞-categories. In particular, if $Y \to X$ is a v-cover, then

$$\mathcal{D}_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \varprojlim \left(\mathcal{D}_{\text{ét}}(Y,\Lambda) \rightrightarrows \mathcal{D}_{\text{ét}}(Y \times_X Y,\Lambda) \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \ldots\right)$$

Remark 14.2. The functor $X \mapsto \mathcal{D}_{\text{ét}}(X,\Lambda)$ is even a hyper-v-sheaf, i.e. you have the analogous statement for hypercovers $Y_\bullet \to X$.

Because of this, it suffices to define $\mathcal{D}_{\text{ét}}(X,\Lambda)$ for X a strictly totally disconnected perfectoid space.

Definition 14.3. Let X be strictly totally disconnected. We define $\mathcal{D}_{\text{ét}}(X,\Lambda) := \mathcal{D}(X_{\text{ét}},\Lambda)$ is the derived ∞-category of the abelian category of étale Λ-modules on X. By general site-theoretic nonsense, this inherits automatically:

- A symmetric monoidal tensor product $- \overset{L}{\otimes}_\Lambda -$.
- $*$-pullback functoriality.

From here, we already get a well-defined category $D_{\text{ét}}(X,\Lambda)$ for any small v-stack X, plus $- \overset{L}{\otimes}_\Lambda -$ and f^*.

Proposition 14.4. *The functor f^* admits a right adjoint Rf_*, and $- \overset{L}{\otimes}_\Lambda -$ admits a partial right adjoint $\mathcal{RH}om_\Lambda(-,-)$.*

Proof. For strictly totally disconnected X, $\mathcal{D}_{\text{ét}}(X,\Lambda)$ is a presentable stable ∞-category because it is the category of sheaves on a site. Then for a general v-stack X, $\mathcal{D}_{\text{ét}}(X,\Lambda)$ is a small limit of presentable ∞-categories and is therefore also presentable. To construct these right adjoints, invoke Lurie's adjoint functor theorem. □

We have now defined $\mathcal{D}_{\text{ét}}(X,\Lambda)$, but we want to understand it more explicitly. Assume X is a locally spatial diamond. Then it has an étale site $X_{\text{ét}}$, defined as follows. A map $f\colon Y \to X$ is *étale* if it is locally separated and for all perfectoid spaces $X' \to X$, the map $Y \times_X X' \to X'$ is representable by a perfectoid space, which is étale over X'. The ideal situation would be that for such X, $\mathcal{D}_{\text{ét}}(X,\Lambda)$ is the derived category of this site. That is *almost* true.

Warning 14.5. Pay close attention to the subscripts in what follows. By definition, $\mathcal{D}(X_{\text{ét}}, \Lambda)$ is the derived ∞-category of abelian categories of étale Λ-modules on X, while $\mathcal{D}_{\text{ét}}(X, \Lambda)$ is the category defined above by descent.

Theorem 14.6. *There is a natural functor*

$$\mathcal{D}(X_{\acute{e}t}, \Lambda) \to \mathcal{D}_{\acute{e}t}(X, \Lambda),$$

which induces an equivalence

$$\mathcal{D}^+(X_{\acute{e}t}, \Lambda) \to \mathcal{D}^+_{\acute{e}t}(X, \Lambda)$$

and realizes $\mathcal{D}_{\acute{e}t}(X, \Lambda)$ as the left-completion

$$\varprojlim_n D^{\geq -n}_{\acute{e}t}(X, \Lambda) \cong \varprojlim_n \mathcal{D}^{\geq -n}(X_{\acute{e}t}, \Lambda).$$

In particular, if $\mathcal{D}(X_{\acute{e}t}, \Lambda)$ is left-complete, for example under "finite cohomological dimension of $X_{\acute{e}t}$", then

$$\mathcal{D}(X_{\acute{e}t}, \Lambda) \xrightarrow{\sim} \mathcal{D}_{\acute{e}t}(X, \Lambda).$$

Sketch. The first observation is that $\mathcal{D}_{\text{ét}}(X, \Lambda)$ always has a natural t-structure (arising by descent from strictly totally disconnected spaces), which is left-complete. (For the left-completeness, you reduce to strictly totally disconnected X, where the cohomological dimension is 0, which makes it easy to prove.)

So it's enough to show that $\mathcal{D}^{\geq 0}(X_{\text{ét}}, \Lambda) \xrightarrow{\sim} \mathcal{D}^{\geq 0}_{\text{ét}}(X, \Lambda)$. We claim that this latter includes as a full subcategory $\mathcal{D}^{\geq 0}_{\text{ét}}(X, \Lambda) \hookrightarrow \mathcal{D}^{\geq 0}(X_v, \Lambda)$. That's because $\mathcal{D}^{\geq 0}_{\text{ét}}(X, \Lambda)$ is a limit over v-hypercovers, and such data tautologically descends to the v-site.

$$\begin{array}{ccc} \mathcal{D}^{\geq 0}(X_{\text{ét}}, \Lambda) & \longrightarrow & \mathcal{D}^{\geq 0}_{\text{ét}}(X, \Lambda) \\ & \searrow & \downarrow \\ & & \mathcal{D}^{\geq 0}(X_v, \Lambda) \end{array}$$

Now the key is to prove that for $\lambda \colon X_v \to X_{\text{ét}}$, λ^* is fully faithful. Equivalently, for all étale Λ-sheaves $\mathcal{F}$,

$$\mathcal{F} \xrightarrow{\sim} \lambda_* \lambda^* \mathcal{F}$$

and $R^i \lambda_* \lambda^* \mathcal{F} = 0$ for $i > 0$. (Slogan: "invariance of cohomology under passage from the étale site to the v-site.) This is an analogue of Grothendieck's results that étale cohomology of étale sheaves coincides with fppf or even fpqc cohomology.

To prove this, we first pass from the étale site to the pro-étale site. This is largely formal: you write pro-étale covers as cofiltered limits of étale covers. Then on cohomology, you get filtered colimits, which are exact.

Next you go from the pro-étale site to the v-site. By pro-étale descent you can assume that X is strictly totally disconnected. If $Y \to X$ is a v-cover by an affinoid perfectoid space, we can write $Y = \varprojlim_i Y_i \to X$ where each $Y_i \to X$ is open in some finite dimensional ball over X. (This is the analogue of approximating any affine scheme as a limit of finite type ones, but here we are even able to arrange this approximation to be by "smooth" objects!) If $X = \operatorname{Spa}(R, R^+)$ and $Y = \operatorname{Spa}(S, S^+)$ there exists $R\langle T_i^{1/p^\infty} : i \in I \rangle \twoheadrightarrow S$ for some I (e.g. take I to be the set of all power-bounded elements of S). That presents $S = V(f_j \colon j \in J) \subset \mathbb{B}^I_X$. This is a limit of

$$\{|f_j| \leq |\epsilon|, j \in J' \subseteq J\} \subset \mathbb{B}^{I'}_X$$

over finite $I' \subseteq I$ and finite $J' \subset J$. So we take these as the Y_i's. Note this is similar to how a Zariski closed immersion was a limit over open neighborhoods.

So descent for $Y \to X$ reduces to descent for $Y_i \to X$. (We are using that the sheaves under consideration come from the étale site, in order to know that their sections on limits are the colimit of sections.) But this has a section, as the $Y_i's$ are "smooth" over X hence have a surjection after an étale cover of X; but since X was strictly totally disconnected, it doesn't have non-trivial étale covers.

Slogan: noetherian descent in the v-world allows things to be smooth!

□

Proposition 14.7. *For all small v-stacks X,*

$$\mathcal{D}_{ét}(X, \Lambda) \hookrightarrow \mathcal{D}(X_v, \Lambda)$$

and $A \in \mathcal{D}(X_v, \Lambda)$ lies in the image if and only if all the $\mathcal{H}^i(A)$ lie in the image. Furthermore, this can be checked after pullback to any locally spatial diamond $Y \to X$, where the meaning is that it comes from an étale sheaf.

14.5. **Base change.**

Theorem 14.8 (qcqs base change)**.** *Consider a cartesian diagram of locally spatial diamonds*

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

where f is qcqs. Then

$$Rf'_*(g')^* \xrightarrow{\sim} g^* Rf_* \colon \mathcal{D}^+_{ét}(Y, \Lambda) \to \mathcal{D}^+_{ét}(X', \Lambda).$$

If f has finite cohomological dimension, then this holds on $\mathcal{D}_{ét}(Y, \Lambda) \to D_{ét}(X', \Lambda)$.

Proof. If $X' \to X$ is pro-étale, then the base change is automatic. Hence we can assume that X, X' are strictly local. Additionally using descent on Y, we can also assume that Y is strictly totally disconnected. Then by passage to connected components, we can also assume that Y is strictly local. So $X = \mathrm{Spa}(C, C^+)$, $X' = \mathrm{Spa}(C', (C')^+)$, $Y = \mathrm{Spa}(\widetilde{C}, \widetilde{C}^+)$, $Y' = \mathrm{Spa}(R, R^+)$ and $R = C' \widehat{\otimes}_C \widetilde{C} \supset R^+$.

Lemma 14.9. *We have*

$$H^i(Y', \mathbf{Z}/n\mathbf{Z}) = \begin{cases} \mathbf{Z}/n\mathbf{Z} & i = 0, \\ 0 & i > -0. \end{cases}$$

This follows from invariance of cohomology under algebraically closed field extension, for which you can reduce to work of Huber in the finite type situation.

□

Note that no properness is needed here. But it is not the kind of base change that you are used to, because "points" of adic spaces are not adic spaces. For example, for strictly local space $\mathrm{Spa}(C, C^+)$, we do not have base change with respect to $\{s\} \hookrightarrow \mathrm{Spa}(C, C^+)$, as $\{s\}$ is not an adic space or even a diamond.

Theorem 14.10 (Proper base change)**.** *If f is in addition proper, then we also get these kinds of base changes.*

Sketch. This is reduced to proper base change for schemes by similar reduction steps. □

14.6. **Proper pushforward.** If $f\colon Y \to X$ is representable in spatial diamonds, compactifiable, and has dim.tr.g.$f < \infty$. Then there is an open immersion

$$\begin{array}{ccc} j\colon Y & \hookrightarrow & \overline{Y}^{/X} \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

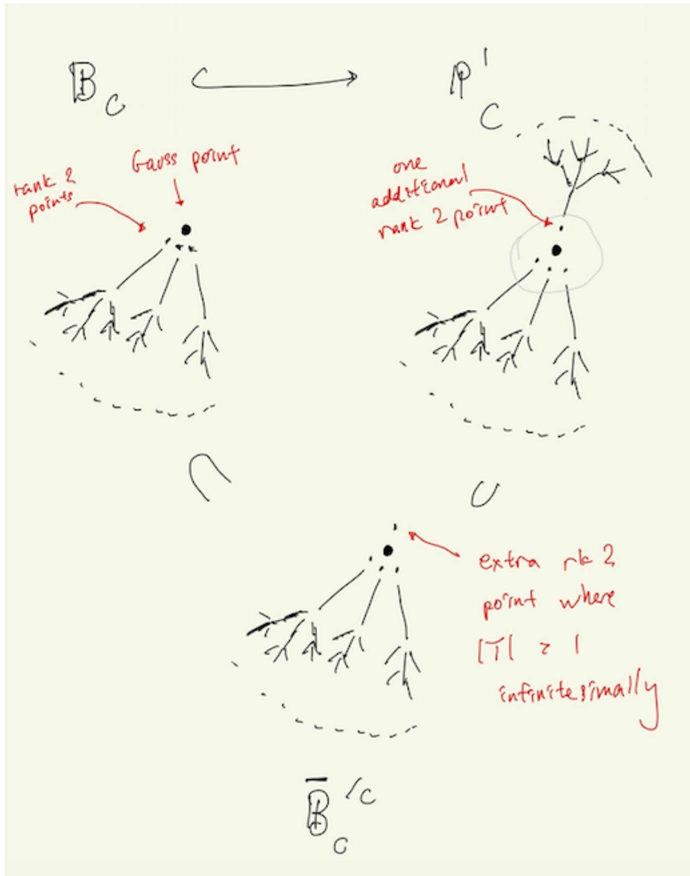
where $\overline{Y}^{/X}(R,R^+) = X(R,R^+) \times_{X(R,R^\circ)} Y(R,R^\circ)$. This can be phrased in terms of the valuative criterion, for extending maps along $\mathrm{Spa}(R,R^\circ) \to \mathrm{Spa}(R,R^+)$. Recall that $Y \to X$ is proper iff

$$\begin{array}{ccc} \mathrm{Spa}(R,R^\circ) & \longrightarrow & Y \\ \downarrow & \overset{\exists !}{\nearrow} & \downarrow \\ \mathrm{Spa}(R,R^+) & \longrightarrow & X \end{array}$$

This is equivalent to $Y \xrightarrow{\sim} \overline{Y}^{/X}$. For general $Y \to X$ as above, $\overline{Y}^{/X}$ is the initial proper diamond over X with a map from Y.

Definition 14.11. We have $Rf_! = R\overline{f}^{/X}_* \circ j_! \colon D_{\text{ét}}(Y,\Lambda) \to D_{\text{ét}}(X,\Lambda)$ where $j_!$ is the extension by zero (the left adjoint of j^*).

Example 14.12. Let $Y = \mathbb{B}_C \to X = \mathrm{Spa}\, C$. Try embedding $\mathbb{B}_C$ into a compactification like $\mathbf{P}^1_C$.



The canonical compactification $\overline{\mathbb{B}}_C^{/C}$ throws in a rank 2 point of radius infinitesimally bigger than 1. Another way to look at this is it shrinks the R^+: we have $\mathbb{B}_C = \mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C\langle T\rangle)$ and $\overline{\mathbb{B}}_C^{/C} = \mathrm{Spa}(C\langle T\rangle, \mathcal{O}_C + \mathfrak{m}_C\langle T\rangle)$.

In general, this compactification "adds all higher rank points". It is completely functorial. More generally, it can be defined if f is only representable in locally spatial diamonds (and compactifiable, with dim.tr.g.$(f) < \infty$).

Theorem 14.13. *$Rf_!$ satisfies the base change and projection formula, and composes.*

This is actually easy from Proper Base Change; it is about the commutation of $j_!$ and Rf_*.

14.7. **Exceptional inverse image.** Using adjoint functor theorems, you check:

Proposition 14.14. *$Rf_!$ has a right adjoint $Rf^!$.*

14.8. **Verdier duality.** This is the trickiest part. We cannot define smoothness directly, since there are no non-reduced perfectoid rings, and therefore no notion of tangent spaces.

Definition 14.15. Let $f\colon Y \to X$ be representable in locally spatial diamonds, compactifiable, and dim.tr.g.$(f) < \infty$. We say that f is *cohomologically smooth* if $Rf^! \cong f^*A \otimes \mathbb{D}_f$ for some $\mathbb{D}_f \in D_{\text{ét}}(Y, \Lambda)$, and this continues to hold after any base change.

We are taking a Theorem about usual ℓ-adic sheaves (Verdier duality) and turning it into a definition. The hard thing is to find examples.

Example 14.16. Let S be a profinite set of infinite cardinality. Then the map $f\colon \underline{S} \times \operatorname{Spa} C = \operatorname{Spa} \operatorname{Cont}(S, C) \to \operatorname{Spa} C$ is pro-étale, but not cohomologically smooth. What is $Rf^!\Lambda$? By definition,

$$R\Gamma(S, Rf^!\Lambda) = \operatorname{RHom}(Rf_!\Lambda, \Lambda) = \operatorname{RHom}(Rf_*\Lambda, \Lambda) = \operatorname{Cont}(S, \Lambda)^\vee.$$

This is the space of Λ-valued *measures* on S. You can similarly compute the stalk

$$(Rf^!\Lambda)_s = \varinjlim_{U \ni s} \mathcal{M}(U, \Lambda)$$

which is "germs of measures at s", rather than Λ.

Example 14.17. We can extend the notion of cohomological smoothness to stacks, by asking for smooth-local smoothness. If $\underline{G}$ is locally pro-p, then the map $[\operatorname{pt}/\underline{G}] \to *$ is cohomologically smooth. This reflects that there is a Haar measure on $\underline{G}$.

15. Smoothness (Dec 18)

15.1. **Recap of last time.** We gave a brief summary of étale cohomology of diamonds. Let Λ be a ring killed by n prime to p. For any small v-stack, we defined $D_{\text{ét}}(X, \Lambda)$. In general this was defined by descent, but it coincides with $D(X_{\text{ét}}, \Lambda)$ if X is a locally spatial diamond with $\dim.\mathrm{tr.g.}(X) < \infty$. It is a (closed) symmetric monoidal triangulated category.

For any map $f\colon Y \to X$, we have f^* and Rf_*.

If f is represented in locally spatial diamonds, compactifiable, and locally $\dim.\mathrm{tr.g.}(f) < \infty$, then there is a functor

$$Rf_!\colon D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(X, \Lambda)$$

with a right adjoint $Rf^!$.

This satisfies all the properties of a "6-functor formalism". In practice, the hard part is to understand $Rf^!$.

15.2. **Why locally spatial diamonds?** One thing we didn't explain last time, which is one of the main motivations for the definition of "locally spatial", is why we demand maps to be representable in locally spatial diamonds. The point is we want $Rf_!$ to commute with all direct sums. This is equivalent to having a right adjoint, which is essentially equivalent to commuting with arbitrary direct sums, by adjoint functor theorems. And ultimately the hypotheses will ensure this.

Example 15.1. We are always considering our functors on bounded derived categories.

If $A_n \in D_{\text{ét}}(Y, \Lambda)$, $n \geq 0$, is concentrated in degree 0, then $\oplus A_n[n] \cong \prod A_n[n]$ (by left-completeness – the natural map can be checked to be an isomorphism by looking at cohomology sheaves). Then

$$Rf_!(\bigoplus A_n[n]) \cong Rf_!(\prod_n A_n[n]) \cong \prod_n (Rf_! A_n)[n].$$

We want this to be isomorphic to $\bigoplus_n (Rf_! A_n)[n]$. This can only be true if $Rf_!$ has finite cohomological dimension.

So it is a general principle that a functor (on a triangulated left-complete category) can only commute with all direct sums if it has finite cohomological dimension.

A key ingredient is:

Theorem 15.2 (Scheiderer '94)**.** *If T is any spectral topological space, then the cohomological dimension of T is $\leq$ the Krull dimension of T.*

Remark 15.3. Under a noetherianity assumption the theorem was proved much earlier by Grothendieck. This is a striking theorem because Krull dimension is a purely local invariant, while cohomological dimension is global. The result drastically fails for compact Hausdorff spaces (they always have Krull dimension 0).

The finiteness properties we impose are through the "geometric transcendence dimension" dim.tr.g., which is a purely punctual condition. This is good enough to detect Krull dimension, but to control the cohomological dimension we need our space to be (locally) spectral, which is what "locally spatial" gives. To summarize, we need "locally spatial" assumption to control the cohomological dimension.

15.3. **Cohomological smoothness.**

Definition 15.4. Fix $\ell \neq p$. Then $f\colon Y \to X$ as above (compactifiable, representable in locally spatial diamonds, with dim.tr.g.$(f) < \infty$ locally) is *ℓ-cohomologically smooth* if, after any base change, $Rf^! \cong \mathbb{D}_f \otimes f^*$ as functors

$$D_{\text{ét}}(X, \mathbf{F}_\ell) \to D_{\text{ét}}(Y, \mathbf{F}_\ell)$$

where $\mathbb{D}_f$ is locally isomorphic to $\mathbf{F}_\ell[n]$ for some $n \in \mathbf{Z}$.

Remark 15.5. If this is satisfied, $\mathbb{D}_f = Rf^!\mathbf{F}_\ell$ and its formation commutes with any base change.

Conversely (and somewhat surprisingly), if $Rf^!\mathbf{F}_\ell$ is invertible and its formation commutes with base change along $Y \to X$, then f is ℓ-cohomogically smooth.

Definition 15.6. We say $f\colon Y \to X$ is *cohomologically smooth* if it's ℓ-cohomologically smooth for all $\ell \neq p$. Then also $Rf^!\Lambda$ is locally isomorphic to $\Lambda[n]$ for any Λ.

Example 15.7. Recall that $\mathbb{B}_C$ is the closed unit disc. The map $\mathbb{B}_C^\diamond \to (\operatorname{Spa} C)^\diamond$ is cohomologically smooth. This follows from results of Huber, whose studies the cohomology of spaces like $\mathbb{B}_C$. A priori we need to check against arbitrary base changes, which could be difficult. Originally the argument was by base change to strictly totally disconnected spaces. However, you can also use the converse direction in Remark 15.5.

Example 15.8. If $f\colon Y \to X$ is a smooth (i.e. locally étale over a ball) map of analytic adic spaces over $\mathbf{Z}_p$, then $f^\diamond\colon Y^\diamond \to X^\diamond$ is cohomologically smooth.

Example 15.9. Let E be a finite extension of $\mathbf{Q}_p$. Then $(\operatorname{Spa} E)^\diamond \to (\operatorname{Spa} \mathbf{F}_q)^\diamond$ is cohomologically smooth. More generally, recall that $\operatorname{Div}^d = (\operatorname{Div}^1)^d/S_d$. Then $\operatorname{Div}^d \to *$ is cohomologically smooth.

In fact, this even works for non-analytic spaces: $(\operatorname{Spa} \mathcal{O}_E)^\diamond \to (\operatorname{Spa} \mathbf{F}_q)^\diamond$ is cohomologically smooth. Recall that $\operatorname{Spa} \mathcal{O}_E$ and $\operatorname{Spa} \mathbf{F}_q$ are not analytic adic spaces, so $(\operatorname{Spa} \mathcal{O}_E)^\diamond$ and $(\operatorname{Spa} \mathbf{F}_q)^\diamond$ are not diamonds. However they are small v-sheaves, so our formalism still applies.

Example 15.10. If $f\colon Y \to X$ is cohomologically smooth and G is a pro-p-group acting freely on Y/X, then $f/\underline{G}\colon Y/\underline{G} \to X$ is still cohomologically smooth. The converse is very much false! The map

$$Y = \operatorname{Spa} C \times \underline{G} \to \operatorname{Spa} C = X$$

is not cohomologically smooth if G is not finite.

Example 15.11. Consider an open Schubert cell $\operatorname{Gr}_{G,\mu}$. The map $\operatorname{Gr}_{G,\mu} \to (\operatorname{Spa} E)^\diamond$ is cohomologically smooth, but not $\operatorname{Gr}_{G,\leq\mu}$ in general.

Example 15.12. If $\mathcal{E}$ is a vector bundle on X_S such that all fibers have only *positive* HN slopes, then $\mathcal{BC}(\mathcal{E}) \to S$ is cohomologically smooth. It is *not* true for slope ≥ 0, e.g. for $\mathcal{E} = \mathcal{O}_{X_S}$, we have $\mathcal{BC}(\mathcal{E}) = \underline{E}$, which is not cohomologically smooth.

Example 15.13. The condition of a map $f\colon Y \to X$ being cohomologically smooth can be checked v-locally on X, given that f is compactifiable, representable in locally spatial diamonds, with dim.tr.g.$(f) < \infty$ locally.

All of these examples are established by starting with $\mathbb{B}_C$, which is essentially contained in work of Huber. Then you observe that cohomological smoothness is stable under quotient by pro-p groups because the ℓ-cohomological dimensions of such groups is 0.

15.4. **Artin stacks.** Analogy:

$$\text{schemes} \hookrightarrow \text{algebraic spaces} \hookrightarrow \text{Artin stacks}$$

$$\text{perfectoid spaces} \hookrightarrow \text{locally spatial diamonds} \hookrightarrow \text{Artin } v\text{-stacks}$$

Definition 15.14. A small v-stack X is *Artin* if

- $\Delta_X : X \to X \times X$ is representable in locally spatial diamonds.
- There exists a cohomologically smooth surjection $f : Y \to X$ with Y a locally spatial diamond.

For an Artin stack X, we can define what it means for $X \to *$ to be "cohomologically smooth". It means that for $f : Y \to X$ as in Definition 15.14, $Y \to X \to *$ is cohomologically smooth. We can extend the notion of dualizing complex to such X, which gives a notion of dimension by looking at the degree of the dualizing complex. (A priori this depends on ℓ.)

Theorem 15.15. Bun_G *is a cohomologically smooth Artin v-stack (of dimension* 0 *in the sense that for* $\pi : \mathrm{Bun}_G \to *$*,* $R\pi^!\Lambda$ *is locally isomorphic to* $\Lambda[0]$*).*

Sketch. Show that the Beauville-Laszlo uniformization

$$\pi : \mathrm{Gr}_{G,\mu} / \underline{G(E)} \to \mathrm{Bun}_G$$

is cohomologically smooth of dimension $\langle 2\rho, \mu\rangle$ and $\mathrm{Gr}_{G,\mu} / \underline{G(E)}$ is also cohomologically smooth of dimension $\langle 2\rho, \mu\rangle$.

The fibers of π are open in $\mathrm{Gr}_{G,\mu^{-1}}$. (The fibers are the modifications of a bundle which are trivial, and we saw that being trivial is an open condition.)

We claim that qcqs cohomologically smooth maps are open, hence the (open) image of this map has the desired property. To justify the claim, suppose $f : Y \to X$ is cohomologically smooth and qcqs. Then $Rf_!$ preserves constructible sheaves, as it preserves compact objects because its right adjoint $Rf^!$ preserves direct sums, and the compact objects are the constructible sheaves. Hence $Rf_!\mathbf{F}_\ell$ is constructible, and therefore its support is open and quasicompact.

Taking $\bigcup_\mu$ of such charts, this covers Bun_G by Theorem 12.25. □

Example 15.16. In particular, $[*/\underline{G(E)}] \cong \mathrm{Bun}_G^1 \subset \mathrm{Bun}_G$ is cohomologically smooth over $*$.

15.5. **Better charts for** Bun_G**.** To study Bun_G, we need better smooth charts.

Example 15.17. Let $G = \mathrm{GL}_2$. We have a specialization $\mathcal{O}(1/2) \rightsquigarrow \mathcal{O} \oplus \mathcal{O}(1)$. These two points form an open substack $U \subset \mathrm{Bun}_G$.

We want a nice atlas for U. Let $b \in B(G)$ be the isocrystal corresponding to $\mathcal{O} \oplus \mathcal{O}(1)$. Let $\mathcal{M}_b$ be the moduli space of extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

where $\mathcal{L}$ is a line bundle of degree 0, and $\mathcal{L}'$ is a line bundle of degree 1.

Then we get a map

$$\pi_b : \mathcal{M}_b \to \mathrm{Bun}_{\mathrm{GL}_2} = \mathrm{Bun}_G$$

sending $(\mathcal{L} \to \mathcal{E} \to \mathcal{L}') \mapsto \mathcal{E}$.

Theorem 15.18. *The map* π_b *is cohomologically smooth.*

What is the structure of $\mathcal{M}_b$? We have $\mathcal{M}_b = \widetilde{\mathcal{M}}_b/(\underline{E}^\times \times \underline{E}^\times)$ where $\widetilde{\mathcal{M}}_b$ parametrizes

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

So $\widetilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1])$ is a negative Banach-Colmez space.

We have a diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{M}}_b^\circ & \hookrightarrow & \widetilde{\mathcal{M}}_b & \hookleftarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_G^{b'} & \hookrightarrow & U & \hookleftarrow & \mathrm{Bun}_G^{b} \end{array}$$

where $b' \leftrightarrow \mathcal{O}(1/2)$. This is because every non-split extension of $\mathcal{O}(1)$ by $\mathcal{O}$ is isomorphic to $\mathcal{O}(1/2)$. We have $\widetilde{\mathcal{M}}_b^\circ = \mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\}$. This can be presented explicitly as $(\operatorname{Spa} k((t)))^\diamond / \mathrm{SL}_1(D_{1/2})$ where $D_{1/2}$ is the quaternion algebra over E. Indeed, $\mathrm{SL}_1(D_{1/2})$ is the group of automorphisms of $\mathcal{O}(1/2)$ preserving the trivialization of its associated graded, and after we choose an isomorphism with this standard extension, it remains to give a nonzero section of $\mathcal{O}(1/2)$, which is parametrized by the Banach-Colmez space $\mathcal{BC}(\mathcal{O}(1/2)) \setminus 0$, which is a punctured open unit disk.

In general we will define a cohomologically smooth map

$$\pi_b \colon \mathcal{M}_b \to \mathrm{Bun}_G .$$

Note that $[*/\underline{G_b}(E)] \underbrace{\to}_{\text{coh. smooth}} \mathrm{Bun}_G^b \to \mathrm{Bun}_G$. So you could wonder if there is a way to extend the cohomologically smooth map $[*/\underline{G_b}(E)] \to \mathrm{Bun}_G^b$ to a small neighborhood. That is what $\mathcal{M}_b$ does. You can think of it as the "strict henselization of Bun_G at $[*/\underline{G_b}(E)] \to \mathrm{Bun}_G^b$". It fits into a diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_b & \longrightarrow & \mathcal{M}_b \\ s \uparrow\!\downarrow & & s \uparrow\!\downarrow \\ * & \longrightarrow & [*/\underline{G_b(E)}] \end{array}$$

where the section s is onto the preimage of $\mathrm{Bun}_G^b \subset \mathrm{Bun}_G$, and $\widetilde{\mathcal{M}}_b^\circ := \widetilde{\mathcal{M}}_b \setminus * \to \mathrm{Bun}_G^{>b}$. The $\widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond.

Warning 15.19. Note that the map $\widetilde{\mathcal{M}}_b \to *$ is representable in locally spatial diamonds, but $\widetilde{\mathcal{M}}_b$ is not itself a locally spatial diamond, because it is not analytic.

Example 15.20. Consider $\operatorname{Spa} k[[t]]$, which represents $\operatorname{Spa}(R, R^+) \mapsto R^{\circ\circ}$. The map $\operatorname{Spa} k[[t]] \to *$ is representable in locally spatial diamonds, because its base change to any S is the open unit disk over S. But $\operatorname{Spa} k[[t]]$ is not a diamond, as it has a non-analytic point. However, $\operatorname{Spa} k[[t]] \setminus \operatorname{Spa} k = \operatorname{Spa} k((t))$ *is* a spatial diamond.

Hence we need to pay attention to the fact that an abstract property is very different from the relative property over $*$. The map $\operatorname{Spa} k((t)) \to *$ is *not* quasi-compact while $\operatorname{Spa} k[[t]]$ is quasi-compact.

In fact this phenomenon is key to the arguments. Ultimately a key for us is to show finiteness properties (e.g. of cohomology groups of moduli spaces of shtukas). The possibility of changing perspective so that a thing becomes quasicompact is doing much of the technical work in the argument.

For GL_n, an isocrystal $b \in B(\mathrm{GL}_n)$ corresponds to a $\mathbf{Q}$-graded vector bundle $\mathcal{E} = \bigoplus_{\lambda \in \mathbf{Q}} \mathcal{E}^\lambda$ where $\mathcal{E}^\lambda$ is semistable of slope λ. Then $\mathcal{M}_b$ parametrizes iterated extension of the $\mathcal{E}^\lambda$.

Definition 15.21. Let $\mathcal{M}$ be the moduli space taking $S \in \mathrm{Perf}_{\overline{\mathbf{F}}_q}$ to exact $\otimes$-functors

$$\mathrm{Rep}_E(G) \to \underbrace{\left\{\mathbf{Q}\text{-filtered vector bundles } \mathcal{E} \supset \mathcal{E}^{\leq \lambda}\right\}}_{\text{exact category}}$$

such that $\mathcal{E}^\lambda = \mathcal{E}^{\leq \lambda} / \bigcup_{\lambda' < \lambda} \mathcal{E}^{\leq \lambda'}$ is semistable of slope λ.

Warning 15.22. Note that the filtration is "opposite" to the HN filtration! (The slopes increase.)

We have a map $\mathcal{M} \to \mathrm{Bun}_G$ sending $(\mathcal{E} \supset \mathcal{E}^{\leq \lambda}) \mapsto \mathcal{E}$.

There is another map on $\mathcal{M} \to \coprod_{b \in B(G)} [*/\underline{G_b(E)}]$, sending $(\mathcal{E} \supset \mathcal{E}^{\leq \lambda}) \mapsto \bigoplus \mathcal{E}^\lambda$. This is an exact $\otimes$-functor valued in $\mathbf{Q}$-*graded* vector bundles, where $\mathcal{E}^\lambda$ is semistable of slope λ. This target category is equivalent to the category of isocrystals, which is equivalent to $[*/\underline{G_b(E)}]$. This induces $\mathcal{M} = \bigcup_{b \in B(G)} \mathcal{M}_b$ where $\mathcal{M}_b$ is the fiber over b.

Theorem 15.23. *The map $\pi_b \colon \mathcal{M}_b \to \mathrm{Bun}_G$ is cohomologically smooth.*

Remark 15.24. For $G = \mathrm{GL}_n$, this can be proved by direct attack. For general G it is quite difficult, and we will need to introduce a "Jacobian criterion" for smoothness.

16. $D_{\text{ÉT}}(\mathrm{Bun}_G)$ (Dec 21)

16.1. **Administration.** The next lecture will be on Friday, January 8. The course will run until Friday, February 12.

16.2. **Where are we?** Let G/E be a reductive group.

We defined the moduli stack Bun_G of G-bundles on the Fargues-Fontaine curve. It represents the functor taking

$$S \in \mathrm{Perf}_{\overline{\mathbf{F}}_q} \mapsto \{G\text{-bundles on } X_S\}.$$

We proved:

Theorem 16.1. *(1)* Bun_G *is an Artin v-stack, cohomologically smooth of dimension* 0.

(2) The map $|\mathrm{Bun}_G| \to B(G)$ *is a continuous bijection.*

(3) For any $b \in B(G)$*, we get a locally closed stratum* $|\mathrm{Bun}_G^b| \subset \mathrm{Bun}_G$*. It has the form* $\mathrm{Bun}_G^b = [*/\mathcal{G}_b]$ *where* $\mathcal{G}_b$ *fits into a short exact sequence*

$$1 \to \underbrace{\text{“unipotent group diamond''}}_{\text{iterated ext'n of positive Banach-Colmez spaces}} \to \mathcal{G}_b \to \underline{G_b(E)} \to 1$$

A choice of representative $b \in G(\breve{E})$ *induces a splitting* $[*/\underline{G_b(E)}] \to [*/\mathcal{G}_b] = \mathrm{Bun}_G^b$ *which is cohomologically smooth as a morphism, of relative dimension* $\langle 2\rho, \nu_b \rangle$*.*

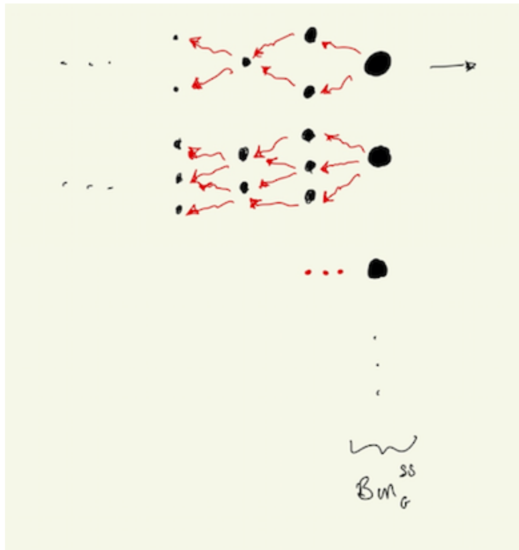
Furthermore, $[*/\underline{G_b(E)}]$ *is cohomologically smooth and isomorphic to* $\mathrm{Bun}_{G_b}^1$*, which is an Artin v-stack of dimension* 0*. So each* Bun_G^b *is also a cohomologically smooth Artin v-stack, of dimension* $-\langle 2\rho, \nu_b \rangle$*.*

Corollary 16.2. *The Kottwitz map induces* $\kappa \colon \pi_0 \mathrm{Bun}_G \xrightarrow{\sim} \pi_1(G)_\Gamma$*. Equivalently, each connected component of* Bun_G *is the closure of* Bun_G^b *for a unique basic* $b \in B(G)$*.*

Proof. We claim that any non-empty open substack $U \subset \mathrm{Bun}_G$ contains a basic (i.e. semistable) point.

Let's conclude the proof assuming the claim. We need to see that the fibers of κ are connected components. Let $b \in B(G)$ be basic and $U \subset \kappa^{-1}(\kappa(b)) \subset \mathrm{Bun}_G$. Then we have $\mathrm{Bun}_G^b \subset U$, so $\kappa^{-1}(\kappa(b))$ must be connected, and b the unique basic point in it.

Now we prove the claim. Take a minimal element $b \in B(G)$ such that $\mathrm{Bun}_G^b \subset U$. By the minimality, $\mathrm{Bun}_G^b \subset U$ is open. (Otherwise there would be a generalization within U.) But U is cohomologically smooth of dimension 0, as it is open in Bun_G, while Bun_G^b is cohomologically smooth of dimension $-\langle 2\rho, \nu_b \rangle$. This means $\langle 2\rho, \nu_b \rangle = 0$, so ν_b is central, i.e. b is basic. □



16.3. **Relation to smooth representation theory of p-adic groups.** We will move on to discuss the relationship between $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ and the representation theory of $G_b(E)$.

Proposition 16.3. *For each $b \in B(G)$, we have*

$$D_{\acute{e}t}(\mathrm{Bun}_G^1, \Lambda) \cong \underbrace{D(G_b(E), \Lambda)}_{\substack{\text{derived cat. of abelian cat.} \\ \text{of smooth rep'ns of } G_b(E) \\ \text{on } \Lambda\text{-modules}}}$$

16.3.1. *Step 0.* Step 0 is to show that $D_{\text{ét}}(*, \Lambda) \cong D(\Lambda)$.

This is not totally obvious. Why? It is easy to show that for a geometric point $\mathrm{Spa}\, C$, $D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda) \cong D(\Lambda)$ as $\mathrm{Spa}\, C$ is a spatial diamond of finite cohomological dimension (in fact 0), so $D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda) \cong D((\mathrm{Spa}\, C)_{\text{ét}}, \Lambda) = D(\Lambda)$ as $(\mathrm{Spa}\, C)_{\text{ét}}$ is the site of finite sets.

But $* = \mathrm{Spa}\, \overline{\mathbf{F}}_q$ is *not* a diamond. To understand what you get here, you have to analyze descent along $\mathrm{Spa}\, C \to \mathrm{Spa}\, \overline{\mathbf{F}}_q$.

Proposition 16.4. *For any small v-stack $X/\overline{\mathbf{F}}_q$, and any complete algebraically closed non-archimedean field C, the pullback $D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(X \times_{\mathrm{Spa}\, \overline{\mathbf{F}}_q} \mathrm{Spa}\, C, \Lambda)$ is fully faithful.*

This implies that

$$D_{\text{ét}}(*, \Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda) \cong D(\Lambda).$$

On the other hand, you can certainly build constant sheaves on any space, which defines a commutative diagram

$$D_{\text{ét}}(*, \Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda)$$
$$D(\Lambda) \to D_{\text{ét}}(*, \Lambda), \qquad D(\Lambda) \to D_{\text{ét}}(\mathrm{Spa}\, C, \Lambda)$$

16.3.2. *Step 1.* We want to show that

$$D_{\text{ét}}([*/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda)$$

and in fact this holds for any locally pro-p group H in place of $G_b(E)$. Similarly, we want to show that

$$D_{\text{ét}}([\operatorname{Spa} C/\underline{G_b(E)}], \Lambda) \cong D(G_b(E), \Lambda).$$

The idea is to use descent along $\operatorname{Spa} C \to [\operatorname{Spa} C/\underline{G_b(E)}]$. You have to take some care to see that you get the notion of *smooth* representation at the end.

A better way of organizing the argument is to show that

$$D_{\text{ét}}([\operatorname{Spa} C/\underline{G_b(E)}], \Lambda) \cong D([\operatorname{Spa} C/\underline{G_b(E)}]_{\text{ét}}, \Lambda)$$

in this case (even though $[\operatorname{Spa} C/\underline{G_b(E)}]$ is not a diamond, for which such a comparison would hold on general grounds). Now, $[\operatorname{Spa} C/\underline{G_b(E)}]_{\text{ét}}$ is equivalent to the site of sets with continuous $G_b(E)$-action. (The statement is that a continuous action map $\underline{G_b(E)} \times \underline{S} \to \underline{S}$ is equivalent to a continuous $G_b(E)$-action on S in the usual sense.) Then, one sees that Λ-modules on that site are smooth $G_b(E)$-representations on Λ-modules (smooth means continuous action on discrete Λ-modules).

16.3.3. *Step 2.* We show that $D_{\text{ét}}(\operatorname{Bun}_G^b, \Lambda) \cong D_{\text{ét}}([*/\underline{G_b(E)}], \Lambda)$. We study the map

$$[*/G_b(E)] \to \operatorname{Bun}_G^b.$$

This map is cohomologically smooth, with fibers having trivial cohomology.

It is a general property that pullback through a cohomologically smooth fibration with acyclic fibers is fully faithful, so that gives

$$D_{\text{ét}}(\operatorname{Bun}_G^b, \Lambda) \hookrightarrow D_{\text{ét}}([*/\underline{G_b(E)}], \Lambda).$$

On the other hand, the map is really the splitting of

$$\begin{array}{ccc} & [*/\underline{G_b(E)}] & \\ \diagup\!\!\diagup & & \nwarrow \\ [*/G_b(E)] & \longrightarrow & \operatorname{Bun}_G^b \end{array}$$

Corollary 16.5 (of the proof). *For any $C/\overline{\mathbf{F}}_q$ complete and algebraically closed, pullback induces*

$$D_{\acute{e}t}(\operatorname{Bun}_G, \Lambda) \cong D_{\acute{e}t}(\operatorname{Bun}_G \times_{\overline{\mathbf{F}}_q} \operatorname{Spa} C, \Lambda).$$

Corollary 16.6. *$D_{\acute{e}t}(\operatorname{Bun}_G, \Lambda) \cong D_{\acute{e}t}(\operatorname{Bun}_G \times_{\overline{\mathbf{F}}_q} \operatorname{Spa} C, \Lambda)$ and admits an infinite semi-orthogonal decomposition with pieces*

$$D_{\acute{e}t}(\operatorname{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda).$$

Proof. Bun_G has a stratification with pieces $i^b \colon \operatorname{Bun}_G^b \hookrightarrow \operatorname{Bun}_G$. The functors $i^b_!, i^{b*}$ induce a semiorthogonal decomposition. This is because in general, for an open-closed decomposition

$$i \colon Z \hookrightarrow X \hookleftarrow U : j$$

you always get a semi-orthogonal decomposition

$$D_{\text{ét}}(Z) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D_{\text{ét}}(X) \underset{j_!}{\overset{j^*}{\rightleftarrows}} D_{\text{ét}}(U)$$

The invariance property follows formally. We know the fully faithfulness $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Bun}_G \times_{\overline{\mathbf{F}}_q} \mathrm{Spa}\, C, \Lambda)$, but we also know that the essential image contains $i^b_! D_{\text{ét}}(\mathrm{Bun}^b_G \times_{\overline{\mathbf{F}}_q} \mathrm{Spa}\, C, \Lambda)$, and then everything by devissage. □

Remark 16.7. How strata interact is encoded in the spaces $\pi_b \colon \mathcal{M}_b \to \mathrm{Bun}_G$ from last lecture. However, we will not elaborate on this today.

16.4. **Structure of the category $D_{\text{ét}}(\mathrm{Bun}_G)$.** Recall that if $\mathcal{C}$ is a triangulated category, then $X \in \mathcal{C}$ is *compact* if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ commutes with all direct sums. We denote by $\mathcal{C}^\omega$ the full subcategory of compact objects.

Theorem 16.8. *$D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compactly generated, and a complex $\mathcal{A} \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compact if and only if all $(i^b)^*\mathcal{A} \in D_{ét}(\mathrm{Bun}^b_G, \Lambda)$ are compact, and almost all are* 0. *Equivalently, they lie in the thick triangulated subcategory generated by* $\text{c-Ind}_K^{G_b(E)}(\Lambda)$ *for $K \subset G_b(E)$ open and pro-p.*

Warning 16.9. The compact objects in $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ are *not* Verdier self-dual. This is familiar at the level of representations: the dual of $\text{c-Ind}_K^{G_b(E)}(\Lambda)$ is of uncountable dimension. The problem is that $\text{c-Ind}_K^{G_b(E)}$ are not admissible.

Theorem 16.10. *On the subcategory of compact objects $D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega \subset D_{ét}(\mathrm{Bun}_G, \Lambda)$, there is a Bernstein-Zelevinsky duality functor*

$$\mathbb{D}_{\mathrm{BZ}} \colon (D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega)^{\mathrm{op}} \to D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$$

such that $\mathrm{RHom}(A, B) \cong \pi_\natural(\mathbb{D}_{\mathrm{BZ}}(A) \overset{L}{\otimes}_\Lambda B)$ *where $\pi \colon \mathrm{Bun}_G \to *$ and $\pi_\natural$ is the left adjoint of π^*. (It is a twist of $R\pi_!$).*

We have $\mathbb{D}_{\mathrm{BZ}}^2 \cong \mathrm{Id}$.

For $b \in B(G)$ basic, $\mathbb{D}_{\mathrm{BZ}}$ restricts to a self-duality on $D_{ét}(\mathrm{Bun}^b_G, \Lambda) \cong D(G_b(E), \Lambda)^\omega$ and agrees with the usual Bernstein-Zelevinsky duality. At the level of objects, it sends

$$\mathbb{D}_{BZ}(\text{c-Ind}_K^{G_b(E)} \Lambda) = \text{c-Ind}_K^{G_b(E)} \Lambda.$$

16.5. **ULA sheaves.** Smooth dual is well-behaved on admissible representations. So we will define a "$D_{\text{ét}}(\mathrm{Bun}_G)$-analogue of admissibility". There is a notion for $A \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ to be *universally locally acyclic* (ULA) [with respect to the map $\mathrm{Bun}_G \to *$][37]. It can be checked on strata, and it is equivalent to: for all $b \in B(G)$, $(i^b)^*\mathcal{A} \in D(G_b(E), \Lambda)$ are *admissible* in the sense that for all open pro-p subgroups $K \subset G_b(E)$,

$$[(i^b)^*\mathcal{A}]^K \in D(\Lambda)$$

is perfect (representable by a finite complex of finite projective Λ-modules).

This class of sheaves is stable under Verdier duality,

$$\mathbb{D}_{\mathrm{Bun}_G}(\mathcal{A}) = \mathcal{RH}om(\mathcal{A}, R\pi^!\Lambda),$$

and satisfy Verdier biduality: $\mathcal{A} \xrightarrow{\sim} \mathbb{D}_{\mathrm{Bun}_G}(\mathbb{D}_{\mathrm{Bun}_G}(\mathcal{A}))$ for any ULA $\mathcal{A}$.

Verdier duality restricts to smooth duality on the strata.

Remark 16.11. Ideally we would like to have a notion of "constructible complexes" on Bun_G; these should be the compact objects, and they should all be universally locally acyclic for $\mathrm{Bun}_G \to *$. However this does not work! The Theorem is a best replacement,

[37] In usual algebraic geometry, one would expect every (ind-)constructible sheaf to be ULA with respect to the map to $*$.

but note that compact does not imply ULA, and vice versa. Both are seen already at the level of representation theory: $\text{c-Ind}_K^{G_b(E)} \Lambda$ is compact, but not admissible (ULA).

Conversely, there are ULA sheaves which are not compact. For example, consider an infinite direct sum of supercuspidals $\bigoplus_{i=1}^{\infty} \pi_i$ with growing conductor. It is admissible but not compact. Indeed, for any fixed open compact K we have

$$\left(\bigoplus_{i=1}^{\infty} \pi_i \right)^K = \bigoplus_{i=1}^{N(K)} \pi_i^K$$

for some $N(K) < \infty$, because there are only finitely many π_i with bounded conductor. This type of representation is exactly what you see when you look at cohomology of locally symmetric space – it is admissible but not compact.

Warning 16.12. There is a notion of constructible complexes on (locally) spatial diamonds, by descent on small v-stacks. (It is generated by $j_! \Lambda$, for $j \colon U \to X$ a qcqs étale map.) This is what we mean by constructible. But this is yet a different notion, and essentially no non-zero $\mathcal{A} \in D_{\text{ét}}(\text{Bun}_G, \Lambda)$ is constructible in this sense.

Example 16.13. Let $X = \mathbb{D}_C$ be the closed unit disk, $i \colon \operatorname{Spa} C \hookrightarrow X$ the inclusion of the origin. Then you might expect that $i_* \Lambda$ should be constructible, but it is not. The problem is that the complementary open $j \colon U = \mathbb{D}_C^* \hookrightarrow \mathbb{D}_C$ is not quasi-compact, as $\mathbb{D}_C^*$ is not quasicompact.

In fact, one can show that constructible sheaves on a rigid-analytic variety X are locally constant in an open neighborhood of any classical point.

Upshot: the notion of "finitely generated" and "admissible" representations, together with Bernstein-Zelevinsky duality and smooth duality, generalize to $D_{\text{ét}}(\text{Bun}_G, \Lambda)$.

16.6. **Coefficients.** Remark about coefficients: so far we only allowed Λ such that $n\Lambda = 0$ for some n prime to p. Ideally, we want $\Lambda = \overline{\mathbf{Q}}_\ell$. For the same reason as why the notion of constructible sheaf is tricky, the passage from $\mathbf{Z}/\ell^n\mathbf{Z}$-coefficients to $\mathbf{Z}_\ell$-coefficients is trickier than usual.

We can define

$$\mathcal{D}_{\text{ét}}(\text{Bun}_G, \mathbf{Z}_\ell) := \varprojlim_n \mathcal{D}_{\text{ét}}(\text{Bun}_G, \mathbf{Z}/\ell^n\mathbf{Z})$$

But this is related to representations on ℓ-adically complete $\mathbf{Z}_\ell$-modules, e.g.

$$\mathcal{D}_{\text{ét}}(*, \mathbf{Z}/\ell^n\mathbf{Z}) = \varprojlim_n \mathcal{D}(\mathbf{Z}/\ell^n\mathbf{Z}).$$

But we don't want representations on ℓ-adically complete vector spaces; we want representations on *discrete* $\mathbf{Z}_\ell$-vector spaces. The usual trick to get around this is to consider

$$\operatorname{Ind}\left(\varprojlim_n \mathcal{D}_{\text{ét}}(\text{Bun}_G, \mathbf{Z}/\ell^n\mathbf{Z})^\omega \right).$$

This works on a point:

$$\operatorname{Ind}\left(\varprojlim_n \mathcal{D}(\mathbf{Z}/\ell^n\mathbf{Z})^\omega \right) = \mathcal{D}(\mathbf{Z}_\ell)$$

is the derived ∞-category of discrete $\mathbf{Z}_\ell$-modules, the point being that finite free $\mathbf{Z}_\ell$-modules are ℓ-adically complete.

However this doesn't work here, because compact objects are not admissible (not "finite enough").

Using the idea of solid modules, developed jointly with Dustin Clausen, we were able to define a version of $D_{\acute{e}t}(\mathrm{Bun}_G, \Lambda)$ for *any* $\mathbf{Z}_\ell$-algebra Λ, for which all assertions in this lecture hold true.

17. Cohomological smoothness (Jan 8)

17.1. **Spaces of sections.** Let S be a perfectoid space and $Z \to X_S$ be a smooth adic space, i.e. locally étale over a finite-dimensional ball over X_S.

Then we can consider the space of sections of $Z \to X_S$.

Definition 17.1. Let $\mathcal{M}_Z \colon \{\text{perfectoid spaces } T/S\} \to \text{Sets}$ be the functor sending T to the space of sections $X_T \dashrightarrow Z$ over X_S.

$$T \mapsto \left\{ \begin{array}{ccc} & & Z \\ & \nearrow & \downarrow \\ X_T & \longrightarrow & X_S \end{array} \right\}$$

Example 17.2. If Z_0/E is smooth we can take $Z = Z_0 \times X_S$, and then

$$\mathcal{M}_Z(T) = \mathrm{Map}(X_T, Z_0).$$

Example 17.3. If $Z = \mathcal{E}$ is a geometric vector bundle over X_S, then $\mathcal{M}_Z = \mathcal{BC}(\mathcal{E})$, so $\mathcal{M}_Z$ in general could be viewed as a "non-linear Banach-Colmez space".

Example 17.4. If $\mathcal{E}$ is a G-torsor on X_S and $P \subset G$ is a parabolic, consider

$$\begin{array}{c} Z := \mathcal{E}/P \\ \downarrow \\ X_S \end{array}$$

Then $\mathcal{M}_Z(T)$ classifies reductions of $\mathcal{E}|_{X_T}$ to $P \subset G$. This will be used for the charts $\pi_b \colon \mathcal{M}_b \to \mathrm{Bun}_G$.

We want to understand the geometry of $\mathcal{M}_Z$.

Proposition 17.5. *If Z/X_S is quasi-projective, i.e. a composition of a Zariski-closed immersion $Z \hookrightarrow U$ and an open embedding $U \subset \mathbf{P}^n_{X_S}$, then $\mathcal{M}_Z$ is representable in locally spatial diamonds and $\mathcal{M}_Z \to S$ is compactifiable, of locally finite* dim.tr.g.

Conjecture 17.6. *This is true for all smooth Z/X_S.*

Proof. Reduce to $Z = \mathbf{P}^n_{X_S}$. (In general, once we have this case, factoring over U is an open condition, and factoring over Z is a closed condition.) This can be made explicit.

$$\mathcal{M}_{\mathbf{P}^n}(T) = \mathrm{Map}(X_T, \mathbf{P}^n) = \left\{ (\mathcal{L}, s_0, \ldots, s_n) \mid {\scriptstyle \mathcal{L} = \text{ line bundle}/X_T \atop s_0, \ldots, s_n \in H^0(X_T, \mathcal{L}) \text{ generating } \mathcal{L}} \right\}.$$

This decomposes according to $\deg \mathcal{L}$. So it can be written as

$$\mathcal{M}_{\mathbf{P}^n} = \coprod_{d \geq 0} \mathcal{M}_{\mathbf{P}^n, \deg = d}.$$

Then there is an $\underline{E}^\times$-torsor $\widetilde{\mathcal{M}}_{\mathbf{P}^n, \deg=d} \to \mathcal{M}_{\mathbf{P}^n, \deg=d}$ parametrizing isomorphisms $\mathcal{L} \cong \mathcal{O}(d)$. The sections $s_0, \ldots, s_n$ are parametrized by $\mathcal{BC}(\mathcal{O}(d))^{n+1}$, and the condition that they generate is an open one. In conclusion,

$$\mathcal{M}_{\mathbf{P}^n} = \coprod_{d \geq 0} \underbrace{\left(\text{open subset of } \mathcal{BC}(\mathcal{O}(d))^{n+1}\right) / \underline{E}^\times}_{\text{locally spatial diamond of finite dimension}}.$$

$\square$

Remark 17.7. We see that $\mathcal{M}_{\mathbf{P}^n}$ is "almost" linear. This is a phenomenon for $G = \mathrm{GL}_n$: spaces in Example 17.4 are "essentially linear". But this does not happen for other groups. For classical groups, we get "essentially quadratic" spaces like

$$\{(x, y, z) \mid x, y \in H^0(\mathcal{O}(1))\colon x^2 + y^2 + z^2 = 0 \in H^0(\mathcal{O}(2))\}.$$

17.2. **The Jacobian criterion.** Goal: find a large open subset $\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z$ such that $\mathcal{M}_Z^{\mathrm{sm}} \to S$ is cohomologically smooth.

We will formulate a "Jacobian criterion", which mimics what you get classically by analyzing tangent spaces. Since Z is smooth over X_S, there is a well-defined relative tangent bundle T_{Z/X_S}. If $s\colon X_T \to Z/X_S$ is a section, we get $s^* T_{Z/X_S}$.

Classically, deformations of s are parametrized by $H^0(X_T, s^* T_{Z/X_S})$, while the obstruction space is $H^1(X_T, s^* T_{Z/X_S})$. The idea is that if $H^1(X_T, s^* T_{Z/X_S})$ vanishes, then s should define a smooth point of $\mathcal{M}_Z$. By the classification of vector bundles and their cohomology, we can formulate what this means by hand.

Definition 17.8. We define $\mathcal{M}_Z^{\mathrm{sm}} \subset \mathcal{M}_Z$ to be the open subfunctor consisting of $s\colon X_T \to Z/X_S$ such that $s^* T_{Z/X_S}$ has everywhere only positive[38] Harder-Narasimhan slopes.

Theorem 17.9. *The map $f\colon \mathcal{M}_Z^{\mathrm{sm}} \to S$ is cohomologically smooth. For $s \in \mathcal{M}_Z^{\mathrm{sm}}(C)$, $(Rf^! \Lambda)_s \cong (R(f_s^{\mathrm{lin}})^! \Lambda)_0$ where*

$$f_s^{\mathrm{lin}}\colon \mathcal{BC}(s^* T_{Z/X_S}) \to \operatorname{Spa} C.$$

(This can be computed to be a shift and twist of the constant sheaf.)

The idea is that infinitesimally near s, we should have $\mathcal{M}_Z^{\mathrm{sm}} \cong \mathcal{BC}(s^* T_{Z/X_S})$ since $\mathcal{BC}(s^* T_{Z/X_S})$ should be the "tangent space" of $\mathcal{M}_Z^{\mathrm{sm}}$ at s. In particular, this does allow to compute the degree in which the dualizing sheaf sits.

17.3. **Application.** Recall that we mentioned the isomorphism between the Lubin-Tate space and the Drinfeld space at infinite level:

$$\mathcal{M}_{\mathrm{LT},\infty}^{\diamond} \cong \mathcal{M}_{\mathrm{Dr},\infty}^{\diamond} / \operatorname{Spa} \breve{E}.$$

These can be thought of as the spaces of maps $\mathcal{O}_{X_S}^n \to \mathcal{O}_{X_S}(1/n)$ with cokernel supported at ∞ (the given untilt).

This is given by some $\mathcal{M}_Z$. A paper of Ivanov-Weinstein [IW20] shows that the Jacobian criterion implies that a connected component of $\mathcal{M}_{\mathrm{LT},\infty} \backslash \{\text{points with extra endomorphisms}\}$ is cohomologically smooth.

Example 17.10. For example, for $n = 2$ this says that the complement of the CM points is cohomologically smooth.

[38]Although "obstructions" vanish for bundles of slope 0, the global sections of $H^0(X_S, \mathcal{O})$ are $\underline{E}$, which is not smooth.

In particular, the cohomology of quasicompact open subsets without CM points stabilizes in the tower.

$$\begin{array}{ccc} \vdots & \hookleftarrow & \vdots \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathrm{LT},2} & \hookleftarrow & U_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathrm{LT},1} & \hookleftarrow & U_1 \\ \downarrow & & \\ \mathcal{M}_{\mathrm{LT},0} & & \end{array}$$

In other words, letting $U_i \subset \mathcal{M}_{\mathrm{LT},i}$ be the complement of the pre-images of CM points, the transition maps in $H^j(U_1, \overline{\mathbf{Q}}_\ell) \to H^j(U_2, \overline{\mathbf{Q}}_\ell) \to \dots$ are eventually isomorphisms. The reason is that the cohomology is finite-dimensional at infinite level, by the cohomological smoothness, so the maps (which are a priori split injections) must stabilize.

17.4. **Proof of the Jacobian criterion.** A naïve idea would be to try to find a direct geometric relation to $\mathcal{BC}(S^*T_{Z/X_S})$, which we know is cohomologically smooth. This seems hard.

The actual method has three steps:

(1) Definition of "formal smoothness" for maps of diamonds.
(2) Definition of "universal local acyclicity". For $f\colon X \to S$ and $A \in D_{\text{ét}}(X, \Lambda)$, we say A is $f-\mathrm{ULA}$ if it is "flat" in an appropriate sense (e.g. cohomology of the fibers is locally constant).

    Then cohomological smoothness is equivalent to: (i) Λ is $f-\mathrm{ULA}$ and (ii) $Rf^!\Lambda$ is invertible.

    This breaks the proof of the cohomological smoothness into two parts, with formal smoothness involved in showing that Λ is $f-\mathrm{ULA}$. More precisely, the formal smoothness plus "geometric" finite-dimensionality imply that Λ is $f-\mathrm{ULA}$.
(3) The final step is to prove that $Rf^!\Lambda$ is invertible. This is done by deformation to the normal cone, from $\mathcal{M}_Z^{\mathrm{sm}} \ni s$ to $\mathcal{BC}(s^*T_{Z/X_S}) \ni 0$. We show that the dualizing complex is constant in this deformation, which rests again on the ULA property.

17.5. **Formal smoothness.** The idea is to replace infinitesimal neighborhoods by small actual neighborhoods. This is reasonable because of the following theorem.

Theorem 17.11. *Let $S_0 \hookrightarrow S$ be a Zariski closed embedding of affinoids, and*

$$\begin{array}{ccc} S_0 & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle smooth} \\ S & \longrightarrow & X \end{array}$$

a commutative diagram of adic spaces. Then there exists an open subset $U \subset S$ containing S_0, and a lift

$$\begin{array}{ccc} S_0 & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle smooth} \\ U & \longrightarrow & X \end{array}$$

The key point is that $\varinjlim_{U\supset S_0}\mathcal{O}(U)$ is henselian along $I=\ker(\mathcal{O}(S)\to\mathcal{O}(S_0))$. Then we essentially want to invoke Hensel's lemma to conclude.

Definition 17.12. Let $f\colon Y\to X$ be a map of small v-stacks. Then f is *formally smooth* if for all Zariski closed immersions of affinoid perfectoid spaces $S_0\hookrightarrow S$, and all diagrams

$$\begin{array}{ccc} S_0 & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow f \\ S & \longrightarrow & X \end{array}$$

there exists an étale map $S'\to S$ whose image contains S_0 and a lift

$$\begin{array}{ccc} S_0\times_S S' & \longrightarrow & Y \\ \big\downarrow & \nearrow & \big\downarrow f \\ S' & \longrightarrow & X \end{array}$$

Remark 17.13. The allowance of an étale map $S'\to S$ implies in particular that it is enough to check the criterion locally on S.

Remark 17.14. This is related to "absolute neighborhood retracts" (ANR) in topology. If X is a compact Hausdorff space, we say that X is ANR if for any closed immersion $Y\hookrightarrow Z$, there exists an open $U\subset Z$ containing Y, and a retraction $U\to Y$.

Now, assume that Y is an affinoid perfectoid space and $X=\operatorname{Spa}C$. We will think about what it means for f to be formally smooth. We will cook up a specific type of test diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow f \\ S & \longrightarrow & \operatorname{Spa}C \end{array}$$

We can embed $Y\hookrightarrow S:=\mathbb{B}_C^I$ via $Y=\operatorname{Spa}(R,R^+)\subset\operatorname{Spa}(C\langle X_i\rangle,\mathcal{O}_C\langle X_i\rangle)$ by choosing $\mathcal{O}_C\langle X_i\mid i\in I\rangle\twoheadrightarrow R^+$. Take $S_0=Y$. The condition that f be formally smooth demands that there exists an étale map $U\to S$ containing Y and a retract $S\to S_0=Y$. In fact it is essentially sufficient to check this condition. So Y is formally smooth if and only if it is a retract of a space over a possibly infinite-dimensional ball.

Definition 17.15. If Y is formally smooth, then we say that it is *geometrically finite-dimensional* if it is a retract of a space étale over a finite-dimensional ball.

Question: assume Y is affinoid perfectoid space over $\operatorname{Spa}C$ that is geometrically finite-dimensional. Is Y cohomologically smooth?

Remark 17.16. The analogue fails for compact Hausdorff spaces. For example, the coordinate axes in $\mathbf{A}^2$ form an ANR, evidentally not smooth. But the analogue is true for schemes.

Theorem 17.17. *$\mathcal{M}_Z^{\mathrm{sm}}\to S$ is formally smooth.*

Remark 17.18. Since we are now concerned with working with "finite-dimensional" objects, we cannot localize to the case of strictly totally disconnected spaces (for example).

Proof sketch. Let $S_0 \subset S$ be a Zariski closed immersion of affinoid perfectoid spaces. We contemplate a test diagram

$$\begin{array}{ccc} S_0 & \longrightarrow & \mathcal{M}_Z^{\mathrm{sm}} \\ \big\downarrow & & \big\downarrow f \\ S & \longrightarrow & S \end{array}$$

This means that we have a section

$$\begin{array}{ccc} X_{S_0} & \xrightarrow{\quad s_0 \quad} & Z \\ & \searrow \quad \swarrow & \\ & X_S & \end{array}$$

such that $s_0^* T_{Z/X_S}$ has only positive HN slopes. Then we want to show that there exists an étale map $U \to S$ with image containing S_0, such that s_0 lifts to $X_U \to Z$.

The idea is to write $X_S = Y_{S,[1,q]}/(Y_{S,[q,q]} \sim Y_{S,[1,1]})$. We can also write compatibly

$$\begin{array}{ccc} Z & = & Z_{[1,q]}/(Z_{[q,q]} \overset{\varphi_Z}{\sim} Z_{[1,1]}) \\ \big\downarrow & & \big\downarrow \\ X_S & = & Y_{S,[1,q]}/(Y_{S,[q,q]} \sim Y_{S,[1,1]}) \end{array}$$

We can arrange that $Z_{[1,q]}$ is affinoid, and even a small ball by passing to a small neighborhood of a point.

Then all information is bound up in the isomorphism φ_Z. We can arrange that it is very close to linear. This situation is close to the one of the Banach-Colmez space. Then do some "Banach-fixed point like" argument to produce φ_Z-invariant sections of

$$\begin{array}{ccc} & & Z_{S,[1,q]} \\ & \nearrow & \big\downarrow \\ Y_{S_0,[1,q]} & \hookrightarrow & Y_{S,[1,q]} \end{array}$$

In fact this comes from the vanishing of H^1 of a vector bundle with positive slopes. The leading term in the failure of invariance comes from a cocycle, which you annihilate using the vanishing of H^1. □

18. Universal local acyclicity (Jan 11)

18.1. **Recap.** In the last few lectures we have discussed:

- $D_{\text{ét}}(\text{Bun}_G)$.
- The Jacobian criterion, which is used to prove some of the fundamental properties of $D_{\text{ét}}(\text{Bun}_G)$ by giving smooth charts.
- ULA sheaves, a technical notion used for both of the above. For example, they give a notion of "admissibility" for $D_{\text{ét}}(\text{Bun}_G)$.

18.2. **Background on ULA sheaves on schemes.** Let $f\colon X \to S$ be a finite type, separated map of noetherian schemes, and $A \in D_c^b(X_{\text{ét}}, \Lambda)$ where $n\Lambda = 0$ for $n \in \mathcal{O}_S^\times$.

Definition 18.1. We say A is *f-locally acyclic* if for all geometric points $\overline{x} \to X$ lying over $\overline{s} \to S$, and all generalizations $\overline{t} \rightsquigarrow \overline{s}$, the map

$$A_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}, A)$$

is an isomorphism.

Remark 18.2. This is an étale analogue of a quasi-coherent sheaf $\mathcal{F}/X$ being flat over S.

It is a highly non-trivial, recent theorem of Gabber that this notion is stable under base change, under certain noetherian assumptions.

Theorem 18.3 (Gabber)**.** *Let A be f-locally acyclic. Then for any base change*

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{g}} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array} \tag{18.2.1}$$

the sheaf $\widetilde{g}^*A$ is f'-locally acyclic.

It is usually better to explicitly ask for the acyclicity after all base changes (although this is automatic a posteriori by Gabber's theorem).

Definition 18.4. We say A is *f-universally locally acyclic* if for any base change as above, $\widetilde{g}^*A$ is f'-locally acyclic.

Example 18.5. If f is smooth, then Λ (or more generally, any locally constant sheaf) is f-ULA.

For $f = \text{Id}\colon X \to X$ the converse holds: A is $\text{Id} - \text{ULA}$ if and only if A is locally constant.

Lemma 18.6. *Consider a commutative triangle*

$$\begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

with h proper and B/Y being $g - \text{ULA}$. Then $A := Rh_*B$ is $f - \text{ULA}$.

Example 18.7. In practice, $Y \to X$ often arises as a resolution of singularities of $f\colon X \to S$. This will happen, for example, in the context of the affine Grassmannian.

Combining Example 18.5 and Lemma 18.6, we get:

Corollary 18.8. *If $f\colon X \to S$ is proper and A is $f - \text{ULA}$, then Rf_*A is locally constant.*

Remark 18.9 (Twisted Verdier duality)**.** If A is $f-$ULA, then there is an A-twisted version of Poincaré duality. More precisely,

$$\mathbb{D}_{X/S}(A) \otimes f^* B \xrightarrow{\sim} \mathcal{RH}om(A, Rf^! B)$$

where $\mathbb{D}_{X/S}(A) = \mathcal{RH}om(A, Rf^! \Lambda)$.

If $A = \Lambda$, this says $Rf^! \Lambda \otimes f^* B \xrightarrow{\sim} Rf^! B$ if Λ is f − ULA, e.g. if f is smooth.

Example 18.10. If A is f − ULA, there is a Verdier duality property recently proved by [LZ20]:

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A)).$$

In fact, they characterize ULA sheaves as dualizable objects is a certain symmetric monoidal category, with the duality given by Verdier duality.

Example 18.11. If S is a geometric point, then all $A \in D^b_c(X_{\text{ét}}, \Lambda)$ are f-ULA.

18.3. **ULA sheaves on diamonds.** We want a variant of this notion for diamonds.

An important point is that we have a good analogue the full unbounded derived category $D_{\text{ét}}(X, \Lambda)$, but "constructibility" is a subtle notion.

Example 18.12. Let $i \colon \operatorname{Spa} C \hookrightarrow \mathbb{B}_C$ be the inclusion of a point into a ball. Then $i_* \Lambda$ is *not* constructible, but should be ULA over $\operatorname{Spa} C$.

Proposition 18.13. *If X is a spatial diamond of finite cohomological dimension (uniformly on $X_{\acute{e}t}$), then $D_{\acute{e}t}(X, \Lambda)$ is compactly generated, and the compact objects are the constructible complexes (locally constant after passing to a constructible*[39] *stratification).*

Example 18.14. Let $j \colon \mathbb{T}_C = \{T \colon |T| = 1\} \hookrightarrow \{T \colon |T| \leq 1\} = \mathbb{B}_C$. Then $j_! \Lambda$ is constructible.

Definition 18.15. Let $f \colon X \to S$ be a map of locally spatial diamonds (compactifiable of locally finite dim.tr.g., so that $Rf_!$ is defined). Let $A \in D_{\text{ét}}(X, \Lambda)$. Then we say A is f*-locally acyclic* if the following two conditions hold.

(1) For all geometric points $\overline{x} \to X$ lying over $\overline{s} \to S$, and generalizations $\overline{t} \rightsquigarrow \overline{s}$,

$$A_{\overline{x}} = R\Gamma(X_{\overline{x}}, A) \xrightarrow{\sim} R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A).$$[40]

(2) For all étale $j \colon U \to X$ such that $f \circ j \colon U \to S$ is qcqs, then

$$R(f \circ j)_!(A|_U) \in D_{\text{ét}}(S, \Lambda)$$

is constructible (meaning constructible after pullback across $S' \to S$ for any spatial diamond S' as in Proposition 18.13).

We say that A is f − ULA if any base change is locally acyclic.

Remark 18.16. For schemes, condition (2) is automatically satisfied, and all the information is in (1). For diamonds it is almost the opposite: (1) is almost automatic, and (2) carries most of the information.

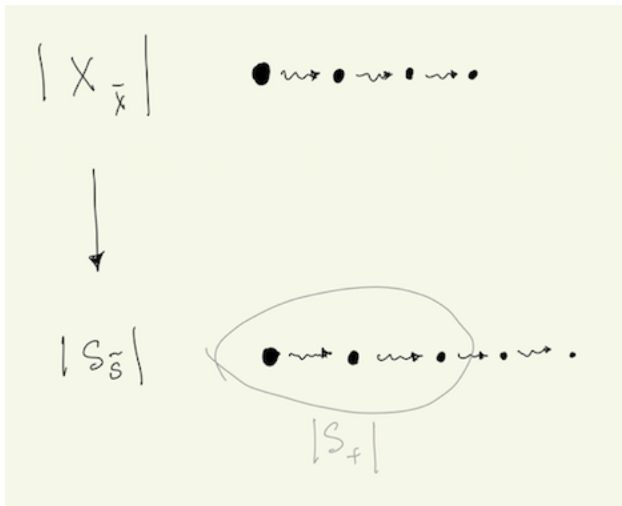
Example 18.17. The analogue of Gabber's theorem fails for diamonds. If $S = \operatorname{Spa} C$ and X is cohomologically smooth over S, then any constructible A is f-locally acyclic but only the locally constant A (among constructible A) are f − ULA.

[39]Constructible subsets are those in the Boolean algebra generated by quasi-compact open subsets.

[40]We could have used $X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}$ instead of $X_{\overline{x}} \times_{S_{\overline{s}}} \overline{t}$ for schemes. For adic spaces, $\overline{t}$ doesn't make sense but $S_{\overline{t}}$ does.

Remark 18.18. The key difference between adic spaces and schemes is that strict henselizations are much easier in the adic setting. Namely, $X_{\overline{x}}$ is represented by $\mathrm{Spa}(C, C^+)$ where C is a complete algebraically closed field and $C^+ \subset C$ is a valuation subring. In particular, $|X_{\overline{x}}|$ is a totally ordered chain of points.

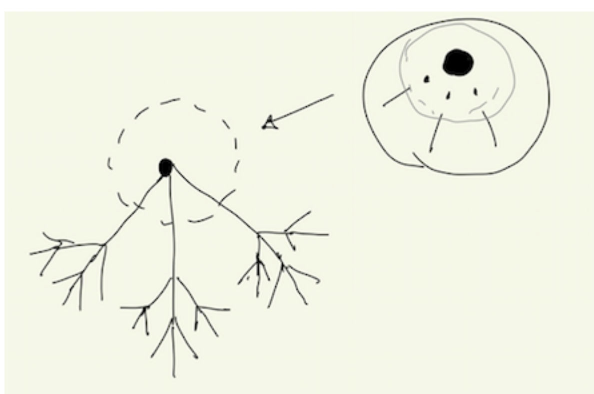
Let us use this to analyze condition (1) in the definition of local acyclicity. Both $|X_{\overline{x}}|$ and $|S_{\overline{s}}|$ are totally ordered chains of points, and $|S_{\overline{t}}|$ is a subset of $|S_{\overline{s}}|$.



This implies that $X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}} = X_{\overline{y}}$ for some specialization $\overline{y} \rightsquigarrow \overline{x}$ lying over $\overline{t} \rightsquigarrow \overline{s}$. The condition is therefore just equivalent to

$$A_{\overline{x}} \xrightarrow{\sim} A_{\overline{y}} = R\Gamma(X_{\overline{y}}, A) \cong R\Gamma(X_{\overline{x}} \times_{S_{\overline{s}}} S_{\overline{t}}, A).$$

A priori, the condition of acyclicity asks this only for certain specializations. But it turns out that asking that (1) holds *universally* is equivalent to A asking that for *all* specializations $\overline{y} \rightsquigarrow \overline{x}$ the map $A_{\overline{x}} \xrightarrow{\sim} A_{\overline{y}}$ is an isomorphism. We say that A is *overconvergent* if this is the case.



Remark 18.19. Overconvergent sheaves on $|\mathbb{B}_C|$ are equivalent to sheaves on the maximal Hausdorff quotient $|\mathbb{B}_C|$, which is the Berkovich space associated to $\mathbb{B}_C$. Similarly, overconvergent étale sheaves are equivalent to étale sheaves on the Berkovich space.

18.4. **Properties.**

Lemma 18.20. *If f is cohomologically smooth, and A locally constant, then A is $f-\mathrm{ULA}$.*

Proof. Overconvergence of A is immediate from the local constancy. To check (2), it is enough to see that if f is qcqs and cohomologically smooth, then $Rf_!$ preserves constructible complexes. We have an adjunction

$$D_{\text{ét}}(X,\Lambda) \underset{Rf_!}{\overset{Rf^!}{\rightleftarrows}} D_{\text{ét}}(S,\Lambda)$$

We know that the constructible complexes are the compact objects in these categories.

Lemma 18.21. *If*

$$\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}$$

is an adjunction of compactly generated triangulated categories, then F preserves compact objects if and only if G commutes with all direct sums.

Proof. This is proved via abstract diagram-chasing, e.g. if G preserves direct sums and $A \in \mathcal{C}$ is compact, then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(F(A), \bigoplus_{i\in I} B_i) &\cong \mathrm{Hom}_{\mathcal{C}}(A, G(\bigoplus_{i\in I} B_i)) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(A, \bigoplus_{i\in I} G(B_i)) \\ &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A, G(B_i)) \\ &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{D}}(F(A), B_i). \end{aligned}$$

□

To complete the proof of Lemma 18.20, it suffices to argue that $Rf^!$ commutes with arbitrary direct sums. But f is cohomologically smooth implies that $Rf^! \cong f^* \otimes Rf^!\Lambda$, which obviously commutes with direct sums. □

Lemma 18.22. *If $f = \mathrm{Id}\colon X \to X$, then A is $f - \mathrm{ULA}$ if and only if A is locally constant with perfect fibers.*

Proof. Suppose A is $f - \mathrm{ULA}$. Condition (2) implies that A is constructible and condition (1) implies that A is overconvergent, and it is a general fact that constructibility plus overconvergence implies local constancy. □

Lemma 18.23. *Consider a commutative triangle*

$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & S & \end{array}$$

with h proper and B/Y being $g - \mathrm{ULA}$. Then $A := Rh_*B$ is $f - \mathrm{ULA}$.

Proof. Direct argument using proper base change. □

Corollary 18.24. *If $f\colon X \to S$ is proper and A is $f-\mathrm{ULA}$, then Rf_*A is locally constant.*

Lemma 18.25 (Twisted Verdier duality)**.** *If A is $f - \mathrm{ULA}$, then for all $B \in D_{\acute{e}t}(S, \Lambda)$ we have*

$$\mathbb{D}_{X/S}(A) \otimes f^*B \xrightarrow{\sim} \mathcal{RH}om(A, Rf^!B) \tag{18.4.1}$$

where $\mathbb{D}_{X/S}(A) = \mathcal{RH}om(A, Rf^!\Lambda)$.

Note that (18.4.1) implies condition (2) in the definition of ULA. Indeed, it implies that $\mathcal{RH}om(A, Rf^!-)\colon D_{\text{ét}}(S, \Lambda) \to D_{\text{ét}}(X, A)$ commute with all direct sums, since the functor on the LHS of (18.4.1) is a composition of left adjoints (namely f^* and $\overset{L}{\otimes}_\Lambda$), which preserve direct sums. Hence by Lemma 18.21, the left adjoint of $Rf^!$ preserves constructibility. The left adjoint is $Rf_!(A \overset{L}{\otimes}_\Lambda -)$. Apply this to $j_!\Lambda$ for $j : U \hookrightarrow X$ the inclusion of a quasi-compact open.

Lemma 18.26 (Verdier biduality)**.** *If A is $f - \mathrm{ULA}$, then $\mathbb{D}_{X/S}(A)$ is $f - \mathrm{ULA}$ and the canonical map*

$$A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$$

is an isomorphism.

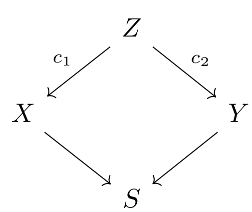
Example 18.27. Let $S = \operatorname{Spa} C$ and $X = X_0^\diamond$ for some algebraic variety X_0/C. Then for any $A_0 \in D^b_c(X_{0,\text{ét}}, \Lambda)$, its analytification $A \in D_{\text{ét}}(X, \Lambda)$ is ULA.

For example, this shows that for $i\colon \operatorname{Spa} C \hookrightarrow \mathbf{A}^1_C$, then $i_*\Lambda$ is ULA. Recall that it is *not* constructible.

18.5. **Proof of Verdier biduality.** There are two proofs of Lemma 18.26, both using a notion of "dualizability" in 2-categories.

18.5.1. *Lu-Zheng approach.* Fix a base S. Consider the symmetric monoidal 2-category LZ_S defined as follows:

- Objects are (X, A) where $X \to S$ is as above and $A \in D_{\text{ét}}(X, \Lambda)$.
- Morphisms $(X, A) \to (Y, B)$ are cohomological correspondences, meaning a diagram



  plus a map $c_1^*A \to Rc_2^!B$.
- The symmetric monoidal structure is given by

$$(X, A) \otimes (Y, B) := (X \times_S Y, A \boxtimes B).$$

  This turns out to define a closed symmetric monoidal structure, with the internal Hom being

$$\mathrm{Hom}_{\mathrm{LZ}_S}((X, A), (Y, B)) = (X \times_S Y, \mathcal{RH}om(p_1^*A, Rp_2^!B)).$$

Theorem 18.28. *The following are equivalent.*

(1) *A is $(X \to S) - \mathrm{ULA}$.*

(2) (X, A) *is dualizable in* LZ_S.
(3) $(X, A) \otimes (X, A)^\vee \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LZ}_S}((X, A), (X, A))$, *i.e.*

$$p_1^* \mathbb{D}_{X/S}(A) \overset{L}{\otimes}_\Lambda p_2^* A \xrightarrow{\sim} \mathcal{RH}om_\Lambda(p_1^* A, Rp_2^! A)$$

for

$$\begin{array}{ccc} X \times_S X & \xrightarrow{p_1} & X \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & S \end{array}$$

In this case, the dual $(X, A)^\vee = (X, \mathbb{D}_{X/S}(A))$ *and therefore* $\mathbb{D}_{X/S}(A)$ *is* $(X \to S) - \mathrm{ULA}$ *and* $A \xrightarrow{\sim} \mathbb{D}_{X/S}(\mathbb{D}_{X/S}(A))$.

Note that (3) above is an instance of A-twisted Poincaré duality.

Corollary 18.29. *(1) The constant sheaf* Λ *is* $f-\mathrm{ULA}$ *if and only if* $p_1^* \mathbb{D}_{X/S} \to \mathbb{D}_{X \times_S X/X} = Rp_2^! \Lambda$ *is an isomorphism.*

(2) f *is cohomologically smooth with respect to* Λ *if and only if* $\mathbb{D}_{X/S}$ *is invertible, and* $p_1^* \mathbb{D}_{X/S} \xrightarrow{\sim} Rp_2^! \Lambda$.

18.5.2. *Second proof.* Define a 2-category $\mathcal{C}_S$ as follows:

- Objects are $X \to S$ as above.
- Morphisms $\mathrm{Fun}_{\mathcal{C}_S}(X, Y) = D_{\text{ét}}(X \times_S Y, \Lambda)$.
- Composition is convolution. For $X, Y, Z \to S$,

$$\begin{array}{ccccc} & & X \times_S Y \times_S Z & & \\ & \swarrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_{13}} & \searrow^{\pi_{23}} & \\ X \times_S Y & & X \times_S Z & & Y \times_S Z \end{array}$$

For $A \in \mathrm{Fun}_{\mathcal{C}_S}$ and $B \in \mathrm{Fun}_{\mathcal{C}_S}(Y, Z)$ we define

$$A \star B := R\pi_{B!}(\pi_{12}^* A \overset{L}{\otimes}_\Lambda \pi_{23}^* B).$$

Proper base change implies associativity. The identity morphism is $\Delta_{X/S!}\Lambda$.

This maps to the 2-category $\mathcal{C}'_S$ with:

- Objects are $X \to S$ as above.
- Morphisms $X \to Y$ are functors $D_{\text{ét}}(X, \Lambda) \to D_{\text{ét}}(Y, \Lambda)$.

Note that this breaks the symmetry between X and Y that was in $\mathcal{C}_S$. The functor $\mathcal{C}_S \to \mathcal{C}'_S$ uses sheaves as kernels.

Recall that in any 2-category, there is a notion of adjoint functors: we say $f\colon X \to Y$ is a left adjoint of $g\colon Y \to X$ if there are $\alpha\colon \mathrm{Id}_X \to gf$ and $\beta\colon fg \to \mathrm{Id}_Y$ such that

$$f \xrightarrow{f\alpha} fgf \xrightarrow{\beta f} f$$

and

$$g \xrightarrow{\alpha g} gfg \xrightarrow{g\beta} g$$

are isomorphic to the respective identity natural transformations.

Theorem 18.30. *The following are equivalent:*

(1) $A \in D_{\acute{e}t}(X, \Lambda)$ *is ULA.*

(2) *$A \in \mathrm{Fun}_{\mathcal{C}_S}(X, S)$ is a left adjoint. In this case, the right adjoint is $\mathbb{D}_{X/S}(A) \in \mathrm{Fun}_{\mathcal{C}_S}(S, X)$.*

(3) *$p_1^* \mathbb{D}_{X/S}(A) \overset{L}{\otimes}_\Lambda p_2^* A \xrightarrow{\sim} \mathcal{RH}om_\Lambda(p_1^* A, Rp_2^! A)$.*

Sketch. For (1) $\implies$ (2) you try to directly build the unit and counit. The map β comes tautologically from the definition, but you need (3) to construct α. □

We use the second approach, as it is well suited for analyzing actions on categories of sheaves.

19. The Jacobian criterion (Jan 15)

Let $E \supset \mathcal{O}_E$, π, $\mathbf{F}_q$ be as usual. Recall the statement of the Jacobian criterion for smoothness:

Theorem 19.1 (Jacobian criterion)**.** *Let S be a perfectoid space over $\mathbf{F}_q$. Let $Z \to X_S = X_{S,E}$ be a smooth map of (sousperfectoid) adic spaces such that Z is quasi-projective*[41]*.*

Let $\mathcal{M}_Z$ be the space of sections of $Z \to X_S$, and let $\mathcal{M}_Z^{\mathrm{smooth}} \subset \mathcal{M}_Z$ be the open subspace where s^*T_{Z/X_S} has everywhere positive HN slopes. Then the map $\mathcal{M}_Z^{\mathrm{smooth}} \to S$ is ℓ-cohomologically smooth for all $\ell \neq p$.

Remark 19.2. Recall that $\mathcal{M}_Z \to S$ is representable in locally spatial diamonds, compactifiable, and of locally finite dim.tr.g.. All of these properties are inherited by the open subset $\mathcal{M}_Z^{\mathrm{smooth}}$.

Strategy of proof:

(1) Prove formal smoothness of $\mathcal{M}_Z^{\mathrm{smooth}} \to S$.
(2) Show that formal smoothness plus "geometric finite dimensionality" imply that $\mathbf{F}_\ell$ is $f-$ULA.
(3) Show that $Rf^!\mathbf{F}_\ell$ is invertible.

19.1. **Step (1).** By definition, the statement that $\mathcal{M}_Z^{\mathrm{smooth}} \to S$ being formally smooth means the following. For any Zariski closed immersion of affinoid perfectoid spaces $T_0 \subset T$, suppose we have a diagram

$$\begin{array}{ccc} T_0 & \longrightarrow & \mathcal{M}_Z^{\mathrm{smooth}} \\ \big\downarrow & & \big\downarrow \\ T & \longrightarrow & S \end{array}$$

Then we need to prove that there exists an étale map $T' \to T$ containing T_0 in the image, and a lift

$$\begin{array}{ccc} T_0 \times_T T' & \longrightarrow & \mathcal{M}_Z^{\mathrm{smooth}} \\ \big\downarrow & \nearrow & \big\downarrow \\ T' & \longrightarrow & S \end{array}$$

The proof is by an involved explicit analysis, using the Banach fixed point theorem.

19.2. **Step (2).** If $\mathcal{M}_Z^{\mathrm{smooth}}$ were a represented by a perfectoid space, we could take T_0 to be an open subset of $\mathcal{M}_Z^{\mathrm{smooth}}$ and use this to see that $\mathcal{M}_Z^{\mathrm{smooth}}$ is a retract of something étale over a perfectoid ball.

Proposition 19.3. *There is a cohomologically smooth and formally smooth surjective map*

$$T_0 \to \mathcal{M}_Z$$

(representable in locally spatial diamonds, compactifiable, and of locally finite dim.tr.g.*) such that T_0 is a perfectoid space that locally admits a Zariski closed embedding into a finite-dimensional perfectoid ball over S.*

Remark 19.4. Of course any diamond has an atlas by a perfectoid space, but not necessarily a cohomologically smooth and formally smooth one. The ones that come up in practice tend to have such an atlas.

[41] This means that there exists a Zariski closed embedding $Z \hookrightarrow U$ which is open in $\mathbf{P}^n_{X_S}$.

Proof. The map $\mathcal{M}_Z \hookrightarrow \mathcal{M}_{\mathbf{P}^n_S}$ is locally Zariski closed. That allows to reduce the statement to the one for projective space. In that case, we have

$$\mathcal{M}_{\mathbf{P}^n_S} = \bigcup_{d \geq 0} \left(\mathcal{BC}(\mathcal{O}(d)^{n+1}) \setminus \{0\} \right) / \underline{E}^{\times}.$$

It's then enough to show that $\mathcal{BC}(\mathcal{O}(d)^{n+1})/\underline{E}^{\times}$ has this property. This can be done explicitly. □

Corollary 19.5. *The constant sheaf $\mathbf{F}_\ell$ is ULA for $\mathcal{M}_Z^{\text{smooth}} \to S$.*

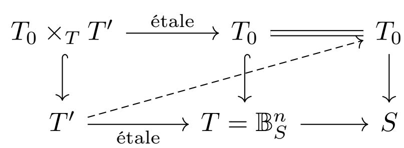
Proof. The property of being ULA can be checked after cohomologically smooth localization. So it suffices to check for $T_0 \to S$. Then it follows from the Lemma below. □

Lemma 19.6. *Let $T_0 \to S$ be a map of affinoid perfectoid spaces such that*

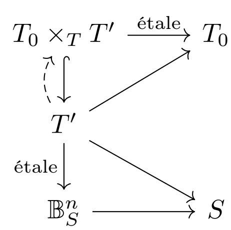
(1) T_0 is formally smooth over S.
(2) $T_0 \hookrightarrow \mathbb{B}^n_S$ is Zariski closed in some finite-dimensional perfectoid ball.

Then $\mathbf{F}_\ell$ is ULA for $T_0 \to S$.

Proof. By the formal smoothness, we can find T' étale over $T := \mathbb{B}^n_S$ plus a diagram



Shrinking T', we can even find a retraction $T' \to T_0 \times_T T'$



hence $T_0 \times_T T' \to S$ is a retract of $T' \xrightarrow{\text{étale}} \mathbb{B}^n_S \to S$. In turn $T_0 \times_T T'$ is an étale cover of T_0.

We claim that the proper of being ULA is preserved under retracts. (Note though that cohomological smoothness is not preserved under retracts.) Granting this, we would obtain that $T_0 \times_T T' \to S$ is ULA. Since being ULA can be checked étale locally, that shows $T_0 \to S$ is also ULA.

Proof of the claim: it can be shown directy from the definition, but a slick argument is that it follows from the categorical characterization of ULA from Theorem 18.30. □

19.3. **Step (3).** We want to show that $Rf^!\mathbf{F}_\ell$ is invertible, i.e. locally isomorphic to $\mathbf{F}_\ell[n]$.

Fact 19.7. *If A is $f-$ULA for $f\colon X \to S$, then $\mathbb{D}_{X/S}(A)$ is again $f-$ULA and its formation commutes with any base change $S' \to S$.*

Since the property of a sheaf being invertible can be checked after a v-cover, it's enough check v-locally. By Fact 19.7, the dual is preserved under any base change. We will reduce to proving this after pullback along a section. Suppose we have a diagram

$$\begin{array}{ccc} S' & \xrightarrow{g} & \mathcal{M}_Z^{\text{smooth}} \\ & \searrow^{h} & \downarrow f \\ & & S \end{array}$$

with g a v-cover. As we said, it's enough to prove that $g^* Rf^! \mathbf{F}_\ell$ is invertible. The data of g is equivalent to a section $s' \colon X_{S'} \to \mathcal{M}_{Z'}$ where $Z' := Z \times_{X_S} X_{S'}$. Then $g = h's'$ where h' is the base change of h, as in the diagram below.

$$\begin{array}{ccc} \mathcal{M}_{Z'}^{\text{smooth}} & \xrightarrow{h'} & \mathcal{M}_Z^{\text{smooth}} \\ {\scriptstyle s'} \uparrow \downarrow {\scriptstyle f'} & & \downarrow f \\ S' & \xrightarrow{h} & S \end{array}$$

Then $g^* Rf^! \mathbf{F}_\ell \cong (s')^*(h')^* Rf^! \mathbf{F}_\ell \cong (s')^* R(f')^! \mathbf{F}_\ell$, invoking Fact 19.7 in the last step.

Renaming things, we see it's enough to prove that for any section $s \colon S \to \mathcal{M}_Z^{\text{smooth}}$, which can be identified with a section $X_S \to Z$, the pullback $s^* Rf^! \mathbf{F}_\ell \in D_{\text{ét}}(S, \mathbf{F}_\ell)$ is invertible.

Now we implement a deformation of the normal cone along this section.

$$\begin{array}{c} Z \\ {\scriptstyle s} \uparrow \downarrow \\ X_S \end{array}$$

From this you can build

$$\begin{array}{c} \widetilde{Z} \\ {\scriptstyle \widetilde{s}} \uparrow \downarrow {\scriptstyle \text{smooth}} \\ X_S \times \mathbf{A}^1 \end{array}$$

such that

- $\widetilde{Z} \times_{\mathbf{A}^1} \{1\} = Z$ and
- $\widetilde{Z} \times_{\mathbf{A}^1} \{0\}$ is the normal cone of s in Z, which is the geometric vector bundle corresponding to $s^* T_{Z/X_S}$.

We would like to realize this whole family as an instance of the same problem, however the base $X_S \times \mathbf{A}^1$ is no longer of the form $X_{\widetilde{S}}$ that we were considering before. Inside $X_S \times \mathbf{A}^1$ we have $X_S \times \underline{E} = X_{S \times \underline{E}}$, which has an $\underline{E}^\times$-action. Pulling back $\widetilde{Z} \to X_S \times \mathbf{A}^1$ gives $\widetilde{Z}' \to X_{S \times \underline{E}}$. This is smooth, and its fiber over $S \times \{1\}$ is Z, and the fiber over $S \times \{0\}$ is $s^* T_{Z/X_S}$.

The $\widetilde{Z}'$ is still quasiprojective, so all the previous results apply. Then we have

$$\mathcal{M}_{\widetilde{Z}'}^{\text{smooth}} \underset{\widetilde{s}}{\overset{\widetilde{f}}{\rightleftarrows}} S \times \underline{E}$$

Then $R\widetilde{f}^! \mathbf{F}_\ell$ is $\widetilde{f}$-ULA (as it is the Verdier dual of the $\widetilde{f}$-ULA sheaf $\mathbf{F}_\ell$), and

- $R\widetilde{f}^! \mathbf{F}_\ell|_{S \times \{1\}} \cong Rf^! \mathbf{F}_\ell$,
- $R\widetilde{f}^! \mathbf{F}_\ell|_{S \times \{0\}}$ is the dualizing complex for $\mathcal{BC}(s^* T_{Z/X_S})$, which we know is invertible by an explicit analysis.

By a tricky spreading out argument, this implies that $s^* Rf^! \mathbf{F}_\ell$ is also invertible. (We know that the whole dualizing complex on the deformation is ULA.)

19.4. **Applications to $D_{\mathrm{\acute{e}t}}(\mathrm{Bun}_G, \Lambda)$.** Recall that we defined charts for Bun_G in the following way.

Definition 19.8. Let $\mathcal{M}$ be the moduli space of $\mathbf{Q}$-filtered G-bundles (with increasing filtration), i.e. exact tensor functors

$$\mathrm{Rep}_E(G) \xrightarrow{\rho} \mathbf{Q}\,\mathrm{Fil}\,\mathrm{Bun}_{X_S}$$

such that for all $V \in \mathrm{Rep}_E(G)$, $\rho(V)^{\leq \lambda} / \bigcup_{\lambda' < \lambda} \rho(V)^{\leq \lambda'}$ is semistable of slope λ. (These are "opposite HN filtrations".)

There is a map $\mathcal{M} \to \mathrm{Bun}_G$ by forgetting the filtration. On the other hand, sending the filtration to its associated graded defines a map $\mathcal{M} \to \bigcup_{b \in B(G)} [*/\underline{G_b(E)}]$, since semistable bundles are equivalent to isocrystals. This induces a decomposition

$$\mathcal{M} = \bigcup_{b \in B(G)} \mathcal{M}_b.$$

Theorem 19.9. *The map $\mathcal{M} \to \mathrm{Bun}_G$ is cohomologically smooth.*

Example 19.10. Let $G = \mathrm{GL}_2$, $b \leftrightarrow \mathcal{O} \oplus \mathcal{O}(1)$. Then $\mathcal{M}_b$ parametrizes extensions

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$$

where $\deg \mathcal{L} = 0$ and $\deg \mathcal{L}' = 1$.

Theorem 19.9 is a consequence of the Jacobian criterion. Take $S \to \mathrm{Bun}_G$, corresponding to a G-bundle $\mathcal{E}/X_S$, and take Z to be the moduli space of $\mathbf{Q}$-filtrations on $\mathcal{E}$. Then $\mathcal{M} \hookrightarrow \mathcal{M}_Z$, and it actually lies in $\mathcal{M}_Z^{\mathrm{smooth}}$ by the condition on slopes.

19.5. **Geometry of $\mathcal{M}_b$.** Now fix $b \in B(G)$, and consider $\pi_b \colon \mathcal{M}_b \to \mathrm{Bun}_G$. We think of this as a "chart for Bun_G near Bun_G^b".

As $\mathcal{M}_b \to [*/\underline{G_b(E)}]$, we can view $\mathcal{M}_b = [\widetilde{M}_b/\underline{G_b(E)}]$ where in $\widetilde{M}_b$, the graded bundle is trivialized.

Example 19.11. Let $G = \mathrm{GL}_2$, $b \leftrightarrow \mathcal{O} \oplus \mathcal{O}(1)$. Then $\widetilde{\mathcal{M}}_b$ parametrizes extensions

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

There is a "base point" $* \in \widetilde{M}_b$ corresponding to the split extension. This gives a section

$$\begin{array}{ccc} [*/\underline{G_b(E)}] & \hookrightarrow & \mathcal{M}_b \\ & \searrow^{\sim} & \downarrow \\ & & [*/\underline{G_b(E)}] \end{array}$$

In fact, the diagram

$$\begin{array}{ccc} [*/\underline{G_b(E)}] & \hookrightarrow & \mathcal{M}_b \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G^b & \hookrightarrow & \mathrm{Bun}_G \end{array}$$

is cartesian. Indeed, if $\mathcal{E} \in \mathrm{Bun}_G^b$, then the HN filtration of $\mathcal{E}$ gives a splitting of the given $\mathbf{Q}$-filtration.

Fact 19.12. *The map $\widetilde{\mathcal{M}}_b \to *$ is representable in locally spatial diamonds, and cohomologically smooth. More precisely, it is a successive extension of negative Banach-Colmez spaces, of total dimension $\langle 2\rho, \nu_b \rangle$.*

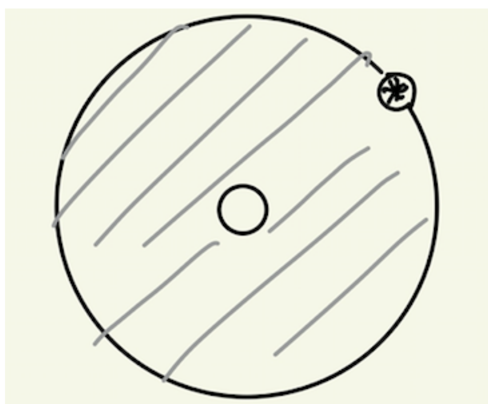
Example 19.13. For b as in Example 19.11, $\mathcal{M}_b = \{0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0\}$ can be identified with $\mathcal{BC}(\mathcal{O}(-1)[1])$.

Fact 19.14. *$\widetilde{\mathcal{M}}_b \setminus \{*\} =: \widetilde{\mathcal{M}}_b^\circ$ is a spatial diamond. (In particular it is qcqs, but not qcqs over $*$.)*

Example 19.15. In Example 19.11, we have $\widetilde{\mathcal{M}}_b^\circ \cong \mathrm{Spa}\, \overline{\mathbf{F}}_q((t^{1/p^\infty}))/\underline{\mathrm{SL}_1(D)}$ where D/E is a quaternion algebra. This is a quotient of an affinoid perfectoid space by a profinite group.

Indeed, on $\widetilde{\mathcal{M}}_b^\circ$, we have $\mathcal{E} \cong \mathcal{O}(1/2)$. Picking such an isomorphism, $\mathcal{O} \hookrightarrow \mathcal{O}(1/2)$ gives a section of $\mathcal{BC}(\mathcal{O}(1/2)) \setminus \{0\} = \mathrm{Spa}\, \overline{\mathbf{F}}_q((t^{1/p^\infty}))$. The space of such torsors (compatible with the chosen trivialization of the associated graded) form a torsor for $\mathrm{SL}_1(D)$, so $\widetilde{\mathcal{M}}_b^\circ$ is the quotient.

Warning 19.16. In Example 19.15, the punctured point $*$ sits at $|t| = 1$, not near $|t| = 0$.



Note that as $t \to 0$, the map is going to 0, not the extension.

$\widetilde{\mathcal{M}}_b$ is "strictly local" in the following sense.

Theorem 19.17. *For any $A \in D_{ét}(\widetilde{\mathcal{M}}_b, \Lambda)$, the restriction*

$$R\Gamma(\widetilde{\mathcal{M}}_b, A) \to R\Gamma(*, A)$$

is an isomorphism.

Sketch. The cone of this map is "cohomology with partial compact support" $R\Gamma_{\partial c}(\widetilde{\mathcal{M}}_b^\circ, A)$ where $R\Gamma_{\partial c}$ means "compact support towards $*$ but no support condition towards the boundary of $\widetilde{\mathcal{M}}_b$. This is a special case of something much more general:

Let X (we have in mind $X = \widetilde{\mathcal{M}}_b^\circ$) be a spatial diamond over $\overline{\mathbf{F}}_q$ of dim.tr.g. $< \infty$. Assume the map $X \to *$ is partially proper, i.e. $X(R, R^+) = X(R, R^\circ)$. Then for any $C/\overline{\mathbf{F}}_q$, X_C has "two ends".

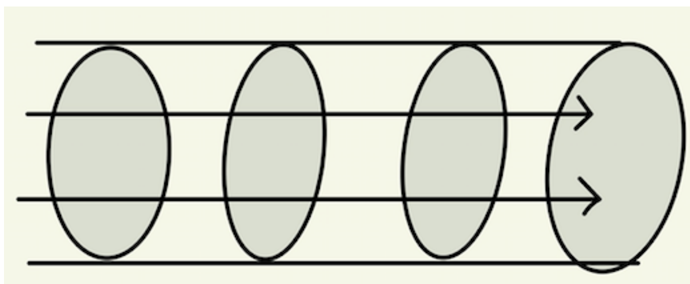
Example 19.18. Suppose $X = \mathrm{Spa}(R, R^+)$ be affinoid perfectoid, $C = \overline{\mathbf{F}_q((t))}^\wedge$. Then X_C is a profinite cover of $X \times_{\mathbf{F}_q} \mathrm{Spa}\, \mathbf{F}_q((t))$, which is the punctured open unit disk over X. This has two ends (the origin and the outer boundary).

So we can define "partial compactly supported cohomology" $R\Gamma_{\partial c}(X_C, A)$ which has compact support at one end but not the other.

Theorem 19.19. *In the above situation* $R\Gamma_{\partial c}(X_C, A) = 0$.

We will give two arguments for this, one purely philosophical and one along the lines of the actual proof. Actual proof: you reduce to the case where $X = \operatorname{Spa} \overline{\mathbf{F}}_q((t^{1/p^\infty}))$ (by using "correspondences", e.g. pushing forward the general case to this one, and using proper base change). Since this space is simple, you can reduce to $A = \Lambda$ and then compute directly.

Intuitive picture: say M is a topological manifold with a free action of $\mathbf{R}$ (a "flow"), such that $\overline{M} = M/\mathbf{R}$ is compact.



There are two boundaries: the "source of the flow" and the "sink of the flow". For all $A \in D(M/\mathbf{R}, \mathbf{Z})$, $R\Gamma_{\partial c}(M, A) = 0$. Indeed, after gluing in a boundary disk, the flow contracts to the boundary.

How is this analogous? Roughly, let $C = \overline{\mathbf{F}}_q((t^{\mathbf{R}}))$. This has an action of $\mathbf{R}_{>0}$ by rescaling (although this is discontinuous). Then X_C has an action of $\mathbf{R}$, and plays the role of M. The quotient $X_C/\mathbf{R}$ is qcqs.

□

20. $D_{\text{ÉT}}(\mathrm{Bun}_G, \Lambda)$ revisited (Jan 18)

Let G/E be a reductive group, with E a local field of residue field $\mathbf{F}_q$ with characteristic $p > 0$. Let Λ be a ring of coefficients, such that $n\Lambda = 0$ for n prime to p.

Recall: $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ has an infinite semi-orthogonal decomposition into $D_{\text{ét}}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$.

20.1. Compact objects.

Recall that an object A in a triangulated category C is *compact* if $\mathrm{Hom}_{\mathcal{C}}(A, -)$ commutes with all direct sums.

Fact 20.1. *If C is the homotopy category of an ∞-category $\mathcal{C}$ possessing all colimits, then $A \in C$ is compact if and only if $A \in \mathcal{C}$ has the property that* $\mathrm{Hom}_{\mathcal{C}}(A, -)$ *commute with* all *colimits.*

Remark 20.2. In abelian 1-categories, compactness is often defined in terms of commuting with filtered colimits. However, since $\mathrm{Hom}_{\mathcal{C}}(A, -)$ is exact in the triangulated / stable ∞-category setting, so it is equivalent to ask for just direct sums.

Fact 20.3. *If such $\mathcal{C}$ is generated under colimits by its compact objects $\mathcal{C}^\omega \subseteq \mathcal{C}$, then* $\mathrm{Ind}(\mathcal{C}^\omega) \xrightarrow{\sim} \mathcal{C}$ *is an equivalence of ∞-categories.*

Proposition 20.4. *$D(G_b(E), \Lambda)$ is compactly generated, with compact generators being* $\text{c-Ind}_K^{G_b(E)} \Lambda$ *for $K \subseteq G_b(E)$ an open pro$-p$ subgroup.*

Proof. We have

$$\mathrm{Hom}_{G_b(E)}(\text{c-Ind}_K^{G_b(E)} \Lambda, A) = \mathrm{Hom}_K(\Lambda, A) = A^K$$

If K is pro$-p$, then taking K-invariants (i.e. K-cohomology) commutes with all direct sums. (If A is represented by a complex $A_\bullet$, then A^K is represented by the level-wise invariants $A_\bullet^K$, and the statement can be verified explicitly.) So $\text{c-Ind}_K^{G_b(E)} \Lambda$ are compact. To see they generate, we just have to note that if A is such that $A^K = 0$ for all K, then $A = 0$. □

Theorem 20.5. *The category $D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compactly generated and $A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is compact if and only if for all $b \in B(G)$, letting $i^b \colon \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$, then $i^{b*}A \in D_{ét}(\mathrm{Bun}_G^b, \Lambda) \cong D(G_b(E), \Lambda)$ is compact, and vanishes for almost all b.*

Proof. First we exhibit compact generators. Fix $b \in B(G)$ and $K \subset G_b(E)$ an open pro-p subgroup. The goal is to show that there exists a complex $A_K^b \in D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ such that $\mathrm{Hom}(A_K^b, B) = (i^{b*}B)^K$.

Such an object is characterized by the Yoneda Lemma, and from the definition is clearly compact if it exists. The collection of such objects generates, if they exist.

To find A_K^b, we use cohomologically smooth charts

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_b & & \\ \downarrow{\scriptstyle G_b(E)} & & \\ \mathcal{M}_b & \xrightarrow{\pi_b} & \mathrm{Bun}_G \end{array}$$

from which we get also $f_K \colon [\widetilde{\mathcal{M}}_b/K] \to \mathrm{Bun}_G$, which is cohomologically smooth.

We take $A_K^b = Rf_{K!}f_K^!\Lambda$. To check that this works, consider

$$\mathrm{RHom}_{\mathrm{Bun}_G}(A_K^b, B) = \mathrm{RHom}_{\mathrm{Bun}_G}(Rf_{K!}Rf_K^!\Lambda, B) = \mathrm{RHom}_{[\widetilde{\mathcal{M}}_b/K]}(Rf_K^!\Lambda, Rf_K^!B).$$

Using that f_K is ULA for Λ (more precisely Lemma 18.25), we can rewrite this latter expression as $\mathrm{RHom}_{[\widetilde{\mathcal{M}}_b/K]}(Rf_K^!\Lambda, f_K^*B\otimes Rf_K^!\Lambda)$. Then using the cohomological smoothness of f (in particular, invertibility of the relative dualizing sheaf $Rf_K^!\Lambda$), we can rewrite this as $R\Gamma([\widetilde{\mathcal{M}}_b/K], f_K^*B)$. Finally, we use that $\widetilde{\mathcal{M}}_b$ is strictly local to equate this with $R\Gamma([*/K], f_K^*B|_{[*/K]}) = (i^{b*}B)^K$.

We still need to check the claimed characterization of compact objects. For this we argue by induction on (quasicompact) open substacks $U\subset \mathrm{Bun}_G$. Let $A\in D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ be compact. Pick some $b\in B(G)$ such that $i^b\colon \mathrm{Bun}_G^b\hookrightarrow U$ is closed (only possible because of the truncation to a quasicompact U, or else there would be infinitely many specializations). Let $j\colon V:=U\setminus \mathrm{Bun}_G^b\hookrightarrow U$. By induction we know the result for $D_{\text{ét}}(V,\Lambda)$.

It is enough to show that j^* preserves compact objects. Indeed, once we know this is the case, if A is compact then j^*A is also compact, and hence $i^{b*}A$ is compact by the exact triangle $j_!j^*A\to A\to i^b_*i^{b*}A$. So compactness of j^*A implies $i^{b'*}A$ is compact for all $b'\neq b$ such that $\mathrm{Bun}_G^{b'}\subset U$ by induction, and $i^{b*}A$ is compact. The fact that all but finitely many such stalks must vanish is proved as in Example 15.1. Similarly for the converse (we can build up A from the $i^{b*}A$; it is formal that $j_!$ preserves compact objects).

Now to show that j^* preserves compactness, it is enough to check on generators, and it is obvious for the generators that "come from" V. So we reduce to checking that $j^*A_K^b\in D_{\text{ét}}(V,\Lambda)$ is compact. Consider the commutative diagram

$$\begin{array}{ccccc} [\widetilde{\mathcal{M}}_b/K] & \xrightarrow{f_K} & U & \hookrightarrow & \mathrm{Bun}_G \\ \uparrow & & \uparrow j & & \\ [\widetilde{\mathcal{M}}_b^\circ/K] & \xrightarrow{f_K^\circ} & V & & \end{array}$$

where $\widetilde{\mathcal{M}}_b^\circ = \widetilde{\mathcal{M}}^\circ\setminus *$ is a spatial diamond of finite dim.tr.g.. So $j^*A_K^b = Rf^\circ_{K!}Rf_K^{\circ!}\Lambda$ by the formula for A_K^b. By a similar computation to the one above,

$$\mathrm{RHom}(j^*A_K^b, B)\cong R\Gamma([\widetilde{\mathcal{M}}_b^\circ/K], f_K^{\circ*}B)$$

for all $B\in D_{\text{ét}}(V,\Lambda)$. We can rewrite this as $R\Gamma(\widetilde{\mathcal{M}}_b^\circ, B)^K$. But $R\Gamma(\widetilde{\mathcal{M}}_b^\circ,-)$ commutes with all direct sums, as $\widetilde{\mathcal{M}}_b^\circ$ is a spatial (hence qcqs!) diamond[42] of finite dim.tr.g.. □

Warning 20.6. The proof that the $j^*A_K^b$ are compact used the specific form of j^*. Not every open restriction will preserve compactness, because general open subsets of $\widetilde{\mathcal{M}}_b^\circ$ will fail to be quasi-compact.

20.2. **Bernstein-Zelevinsky duality.** We will explain a duality on compact objects.

Proposition 20.7 (Bernstein-Zelevinsky duality)**.** *For any $A\in D(G_b(E),\Lambda)^\omega$, there exists a unique $A'\in D(G_b(E),\Lambda)^\omega$ such that*

$$\mathrm{RHom}(A',B) = (A\otimes B)_{hG_b(E)}.$$

(The derived homology functor $(-)_{hG_b(E)}$ is the left adjoint of the pullback $D(\Lambda)\to D(G_b(E),\Lambda)$.)

For $A = \text{c-Ind}_K^{G_b(E)}\Lambda$, we have $A' = \text{c-Ind}_K^{G_b(E)}\Lambda$; in general

$$A' = \mathrm{RHom}_{G_b(E)}(A, \mathcal{H}(G_b(E))).$$

The biduality map $A'' := (A')'\to A$ is an isomorphism.

[42] We have not yet given the proof of this property in general. It is a bit of tricky point-set topology.

Proof. By Yoneda, A' is unique if it exists. To get existence, it is enough to take $A = \text{c-Ind}_K^{G_b(E)} \Lambda$. Then we have

$$(A \otimes B)_{G_b(E)} = B_K \underbrace{\xrightarrow{\sim}}_{\text{Avg}} B^K = \text{RHom}(\text{c-Ind}_K^{G_b(E)} \Lambda, B).$$

Hence the claimed formula for A' works in this case. Then use that these A's generate. □

Theorem 20.8. *For any $A \in D_{ét}(\text{Bun}_G, \Lambda)^\omega$, there exists a unique $A' = \mathbb{D}_{\text{BZ}}(A) \in D_{ét}(\text{Bun}_G, \Lambda)$ such that*

$$\text{RHom}(\mathbb{D}_{\text{BZ}}(A), B) = \pi_\natural(A \overset{L}{\otimes}_\Lambda B)$$

where $\pi_\natural \colon D_{ét}(\text{Bun}_G, \Lambda) \to D(\Lambda)$ is left adjoint to π^* for $\pi \colon \text{Bun}_G \to *$.[43]

The biduality map $\mathbb{D}_{\text{BZ}}(\mathbb{D}_{\text{BZ}}(A)) \to A$ is an isomorphism.

If $U \subset \text{Bun}_G$ is an open substack, then $\mathbb{D}_{\text{BZ}}$ respects $D_{ét}(U, \Lambda)^\omega \subseteq D_{ét}(\text{Bun}_G, \Lambda)^\omega$. For $U = \text{Bun}_G^b$, with b basic, it reduces to the usual Bernstein-Zelevinsky duality on $D_{ét}(\text{Bun}_G^b, \Lambda)^\omega = D(G_b(E), \Lambda)^\omega$.

Remark 20.9. The Bernstein-Zelevinsky dual of a sheaf on a deeper stratum Bun_G^b (for b not basic) will "spread out" to the open strata. This is in contrast to the Verdier dual, which only spreads out to more special strata.

Proof. We check existence for a class of generators. Take $i_!^b(\text{c-Ind}_K^{G_b(E)} \Lambda)$ for $i^b \colon \text{Bun}_G^b \hookrightarrow \text{Bun}_G$. We claim that $\mathbb{D}_{\text{BZ}}(i_!^b \,\text{c-Ind}_K^{G_b(E)} \Lambda)$ is A_K^b.

To prove the claim, we check:

$$\text{RHom}(A_K^b, B) = (i^{b*}B)^K \underbrace{\xrightarrow{\sim}}_{\text{Avg}} (i^{b*}B)_K = {}^{44}\pi_\natural(i_!^b \,\text{c-Ind}_K^{G_b(E)} \Lambda \otimes B). \tag{20.2.1}$$

To prove biduality, we need to compute

$$\mathbb{D}_{\text{BZ}}(A_K^b) \stackrel{?}{=} i_!^b[\text{c-Ind}_K^{G_b(E)} \Lambda].$$

This is easy to see on the stratum Bun_G^b, so we need to check it after pullback to the complement. That amounts to the statement that the LHS vanishes after such a pullback.

Let $j \colon U \hookrightarrow \text{Bun}_G$ be an open substack consisting of proper generalizations of b. We need to see that $j^*\mathbb{D}_{\text{BZ}}(A_K^b) = 0$, or equivalently that for all $B \in D_{ét}(U, \Lambda)$, $\text{RHom}(\mathbb{D}_{\text{BZ}}(A), Rj_*B) = 0$. By definition, this is $\pi_\natural(A_K^b \otimes Rj_*B)$. We can compute this in terms of the formula for A_K^b:

$$\begin{aligned} \pi_\natural(A_K^b \otimes Rj_*B) &\cong \pi_\natural f_{K!}(f_K^! \Lambda \otimes f_K^* Rj_*B) \\ &\cong (\pi \circ f_K)!(f_K^* Rj_*B) \\ &\cong R\Gamma_c(\widetilde{\mathcal{M}}_b/K, f^*Rj_*B). \end{aligned}$$

We can think of $R\Gamma_c(\widetilde{\mathcal{M}}_b/K, f^*Rj_*B)$ as the cohomology of $[\widetilde{\mathcal{M}}_b^\circ/K]$ with compact support towards the boundary of $[\widetilde{\mathcal{M}}_b/K]$, and no support condition near $[*/K] \hookrightarrow [\widetilde{\mathcal{M}}_b/K]$. This vanishes by "vanishing of cohomology with partial compact support", Theorem 19.19. □

[43] If you like, $\pi_\natural = R\pi_!(- \otimes R\pi^!\Lambda)$.

[44] up to shifts?

20.3. **Verdier duality.** Verdier duality is the contravariant endofunctor $A \mapsto \mathrm{RHom}(A, R\pi^! \Lambda)$. It is a contravariant endofunctor on $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.

Verdier duality on $D_{\text{ét}}(\mathrm{Bun}_G^b) \cong D(G_b(E), \Lambda)$ is just smooth duality (up to shift and twist).

Example 20.10. The dualizing complex for $D(G_b(E), \Lambda)$ is the module of Haar measures.

Theorem 20.11. *For any open immersion $j\colon U \hookrightarrow V$ of open substacks of Bun_G, and any $A \in D_{ét}(U, \Lambda)$, we have*

(1) $Rj_* \mathcal{RH}om(A, \mathbb{D}_U) \cong \mathcal{RH}om(j_! A, \mathbb{D}_V)$.
(2) $j_! \mathcal{RH}om(A, \mathbb{D}_U) \cong \mathcal{RH}om(Rj_* A, \mathbb{D}_V)$.

Proof. (1) is clear by adjunction.

(2) We can assume by induction that $U = V \setminus \mathrm{Bun}_G^b$ for some $b \in B(G)$. Let $j\colon U \hookrightarrow \mathrm{Bun}_G^b$. The statement is clear after applying j^*. So it is enough to show it's an isomorphism after applying $\mathrm{RHom}(A_K^b, -)$. As $\mathrm{RHom}(A_K^b, B) = (i^{b*} B)^K$, the LHS vanishes. The RHS is (up to twist)

$$\mathrm{RHom}(A_K^b, \mathcal{RH}om(Rj_* A, \Lambda)) \cong \mathrm{RHom}(A_K^b \overset{L}{\otimes}_\Lambda Rj_* A, \Lambda) \cong \mathrm{RHom}(\pi_\natural(A_K^b \overset{L}{\otimes}_\Lambda Rj_* A), \Lambda).$$

It is therefore enough to show that $\pi_\natural(A_K^b \overset{L}{\otimes}_\Lambda Rj_* A)$ vanishes. Using that $\mathbb{D}_{\mathrm{BZ}}(A) = i_!^b(\text{c-Ind}_K^{G_b(E)} \Lambda)$, we can rewrite this as $\mathrm{RHom}(i_!^b(\text{c-Ind}_K^{G_b(E)} \Lambda), Rj_* A)$, which vanishes because $j^* i_!^b = 0$. $\square$

Corollary 20.12. *$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is reflexive, i.e. $A \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(A))$, if and only if for all $b \in B(G)$, $i^{b*} A \in D_{ét}(\mathrm{Bun}_G^b, \Lambda)$ is reflexive, i.e. $(i^{b*} A)^K \in D(\Lambda)$ is reflexive for all open pro-p subgroups $K \subset G_b(E)$.*

Proof. The Theorem implies that i^{b*} commutes with $\mathbb{D}(\mathbb{D}(-))$, by an inductive argument. $\square$

20.4. **ULA sheaves.** As Bun_G is an Artin v-stack, there is a notion of ULA sheaves for $\pi\colon \mathrm{Bun}_G \to *$. (Being ULA is cohomologically smooth local on the source.) The Proposition below is a consequence of the "dualizability" characterization of being ULA.

Proposition 20.13. *$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is ULA if and only if*

$$p_1^* \mathcal{RH}om(A, \Lambda) \overset{L}{\otimes}_\Lambda p_2^* A \to \mathcal{RH}om(p_1^* A, p_2^* A)$$

is an isomorphism, where $p_1, p_2\colon \mathrm{Bun}_G \times \mathrm{Bun}_G \to \mathrm{Bun}_G$ are the projection maps.

(Previously, a characterization was stated involving upper !, but as Bun_G is cohomologically smooth over a point, it can be reformulated in terms of upper $*$.)

Theorem 20.14. *$A \in D_{ét}(\mathrm{Bun}_G, \Lambda)$ is ULA if and only if for all $b \in B(G)$, and every open pro-p subgroup $K \subset G_b(E)$, $(i^{b*} A)^{hK} \in D(\Lambda)$ is a perfect complex.*

Proof. We need to figure out whether a certain map of sheaves on $\mathrm{Bun}_G \times \mathrm{Bun}_G$ is an isomorphism. We invoke our knowledge of the category of sheaves on $\mathrm{Bun}_G \times \mathrm{Bun}_G$.

Lemma 20.15. *The exterior $\boxtimes$-product*

$$(-) \boxtimes (-)\colon \mathcal{D}_{ét}(\mathrm{Bun}_G, \Lambda) \boxtimes_{\mathcal{D}(\Lambda)} \mathcal{D}_{ét}(\mathrm{Bun}_G, \Lambda) \to \mathcal{D}_{ét}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$$

is an equivalence of ∞-categories. More precisely, for $A_1, A_2 \in D_{ét}(\mathrm{Bun}_G, \Lambda)^\omega$, the exterior tensor product $A_1 \boxtimes A_2 \in \mathcal{D}_{ét}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$ is compact, such objects form compact generators of $\mathcal{D}_{ét}(\mathrm{Bun}_G \times \mathrm{Bun}_G, \Lambda)$, and for all $B_1, B_2 \in D_{ét}(\mathrm{Bun}_G, \Lambda)$,

$$\mathrm{RHom}(A_1 \boxtimes A_2, B_1 \boxtimes B_2) \xleftarrow{\sim} \mathrm{RHom}(A_1, B_1) \overset{L}{\otimes}_\Lambda \mathrm{RHom}(A_2, B_2).$$

Proof. Analyze the compact generators A_K^b. □

Now we complete the proof of Theorem 20.14. We need to figure out when

$$p_1^*\mathcal{RH}om(A, \Lambda) \otimes p_2^* A \xrightarrow{\sim} \mathcal{RH}om(p_1^* A, p_2^* A)$$

is an isomorphism. We apply $\mathrm{RHom}(A_1 \boxtimes A_2, -)$ to both sides and use Lemma 20.15. It boils down to proving that

$$\mathrm{RHom}(\pi_\natural(A_1 \overset{L}{\otimes}_\Lambda A), \Lambda) \overset{L}{\otimes}_\Lambda \mathrm{RHom}(A_2, A) \to \mathrm{RHom}(\pi_\natural(A_1 \overset{L}{\otimes}_\Lambda A), \mathrm{RHom}(A_2, A))$$

is an isomorphism. It is satisfied if and only if $\pi_\natural(A_1 \overset{L}{\otimes}_\Lambda A) \in D(\Lambda)$ is perfect. Use $A_1 = i_{b!}\, \text{c-Ind}_K^{G_b(E)} \Lambda$ and (20.2.1) to see that this translates to $(i^{b*}A)^K \in D(\Lambda)$ being perfect. □

21. Geometric Satake (Jan 22)

21.1. **More on $D_{\text{ét}}(\mathrm{Bun}_G)$.** So far, for G/E as usual we have defined an "Artin v-stack" Bun_G on $\mathrm{Perf}_{\overline{\mathbf{F}}_q}$, and a derived category of étale sheaves

$$D_{\text{ét}}(\mathrm{Bun}_G, \mathbf{Z}/\ell^n\mathbf{Z}). \tag{21.1.1}$$

Question 21.1 (Drinfeld)**.** Can one define a category $D(\mathrm{Bun}_G, \mathbf{Z}[1/p])$ such that

$$D(\mathrm{Bun}_G, \mathbf{Z}[1/p]) \otimes_{\mathbf{Z}[1/p]} \mathbf{Z}/\ell^n\mathbf{Z} = D_{\text{ét}}(\mathrm{Bun}_G, \mathbf{Z}/\ell^n\mathbf{Z})?$$

Usually one would approach this through the theory of motives, but that does not apply here.

As $D_{\text{ét}}(\mathrm{Bun}_G, \mathbf{Z}/\ell^n\mathbf{Z})$ is stratified into pieces $D_{\text{ét}}(\mathrm{Bun}_G^b, \mathbf{Z}/\ell^n) \cong D(G_b(E), \mathbf{Z}/\ell^n)$, one could hope that the hypothetical $D(\mathrm{Bun}_G, \mathbf{Z}[1/p])$ admits such a stratification into pieces $D(\mathrm{Bun}_G^b, \mathbf{Z}[1/p]) \cong D(G_b(E), \mathbf{Z}[1/p])$.

Partial answer: such categories exist with $\mathbf{Z}_\ell$-coefficients, and then also $\overline{\mathbf{Q}}_\ell$-coefficients. But it is unclear whether there are equivalences $D(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell) \cong D(\mathrm{Bun}_G, \overline{\mathbf{Q}}_{\ell'})$ given $\iota\colon \overline{\mathbf{Q}}_\ell \cong \overline{\mathbf{Q}}_{\ell'}$.

We can only work canonically with $\mathbf{Z}_\ell$-coefficients, as the category implicitly knows about the Tate twist $\mathbf{Z}_\ell(1) \cong \mathrm{Hom}(\mathbf{Q}_\ell/\mathbf{Z}_\ell, \overline{\mathbf{F}}_q^\times)$, which is a free $\mathbf{Z}_\ell$-module of rank 1. So we would need a $\mathbf{Z}[1/p]$-structure on $\mathbf{Z}_\ell(1)$.

We could for example choose an isomorphism $\mathbf{Z}_\ell(1) \cong \mathbf{Z}_\ell$ for all $\ell \neq p$. Fixing such choices, it seems that such a category ought to exist. A related fact is that on the Langlands dual side, we have a canonical Artin stack Par_G over $\mathbf{Z}_\ell$ of L-parameters, i.e.,

$$\mathrm{Par}_G(A) = \frac{\{\text{continuous } W_E \to \widehat{G}(A)\}}{\widehat{G} - \text{conj}} \quad \text{for } A/\mathbf{Z}_\ell.$$

This needs to be defined over $\mathbf{Z}_\ell$, because the tame inertia $\prod_{\ell \neq p} \mathbf{Z}_\ell(1)$ can only map non-trivially to $\mathbf{Z}_\ell$-algebras. However, upon fixing a topological generator $\tau \in \prod_{\ell \neq p} \mathbf{Z}_\ell(1)$, one can form a partially discretized version $W_E^\tau \subset W_E$ of the Weil group, replacing the tame inertia by $\mathbf{Z}[1/p] \cdot \tau$. Then there is an Artin stack parametrizing $\{W_E^\tau \to \widehat{G}\}/\widehat{G}$ over $\mathbf{Z}[1/p]$, base changing to all the canonical ones over $\mathbf{Z}_\ell$ [DHKM, Zhu20].

Question 21.2 (Drinfeld)**.** Can one make (21.1.1) explicit when $G = \mathrm{SL}_2$?

The key problem is as follows. Consider

$$\mathrm{Bun}_G^{b_1} \xhookrightarrow{i_1} \mathrm{Bun}_G \xhookleftarrow{i_2} \mathrm{Bun}_G^{b_2}$$

What is $i_2^* Ri_{1*}\colon D(G_{b_1}(E), \Lambda) \to D(G_{b_2}(E), \Lambda)$?

Abstract answer (following from the structure of local charts): consider

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{b_2} & \longrightarrow & \mathrm{Bun}_G \\ \uparrow & & \uparrow \\ \widetilde{\mathcal{M}}_{b_2}^{b_1} & \xrightarrow{f} & \mathrm{Bun}_G^{b_1} \end{array}$$

There is a $G_{b_2}(E)$-action on $\widetilde{\mathcal{M}}_{b_2}^{b_1}$. For $\pi \in \mathrm{Rep}(G_{b_2}(E))$, we let $[\pi]$ be the corresponding sheaf on $\mathrm{Bun}_G^{b_1}$. Then the functor $i_2^* Ri_{1*}$ is given by (maybe up to shift)

$$\pi \mapsto R\Gamma(\widetilde{\mathcal{M}}_{b_2}^{b_1}, f^*[\pi])$$

Example 21.3. For $G = \mathrm{SL}_2$, $b_1 \leftrightarrow \mathcal{O}^2$, $b_2 \leftrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$, $\widetilde{\mathcal{M}}_{b_2}^{b_1}$ parametrizes extensions of $\mathcal{O}(1)$ by $\mathcal{O}(-1)$ which are isomorphic to $\mathcal{O}^2$. Alternatively, choosing an isomorphism with $\mathcal{O}^2$ gives a cover $\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1}$ parametrizing injections $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^2$ with cokernel $\mathcal{O}(1)$.

In other words, these are the saturated injections $\mathcal{O}(-1) \hookrightarrow \mathcal{O}^2$. The non-saturated ones extend to maps $\mathcal{O} \hookrightarrow \mathcal{O}^2$. These are easy to understand, as the global sections of $\mathcal{O}$ are E. So

$$\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1} = \mathcal{BC}(\mathcal{O}(1))^2 \setminus (\mathrm{GL}_2(E) \cdot \Delta\mathcal{BC}(\mathcal{O}(1))).$$

We need to compute $R\Gamma(\mathrm{SL}_2(E), \pi \otimes R\Gamma(\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1}, \Lambda))$. In particular we need to compute $R\Gamma(\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1}, \Lambda)$. As $\mathcal{BC}(\mathcal{O}(1))^2 \cong \mathbb{D}^2$, and $\mathrm{GL}_2(E) \cdot \Delta\mathcal{BC}(\mathcal{O}(1)))$ is $\mathbf{P}^1(E)$ copies of $\mathbb{D}$ glued at 0, you can analyze this by excision. You get copies of the trivial and Steinberg representations, up to twists.

Example 21.4. Suppose $b_1 \leftrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and $b_2 \cong \mathcal{O}(-2) \oplus \mathcal{O}(2)$, then $\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1}$ parametrizes saturated injections $\mathcal{O}(-2) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$. The non-saturated ones extend to $\mathcal{O}(-1) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(1)$, so

$$\widetilde{\widetilde{\mathcal{M}}}_{b_2}^{b_1} = \mathcal{BC}(\mathcal{O}(1)) \times \mathcal{BC}(\mathcal{O}(3)) \setminus (\text{image of } \underline{E} \times \mathcal{BC}(\mathcal{O}(2)) \times \mathcal{BC}(\mathcal{O}(1)))$$

where the map is $(xz, yz) \leftarrow\!\shortmid (x, y, z)$.

Again, this can be computed by excision.

21.2. **Where are we going?** We want to extract L-parameters

$$\varphi \colon W_E \to \widehat{G} \in H^1(W_E, \widehat{G}).$$

We still need to make the dual group $\widehat{G}$ appear.

Idea: the L-parameter is "spectral information" arising as "eigenvalues" of Hecke operators acting on $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$.

The key fact is that Hecke operators are enumerated by $\mathrm{Rep}\, \widehat{G}$.

Warning 21.5. There is a classical notion of Hecke operators in the representation theory of p-adic groups. This is *not* what we are talking about.

21.3. **Hecke operators.**

Definition 21.6. Let Hecke_G be the small v-stack on $\mathrm{Perf}_{\overline{\mathbf{F}}_q}$ with functor of points

$$\mathrm{Hecke}_G(S) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, S^\#, f) \colon \begin{array}{r} \mathcal{E}_1, \mathcal{E}_2 \in \mathrm{Bun}_G(S) \\ S^\# \in \mathrm{Div}^1_X(S) \text{ untilt of } S \text{ over } E \text{ up to Frob} \\ f \colon \mathcal{E}_1|_{X_S \setminus S^\#} \cong \mathcal{E}_2|_{X_S \setminus S^\#} \text{ meromorphic at } S^\# \end{array} \right\}.$$

As defined Hecke_G is "infinite-dimensional" because the modification f is not "bounded". For dominant coweights μ of G, there are substacks $\mathrm{Hecke}_{G, \leq \mu}$ bounding the pole of the modification, such that the maps h_1, h_2 to Bun_G are proper and representable in spatial diamonds.

$$\mathrm{Bun}_G \xleftarrow{\;h_1\;} \mathrm{Hecke}_{G, \leq \mu} \xrightarrow{\;h_2\;} \mathrm{Bun}_G \times \mathrm{Div}^1$$

Hence we get operators like

$$Rh_{2*}h_1^*\colon D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)\to D_{\text{ét}}(\mathrm{Bun}_G\times\mathrm{Div}^1,\Lambda).$$

As $\mathrm{Div}^1\cong(\mathrm{Spa}\,\widehat{\overline{E}})^\diamond/\underline{W_E}$, we have $D_{\text{ét}}(\mathrm{Bun}_G\times\mathrm{Div}^1,\Lambda)\cong D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^{W_E}$. Here we implicitly used the invariance of $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ under base change along $(\mathrm{Spa}\,\widehat{\overline{E}})^\diamond\to(\mathrm{Spa}\,\overline{\mathbf{F}}_q)^\diamond$.

Actually it is better to allow kernels on the correspondences.

Theorem 21.7 (Geometric Satake, first incarnation)**.** *There exists a canonical exact monoidal functor*

$$\mathrm{Rep}_{\mathbf{Z}_\ell}(\widehat{G})\to D_{\acute{e}t}(\mathrm{Hecke}_G,\Lambda)$$

denoted $V\mapsto\mathcal{S}_V$*.*

Hence we get Hecke operators

$$T_V\colon Rh_{2*}(h_1^*\otimes\mathcal{S}_V)\colon D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)\to D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)^{W_E}.$$

The monoidality of $V\mapsto\mathcal{S}_V$ implies that $T_W\circ T_V\cong T_{V\otimes W}$.

21.4. **Classical Geometric Satake.** We begin with the classical setup (developed by Mirkovic-Vilonen, Lusztig, Ginzburg, ...)

Let $G/\mathbf{C}$ be a reductive group. The *positive loop group* L^+G has functor of point $L^+G(A)=G(A[[t]])$. This defines an infinite-dimensional affine scheme. The *loop group* LG has functor of points $LG(A)=G(A((t)))$. It is an ind-scheme.

Definition 21.8. The *affine Grassmannian* is $\mathrm{Gr}_G=LG/L^+G$. Its functor of points takes A to the set of G-torsors $\mathcal{E}$ on $A[[t]]$ trivialized over $A((t))$.

This is an ind-projective scheme, with transition maps maps being closed immersions.

Definition 21.9. We define the *Satake category for* G to be $\mathrm{Sat}_G:=\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G;\mathbf{Z})$.

The orbit spaces $L^+G\backslash LG/L^+G$ can be identified with X_*^+, the dominant cocharacters of G. This bijection takes $\mu\in X_*^+$ to the double coset represented by $\mu(t)$. The L^+G-orbit of $\mu(t)$ is denotd Gr_μ, and its closure is a projective scheme $\mathrm{Gr}_{G,\le\mu}\subset\mathrm{Gr}_G$, an (affine) Schubert variety.

In particular, for each μ we have an intersection complex $\mathrm{IC}_\mu=\mathrm{IC}_{\mathrm{Gr}_{G,\le\mu}}\in\mathrm{Sat}_G$. By definition,

$$\mathrm{IC}_\mu=\mathrm{Im}\left(\underbrace{{}^{\mathfrak{p}}j_{\mu!}\mathbf{Z}[d_\mu]}_{\text{“standard” }\Delta_\mu}\to\underbrace{{}^{\mathfrak{p}}j_{\mu*}\mathbf{Z}[d_\mu]}_{\text{“costandard” }\nabla_\mu}\right).$$

where $j_\mu\colon\mathrm{Gr}_\mu\hookrightarrow\mathrm{Gr}_{G,\le\mu}$. Here $d_\mu:=\dim\mathrm{Gr}_\mu=\langle2\rho,\mu\rangle$.

Remark 21.10. With $\mathbf{Q}$-coefficients, ${}^{\mathfrak{p}}j_{\mu!}\mathbf{Q}[d_\mu]\xrightarrow{\sim}\mathrm{IC}_{\mu,\mathbf{Q}}\xrightarrow{\sim}{}^{\mathfrak{p}}j_{\mu*}\mathbf{Q}[d_\mu]$.

Remark 21.11. The structure of the $\Delta_\mu,\mathrm{IC}_\mu,\nabla_\mu$ give Sat_G the structure of a “highest weight category”, with weights given by X_*^+. With $\mathbf{Q}$-coefficients, it is semi-simple.

Definition 21.12. There is a convolution monoidal structure on Sat_G given as follows. Consider the diagram

$$\begin{array}{c} (L^+G\backslash LG/L^+G) \times (L^+G\backslash LG/L^+G) \\ \uparrow \pi \\ L^+G\backslash LG \times^{L^+G} LG/L^+G \\ \downarrow m \\ L^+G\backslash LG/L^+G \end{array}$$

For $A, B \in \mathrm{Sat}_G$, we define $A \star B = Rm_* \pi^*(A \boxtimes B)$.

Theorem 21.13 (Mirkovic-Vilonen [MV07])**.** *The cohomology functor*

$$(\mathrm{Sat}_G, \star) \xrightarrow{\oplus H^i(\mathrm{Gr}_G, -)} (\mathrm{Vect}, \otimes)$$

is a fiber functor, and $(\mathrm{Sat}_G, \star)$ *can be upgraded to a symmetric monoidal structure, making* $\oplus H^i(\mathrm{Gr}_G, -)$ *into a symmetric monoidal functor. The corresponding Tannaka group is* $\widehat{G}$*, so we have a commutative triangle*

$$\begin{array}{ccc} (\mathrm{Sat}_G, \star) & \xrightarrow{\sim} & (\mathrm{Rep}\, \widehat{G}, \otimes) \\ & {\scriptstyle \oplus H^i(\mathrm{Gr}_G, -)} \searrow \quad \swarrow {\scriptstyle forget} & \\ & (\mathrm{Vect}, \otimes) & \end{array}$$

- *With* $\mathbf{Q}$*-coefficients,* IC_μ *corresponds to* V_μ*, the highest weight representation of* $\widehat{G}$ *of weight* $\mu \in X_*^+ = X_+^*(\widehat{G})$*.*
- *With* $\mathbf{F}_p$*-coefficients,* IC_μ *corresponds to* L_μ*, an irreducible representation, and* ${}^{\mathfrak{p}}j_{\mu !}\mathbf{F}_p[d_\mu]$ *corresponds to* Δ_μ*,* ${}^{\mathfrak{p}}j_{\mu *}\mathbf{F}_p[d_\mu]$ *corresponds to* ∇_μ*.*

21.5. **p-adic Geometric Satake.** We want a version of this story for the $\mathrm{B}_{\mathrm{dR}}^+$-affine Grassmannian.

Definition 21.14. Let G/E, Div^1 be as usual. We define a functor $\mathrm{Gr}_G \to \mathrm{Div}^1$ parametrizing $S^\# \in \mathrm{Div}^1(S)$ and a G-torsor $\mathcal{E}$ on the completion of X_S at $S^\#$, plus a trivialization on $(X_S)^\wedge_{S^\#} \setminus S^\#$.

Remark 21.15. There is a subtlety: what does $(X_S)^\wedge_{S^\#}$ mean? We only define it for $S = \mathrm{Spa}(R, R^+)$ affinoid. Then $S^\# = \mathrm{Spa}(R^\#, R^{\#+})$ is also affinoid, and we have

$$\theta \colon W_{\mathcal{O}_E}(R^+)[1/[\varpi]] \twoheadrightarrow R^\#.$$

We define $\mathrm{B}_{\mathrm{dR}}^+(R^\#)$ to be the $(\ker\theta)$-adic completion of $W_{\mathcal{O}_E}(R^+)[1/[\varpi]]$. (This is the same as the completion of any open affinoid subset $\mathrm{Spa}(A, A^+) \subset X_S$, such that $\mathrm{Spa}(A, A^+) \supset S^\#$, along $S^\#$.) One shows that $\ker\theta = (\xi)$ is principal; then define $\mathbb{B}_{\mathrm{dR}}(R^\#) := B_{\mathrm{dR}}^+(R^\#)[1/\xi]$. We define $(X_S)^\wedge_{S^\#} := \mathrm{Spa}\,\mathrm{B}_{\mathrm{dR}}^+(R^\#)$ and $(X_S)^\wedge_{S^\#} \setminus S^\# := \mathrm{Spa}\,\mathrm{B}_{\mathrm{dR}}(R^\#)$.

So $\mathrm{Gr}_G(S) = \{G\text{-torsors on } \mathrm{B}_{\mathrm{dR}}^+(R^\#), \text{ trivialized over } \mathrm{B}_{\mathrm{dR}}(R^\#)\}$. We have $\mathrm{Gr}_G = LG/L^+G$ where LG is the functor sending $R^\# \mapsto G(\mathrm{B}_{\mathrm{dR}}(R^\#))$ and L^+G is the functor sending $R^\# \mapsto G(\mathrm{B}_{\mathrm{dR}}^+(R^\#))$.

Definition 21.16. We define the local Hecke stack

$$\mathrm{Hk}_G = L^+G\backslash \mathrm{Gr}_G = L^+G\backslash LG/L^+G.$$

We will define Sat_G = "$\mathrm{Perv}(\mathrm{Hk}_G, \Lambda)$", endowed with a convolution monoidal structure. These are all $\mathrm{Rep}_{W_E}(\Lambda)$-linear categories, because everything lives over Div^1. So the $\mathrm{Rep}_{W_E}(\Lambda)$-linear structure comes from tensoring with pullback of local systems on Div^1.

Theorem 21.17. *The monoidal category* $(\mathrm{Sat}_G(\Lambda), \star)$ *upgrades naturally to a symmetric monoidal category such that* $\oplus H^i(\mathrm{Gr}_{G,\mathrm{Spa}\,\overline{\eta}}, -)\colon \mathrm{Sat}_G(\Lambda) \to \mathrm{Rep}_{W_E}(\Lambda)$ *is a fiber functor, with corresponding Tannaka group given by* $\widehat{G}$.

Remark 21.18. Since the Tananaka group is internal in Rep_{W_E}, it can be viewed as a reductive group $\widehat{G}$ with equipped with a natural W_E-action.

Fix $G/\mathcal{O}_E$ a reductive group. There is also the Witt vector affine Grassmannian $\mathrm{Gr}_G^{\mathrm{Witt}}$, a functor sending perfect $\overline{\mathbf{F}}_q$-algebras A to $\{G$-torsors on $W_{\mathcal{O}_E}(A)$ trivialized on $W_{\mathcal{O}_E}(A)[1/\pi]\}$. This has an action of $L^+_{\mathrm{Witt}}G$, the functor sending A to $G(W_{\mathcal{O}_E}(A))$. We deduce the following result which had previously been obtained by Zhu.

Corollary 21.19 (Zhu)**.** *We have an equivalence of symmetric monoidal categories*

$$(\mathrm{Perv}_{L^+_{\mathrm{Witt}}G}(\mathrm{Gr}_G^{\mathrm{Witt}}), \star) \cong \mathrm{Rep}(\widehat{G}, \otimes).$$

To deduce this, we use a degeneration

$$\begin{array}{ccccc} (\mathrm{Gr}_G^{\mathrm{Witt}})^\diamond & \longrightarrow & \mathrm{Gr}_G^{\mathrm{BD}} & \longleftarrow & \mathrm{Gr}_G \times_{\mathrm{Div}^1} \mathrm{Spa}\,\breve{E} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spa}\,\overline{\mathbf{F}}_q & \longrightarrow & \mathrm{Spa}\,\mathcal{O}_{\breve{E}} & \longleftarrow & \mathrm{Spa}\,\breve{E} \end{array}$$

Then there is a way to "specialize" perverse sheaves using the ULA formalism.

The new proof is very different from Zhu's. The issue is that the symmetric monoidal structure is not easy to see geometrically. In the classical situation, it comes from "fusion", in which two points on a global curve collide. In the p-adic situation, this would require a space like "Spec $\mathbf{Q}_p \times$ Spec $\mathbf{Q}_p$". One cannot make this in an interesting way in the world of schemes, but it can be done in the world of diamonds.

22. Perverse sheaves and hyperbolic localization (Jan 25)

Our goal is to get the Geometric Satake Theorem, which said (roughly)

$$(\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, \mathbf{Z}_\ell), \star) \cong (\mathrm{Rep}\, \widehat{G}, \otimes).$$

However, today we will spend most of our time on some foundational material.

22.1. **Reminder on perverse sheaves.** We will begin by recalling the classical theory, in the setting of algebraic geometry. Let X be a separated scheme of finite type over an algebraically closed field k.

Let Λ be a Noetherian ring killed by $n \in k^\times$. We could also consider $\Lambda = \overline{\mathbf{Q}}_\ell$, but we want to focus on the torsion case.

We have a derived category of étale sheaves $D_{\text{ét}}(X, \Lambda) = D(X_{\text{ét}}, \Lambda)$. It is compactly generated, with compact objects being $D^b_{c,\mathrm{ftor}}(X_{\text{ét}}, \Lambda)$, the bounded complexes with constructible cohomology sheaves, of finite Tor dimension over Λ.

Remark 22.1. We have $D^b_{c,\mathrm{ftor}}(X_{\text{ét}}, \Lambda) \subset D^b_c(X_{\text{ét}}, \Lambda)$, the subcategory of all bounded complexes with constructible cohomology sheaves.

Definition 22.2. (1) The full subcategory ${}^{\mathfrak{p}}D^{\leq 0}_{\text{ét}}(X, \Lambda) \subseteq D_{\text{ét}}(X, \Lambda)$ consists of $A \in D_{\text{ét}}(X, \Lambda)$ such that for all geometric points $\overline{x} \to X$, $A_{\overline{x}} \in D^{\leq -d(\overline{x})}(\Lambda)$ where $d(\overline{x}) = \dim \overline{\{x\}} = \mathrm{trdeg}\, k(\overline{x})/k$.

(2) The full subcategory ${}^{\mathfrak{p}}D^{\leq n}_{\text{ét}}(X, \Lambda) \subseteq D_{\text{ét}}(X, \Lambda)$ is ${}^{\mathfrak{p}}D^{\leq 0}_{\text{ét}}(X, \Lambda)[-n]$.

(3) The full subcategory ${}^{\mathfrak{p}}D^{\geq 0}_{\text{ét}}(X, \Lambda) \subseteq D_{\text{ét}}(X, \Lambda)$ is the right orthogonal of ${}^{\mathfrak{p}}D^{\geq -1}_{\text{ét}}(X, \Lambda)$, i.e. $B \in {}^{\mathfrak{p}}D^{\geq 0}$ if and only if for all $A \in {}^{\mathfrak{p}}D^{\leq -1}$, we have $\mathrm{Hom}(A, B) = 0$.

(4) The full subcategory ${}^{\mathfrak{p}}D^{\geq n}_{\text{ét}}(X, \Lambda) \subseteq D_{\text{ét}}(X, \Lambda)$ is ${}^{\mathfrak{p}}D^{\geq 0}_{\text{ét}}(X, \Lambda)[-n]$.

Theorem 22.3. *(1) The pair $({}^{\mathfrak{p}}D^{\leq 0}_{\acute{e}t}, {}^{\mathfrak{p}}D^{\geq 0}_{\acute{e}t})$ defines a t-structure on $D_{\acute{e}t}(X, \Lambda)$. In particular, there exist truncation functors*

$$ {}^{\mathfrak{p}}\tau^{\geq 0} D_{\acute{e}t}(X, \Lambda) \to {}^{\mathfrak{p}}D^{\leq 0}_{\acute{e}t}(X, \Lambda)$$
$$ {}^{\mathfrak{p}}\tau^{\leq 0} \colon D_{\acute{e}t}(X, \Lambda) \to {}^{\mathfrak{p}}D^{\geq 0}_{\acute{e}t}(X, \Lambda)$$

which are left/right adjoint to the inclusions, and

$$ {}^{\mathfrak{p}}\tau^{\leq 0} A \to A \to {}^{\mathfrak{p}}\tau^{\geq 1} A$$

is a distinguished triangle.

(2) $A \in D_{\acute{e}t}(X, \Lambda)$ lies in ${}^{\mathfrak{p}}D^{\geq 0}_{\acute{e}t}$ if and only if for all geometric points $i_{\overline{x}} \colon \overline{x} \to X$,

$$ Ri^!_{\overline{x}} A \in D^{\geq -d(\overline{x})}(\Lambda).$$

By definition, if we factorize $i_{\overline{x}}$ as

$$ \overline{x} \xrightarrow{j_{\overline{x}}} \overline{\{x\}} \xrightarrow{i} X$$

then $Ri^!_{\overline{x}} A := j^*_{\overline{x}} Ri^! A$.

(3) It induces a t-structure on $D^b_c(X, \Lambda)$ (equivalently, the truncations ${}^{\mathfrak{p}}\tau^{\geq 0}, {}^{\mathfrak{p}}\tau^{\leq 0}$ preserve this subcategory).

Warning 22.4. The truncation functors ${}^{\mathfrak{p}}\tau^{\geq 0}, {}^{\mathfrak{p}}\tau^{\leq 0}$ do *not* preserve $D^b_{c,\mathrm{ftor}}(X, \Lambda)$ in general. This subtlety appears already for $X = \mathrm{Spec}\ k$: truncations of perfect Λ-complexes need not be perfect. However, it is OK if Λ is regular.

Definition 22.5. The category of *perverse sheaves* is the heart of the t-structure, i.e.

$$\mathrm{Perv}(X,\Lambda) := ({}^{\mathfrak{p}}D^{\leq 0}_{\text{ét}} \cap {}^{\mathfrak{p}}D^{\geq 0}_{\text{ét}}).$$

It is an abelian category (by general facts about t-structures).

Example 22.6. If $i\colon \operatorname{Spec} k \hookrightarrow X$, then $i_*\Lambda$ is perverse.

Example 22.7. If X is smooth of dimension d, then $\Lambda[d]$ is perverse. This ultimately comes from purity.

Theorem 22.8. *If $\Lambda = \overline{\mathbf{F}}_\ell$, then $\mathrm{Perv}(X,\Lambda) \cap D^b_c(X,\Lambda)$ is an artinian category, i.e. every object has finite length. The irreducible objects are in bijection with pairs (Z,ρ) where Z is a closed irreducible subset of X, and ρ is an irreducible representation of the absolute Galois group of $k(Z)$ on a $\overline{\mathbf{F}}_\ell$-vector space.*

Sketch. Given Z and ρ, we get a dense open $j\colon U \hookrightarrow Z$ and an irreducible $\overline{\mathbf{F}}_\ell$-local system $\mathbb{L}$ on U. By restricting further, we may assume that U is smooth.

Write $d_Z := \dim Z$. Then $\mathbb{L}[d_Z] \in \mathrm{Perv}(U, \overline{\mathbf{F}}_\ell)$ by Example 22.7. It is easy to check that $j_!\mathbb{L}[d_Z] \in {}^{\mathfrak{p}}D^{\leq 0}(Z, \overline{\mathbf{F}}_\ell)$ and $Rj_*\mathbb{L}[d_Z] \in {}^{\mathfrak{p}}D^{\geq 0}(Z, \overline{\mathbf{F}}_\ell)$. We define

$$\begin{aligned} {}^{\mathfrak{p}}j_!\mathbb{L}[d_Z] &:= {}^{\mathfrak{p}}\tau^{\geq 0}(j_!\mathbb{L}[d_Z]) \\ {}^{\mathfrak{p}}Rj_*\mathbb{L}[d_Z] &:= {}^{\mathfrak{p}}\tau^{\leq 0}(Rj_*\mathbb{L}[d_Z]). \end{aligned}$$

Then we define the *intersection complex* $\mathrm{IC}(Z,\mathbb{L})$ to be the image of ${}^{\mathfrak{p}}j_!\mathbb{L}[d_Z] \to {}^{\mathfrak{p}}Rj_*\mathbb{L}[d_Z]$. Then we get a perverse sheaf $i_*\,\mathrm{IC}(Z,\mathbb{L}) \in \mathrm{Perv}(X, \overline{\mathbf{F}}_\ell)$. These are the irreducible objects. The crucial point is that this is well-defined, independent of the choice of U. (No such independence property holds for constructible sheaves, which is why constructible sheaves don't form an artinian category.) □

22.2. **Relative perverse sheaves.** Let $f\colon X \to S$ be a separated finite type morphism, S an arbitrary scheme. Goal: define a notion of "perversity relative to S".

Definition 22.9. (1) ${}^{\mathfrak{p}/S}D^{\leq 0}_{\text{ét}}(X,\Lambda) \subset D_{\text{ét}}(X,\Lambda)$ is the full subcategory of $A \in D_{\text{ét}}(X,\Lambda)$ such that for all geometric points $\overline{s} \to S$, $A|_{X_{\overline{s}}} \in {}^{\mathfrak{p}}D^{\leq 0}(X_{\overline{s}}, \Lambda)$.

Equivalently, for all geometric points $\overline{x} \to X$ lying over geometric points $\overline{s} \to S$, we have $A_{\overline{x}} \in D^{\leq -d(\overline{x}/\overline{s})}(\Lambda)$.

(2) ${}^{\mathfrak{p}/S}D^{\geq 0}(X,\Lambda)$ is the right orthogonal of ${}^{\mathfrak{p}/S}D^{\leq -1}$.

Theorem 22.10 (Hansen-S)**.** *(1) This defines a t-structure on $D_{ét}(X,\Lambda)$.*

(2) $A \in D_{ét}(X,\Lambda)$ lies in ${}^{\mathfrak{p}/S}D^{\geq 0}(X,\Lambda)$ if and only if for all geometric points $\overline{s} \to S$, the $*$-restriction $A|_{X_{\overline{s}}} \in {}^{\mathfrak{p}/S}D^{\geq 0}(X_{\overline{s}},\Lambda)$.

(3) This induces a t-structure on $D^b_c(X,\Lambda)$.

Remark 22.11. When S is a DVR, the content of the Theorem is essentially equivalent to Gabber's results on good behavior of perverse sheaves under nearby cycles.

Corollary 22.12. *Pullback under $S' \to S$ induces t-exact functors. More precisely, given a cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

the $*$-pullback sends

$${}^{\mathfrak{p}/S}D^{\leq 0}(X,\Lambda) \to {}^{\mathfrak{p}/S'}D^{\leq 0}(X',\Lambda)$$

and

$$^{\mathfrak{p}/S}D^{\geq 0}(X,\Lambda) \to {}^{\mathfrak{p}/S'}D^{\geq 0}(X',\Lambda)$$

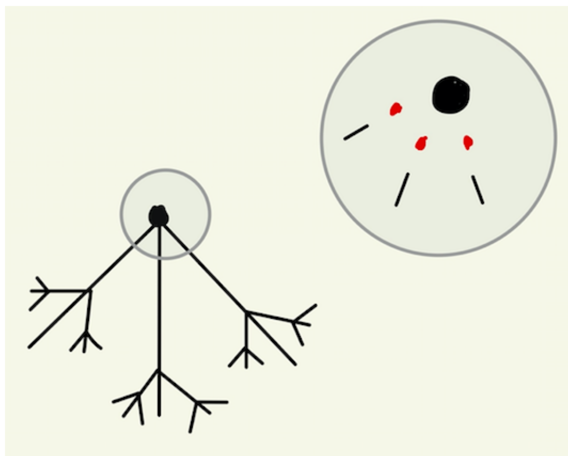
Corollary 22.13. *There is a notion of "family of perverse sheaves on X/S", namely*

$$\mathrm{Perv}(X/S,\Lambda) := {}^{\mathfrak{p}/S}D^{\leq 0}(X,\Lambda) \cap {}^{\mathfrak{p}/S}D^{\geq 0}(X,\Lambda)$$

22.3. **Perverse sheaves in p-adic geometry.** The story in p-adic geometry is subtle. One immediate problem is how to define the dimension of a point.

Warning 22.14. I do not know how to define the "correct" dimension of a point of $\mathbb{B}^2_{\mathbf{C}_p}$.

Example 22.15. Consider $|\mathbb{B}^{\mathrm{ad}}_{\mathbf{C}_p}|$.



The classical points have dimension 0. All the other rank 1 points should be of dimension 1.

What about the rank 2 points? These look either 0 or 1-dimensional, depending on perspective. The choices are exchanged under Verdier duality.

Example 22.16. Consider $|\mathbb{B}^2_{\mathbf{C}_p}|$. There is no classification of rank 1 points. The "topological transcendence degree" has weird behavior, e.g. there exist towers $\mathbf{C}_p \subset K_1 \subset K_2$ where $K_1/\mathbf{C}_p$ has topological transcendence degree 1, K_2/K_1 has topological transcendence degree 1, and $K_2/\mathbf{C}_p$ also has topological transcendence degree 1 [Tem]. If a point has residue field K_2, should it have dimension 1 or 2?

Upshot: there seems to be no hope for a completely general theory of perverse sheaves in p-adic geometry.

On the other hand, we won't need a completely general theory. We only need a theory of "relatively perverse sheaves" for $\mathrm{Hk}_G \to \mathrm{Div}^1$. So we only need to define dimensions of points of $\mathrm{Hk}_G \times_{\mathrm{Div}^1} \mathrm{Spa}\, C$. For this we use the Cartan stratification, and define the dimension by hand. We have $\mathrm{Hk}_G = L^+G \backslash \mathrm{Gr}_G$, and the L^+G-orbits on Gr_G induce a stratification

$$\mathrm{Gr}_G = \bigcup_{\mu \in X_*(T)_+} \mathrm{Gr}_{G,\mu}$$

with $\dim \mathrm{Gr}_{G,\mu} = \langle 2\rho, \mu \rangle$.

22.4. **Hyperbolic localization.** We start with the classical background for schemes. Let k be an algebraically closed field, X/k a proper scheme with a $\mathbf{G}_m$-action.

The fixed points $X^0 = X^{\mathbf{G}_m}$ form a closed subset of X. Write

$$X^0 = \sqcup_{i=1}^m X_i^0,$$

with each X_i^0 open and closed in X^0.[45] With respect to this, we can define two $\mathbf{G}_m$-stable stratifications:

(1) A stratification $X = \bigcup_{i=1}^m X_i^+$, where X_i^+ consists of points x such that "$\lim_{t\to 0} t\cdot x$ exists, and lies in X_i^0."
(2) A stratification $X = \bigcup_{i=1}^m X_i^-$, where X_i^- consists of points x such that "$\lim_{t\to\infty} t\cdot x$ exists, and lies in X_i^0."

More precisely, let $(\mathbf{A}^1)^+ \hookrightarrow \mathbf{P}^1$ be the complement of ∞ and $(\mathbf{A}^1)^- \hookrightarrow \mathbf{P}^1$ be the complement of 0. The $\mathbf{G}_m$-action on X_i^+ extends to

$$(\mathbf{A}^1)^+ \times X_i^+ \to X_i^+$$

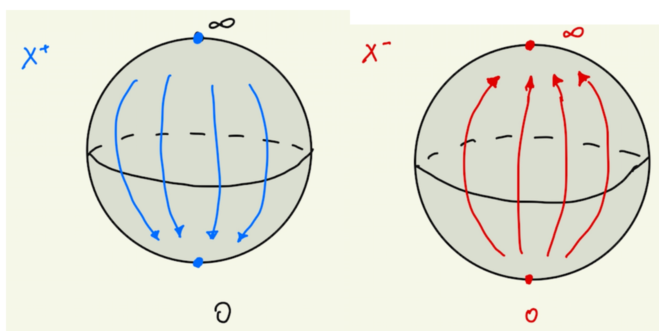
sending $0 \times X_i^+ \to X_i^0$, and the $\mathbf{G}_m$-action on X_i^- extends to

$$(\mathbf{A}^1)^- \times X_i^- \to X_i^-$$

sending $0 \times X_i^- \to X_i^0$.

We write $X^+ := \coprod X_i^+$ and $X^- := \bigcup X_i^-$.

Example 22.17. Consider $\mathbf{G}_m$ acting on $X = \mathbf{P}^1$ in the standard way.



We take the decomposition

$$X^0 = \{0,\infty\} = \{0\} \sqcup \{\infty\},$$

for which

$$X^+ = \mathbf{A}^1 \sqcup \{\infty\},$$
$$X^- = \{0\} \sqcup (\mathbf{A}^1)^-.$$

Exercise 22.18. Consider $\mathbf{G}_m$ acting on $\mathbf{P}^1 \times \mathbf{P}^1$ by $t\cdot(a_1,a_2) = (t^{-1}a_1, ta_2)$. Figure out the stratifications for

$$(\mathbf{P}^1 \times \mathbf{P}^1)^0 = (0,0) \sqcup (0,\infty) \sqcup (\infty,0) \sqcup (\infty,\infty).$$

The goal of hyperbolic localization is to describe the cohomology of $\mathbf{G}_m$-equivariant sheaves on X in terms of local information at X^0.

[45] One would usually just take the X_i^0 to be exactly the connected components for X^0. However, presenting it in this more flexible way will adapt better to p-adic geometry

Theorem 22.19 (Braden)**.** *There exists a functor*

$$L\colon D_{ét}(X/\mathbf{G}_m, \Lambda) \to D_{ét}(X^0, \Lambda)$$

such that $R\Gamma(X, A)$ has a filtration, with associated graded $R\Gamma(X^0, L(A))$.

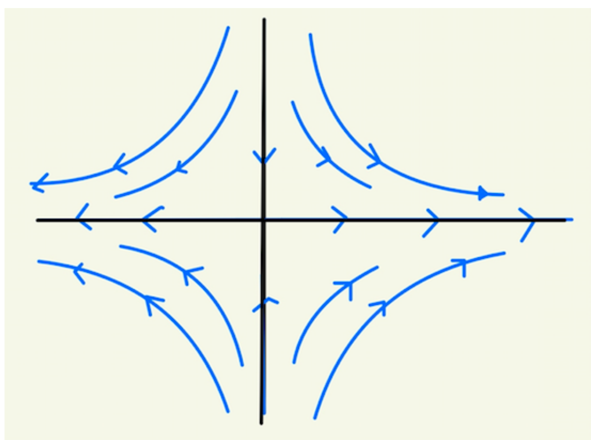
In fact, L admits four explicit descriptions. Consider the Cartesian diagram

$$\begin{array}{ccc} X_i^0 & \xrightarrow[i^-]{} & X_i^- \\ {\scriptstyle p^+}\uparrow\downarrow{\scriptstyle i^+} & \overset{p^-}{\curvearrowleft} & \downarrow{\scriptstyle q^-} \\ X_i^+ & \xrightarrow[q^+]{} & X \end{array}$$

The functor L admits the four naturally isomorphic descriptions below:

$$R(p^-)_!(q^-)^* \xleftarrow{\sim} R(i^-)^!(q^-)^* \xleftarrow{\sim} (i^+)^* R(q^+)^! \xleftarrow{\sim} R(p^+)_* R(q^+)^!.$$

Example 22.20. Consider $\mathbf{G}_m$-equivariant sheaves A on $\mathbf{A}^2$ with the hyperbolic $\mathbf{G}_m$-action, $t \cdot (a_1, a_2) = (t^{-1}a_1, ta_2)$.



$$\begin{array}{ccc} 0 & \xrightarrow{i^-} & \mathbf{A}^1 \times \{0\} \\ \downarrow{\scriptstyle i^+} & & \downarrow{\scriptstyle q^-} \\ 0 \times \mathbf{A}^1 & \xrightarrow{q^+} & \mathbf{A}^2 \end{array}$$

Then the hyperbolic localization functor $D(X/\mathbf{G}_m, \Lambda) \to D(\Lambda)$ can be taken to be $(i^+)^* R(q^+)^! A \xrightarrow{\sim} R(i^-)^!(q^-)^+ A$.

Example 22.21. Consider $X = \mathbf{P}^1$ with the usual $\mathbf{G}_m$-action, and $A = \Lambda$. Then $R\Gamma(\mathbf{P}^1, \Lambda) = \Lambda[0] \oplus \Lambda[-2]$. Then $L(A)_{\{\infty\}} = R\Gamma_c(\mathbf{A}^1, \Lambda) = \Lambda[-2]$, which is also identified with $R\Gamma_{\{\infty\}}(\mathbf{A}^1, \Lambda) = \Lambda[-2]$. On the other hand, $L(A)_{\{0\}} = R\Gamma_c(\{0\}, \Lambda) = \Lambda[0]$. So we explicitly confirm Theorem 22.19 in this case.

Example 22.22. Let X be a flag variety G/P. It has an action of G, which we can inflate via dominant regular cocharacter to $\mathbf{G}_m$. Then $X^{\mathbf{G}_m} = X^T = W/W_P$. Theorem 22.19

implies that

$$R\Gamma(X,\Lambda) = \bigoplus_{w\in \underbrace{W^P}_{W/W_P}} \Lambda[-2\ell(w)].$$

We will use this for $X = \mathrm{Gr}_{G,\leq\mu} \subset \mathrm{Gr}_G$ and $\mathbf{G}_m \subset L^+G$, in order to understand the cohomology of L^+G-equivariant perverse sheaves on Gr_G.

Remark 22.23. It is very useful that L has different descriptions. In particular, since it has a description in terms of left adjoint and another description in terms of right adjoints, it commutes with all limits and colimits. There is a relative version.

22.5. **Hyperbolic localization for diamonds.** Setup: for $f\colon X \to S$ a proper map of small v-stacks, representable in spatial diamonds, with $\dim.\mathrm{tr.g.} f < \infty$. Suppose X has a $\mathbf{G}_m$-action[46], and f is equivariant for it (with the trivial action on S).

Assume we have $\mathbf{G}_m$-equivariant stratifications $X = \bigcup X_i^+$ and $X = \bigcup X_i^-$ as above.[47]

Theorem 22.24. *In this situation, for all $A \in D_{ét}(X/\mathbf{G}_m,\Lambda)$ the maps*

$$R(p^-)_!(q^-)^* \xleftarrow{\sim} R(i^-)^!(q^-)^* \xleftarrow{\sim} (i^+)^*R(q^+)^! \xleftarrow{\sim} R(p^+)_*R(q^+)^!.$$

are isomorphisms, defining "hyperbolic localization functor"

$$L_{X/S}\colon D_{ét}(X/\mathbf{G}_m,\Lambda) \to D_{ét}(X^0,\Lambda).$$

The functor $L_{X/S}$ commutes with all (co)limits (in the ∞-categorical setting) and all base changes $S' \to S$, and for

$$\begin{array}{ccc} X & \xrightarrow{f} & S \\ \uparrow & \nearrow_{f^0} & \\ X^0 & & \end{array}$$

there is a filtration on Rf_*, with associated graded being $Rf^0_* L_{X/S}$.

Proof. Everything follows from the following geometric principle: if Y is a locally spatial diamond of finite dimension, with a $\mathbf{G}_m$-action, partially proper over S, and $[Y/\mathbf{G}_m]$ is qcqs over S, then Y has two ends, so for all $A \in D_{\text{ét}}([Y/\mathbf{G}_m],\Lambda)$ we have $R\Gamma_{\partial c}(Y,A) = 0$ by Theorem 19.19. This is not hard to prove. You reduce to the case of $\mathbf{G}_m$, which you do explicitly ☐

Example 22.25. Consider $\mathbf{G}_m \curvearrowright \mathbf{P}^1$ in the usual way. The difference between $R\Gamma_c(\mathbf{A}^1,A)$ and $R\Gamma_{\{0\}}(\mathbf{A}^1,A)$ is $R\Gamma_{\partial c}(\mathbf{G}_m,A)$.

Remark 22.26. We are very much using the analytic world, in which $\mathbf{G}_m$ is not quasi-compact and has two ends.

[46] The adic $\mathbf{G}_m$ has functor of points $(R,R^+) \mapsto R^\times$.

[47] The existence of such stratifications may not be automatic, as it is for *normal* schemes.

23. The Beilinson-Drinfeld Grassmannian (Jan 29)

23.1. **Beilinson-Drinfeld Grassmannian.** Assume $G/\mathcal{O}_E$ is split reductive. (In general, use étale localizations to reduce to this case.[48])

Let $S = \mathrm{Spa}(R, R^+)$, $\varpi \in R$ a pseudouniformizer, $\mathcal{Y}_S = \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus \{[\varpi] = 0\}$.[49] Recall that we have a moduli space of degree d Cartier divisors on $\mathcal{Y}_S = S \times \mathrm{Spa}\, \mathcal{O}_E$,

$$\mathrm{Div}^d_{\mathcal{Y}} = (\mathrm{Div}^1_{\mathcal{Y}})^d/\Sigma_d = (\mathrm{Spa}\, \mathcal{O}_E)^{\diamond, d}/\Sigma_d.$$

This is a small v-stack, and $\mathrm{Div}^d_{\mathcal{Y}} \to *$ is representable in locally spatial diamonds. It "parametrizes d points on $\mathrm{Spa}\, \mathcal{O}_E$".

Given S and a section of $\mathrm{Div}^d_{\mathcal{Y}}(S)$, we get a relative Cartier divisor $D_S \subset \mathcal{Y}_S$, with ideal sheaf $\mathcal{I}(D_S)$. If $S = \mathrm{Spa}(R, R^+)$ is affinoid, let B^+ be the completion of $\mathcal{O}(\mathcal{Y}_S)$ along $\mathcal{I}(D_S)$. In other words, it is

$$B^+ := W_{\mathcal{O}_E}(R^+)[1/[\varpi]]^\wedge_\xi$$

where $D_S = V(\xi)$. Define $B := B^+[1/\xi]$. (Earlier these were called $\mathrm{B}^+_{\mathrm{dR}}$ and B_{dR}, in the $d = 1$ case.) Informally, "$B = \mathcal{O}((\mathcal{Y}_S)^\wedge_{D_S} \setminus D_S)$."

Warning 23.1. This differs from the notation B, B^+ appearing in [FF].

Definition 23.2. We define $L^+G/\mathrm{Div}^d_{\mathcal{Y}}$ to be the small v-sheaf (over $\mathrm{Div}^d_{\mathcal{Y}}$)

$$S = \mathrm{Spa}(R, R^+)/\mathrm{Div}^d_{\mathcal{Y}} \mapsto G(B^+)$$

and $LG/\mathrm{Div}^d_{\mathcal{Y}}$ to be the small v-sheaf (over $\mathrm{Div}^d_{\mathcal{Y}}$)

$$S = \mathrm{Spa}(R, R^+)/\mathrm{Div}^d_{\mathcal{Y}} \mapsto G(B).$$

We define the *Beilinson-Drinfeld Grassmannian*

$$\mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}} := LG/L^+G$$

and the *local Hecke stack*

$$\mathrm{Hk}_{G, \mathrm{Div}^d_{\mathcal{Y}}} = L^+G \backslash \mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}} = L^+G \backslash LG/L^+G.$$

This $\mathrm{Hk}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$ is a small v-stack over $\mathrm{Div}^d_{\mathcal{Y}}$.

Proposition 23.3. *$\mathrm{Gr}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$ parametrizes G-torsors $\mathcal{E}$ over B^+ equipped with a trivialization over B. Equivalently[50], it parametrizes G-torsors $\mathcal{E}$ over $\mathcal{Y}_S$ plus a meromorphic trivialization over $\mathcal{Y}_S \setminus D_S$.*

$\mathrm{Hk}_{G, \mathrm{Div}^d_{\mathcal{Y}}}$ parametrizes G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over B^+ plus an isomorphism between them after tensoring up to B. Equivalently, it parametrizes G-torsors $\mathcal{E}_1, \mathcal{E}_2$ over $\mathcal{Y}_S$ plus an isomorphism between their restrictions to $\mathcal{Y}_S \setminus D_S$.

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[48]This is because the Grassmannian is local in nature. On an object of global nature, e.g. a Fargues-Fontaine curve, one could not do this.

[49]Earlier we were mostly working on Y_S, where we also set $\pi \neq 0$, but for this part we want to keep the degeneration to characteristic p.

[50]The equivalence is an analogue of the "Beauville-Laszlo gluing Lemma".

23.2. **Schubert varieties.** For $S \to \mathrm{Div}^d_{\mathcal{Y}}$ any small v-stack, let $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} := \mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$.

For example, $S = \mathrm{Spa}\, C$ could be a geometric point, with $S \to \mathrm{Div}^d_{\mathcal{Y}}$ corresponding to a collection of d untilts $C_1^\#, \ldots, C_d^\#$ of C. We permit coincidences among the C_i, but removing multiplicities does not change $\mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}}$. So for the purpose of understanding the fibers, we may as well assume that the untilts are distinct. Let $\xi_1, \ldots, \xi_d \in W_{\mathcal{O}_E}(\mathcal{O}_C)$ such that $\mathcal{O}_{C_i^\#} = W_{\mathcal{O}_E}(\mathcal{O}_C)/(\xi_i)$. Write $\xi = \xi_1 \cdot \ldots \cdot \xi_d$.

Proposition 23.4. *Let $T \subset B \subset G$. Then there is a bijection*

$$X_*^+(T)^d \xrightarrow{\sim} |\,\mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}}\,|$$

sending $(\mu_1, \ldots, \mu_d) \in X_*^+(T)^d$ to the orbit of $(\mu_1(\xi_1) \ldots \mu_d(\xi_d)) \in LG(C) = G(B)$.

Remark 23.5. In fact,

$$\mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}} = \left(\prod_{i=1}^m\right)^{/S} \mathrm{Hk}_{G,C/\mathrm{Div}^1_{\mathcal{Y}}},$$

so to understand geometric fibers we can reduce to the case $d = 1$. However, unlike in the classical case the $\mathrm{Hk}_{G,C/\mathrm{Div}^1_{\mathcal{Y}}}$ are not all the same, as they depend on the map $\mathrm{Spa}\, C \to \mathrm{Div}^1_{\mathcal{Y}}$ which is given by $C_i^\#$.

We can define L^+G-orbits

$$\mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},(\mu_1,\ldots,\mu_d)} \subset \mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}}$$

which satisfy the usual closure relations

$$\mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},\leq(\mu_1,\ldots,\mu_d)} := \overline{\mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},(\mu_1,\ldots,\mu_d)}} = \bigcup_{(\mu_1',\ldots,\mu_d')\leq(\mu_1,\ldots,\mu_d)} \mathrm{Gr}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},(\mu_1',\ldots,\mu_d')}$$

where the ordering is the usual dominance order. Taking quotients by L^+G, we have also

$$\mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},(\mu_1,\ldots,\mu_d)} \subset \mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}}$$

and its closure

$$\mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}},\leq(\mu_1,\ldots,\mu_d)} \subset \mathrm{Hk}_{G,C/\mathrm{Div}^d_{\mathcal{Y}}}\,.$$

We can also define this in families, by applying the preceding definition fiberwise. Over $S = (\mathrm{Div}^1_{\mathcal{Y}})^d \to \mathrm{Div}^d_{\mathcal{Y}}$, and $\mu_1, \mu_2, \ldots, \mu_d \in X_*^+$, we can define

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},(\mu_1,\ldots,\mu_d)} \subset \mathrm{Gr}_{G,S/\mathrm{Div}_{\mathcal{Y}}}$$

and

$$\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq(\mu_1,\ldots,\mu_d)} \subset \mathrm{Gr}_{G,S/\mathrm{Div}_{\mathcal{Y}}}\,.$$

Warning 23.6. When the untilts collide, one must add the corresponding μ_i in the definition.

Proposition 23.7 ([Berk])**.** *The closed substack $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq(\mu_1,\ldots,\mu_d)} \subset \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ is proper and representable in spatial diamonds over S, of finite* dim.tr.g.

Remark 23.8. No explicit pro-étale charts are known! One can give a more explicit proof after base change to $(\mathrm{Spa}\, E)^{\diamond,d} = \mathrm{Div}^d_Y$ [Master thesis of Bence Hevesi]. However, in the generality above (incorporating the degeneration to characteristic p), the only known argument is indirect, using a version of Artin's criterion.

Proposition 23.9. *On open Schubert cells, away from the diagonals, the L^+G-action is transitive.*

This can be reduced to the case of GL_n by the Tannakian formalism. In the case of GL_n, it amounts to an explicit statement about lattices, which one analyzes directly.

Corollary 23.10. *The strata of $\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ are, away from the diagonals, of the form $[S/\text{some large group}]$. The group is an extension of a finite-dimensional cohomologically smooth group (like $G^\diamond$) plus an infinite-dimensional "unipotent group" (e.g., $\ker(L^+G \to G^\diamond)$).*

Hence, on the level of $D_{\text{ét}}$, all strata behave like Artin v-stacks (that is, you can forget about the unipotent part). Indeed, the L^+G-action on each $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq(\mu_1,\dots,\mu_d)}$ factors over a quotient $(L^+G)_{\leq(\mu_1,\dots,\mu_d)}$ which is finite-dimensional and cohomologically smooth, and the kernel is "unipotent". So

$$D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet},\Lambda) \cong D_{\text{ét}}((L^+G)_{\leq\mu_\bullet}\backslash \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet},\Lambda).$$

Definition 23.11. We define

$$D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{\mathrm{bd}} := \bigcup_{\mu} D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet},\Lambda) \subset D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu_\bullet},\Lambda).$$

The category $D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}},\Lambda)^{\mathrm{bd}}$ is monoidal under *convolution*. This is defined as follows. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hk}\times\mathrm{Hk} & \xleftarrow{\pi} & L^+G\backslash LG \times^{L^+G} LG/L^+G \\ & & \downarrow m \\ & & L^+G\backslash LG/L^+G = \mathrm{Hk} \end{array}$$

The convolution of $A, B \in D_{\text{ét}}(\mathrm{Hk},\Lambda)^{\mathrm{bd}}$ is

$$A \star B := Rm_* i^*(A \boxtimes B).$$

Remark 23.12. The map m is ind-proper, as the fibers are affine Grassmannians, so proper base change applies to ensure that this is associative.

23.3. **Perverse sheaves on** Hk**.**

Definition 23.13. Fix $S \to \mathrm{Div}^d_{\mathcal{Y}}$. Define ${}^{\mathfrak{p}}D^{\leq 0}_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu},\Lambda)^{\mathrm{bd}} \subset D(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu},\Lambda)^{\mathrm{bd}}$ to be the full subcategory of all $A \in D(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu},\Lambda)^{\mathrm{bd}}$ such that for all geometric points $\mathrm{Spa}(C,C^+) \to S$, and all $\mu_1,\dots,\mu_m \in X_*^+$ (where m is the number of distinct untilts of $\mathrm{Spa}(C,C^+)$ corresponding to $\mathrm{Spa}(C,C^+) \to S \to \mathrm{Div}^d_{\mathcal{Y}}$), we have

$$A|_{\mathrm{Hk}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu}} \in D^{\leq -d(\mu)}(\Lambda)$$

where $d(\mu) = \sum_{i=1}^m \langle 2\rho,\mu_i\rangle$. (This is the direct analogue of "relative perversity".) We define ${}^{\mathfrak{p}}D^{\geq 0}$ to be the right orthogonal of ${}^{\mathfrak{p}}D^{\leq 0}[1]$.

Theorem 23.14. *This defines a t-structure. Moreover, $A \in {}^{\mathfrak{p}}D^{\geq 0}$ if and only if for all geometric points $\mathrm{Spa}(C,C^+) \to S$ as above, $A|_{\mathrm{Hk}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu}} \in {}^{\mathfrak{p}}D^{\geq 0}$. This in turn is equivalent to the $!$-restriction to all Schubert cells $\mathrm{Hk}_{G,\mathrm{Spa}(C,C^+)/\mathrm{Div}^d_{\mathcal{Y}},\leq\mu}$ lying in $D^{\geq -d(\mu)}$. In particular, pullback under $S' \to S$ is t-exact.*

Warning 23.15. For schemes, the analogous result uses perversity of nearby cycles. However, this *fails* in p-adic geometry, without serious assumptions (e.g. Zariski constructibility). The closely related result, Artin vanishing, is also false in p-adic geometry.

Example 23.16. Take the p-adic completion of $\mathbf{A}^1_{\mathcal{O}_C}$. The generic fiber is $\mathbb{B}_C$, and the special fiber is $\mathbf{A}^1_k$. Artin vanishing would suggest that for any perverse sheaf A on $\mathbb{B}_C$, we should have $R\Gamma(\mathbb{B}_C, A) \in D^{\leq 1}$. But this fails for $A = j'_!\Lambda$ where $j' \colon \mathbb{B}'_C \hookrightarrow \mathbb{B}_C$ is the inclusion of the ball of radius $1/2$, as $R\Gamma(\mathbb{B}_C, A) = R\Gamma_c(\mathbb{B}'_C, \Lambda) = \Lambda[-2]$.

Let $i \colon \mathbf{A}^1_k \hookrightarrow \widehat{\mathbf{A}}^1_{\mathcal{O}_C}$. We claim that $R\psi(A) = i^* Rj_* A$ is the skyscraper sheaf $\Lambda[-2]$ at origin. This is because the tubular neighborhood of any point other than the origin does not intersect $\mathbb{B}'_C$.

This makes the proof of Theorem 23.14 a challenge, as the good properties of nearby cycles are used heavily in the classical situation.

Proof sketch of Theorem 23.14. We use hyperbolic localization. We first note that the Theorem is easy when $G = T$ is a torus. In this case $\mathrm{Gr}_{T,S/\operatorname{Div}^d_{\mathcal{Y}}, \leq(\mu_1,\ldots,\mu_d)} \to S$ is finite. So the t-structure is just the usual t-structure.

Consider the diagram

$$\begin{array}{ccc} & \mathrm{Gr}_B & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathrm{Gr}_G & & \mathrm{Gr}_T \end{array}$$

We define $\mathrm{CT}_B := Rp_! q^*[\langle 2\rho, \nu\rangle]$. If you set up the equivariance properly, you can show that it induces

$$\mathrm{CT}_B \colon D_{\text{ét}}(\mathrm{Hk}_G, \Lambda)^{\mathrm{bd}} \to D_{\text{ét}}(\mathrm{Hk}_T, \Lambda)^{\mathrm{bd}}$$

Lemma 23.17 (Key Lemma)**.** *The functor* CT_B *is t-exact plus conservative.*

Using Lemma 23.17, we deduce the desired result on G from the results on T.

□

Sketch of proof of Lemma 23.17. We can reduce to the case of geometric points. Then $D_{\text{ét}}(\mathrm{Hk}_G, \Lambda)^{\mathrm{bd}}$ has stratification in terms of $D_{\text{ét}}(\mathrm{Hk}_{G,(\mu_1,\ldots\mu_d)}, \Lambda)^{\mathrm{bd}}$. This is generated by $D(\Lambda)$ (as $\mathrm{Hk}_{G,(\mu_1,\ldots\mu_d)}$ is the quotient stack of a point by a connected group). So it suffices to check that $\mathrm{CT}_B({}^{\mathfrak{p}}D^{\leq 0}) \subset {}^{\mathfrak{p}}D^{\leq 0}$ and $\mathrm{CT}({}^{\mathfrak{p}}D^{\geq 0}) \subset {}^{\mathfrak{p}}D^{\geq 0}$ which are generated respectively by the standard objects $j_{\mu!}\Lambda[d_\mu]$ and the costandard objects $Rj_{\mu*}\Lambda[d_\mu]$.

We didn't explain this yet, but it turns out these sheaves are ULA on $\mathrm{Gr}_{G,S}/S$. There is a $\mathbf{G}_m$-action on $X = \mathrm{Gr}_G$ such that Gr_B is X^+ and Gr_T is X^0. Then we can apply hyperbolic localization, which preserves ULA-ness. This implies that the cohomology is locally constant, and we can reduce to the case of geometric points in characteristic p. Then we are in the setting of the Witt vector affine Grassmannian: for $S = (\operatorname{Spa} \mathbf{F}_q)^\diamond \to \operatorname{Div}^d_{\mathcal{Y}}$,

$$\mathrm{Gr}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}} = (\mathrm{Gr}^{\mathrm{Witt}}_G)^\diamond.$$

Also the six operations are compatible for schemes vs. the associated v-sheaves. This reduces to the statement for the Witt vector affine Grassmannian, in which case the result is proved by Zhu. (This comes down to the usual bounds for the dimension of MV cycles.)

The conservativity follows from examining the geometry of MV cycles. □

24. Geometric Satake, continued (Feb 1)

24.1. **Recap.** Currently we are considering a *split* reductive group $G/\mathcal{O}_E$. We choose $T \subset B \subset G$. We defined a moduli space $\mathrm{Div}^d_{\mathcal{Y}}$ of degree d Cartier divisors on $\mathcal{Y}_S$, which could be viewed as a deformed "product", "$S \dot{\times} \operatorname{Spa} \mathcal{O}_E$."

We defined a Beilinson-Drinfeld Grassmannian

$$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} \longrightarrow \mathrm{Div}^d_{\mathcal{Y}}$$

$\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ parametrizes G-torsors on $\mathcal{Y}_S$ with a meromorphic trivialization on $\mathcal{Y}_S \setminus D_S$, or equivalently by Beauville-Laszlo it parametrizes on $(\mathcal{Y}_S)^\wedge_{D_S}$ with a meromorphic trivialization on $(\mathcal{Y}_S)^\wedge_{D_S} \setminus D_S$. In particular, there is an action of L^+G on $\mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}}$ by change of trivialization.

We often want to consider families over it,

$$\begin{array}{ccc} \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{G,\mathrm{Div}^d_{\mathcal{Y}}} & \longrightarrow & \mathrm{Div}^d_{\mathcal{Y}} \end{array}$$

The local Hecke stack is $\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = L^+G \backslash \mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} = (L^+G\backslash LG/L^+G) \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$. We defined $D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$. It has a relative/$S$ perverse t-structure. In particular, we get an abelian category $\mathrm{Perv}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$, functorial in S.

24.2. **Convolution on the Satake category.** We saw that there was a convolution monoidal structure on $D_{\text{ét}}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)$. We want to show that it preserves the relative$^{/S}$ perverse heart.

Proposition 24.1. *Pullback to* Gr *induces a fully faithful funtor*

$$P(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda) \hookrightarrow D_{\acute{e}t}(\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Proof. The map $\mathrm{Gr}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}} \to \mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$ is a torsor for L^+G, which is a connected group. □

Proposition 24.2. *If $A, B \in {}^{\mathfrak{p}}D^{\leq 0}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}})^{\mathrm{bd}}$, then also $A \star B \in {}^{\mathfrak{p}}D^{\leq 0}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}})^{\mathrm{bd}}$.*

If A, B are perverse and A is flat perverse (i.e. $A \overset{L}{\otimes}_\Lambda M$ is perverse for any Λ-module M)[51] *then also $A \star B$ is perverse.*

Sketch. By our definition of the relative perverse t-structure, the statement can be proved on geometric fibers. The geometric fibers decompose as products over untilts, so we reduce to the case $d = 1$, and $A = j_{\mu_1 !}\Lambda$, $B = j_{\mu_2 !}\Lambda$.

As everything commutes with base change, we reduce to $S = \mathrm{Div}^1_{\mathcal{Y}}$. Then everything is ULA, so we can reduce to the special fiber, where it follows from work of Zhu on the Witt vector affine Grassmannian [Zhu17].

□

The above sketch will be fleshed out in the next couple subsections.

[51] The flatness is clearly necessary, just by the case where G is the trivial group.

Remark 24.3. Alternatively, we can use the fusion product, so we do not need to use the Witt vector affine Grassmannian or "semi-smallness of convolution" here.

So we get a convolution product on flat perverse sheaves.

24.3. **ULA sheaves on** $\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}$**.**

Definition 24.4. An object $A \in D_{\text{ét}}(\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ is *ULA over S* if its pullback to $\mathrm{Gr}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}$ is ULA/S.

Warning 24.5. This introduces a small asymmetry. Indeed, $\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}} = L^+G\backslash LG/L^+G$ has a switching symmetry, which this definition breaks. It turns out that the property of being ULA over S is symmetric, although this is not evident from the definition.

Proposition 24.6. *For $d=1$, $A \in D_{\acute{e}t}(\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ is ULA if and only if for all μ and $S \xrightarrow{[\mu]} \mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}$, the restriction $A|_S \in D_{\acute{e}t}(S,\Lambda)$ is locally constant with perfect fibers.*

Corollary 24.7. *The class of ULA sheaves in $D_{\acute{e}t}(\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}$ is preserved under all the six operations one can build from $\mathrm{Hk}_{G,S/\operatorname{Div}^d_{\mathcal{Y}}}$, and its strata (e.g. $Rj_*, Ri^!, \mathcal{RH}om$, ...)*

Corollary 24.8. *Consider*

$$\begin{array}{ccccc} (\operatorname{Spa} C)^\diamond & \hookrightarrow & (\operatorname{Spa} \mathcal{O}_C)^\diamond & \hookleftarrow & (\operatorname{Spa} k)^\diamond \\ & \searrow & \downarrow & \swarrow & \\ & & \operatorname{Div}^1_{\mathcal{Y}} = (\operatorname{Spa} \mathcal{O}_E)^\diamond & & \end{array}$$

This induces (perverse$^{/S}$ t-exact) equivalences

$$D^{\mathrm{ULA}}_{\acute{e}t}(\mathrm{Hk}_{G,\operatorname{Spa} C}, \Lambda)^{\mathrm{bd}}) \xleftarrow{\sim} D^{\mathrm{ULA}}_{\acute{e}t}(\mathrm{Hk}_{G,\operatorname{Spa} \mathcal{O}_C}, \Lambda)^{\mathrm{bd}} \xrightarrow{\sim} D^{\mathrm{ULA}}_{\acute{e}t}(\mathrm{Hk}_{G,\operatorname{Spa} k}, \Lambda)^{\mathrm{bd}}$$

In particular, the categories of perverse sheaves are equivalent in the respect categories.

Proof. In all three cases, the categories are built from a semi-orthogonal decomposition of sheaves on strata which are "strictly henselian", so the category coming from each stratum is just $D(\Lambda)$. This lets one match up generators, and then Ext's are matched by ULAness. □

Sketch of proof of Proposition 24.6. The key point is that $j_{\mu!}\Lambda$ is ULA. To check this we pass to the Grassmannian, and consider instead $j_\mu \colon \mathrm{Gr}_{G,\mu} \hookrightarrow \mathrm{Gr}_{G,\leq\mu}$. We can reduce to the universal case, where the base is $S = \operatorname{Div}^1_{\mathcal{Y}}$. We can then check this after pullback to the affine flag variety

$$\mathcal{F}\ell_G = LG/\mathrm{Iw} \xrightarrow{(G/B)^\diamond} \mathrm{Gr}_G = LG/L^+G.$$

The Iw-action on $\mathcal{F}\ell_G$ induces a stratification

$$\mathcal{F}\ell_G = \bigcup_{w \in \widetilde{W}} \mathcal{F}\ell_{G,w}$$

where $\widetilde{W} = N(T)(\mathrm{B_{dR}})/T(\mathrm{B}^+_{\mathrm{dR}})$ is the extended affine Weyl group. So there is a short exact sequence

$$1 \to \underbrace{X_*(T)}_{T(\mathrm{B_{dR}})/T(\mathrm{B}^+_{\mathrm{dR}})} \to \widetilde{W} \to W \to 1.$$

The advantage of the $\mathcal{F}\ell_{G,w}$ is that they have nice resolution of singularities. So we can reduce instead to $j_{w!}\Lambda$ where $j_w \colon \mathcal{F}\ell_{G,w} \hookrightarrow \mathcal{F}\ell_G$.

Then the key point is that there exists a Demazure-Bott-Samelson resolution

$$\begin{array}{ccc} & \widetilde{\mathcal{F}\ell}_{G,\dot{w}} & \\ {}^{\widetilde{j}_w}\nearrow & & \searrow \\ \mathcal{F}\ell_{G,w} & \xhookrightarrow{j_w} & \mathcal{F}\ell_{G,\leq w} \end{array}$$

for $w = s_{i_1} \ldots s_{i_l}\omega$ where $l = \text{length}(w)$, $\omega \in \Omega \subset \widetilde{W}$. The Ω parametrizes closed orbits (so $\mathcal{F}\ell_{G,\omega}$ is a point for $\omega \in \Omega$). We have

$$\widetilde{\mathcal{F}\ell}_{G,\dot{w}} = \mathcal{P}_{s_{i_1}} \overset{\text{Iw}}{\times} \mathcal{P}_{s_{i_2}} \overset{\text{Iw}}{\times} \ldots \overset{\text{Iw}}{\times} \mathcal{P}_{s_{i_l}} / \,\text{Iw}$$

and each $\mathcal{P}_{s_{i_j}}/\,\text{Iw} \cong (\mathbf{P}^1)^\diamond$, so that $\widetilde{\mathcal{F}\ell}_{G,\dot{w}}$ is an iterated $(\mathbf{P}^1)^\diamond$-bundle. Here $\mathcal{P}_s \subset L^+G$ is the parahoric subgroup corresponding to the simple affine reflection s.

How does the logic work? We define the parahoric groups by hand, and check that the resulting Demazure-Bott-Samelson resolution are iterated $(\mathbf{P}^1)^\diamond$-bundles. To identify the image in the affine flag variety, we can check on geometric points, where B_{dR} becomes a Laurent series ring over $\mathbf{C}_p$ (in characteristic 0). To check the existence of the lift $\widetilde{j}_w$, we can also check a certain map is an isomorphism on points.

Since proper pushforward preserves ULA, it is enough to show that $\widetilde{j}_{w!}\Lambda$ is ULA on $\widetilde{\mathcal{F}\ell}_{G,\dot{w}}$. Now $\widetilde{\mathcal{F}\ell}_{G,\dot{w}}$ is smooth, and its boundary in $\widetilde{\mathcal{F}\ell}_G$ behaves like a normal crossings divisor. The boundary is composed of similar cells obtained by removing factors from $\dot{w}$. So that gives a resolution of $\widetilde{j}_{w!}\Lambda$ by constant sheaves on strata, all of which are smooth. □

Proposition 24.9. *Let $S \to \mathrm{Div}^d_{\mathcal{Y}}$ be arbitrary. Consider the constant term functor*

$$\mathrm{CT}_B \colon D_{\acute{e}t}(\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}} \to D_{\acute{e}t}(\mathrm{Hk}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}}, \Lambda)^{\mathrm{bd}}.$$

Then A is ULA if and only if $\mathrm{CT}_B(A)$ is ULA, if and only if for $\pi_T \colon \mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}} \to S$, $R\pi_{T*}\,\mathrm{CT}_B(A) \in D_{\acute{e}t}(S,\Lambda)$ is locally constant with perfect values.

Remark 24.10. The functor CT_B could be interpreted as hyperbolic localization for a certain $\mathbf{G}_m$-action on $\mathrm{Hk}_{G,S/\mathrm{Div}^d_{\mathcal{Y}}}$, and hyperbolic localization preserves ULA-ness in general. The other thing being used here is that $\mathrm{Gr}_{T,S/\mathrm{Div}^d_{\mathcal{Y}}} \to S$ is actually finite, and for *finite* maps ULA can be checked after pushforward to the base.

24.4. **Back to proof of Proposition 24.2.** We want to show that

$$j_{\mu_1!}\Lambda \star j_{\mu_2!}\Lambda \in {}^{\mathfrak{p}}D^{\leq 0}(\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda).$$

Consider the convolution Grassmannian $\widetilde{\mathrm{Gr}}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2} \xrightarrow{\pi_2} \mathrm{Gr}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2}$ parametrizing $(\mathcal{E}_1, \mathcal{E}_2, S_1^\#, S_2^\#)$, a trivialization of $\mathcal{E}_1$ away from $S_1^\#$, and an isomorphism between $\mathcal{E}_1$ and $\mathcal{E}_2$ away from $S_2^\#$. The map π_2 forgets $\mathcal{E}_1$ and composes the two isomorphisms. It is an isomorphism away from the diagonal. Over the diagonal, it restricts to the convolution affine Grassmannian

$$\widetilde{\mathrm{Gr}}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \xrightarrow{\pi} \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}}$$

We want $R\pi_*(j_{\mu_1!}\Lambda \widetilde{\boxtimes} j_{\mu_2!}\Lambda) \in {}^{\mathfrak{p}}D^{\leq 0}(\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}})$. This globalizes to $R\pi_{2!}(j_{\mu_1!}\Lambda \widetilde{\boxtimes} j_{\mu_2!}\Lambda)$. Now, $j_{\mu_1!}\Lambda \widetilde{\boxtimes} j_{\mu_2!}\Lambda$ is ULA on $\widetilde{\mathrm{Gr}}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2}$. So the (proper) pushforward is ULA on $\mathrm{Gr}_{G,(\mathrm{Div}^1_{\mathcal{Y}})^2}$. We want it to be in ${}^{\mathfrak{p}}D^{\leq 0}$. This can be checked after applying the t-exact functor $R\pi_{T*}\,\mathrm{CT}_B[\deg]$.

The result is in $D_{\text{ét}}((\mathrm{Div}^1_{\mathcal{Y}})^2, \Lambda)$, and locally constant by ULAness. Away from the diagonal, it is just the tensor product by Künneth, namely $R\pi_{T*}\,\mathrm{CT}_B(j_{\mu_1!}\Lambda) \overset{L}{\otimes} R\pi_{T*}\,\mathrm{CT}_B(j_{\mu 2!}\Lambda) \in D^{\leq 0}((\mathrm{Div}^1_{\mathcal{Y}})^2, \Lambda)$. As the complement of the diagonal is dense, the same is true on the diagonal.

Remark 24.11. Here we used the idea of "fusion", reinterpreting the convolution in terms of something more symmetric.

24.5. **Satake category.** From now on, we work again over $(\mathrm{Div}^1_X)^d$. Note that we replaced $\mathcal{Y}$ by X here; in particular, it lives in characteristic 0.

Let G/E be any reductive group.

We will use the previous results via implicit étale localization to reduce to the split case.

Definition 24.12. Let I be a finite set, and Λ any ring killed by some n prime to p. We define the *Satake category*

$$\mathrm{Sat}^I_G(\Lambda) := \mathrm{Perv}^{\mathrm{ULA}}_{\mathrm{flat}}(\underbrace{\mathrm{Hk}_{G,(\mathrm{Div}^1_X)^I}}_{\text{“}\mathrm{Hk}^I_G\text{”}}, \Lambda),$$

the full subcategory of flat perverse sheaves A on Hk^I_G that are ULA, with fiber functor

$$F\colon\ \mathrm{Sat}^I_G(\Lambda) \to \mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \overset{\mathrm{Thm}}{\cong} \mathrm{Rep}_{W^I_E}(\Lambda)$$

given by $A \mapsto \bigoplus_{i\in\mathbf{Z}} R^i\pi_{G*}A$, where $\pi_G\colon\ \underbrace{\mathrm{Gr}_{G,(\mathrm{Div}^1_X)^I}}_{\mathrm{Gr}^I_G} \to (\mathrm{Div}^1_X)^I$.

Remark 24.13. If G is split, using hyperbolic localization one can show that $F(A) = R\pi_{T*}\,\mathrm{CT}_B(A)$. We are using here that the spectral sequence coming from the hyperbolic localization filtration (cf. Theorem 22.19) on $F(A)$ degenerates, by concentration of degree.

Why does $R\pi_{G*}$ send $\mathrm{Sat}^I_G(\Lambda)$ to $\mathrm{LocSys}(\mathrm{Div}^1_X)^I, \Lambda)$, and why is it exact (and conservative)? We can check this étale locally, so reduce to the case where G is split. Then we can apply the hyperbolic localization description of Remark 24.13. We know the constant term functor is t-exact and preserves ULAness. So $F(A)$ is a perfect complex concentrated in degree 0 and flat.

Remark 24.14. The Theorem identifying $\mathrm{LocSys}((\mathrm{Div}^1_X)^I, \Lambda) \cong \mathrm{Rep}_{W^I_E}(\Lambda)$ is a version of Drinfeld's Lemma. This is quite subtle when Λ is not finite.

24.6. **Fusion product.** We will define a functor

$$\mathrm{Sat}^{I_1}_G(\Lambda) \times \ldots \times \mathrm{Sat}^{I_m}_G(\Lambda) \to \mathrm{Sat}^{I_1 \sqcup \ldots \sqcup I_m}_G(\Lambda) \tag{24.6.1}$$

fitting into a diagram

$$\begin{array}{ccc} *\colon\ \mathrm{Sat}^{I_1}_G(\Lambda) \times \ldots \times \mathrm{Sat}^{I_m}_G(\Lambda) & & \\ \downarrow & \searrow & \\ \mathrm{Sat}^{I_1\sqcup\ldots\sqcup I_m}_G(\Lambda) & \hookrightarrow & \mathrm{Sat}^{I_1,\ldots,I_m}_G(\Lambda) \end{array} \tag{24.6.2}$$

where $\mathrm{Sat}^{I_1,\ldots,I_m}_G(\Lambda)$ is defined like $\mathrm{Sat}^{I_1\sqcup\ldots\sqcup I_m}_G(\Lambda)$, but away from the locus in $(\mathrm{Div}^1_X)^I$ where $x_i = x_j$ when i, j lie in different I_k's (these are partial diagonals). The restriction of

$\mathrm{Gr}_G^{I_1 \sqcup \ldots \sqcup I_m}$ to this locus factors as $\mathrm{Gr}_G^{I_1} \times \ldots \times \mathrm{Gr}_G^{I_m}$. So the diagonal arrow in (24.6.2) is just exterior tensor product.

The restriction from $\mathrm{Sat}_G^{I_1 \sqcup \ldots \sqcup I_m}(\Lambda)$ to $\mathrm{Sat}_G^{I_1, \ldots, I_m}(\Lambda)$ is fully faithful. You can check this by applying hyperbolic localization, and thereby reducing to the statement that the category of local systems on $(\mathrm{Div}_X^1)^I$ embeds fully faithfully into the category of local systems on the complement of the partial diagonals. This is easy.

To construct (24.6.1), it suffices to show that the essential image of the diagonal arrow lies in the subcategory $\mathrm{Sat}_G^{I_1 \sqcup \ldots \sqcup I_m}(\Lambda)$. For this it suffices to produce a sheaf with the correct restriction properties. This can be done using the convolution affine Grassmannian. This also implies that convolution = fusion.

Composing (24.6.1) with restriction to the diagonal, we now have a *fusion product*

$$*\colon \mathrm{Sat}_G^I(\Lambda) \times \ldots \times \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_G^{I \sqcup \ldots \sqcup I}(\Lambda) \xrightarrow{\Delta^*} \mathrm{Sat}_G^I(\Lambda).$$

This turns each $\mathrm{Sat}_G^I(\Lambda)$ into a symmetric monoidal category. Moreover, the fusion product commutes with the convolution product. Now, whenever you have a symmetric monoidal structure commuting with the monoidal structure, they must coincide (Eckmann-Hilton argument), compatibly with the rigid monoidal functor F.

Proposition 24.15. *The functor $F = \oplus R^i \pi_{G*}$ is a symmetric monoidal functor*[52]

$$\mathrm{Sat}_G^I(\Lambda) \to \mathrm{Rep}_{W_E^I}(\Lambda).$$

Theorem 24.16. *The functor $F\colon \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Rep}_{W_E^I}(\Lambda)$ satisfies all required properties for Tannakian reconstruction, hence there exists a Hopf algebra $\mathcal{H} \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda)$ such that*

$$\mathrm{Sat}_G^I(\Lambda) = \mathrm{CoMod}_{\mathcal{H}}(\mathrm{Rep}_{W_E^I}(\Lambda)).$$

The statement would be clear in characteristic 0. Integrally, it requires a bit of work. We need to use that $\mathrm{Sat}_G^I(\Lambda)$ is a highest weight category, which includes a non-trivial statement about the vanishing of the Ext^2 group.

Proposition 24.17. *We have $\mathcal{H}^I \cong \bigotimes_{i \in I} \mathcal{H}^{\{i\}}$, and $\mathcal{H}^{\{1\}}$ corresponds to an affine group scheme $\check{G}/\Lambda$ with continuous W_E-action.*

Theorem 24.18. *There exists a canonical isomorphism $\check{G} \cong \widehat{G}$, which is W_E-equivariant if the pinning of $\widehat{G}$ includes a cyclotomic twist, e.g. $\mathrm{Lie}\,\widehat{U}_a \cong \Lambda(1)$, to be elaborated upon next time.*

Corollary 24.19. $\mathrm{Sat}_G(\Lambda) \cong \mathrm{Rep}(\widehat{G})$, *and* $\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}(\widehat{G}^I)$ *internally in* $\mathrm{Rep}_{W_E^I}(\Lambda)$.

[52]One needs to insert some signs in the commutativity constraints. The point is that $R^i \pi_{G*}$ should be in degree i.

25. Geometric Satake, finale (Feb 5)

25.1. **Recap.** Let E be a nonarchimedean local field, with residue field $\mathbf{F}_q \subset \overline{\mathbf{F}}_q$, $\breve{E}$ as usual. Let G/E be any reductive group.

We defined the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_G^I \to (\mathrm{Div}_X^1)^I$, where $\mathrm{Div}_X^1 = (\mathrm{Spa}\,\breve{E})^\diamond/\varphi^{\mathbf{Z}}$ is a v-sheaf on $\mathrm{Perf}_{\overline{\mathbf{F}}_q}$. There is an action of $(L^+G)^I$ on Gr_G^I, and we defined $\mathrm{Hk}_G^I = (L^+G)^I \backslash \mathrm{Gr}_G^I$ over $(\mathrm{Div}_X^1)^I$. We also defined a notion of perverse and ULA sheaves.

Definition 25.1 (Satake category)**.** Let Λ be a ring such that $n\Lambda = 0$ for some $(n,p)=1$. Then

$$\mathrm{Sat}_G^I(\Lambda) := \mathrm{Perv}_{\mathrm{flat}}^{\mathrm{ULA}}(\mathrm{Hk}_G^I, \Lambda)$$

with the fiber functor

$$F^I = \bigoplus_i R^i\pi_{G*} \colon \mathrm{Sat}_G^I(\Lambda) \to \mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda) \cong \mathrm{Rep}_{W_E^I}(\Lambda).$$ [53]

Remark 25.2. There is a "better" description of F^I, at least G has a Borel subgroup B. (In general, one uses descent to bootstrap from this case.) We have the constant term functor

$$\mathrm{CT}_B = Rp_! q^*[\deg] \colon \mathrm{Sat}_G^I(\Lambda) \to \mathrm{Sat}_T^I(\Lambda)$$

where p, q are as in the diagram

$$\mathrm{Gr}_G^I \xleftarrow{\ q\ } \mathrm{Gr}_B^I \xrightarrow{\ p\ } \mathrm{Gr}_T^I$$

This functor turns out to be independent of B. Indeed, varying B, we get a functor to

$$\mathrm{LocSys}((\mathrm{Div}_X^1)^I \times \mathrm{Fl}_G^\diamond, \Lambda)$$

but as the flag variety $\mathrm{Fl}_G^\diamond$ is simply connected, all such local systems are pulled back from the base $\mathrm{LocSys}((\mathrm{Div}_X^1)^I, \Lambda)$. In particular, CT_B descends to all G. We have

$$\pi_{T*}\,\mathrm{CT}_B \cong \bigoplus R^i\pi_{G*} = F^I$$

by hyperbolic localization.

The fusion product makes Sat_G^I symmetric monoidal, and $F^I \colon \mathrm{Sat}_G^I \to \mathrm{Rep}_{W_E^I}(\Lambda)$ is a symmetric monoidal fiber functor. A tiny bit more work shows that there exists a Hopf algebra $\mathcal{H}^I \in \mathrm{Ind}\,\mathrm{Rep}_{W_E^I}(\Lambda)$ such that $\mathrm{Sat}_G^I \cong \mathrm{CoMod}_{\mathcal{H}^I}(\mathrm{Rep}_{W_E^I}(\Lambda))$.

The "Künneth formula" implies that $\mathcal{H}^I \cong \bigotimes_{i\in I} \mathcal{H}^{\{i\}}$, so it suffices to determine $\mathcal{H}^{\{1\}}$. It is equivalent to the data of an affine flat group scheme $\check{G}_\Lambda/\Lambda$ plus a continuous W_E-action.

The formation of $\mathcal{H}^{\{1\}}$ is compatible with base change in Λ, so we may assume that $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$.

Theorem 25.3. *Let $\widehat{G}$ be the Langlands dual group (to be explained). There is a canonical isomorphism $\check{G}_{\mathbf{Z}/\ell^n\mathbf{Z}} \cong \widehat{G}_{\mathbf{Z}/\ell^n\mathbf{Z}}$, which is W_E-equivariant if the W_E-action on $\widehat{G}$ has a cyclotomic twist (to be explained).*

Remark 25.4. Notationally, we distinguish between the geometrically constructed group $\check{G}$ coming from Geometric Satake, and the abstractly constructed dual group $\widehat{G}$.

[53] By LocSys and Rep, we mean on finite projective Λ-modules.

Corollary 25.5. *We have a canonical equivalence* $\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}_{\mathrm{Rep}_{W_E^I}(\Lambda)}(\widehat{G}_\Lambda^I)$, *commuting with the respective forgetful functors to* $\mathrm{Rep}_{W_E^I}(\Lambda)$.

In other words, $\mathrm{Sat}_G^I(\Lambda) \cong \mathrm{Rep}((\widehat{G}_\Lambda \rtimes W_E)^I)$ *once the latter is appropriately defined.*

Remark 25.6. That the full L-group arises from Geometric Satake over non algebraically closed fields was first observed by Timo Richarz and Xinwen Zhu.

25.2. **The dual group.** Let G/E (or any field).

25.2.1. *Universal Cartan.* There is a "universal Cartan" T of G. This can be thought of as follows: there is a flag variety Fl_G/E parametrizing Borel subgroups $B \subset G$, and each B has a torus quotient T. This family corresponds to a $\mathbf{Z}$-local system $X^*(T)$ over Fl. (It is a general fact that tori over a scheme correspond to $\mathbf{Z}$-local systems, by associating the character group.)

Now, since the geometric flag variety is simply connected, any $\mathbf{Z}$-local system on Fl_G is pulled back from the base, which in this case is $(\mathrm{Spec}\ E)_{\text{ét}}$. So the family of T arises uniquely by pullback from a torus T/E.

In particular, we have an action of $\mathrm{Gal}(\overline{E}/E)$ on $X^*(T_{\overline{E}})$. So from G/E, we get a finite free $\mathbf{Z}$-module X^* equipped with a $\mathrm{Gal}(\overline{E}/E)$-action.

25.2.2. *Root datum.* But in fact we get a bit more. Indeed, the universal Cartan is identified with the quotient of any Borel. So it comes with a canonical notion of dominant cocharacters X^*_+. This is completely canonical, hence stable by the action of $\mathrm{Gal}(\overline{E}/E)$. There is also an action of the Weyl group W on X^*.

As we have dominant cocharacters, hence a dominant Weyl chamber, we also get a set of simple reflections $S \subset W$ preserved by the $\mathrm{Gal}(\overline{E}/E)$-action. For any simple reflection $s \in S$, we have:

- A simple root $\alpha_s \in X^*$. Given a Borel B, we get a root space $U_{\alpha_s} \subseteq G_{\overline{E}}$.
- A simple coroot $\alpha_s^\vee \in X_* := (X^*)^\vee$. Given a Borel B, we can think of this coming from a map $\mathrm{SL}_2 \to G_{\overline{E}}$ corresponding to s.

So, independently of *any* choice, we have a root datum $(X^*, X_*, \Phi, \Phi^\vee, X^*_+, X_*^+)$ with an action of $\mathrm{Gal}(\overline{E}/E)$.

25.2.3. *Langlands dual group.* Now, Langlands made the observation that exchanging X^* and X_* also gives a root datum. The functor from (reductive groups) to (root data) has a canonical splitting, given by Chevalley group schemes. We let $\widehat{G}/\mathbf{Z}$ be the Chevalley group scheme corresponding to $(X_*, X^*, \Phi^\vee, \Phi, X_*^+, X^*_+)$. Since the Chevalley group scheme was a functorial construction, this inherits the action of $\mathrm{Gal}(\overline{E}/E)$. This is the "dual group" of G.

Note that $\widehat{G}$ comes with canonical $\widehat{T} \subset \widehat{B} \subset \widehat{G}$, which are in particular stable under the $\mathrm{Gal}(\overline{E}/E)$-action. We also have canonical identifications $X^*(\widehat{T}) = X_*$, and as part of the Chevalley construction, for any simple reflection $s \in S$ we have isomorphisms $\psi_s \colon \mathrm{Lie}\,\widehat{U}_{\check{\alpha}_s} \cong \mathbf{Z}$ which are $\mathrm{Gal}(\overline{E}/E)$-invariant.

25.3. **Cyclotomic twists.** We will now work over $\mathbf{Z}_\ell$ or $\mathbf{Z}/\ell^n\mathbf{Z}$ instead of $\mathbf{Z}$, and then replace ψ_s with isomorphisms $\psi_s' \colon \mathrm{Lie}\,\widehat{U}_{\check{\alpha}_s} \cong \mathbf{Z}_\ell(1)$. This is a Tate twist representation of $\mathrm{Gal}(\overline{E}/E)$. This is the same abstract group, but we changed the W_E-action.

To prove Theorem 25.3 we need to find $\check{T} \subset \check{B} \subset \check{G}$, which are W_E-stable, such that:

- $X^*(\check{T}) = X_*$.
- $\check{G}$ is reductive of the correct type.

- We have isomorphisms $\operatorname{Lie} \check{U}_{\check{\alpha}_s} \cong \mathbf{Z}/\ell^n\mathbf{Z}(1)$.

25.4. **Proof of Theorem 25.3.** Since we will produce all the requisite structure canonically, we don't really need to keep track of the Galois equivariance. We are free to make a base extension and assume G is split, as long as our constructions are independent of the splitting. The independence is proved as above, by phrasing things in terms of local systems on the flag variety, and then using that it is simply connected. Since we have already seen this argument several times, we will suppress it from now on.

25.4.1. *The case of tori.* We have

$$\mathrm{Sat}_T \cong \mathrm{Perv}^{\mathrm{ULA}}_{\mathrm{flat}}(\mathrm{Hk}_T).$$

As

$$\mathrm{Hk}_T = \bigcup_{X_*(T)} [\mathrm{Div}^1_X / L^+T],$$

we see explicitly that

$$\mathrm{Perv}^{\mathrm{ULA}}_{\mathrm{flat}}(\mathrm{Hk}_T) = \bigoplus_{X_*(T)} \mathrm{Rep}_{W_E}(\Lambda).$$

Now, the category $\bigoplus_{X_*(T)} \mathrm{Rep}_{W_E}(\Lambda)$ is canonically identified with $\mathrm{Rep}(\check{T})$ where $\check{T}$ is the torus with $X^*(\check{T}) = X_*(T)$. So we have a canonical identification

$$\mathrm{Sat}_T \cong \mathrm{Rep}(\check{T}).$$

25.4.2. The functor $\mathrm{CT}_B \colon \mathrm{Sat}_G \to \mathrm{Sat}_T$ is symmetric monoidal and commutes with the respective fiber functors. This induces a map $\check{T} \to \check{G}$. You check that it is injective by looking at the standard objects $j_{\mu!}\Lambda$, computing the fiber functor F and observing that the highest weight space is 1-dimensional.

Furthermore, the cohomological $\mathbf{Z}$-grading on the fiber functor $F = \bigoplus_i R^i\pi_{G*}$ gives a map $\mathbf{G}_m \to \check{G}$, and you can check that it canonically factors as

$$\begin{array}{ccc} \check{T} & \longrightarrow & \check{G} \\ & \nwarrow \quad \nearrow & \\ & \mathbf{G}_m & \end{array}$$

(This comes down to the fact that the cohomological grading coincides with the degree shifts in the definition of CT.) The $\mathbf{G}_m$ defines an attracting "parabolic" $\check{T} \subset \check{B} \subset \check{G}$ (but a priori we don't know that $\check{B}$ is a parabolic group).

25.4.3. *Rank 1 groups.* We now suppose that G has rank 1, so $G_{\mathrm{ad}} \cong \mathrm{PGL}_2$. The map $G \twoheadrightarrow G_{\mathrm{ad}}$ induces

$$\mathrm{Gr}_G = \mathrm{Gr}_{G_{\mathrm{ad}}} \times_{\mathbf{Z}/2\mathbf{Z}} \pi_1(G).$$

(In particular, there is no difference between Gr_G and $\mathrm{Gr}_{G_{\mathrm{ad}}}$ on connected components.) Hence $\check{G} \cong \check{G}_{\mathrm{ad}} \times^{\mu_2} \check{Z}$, where $\check{Z}$ is the torus with $X^*(\check{Z}) = \pi_1(G)$.

Remark 25.7. In fact, the same trick allows to reduce from G to G_{ad} in general.

Anyway, we have reduced to analyzing $G = \mathrm{PGL}_2$. This has a minuscule cocharacter μ, and $\mathrm{Gr}_{\mathrm{PGL}_2,\mu} \cong (\mathbf{P}^1)^\diamond$. Then $j_{\mu*}\Lambda[1] \in \mathrm{Sat}_G(\Lambda) = \mathrm{Rep}(\check{G})$. So

$$F(A) = H^0(\mathbf{P}^1, \Lambda) \oplus H^2(\mathbf{P}^1, \Lambda) = \Lambda \oplus \Lambda(-1).$$

There is a tautological map $\check{G} \to \mathrm{Aut}(F(A)) = \mathrm{GL}_2/\Lambda$.

We know that over $\mathbf{Q}_\ell$ and $\mathbf{F}_\ell$,

$$\mathrm{IrrRep}(\check{G}) \leftrightarrow X_*^+ = \mathbf{Z}_{\geq 0}$$

with $\lambda \in X_*^+$ corresponding to IC_λ. Here, we define

$$\mathrm{Sat}_G(\mathbf{Z}_\ell) := \varprojlim_n \mathrm{Sat}_G(\mathbf{Z}/\ell^n \mathbf{Z})$$

and

$$\mathrm{Sat}_G(\mathbf{Q}_\ell) := \mathrm{Sat}_G(\mathbf{Z}_\ell)[1/\ell].$$

We also know that $\mathrm{Sat}_G(\mathbf{Q}_\ell)$ is *semi-simple* (all objects are direct sums of IC_λ), via comparison to the Witt vector affine Grassmannian. This semi-simplicity implies that $\check{G}_{\mathbf{Q}_\ell}$ is reductive (including connected, by checking that the category of representations has no proper sub tensor category, by classification of the irreducibles), with rank 1 (again by the classification of the irreducibles). This means that the map from $\check{G}_{\mathbf{Q}_\ell}$ to GL_2 must factor over $\check{G} \to \mathrm{SL}_2 \subset \mathrm{GL}_2$ over $\mathbf{Q}_\ell$, and therefore also over $\mathbf{Z}_\ell$. By examining what happens on the torus, which is dictated by the grading on $F(A)$,

$$\begin{array}{ccc} \check{G} & \longrightarrow & \mathrm{SL}_2 \\ & \nwarrow \quad \nearrow & \\ & \mathbf{G}_m & \end{array}$$

we see that $\check{G} \to \mathrm{SL}_2$ is an isomorphism over $\mathbf{Q}_\ell$.

Over $\mathbf{F}_\ell$, $\check{G}_{\mathbf{F}_\ell} \to \mathrm{SL}_{2,\mathbf{F}_\ell}$ must be surjective, as otherwise the image is some subgroup containing the torus, for which the only other options are the torus or Borel or normalizer of a torus, all of which possibilities would lead to too many irreducible representations.

Hence the map $\mathcal{O}(\mathrm{SL}_2) \to \mathcal{O}(\check{G})$ over $\mathbf{Z}_\ell$ is an isomorphism in characteristic 0, both are flat over $\mathbf{Z}_\ell$, and it is injective mod ℓ. Then it is an isomorphism by general algebra. (We need to check that the map is surjective; any element lies in the image after multiplying by ℓ^n, but the reduction mod ℓ is 0...)

So $\check{G} \xrightarrow{\sim} \mathrm{SL}_2(F(A)) = \mathrm{SL}_2(\Lambda \oplus \Lambda(-1))$. This is how you see the cyclotomic twist in the root group.

25.5. **General G.** The CT_P exists for any parabolic $P \twoheadrightarrow M$. This defines a symmetric monoidal functor

$$\begin{array}{ccccc} \mathrm{Sat}_G & \xrightarrow{\mathrm{CT}_P} & \mathrm{Sat}_M & \xrightarrow{\mathrm{CT}_{B_M}} & \mathrm{Sat}_T \\ & \searrow & \downarrow & \swarrow & \\ & & \mathrm{Rep}_{W_E} & & \end{array}$$

Use this for minimal parabolics, so that M has rank 1. Then we know

$$\begin{array}{ccccc} \check{T} & \hookrightarrow & \check{M} & \hookrightarrow & \check{G} \\ \| & & \| & & \\ \widehat{T} & & \widehat{M} & & \end{array}$$

In particular, for any simple reflection s we get

$$\mathbf{G}_m \hookrightarrow \mathrm{SL}_2 \longrightarrow \check{G}, \qquad \mathbf{G}_m \xrightarrow{\alpha_s \in X^* = X_*(\check{T})} \check{G} \tag{25.5.1}$$

At this point we know that $\check{G}_{\mathbf{Q}_\ell}$ is connected and reductive, and has at least as many roots and coroots as $\widehat{G}$. We need to check that it does not have more than these. For this we use a combinatorial argument, looking at the weights appearing in $F(\mathrm{IC}_\lambda)$, i.e. the weights appearing in the corresponding irreducible representation of $\check{G}$. The number is already the same as what one would expect for the representation of $\widehat{G}$ with highest weight λ. This shows that $\check{G}_{\mathbf{Q}_\ell} \cong \widehat{G}_{\mathbf{Q}_\ell}$ canonically, as the simple root subgroups are pinned by (25.5.1).

We also have $\widehat{M} \to \check{G}$ defined integrally. In particular, $\check{G}(\check{\mathbf{Z}}_\ell) \subset \check{G}(\check{\mathbf{Q}}_\ell) \cong \widehat{G}(\check{\mathbf{Q}}_\ell)$ contains $\widehat{M}(\check{\mathbf{Z}}_\ell) \subset \widehat{G}(\check{\mathbf{Q}}_\ell)$ for any minimal Levi. These generate all of $\widehat{G}(\check{\mathbf{Z}}_\ell)$, so $\widehat{G}(\check{\mathbf{Z}}_\ell)$ must be contained in $\check{G}(\check{\mathbf{Z}}_\ell) \subset \widehat{G}(\check{\mathbf{Q}}_\ell)$. But as $\check{G}(\check{\mathbf{Z}}_\ell)$ is bounded and $\widehat{G}(\check{\mathbf{Z}}_\ell)$ is a hyperspecial maximal compact subgroup, we must then we have $\widehat{G}(\check{\mathbf{Z}}_\ell) = \check{G}(\check{\mathbf{Z}}_\ell)$. A bit more work shows that $\check{G} = \widehat{G}$ as integrals models of $\check{G}_{\mathbf{Q}_\ell} \cong \widehat{G}_{\mathbf{Q}_\ell}$. Namely, we use:

Lemma 25.8 (Prasad-Yu)**.** *Let H be reductive over $\mathbf{Z}_\ell$, H' an affine flat group scheme of finite type over $\mathbf{Z}_\ell$, $\rho\colon H \to H'$ a homomorphism that is a closed immersion in the generic fiber. Assume that $\ell \neq 2$ or that no almost simple factor of $H_{\overline{\mathbf{Q}}_\ell}$ is isomorphic to* SO_{2n+1} *(this is satisfied if e.g., the derived group of H is simply connected). Then ρ is a closed embedding.*

How to apply this? We can assume G is adjoint, so $\widehat{G}$ is simply connected. (This means we don't meet the forbidden case in the Lemma.) Pick a representation $\check{G} \to \mathrm{GL}_N$ that is a closed immersion on the generic fiber. (One enemy to keep in mind here is that we don't know a priori that $\check{G}$ is of finite type. Embedding into GL_N removes this concern.) Then we get $\widehat{G}_{\mathbf{Q}_\ell} \cong \check{G}_{\mathbf{Q}_\ell} \hookrightarrow \mathrm{GL}_{N,\mathbf{Q}_\ell}$ and $\widehat{G}(\check{\mathbf{Z}}_\ell) \cong \check{G}(\check{\mathbf{Z}}_\ell) \subset \mathrm{GL}_N(\check{\mathbf{Z}}_\ell)$. Since $\widehat{G}$ and GL_N are *smooth* group schemes, we can check the integral structure on points, and therefore we get a map $\widehat{G} \to \mathrm{GL}_N / \mathbf{Z}_\ell$. Then the Lemma implies that it is a closed immersion. So now we have

$$\begin{array}{ccc} \check{G} & \longrightarrow & \mathrm{GL}_N \\ & {\scriptstyle \exists}\searrow & \nearrow \\ & \widehat{G} & \end{array}$$

The map extends over $\check{G} \to \widehat{G}$ over $\mathbf{Q}_\ell$, so it extends integrally as well (by flatness). Now we run the same argument as in the rank 1 case for SL_2: it's an isomorphism on the generic fiber, surjective over $\mathbf{F}_\ell$, so integrally an isomorphism over $\mathbf{Z}_\ell$. ☐

26. L-parameters (Feb 8)

26.1. **Setup.** Let E be a non-archimedean local field, G/E a reductive group.

The Local Langlands correspondence predicts that there is a canonical map

$$\{\text{irred. smooth } G(E)\text{-rep.}\} \to \{L\text{-parameters}\}$$

which we denote $\pi \mapsto \varphi_\pi$.

Usually you would set this up with $\mathbf{C}$-coefficients. Note that there is a "canonical" $\sqrt{q} \in \mathbf{C}$, and the choice of it is implicit in the Local Langlands correspondence – over a general algebraically closed field, one must make this choice to get the correspondence.

26.2. **The L-group.** For G/E, we have a dual group $\widehat{G}/\mathbf{Z}$ as discussed last time. It has an action of $\mathrm{Gal}(\overline{E}/E)$, which factors over a finite quotient Q.

Definition 26.1. The Langlands L-group is ${}^LG := \widehat{G} \rtimes Q$, viewed as an algebraic group over $\mathbf{Z}$.

Remark 26.2. The definition of ${}^LG$ depends on the choice of Q, although what is used about it in practice is often independent. One could try to work with $\widehat{G} \rtimes \mathrm{Gal}(\overline{E}/E)$ which is independent of any choice, but then it is more challenging to articulate what kind of object this is (e.g., it is not an algebraic group).

Definition 26.3 (L-parameters, Take 1)**.** An *L-parameter* is a continuous map $W_E \to {}^LG(\mathbf{C})$ making the diagram below commute:

$$\begin{array}{ccc} W_E & \longrightarrow & {}^LG(\mathbf{C}) \\ & \searrow \quad \swarrow & \\ & Q & \end{array}$$

Equivalently, it is a continuous 1-cocycle $W_E \to \widehat{G}(\mathbf{C})$.

Remark 26.4. The continuity condition is equivalent to asking that the cocycle factors over a discrete quotient W_E/I' where $I' \subset I_E$ is an open finite index subgroup (because the topology of the $\widehat{G}(\mathbf{C})$ is incompatible with the topology of I_E).

Deligne observed that it is better to also keep track of a monodromy operator.

Definition 26.5 (L-parameters, Take 2)**.** An *L-parameter over* $\mathbf{C}$ is a pair (φ, N) where $\varphi\colon W_E \to {}^LG(\mathbf{C})$ is as in Definition 26.3, and $N \in \mathrm{Lie}\,\widehat{\mathfrak{g}} \otimes \mathbf{C}$ such that for all $w \in W_E$, $\mathrm{Ad}(\varphi(w))N = q^{|w|}N$.[54]

For $G = \mathrm{GL}_n$, these are also called *Weil-Deligne representations.*

Definition 26.6 (L-parameters, Take 3)**.** An *L-parameter over* $\mathbf{C}$ is a pair (φ, r) where $\varphi\colon W_E \to {}^LG(\mathbf{C})$ is a continuous group homomorphism, and $r\colon \mathrm{SL}_2 \to \widehat{G}/\mathbf{C}$ is an algebraic representation such that

$$\varphi' := \varphi(w) r \begin{pmatrix} q^{|w|/2} & \\ & q^{-|w|/2} \end{pmatrix}$$

with $N = (\mathrm{Lie}\, r)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ defines an L-parameter in the sense of Definition 26.5.

[54] It might be $q^{-|w|}N$, depending on the normalization.

Remark 26.7. Each of Take 1, Take 2, and Take 3 naturally gives rise to an algebraic variety of L-parameters, which are all distinct.

Parameters in the sense of Take 2 and Take 3 are, up to $\widehat{G}(\mathbf{C})$, in bijection, but the scheme structures are different. The reason is that in Take 2, $N \neq 0$ can degenerate to $N = 0$. But since the representation theory of SL_2 is semi-simple, different representations of SL_2 cannot degenerate to each other. We *want* to have the degenerations, so Take 2 is the correct one for us.

26.3. **Deligne's motivation.** Fix an isomorphism $\mathbf{C} \cong \overline{\mathbf{Q}}_\ell$.

Definition 26.8 (L-parameters, Take 2'). An *L-parameter over* $\overline{\mathbf{Q}}_\ell$ is a continuous group homomorphism $\varphi_\ell \colon W_E \to {}^L G(\overline{\mathbf{Q}}_\ell)$ fitting into a commutative diagram

$$\begin{array}{ccc} \varphi\colon W_E & \longrightarrow & {}^L G(\overline{\mathbf{Q}}_\ell) \\ & \searrow \quad \swarrow & \\ & Q & \end{array}$$

or in other words, a continuous 1-cocycle

$$W_E \to \widehat{G}(\overline{\mathbf{Q}}_\ell)$$

Take 2 and Take 2' are equivalent in the following sense. Fix a trivialization $\mathbf{Z}_\ell(1) \cong \mathbf{Z}_\ell$, and a Frobenius element $\Phi \in W_E$. This induces a retract $W_E \to I_E$, and then composing with the projection to the pro-ℓ part and the trivialization gives $t_\ell \colon W_E \to \mathbf{Z}_\ell$.

Theorem 26.9 (Grothendieck, Deligne). *With the notation above, any $\varphi_\ell \colon W_E \to {}^L G(\overline{\mathbf{Q}}_\ell)$ fitting into*

$$\begin{array}{ccc} \varphi_\ell\colon W_E & \longrightarrow & {}^L G(\overline{\mathbf{Q}}_\ell) \\ & \searrow \quad \swarrow & \\ & Q & \end{array}$$

is of the form $\varphi_\ell(w) = \varphi(w)\exp(t_\ell(w) \cdot N)$ for a unique L-parameter (φ, N) in the sense of Definition 26.5.

Key point: Homomorphisms $W_E \to \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ need not be trivial on an open subgroup $I' \subset I_E$; rather, the statement is that we can find such I' so that it factors over $I' \twoheadrightarrow \mathbf{Z}_\ell$. Then $\mathrm{Hom}(\mathbf{Z}_\ell, \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell))$ are, on an open subgroup, given by $x \mapsto \exp(xN)$ for a nilpotent matrix N. (The argument is that the eigenvalues must be 1 on an open subgroup, and then N can be extracted from the logarithm.)

Deligne observed that one does get the monodromy operator N in the Galois representations coming from cohomology of algebraic varieties.

Warning 26.10. The bijection of Theorem 26.9 really depends on the auxiliary choices.

We will adopt Definition 26.8 as our definition of L-parameters. Then we are forced to work over $\mathbf{Z}_\ell$.

26.4. **Moduli of L-parameters.** Our goal is to construct a moduli space of L-parameters i.e. a scheme $\mathcal{Z}^1(W_E, \widehat{G})$ over $\mathbf{Z}_\ell$, such that the A-valued points (for any $\mathbf{Z}_\ell$-algebra A) are the continuous group homomorphisms

$$\begin{array}{ccc} \varphi\colon W_E & \longrightarrow & {}^L G(A) \\ & \searrow \quad \swarrow & \\ & Q & \end{array}$$

i.e. continuous 1-cocycles $W_E \to \widehat{G}(A)$. (Reference: [DHKM].)

Regarding "continuity", what topology do we put on A? We don't want the discrete topology, but we also don't want the ℓ-adic topology because we want to include $A = \mathbf{Q}_\ell$.

Definition 26.11. Any $\mathbf{Z}_\ell$-module M can be endowed with the *filtered colimit* topology

$$M = \varinjlim_{\mathbf{Z}_\ell\text{-fin. gen. } M' \subset M} (M', \ell\text{-adic topology}).$$

Remark 26.12. In the language of condensed mathematics, the corresponding condensed group can be expressed as

$$\underline{M} = M^\delta \otimes_{\mathbf{Z}_\ell^\delta} \mathbf{Z}_\ell.$$

or in words, "tensor M with the discrete topology against $\mathbf{Z}_\ell$ with the ℓ-adic topology, over $\mathbf{Z}_\ell$ with the discrete topology". What is the content of the equivalence? Since the constructions are all compatible with filtered colimits, it comes down to seeing that any map from a profinite set to such an M factors through a finitely generated $M' \subset M$.

Remark 26.13. One advantage of the condensed formalism is that it incorporates derived rings well (where a classical framework would have difficulty mixing derived rings with the ℓ-adic topology). However, it turns out that the derived version of $\mathcal{Z}^1(W_E, \widehat{G})$ would already be classical.

Theorem 26.14. *There is a scheme $\mathcal{Z}^1(W_E, \widehat{G})/\mathbf{Z}_\ell$ parametrizing L-parameters for G, and it is a disjoint union of affine schemes of finite type over $\mathbf{Z}_\ell$ that are flat, complete intersections, and of dimension* $\dim G = \dim \widehat{G}$.

Remark 26.15. Quotienting by the conjugation action of $\widehat{G}$ gives an Artin stack "$\mathrm{LocSys}_{\widehat{G}}$". But it has infinitely many connected components, unlike the classical $\mathrm{LocSys}_{\widehat{G}}$ occurring in Geometric Langlands.

Proof sketch. The first point is that any continuous 1-cocycle $\varphi\colon W_E \to \widehat{G}(A)$ is trivial on an open subgroup P of wild inertia. Hence

$$\mathcal{Z}^1(W_E, \widehat{G}) = \bigcup_P \mathcal{Z}^1(W_E/P, \widehat{G}),$$

and moreover the transition maps being open and closed immersions. This is because elements of P must map to p-power roots of 1 in $\widehat{G}(A)$, which are discrete.

So it is enough to prove that upon fixing P, all the $\mathcal{Z}^1(W_E/P, \widehat{G})$ are affine, flat, complete intersection of dimension $= \dim \widehat{G}$.

We need to reformulate the definition (which involved continuity) in terms of something more algebraic. For this there is a trick: we will construct a dense discrete subgroup $W \subset W_E/P$ in the following way. Pick $\sigma \in W_E$ a Frobenius element, $\tau \in I_E$ a generator of tame inertia. Then take the subgroup generated by σ, τ, and wild inertia. Then we build a short exact sequence

$$1 \to I \to W \to \mathbf{Z}\langle\sigma\rangle \to 1$$

$$1 \to (\text{finite } p\text{-group}) \to I \to \mathbf{Z}[1/p]\langle\tau\rangle \to 1$$

The point is that W is a discretization of W_E/P.

Claim: $\mathcal{Z}^1(W_E/P, \widehat{G}) \to \mathcal{Z}^1(W, \widehat{G})$ is an isomorphism.

The LHS is obviously an affine algebraic variety, as W_E/P is a finitely generated discrete group. We can turn any such presentation into a presentation of $\mathcal{Z}^1(W_E/P, \widehat{G})$ as an affine algebraic variety.

Proof of claim: for any A, we need to see that a cocycle $\varphi_0 \colon W \to \widehat{G}(A)$ extends uniquely to a continuous cocycle $\varphi \colon W_E/P \to \widehat{G}(A)$. The uniqueness is clear by density of W in W_E/P. We need to prove existence. For this we can replace E be a finite extension, to kill off the finite p-group. Then what we have to see is that a representation

$$\mathbf{Z}[1/p]\langle\tau\rangle \rtimes \mathbf{Z}\langle\sigma\rangle \to \mathrm{GL}_n(A)$$

with the action $\sigma^{-1}\tau\sigma = \tau^q$, then the map $\mathbf{Z}[1/p] \to \mathrm{GL}_n(A)$ sending $n \mapsto (\mathrm{Im}\ \tau)^n$ extends continuously to $\prod_{\ell \neq p} \mathbf{Z}_\ell$.

Note that the image of $\mathrm{Im}\ \tau$ are conjugate to $\mathrm{Im}\,(\tau)^q$, which implies that all eigenvalues are roots of unity of order prime to p. So some power is unipotent. But for unipotent matrices, all $\mathbf{Z}_\ell$-powers are well-defined.

As explained above, the claim implies that $\mathcal{Z}^1(W_E/P, \widehat{G})$ is an affine scheme. We can also measure the deformation theory, in terms of Galois cohomology. That makes it clear it is a complete intersection (as W_E/P has cohomological dimension 2).

To get flatness of the correct dimension, it is enough to bound the dimension of the special fiber. The key input for this is a Theorem of Lusztig, that there are only finitely many unipotent conjugacy classes. The point is that τ is sent to a unipotent matrix. Once you fix the conjugacy class, the choices for the image of σ are a torsor for the centralizer. In this way one gets the desired dimension bound. □

26.5. **A presentation of $\mathcal{O}(\mathcal{Z}^1(W_E/P, \widehat{G}))$.** Fix a discretization $W \subset W_E/P$.

Then for any map $F_n \to W$ where F_n is a free group, we get a map

$$\mathcal{Z}^1(W_E/P, \widehat{G}) = \mathcal{Z}^1(W, \widehat{G}) \to \mathcal{Z}^1(F_n, \widehat{G}) = \widehat{G}^n.$$

Proposition 26.16. *We have*

$$\varinjlim_{(n, F_n \to W)} \mathcal{O}(\widehat{G}^n) \xrightarrow{\sim} \mathcal{O}(\mathcal{Z}^1(W_E/P, \widehat{G}).$$

Remark 26.17. The colimit $\varinjlim_{(n,F_n \to W)} \mathcal{O}(\widehat{G}^n)$ is sifted, so the answer is the same in modules/algebras/... It will appear later as the "algebra of excursion operators".

Corollary 26.18. *The map*

$$\varinjlim_{(n, F_n \to W)} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to \mathcal{O}(\mathcal{Z}^1(W_E/P, \widehat{G}))^{\widehat{G}} = \mathcal{O}(\mathcal{Z}^1(W_E/P, \widehat{G})/\widehat{G}) \tag{26.5.1}$$

is a universal homeomorphism on spectra, and an isomorphism after inverting ℓ.

Proof. Since $(-)^{\widehat{G}}$ is exact in characteristic 0, it follows from Proposition after inverting ℓ. In positive characteristic, use "geometric reductivity" of Haboush [Hab75], which says that up to universal homemorphisms it behaves as if it were semisimple. □

Remark 26.19. The action of $\widehat{G}$ on $\mathcal{O}(\widehat{G}^n)$ is through the identification $\mathrm{Hom}(F_n, \widehat{G}) \cong \widehat{G}^n$, so it is a kind of "twisted conjugation" which depends on how F_n maps to $\widehat{G}$.

Definition 26.20. We call $\mathcal{O}(\mathcal{Z}^1(W_E/P, \widehat{G})/\widehat{G})$ the "spectral Bernstein center".

Remark 26.21. The algebra $\varinjlim_{(n,F_n\to W)} \mathcal{O}(\widehat{G}^n)^{\widehat{G}}$ is the "algebra of excursion operators".

Theorem 26.22. *Actually, the map* (26.5.1) *is an isomorphism if* $Z(G)$ *is connected and* ℓ *"is not too small": (i.e.* ℓ *is a "good prime")*

- *All* ℓ *in type* A.
- *All* $\ell \neq 2$ *in type* $A_n, B_n, C_n, D_n, {}^2D_n$.
- *All* $\ell \neq 2, 3$ *in type* ${}^3D_4, {}^6D_4, E_6, E_7, F_4, G_2, {}^2E_6$.
- *All* $\ell \neq 2, 3, 5$ *in type* E_8.

The conditions imply that $\mathcal{O}(\mathcal{Z}^1)$ has a good filtration, so the higher $\widehat{G}$-cohomology vanishes, and the invariants commute with any base change.

Remark 26.23. We expect the "good prime" assumption to be unnecessary. The assumption that $Z(G)$ is connected does not cause problems for later applications.

27. Construction of L-parameters (Feb 12)

27.1. **Setup.** Let E be a non-archimedean local field, G/E a reductive group. Fix a prime $\ell \neq p$. Let $\widehat{G}/\mathbf{Z}_\ell$ be the dual group. As we discussed, there are two normalizations of the action of W_E on $\widehat{G}$. We suppress this by fixing a choice of $\sqrt{q}$, and pretending it lies in $\mathbf{Z}_\ell$.

27.2. **Local Langlands correspondence.**

27.2.1. *Representation theory side.* We are interested in the category of smooth $G(E)$-representations, $D(G(E), \mathbf{Z}_\ell)$. This embeds fully faithfully in $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbf{Z}_\ell)$, a variant of $D_{\text{ét}}$ that works for all $\mathbf{Z}_\ell$-algebras Λ. (This uses some elements of condensed mathematics, especially the "solid 6-functor formalism".)

27.2.2. *Galois side.* We consider the moduli stack of L-parameters, $\mathcal{Z}^1(W_E, \widehat{G})/\widehat{G}$ (an Artin stack).

Local Langlands concerns the relation between these two sides. In the more classical formulations, one looks at irreducible objects on the representation theoretic side, which are matched with points of $\mathcal{Z}^1(W_E, \widehat{G})/\widehat{G}$.

27.3. **Bernstein centers.** One wants to express the property that

$$\text{“}\pi \mapsto \varphi_\pi \text{ varies algebraically”.} \tag{27.3.1}$$

Definition 27.1. The Bernstein center $Z(G)$ is the (commutative) algebra of endomorphisms of the identity functor on the category of smooth $G(E)$-representations.

Concretely, this means that $f \in Z(G)$ represents induces an endomorphism $f(\pi)\colon \pi \to \pi$ for each π, which commutes with all maps $\pi \to \pi'$.

In particular, if $f \in Z(G)$ and $\pi \in \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell}(G)$, we get a scalar $f(\pi) \in \overline{\mathbf{Q}}_\ell$. This gives a map $Z(G)_{\overline{\mathbf{Q}}_\ell} \hookrightarrow \{\text{functions on } \mathrm{Irr}_{\overline{\mathbf{Q}}_\ell}(G)\}$, and $Z(G)_{\overline{\mathbf{Q}}_\ell}$ should be thought of as "the algebraic functions on the set $\mathrm{Irr}_{\overline{\mathbf{Q}}_\ell}(G)$".

One way to express (27.3.1) is that we want that for any $f \in \mathcal{O}(\mathcal{Z}^1(W_E, \widehat{G}))^{\widehat{G}}$, the map $\pi \mapsto f(\varphi_\pi)$ should be "algebraic", i.e. lie in the Bernstein center $Z(G)$.

Definition 27.2. The *spectral Bernstein center* is

$$Z^{\mathrm{spec}}(G) := \mathcal{O}(\mathcal{Z}^1(W_E, \widehat{G}))^{\widehat{G}}.$$

We also consider $Z^{\mathrm{geom}}(G)$, the "Bernstein center of $D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbf{Z}_\ell)$", i.e. $\mathrm{End}(\mathrm{Id}_{D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbf{Z}_\ell)})$. By restricting the action to smooth representations of G, we get a map

$$Z^{\mathrm{geom}}(G) \to Z(G).$$

27.4. **A semisimplified correspondence.**

Theorem 27.3 (Fargues-S). *If ℓ is a good prime for $\widehat{G}$, then there exists a canonical map*

$$\psi\colon Z^{\mathrm{spec}}(G) \to Z^{\mathrm{geom}}(G) \quad \textit{over } \mathbf{Z}_\ell.$$

(Over $\mathbf{Q}_\ell$, no assumption on ℓ is needed.)

In particular, if $L/\mathbf{Z}_\ell$ is an algebraically closed field, $A \in D_{\mathrm{lis}}(\mathrm{Bun}_G, L)$ with $\mathrm{End}(A) = L$, then there exists a unique up to $\widehat{G}(L)$-conjugation homomorphism

$$\varphi_A\colon W_E \to \widehat{G}(L)$$

which is "semisimple", and such that for all $f \in Z^{\mathrm{spec}}(G)$,

$$f(\varphi_A) = \psi(f)A \in L.$$

Example 27.4. If π is an irreducible representation of $G(E)$, then we can take $A = j_!\pi$.

Remark 27.5. We only get "semi-simple" L-parameters, as functions only detect closed orbits. However, in general one would expect that L-parameters should arise which are not semi-simple.

Proposition 27.6 (Properties of the correspondence)**.** *The map $\pi \mapsto \varphi_\pi$ has the following properties.*

(1) *For tori, it agrees with the usual LLC.*
(2) *It is compatible with twisting and central characters.*
(3) *It is compatible with duals.*
(4) *If $G' \to G$ is a map inducing an isomorphism of adjoint groups, and π is an irreducible representation of $G(E)$, and π' is an irreducible constituent of $\pi|_{G'(E)}$, then $\varphi_{\pi'}$ is the image of φ_π under $\widehat{G} \to \widehat{G}'$.*
(5) *It is compatible with products.*
(6) *It is compatible with Weil restrictions of scalars.*
(7) *It is compatible with parabolic induction.*
(8) *It agrees with the usual LLC for GL_n.*[55]
(9) *It is compatible with Hecke functors on Bun_G. In particular, one can compute the L-parameters of Hecke modifications of A.*
(10) *It is compatible with cohomology of moduli space of local shtukas, e.g. local Shimura varieties (in paticular, Rapoport-Zink spaces).*

Corollary 27.7 (Theorem of Helm-Moss)**.** *For $G = \mathrm{GL}_n$, the map*

$$Z^{\mathrm{spec}}(G)_{\mathbf{Q}_\ell} \to Z(G)_{\mathbf{Q}_\ell}$$

defined by the usual LLC, is defined integrally, i.e. induces $Z^{\mathrm{spec}}(G) \to Z(G)$. This expresses "compatibility of LLC with congruences".

27.5. **Construction of $\psi\colon Z^{\mathrm{spec}}(G) \to Z^{\mathrm{geom}}(G)$.** We begin by summarizing the data we have.

- We have an ∞-category $\mathcal{C} = D_{\mathrm{lis}}(\mathrm{Bun}_G, \mathbf{Z}_\ell)$ and for any finite set I, an exact monoidal functor (picking a finite quotient $W_E \twoheadrightarrow Q \curvearrowright \widehat{G}$ through which the action on $\widehat{G}$ factors)

$$\mathrm{Rep}_{\mathbf{Z}_\ell}(\widehat{G} \rtimes Q)^I \to \mathrm{End}(\mathcal{C})^{W_E^I}$$

  denoted $V \mapsto T_V$, which is linear over $\mathrm{Rep}_{\mathbf{Z}_\ell}(Q^I)$, and functorial in I. This comes from the Hecke action.

We will only use this kind of abstract data.

Proposition 27.8. *For any $A \in \mathcal{C}^\omega$, there exists an open subgroup of the wild inertia subgroup $P \subset I_E$, such that for all I and $V \in \mathrm{Rep}(\widehat{G} \rtimes Q)^I$, the W_E^I-action on $T_V(A)$ factors over $(W_E/P)^I$.*

This means that we can replace W_E by W_E/P above. Then (as last time) we can replace W_E/P by a discretization $W \subset W_E/P$.

Last time we stated:

[55] This is the only place where global methods are used; indeed the usual LLC is proved by global methods.

Theorem 27.9. *Under mild technical conditions, the map*

$$\varinjlim_{(n,F_n\to W)} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to \mathcal{O}(\mathcal{Z}^1(W_E/P,\widehat{G}))^{\widehat{G}} \tag{27.5.1}$$

is an isomorphism.

We want to find $\mathcal{O}(\mathcal{Z}^1(W_E/P,\widehat{G}))^{\widehat{G}} \to Z^{\mathrm{geom}}(G) = \mathrm{End}(\mathrm{Id}_G)$. By the Theorem, it suffices to produce the map $\varinjlim_{(n,F_n\to W)} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to Z^{\mathrm{geom}}(G)$. This will be done by "excursion operators" (V. Lafforgue).

Definition 27.10. (1) An *excursion datum* is a tuple $(I,V,\alpha,\beta,(\gamma_i)_{i\in I})$ where

- I is a finite set.
- $V \in \mathrm{Rep}(\widehat{G}\rtimes Q)^I$.
- $\alpha\colon \mathbb{1} \to V|_{\Delta(\widehat{G})}$ and $\beta\colon V|_{\Delta(\widehat{G})} \to \mathbb{1}$,
- $\gamma_i \in W_E$.

(2) Given excursion data, the excursion operator is the following element of $\mathrm{End}(\mathrm{Id}_G)$. For any $A\in\mathcal{C}$, the induced endomorphism of A is

$$A = T_{\mathbb{1}}(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_{\mathbb{1}}(A) = A$$

Proposition 27.11. *The collection of excursion operators defines a map*

$$\varinjlim_{(n,F_n\to W)} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to Z^{\mathrm{geom}}(G)$$

Corollary 27.12. *The L-parameter φ_A is characterized as follows. For all excursion data, the scalar*

$$L \xrightarrow{\alpha} V \xrightarrow{(\varphi_A(\gamma_i))_{i\in I}} V \xrightarrow{\beta} L$$

agrees with the scalar

$$A = T_{\mathbb{1}}(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i\in I}} T_V(A) \xrightarrow{\beta} T_{\mathbb{1}}(A) = A$$

27.6. **The spectral action.**

Theorem 27.13. *The ∞-categorical data from above are equivalent to an action of* $\mathrm{Perf}(\mathcal{Z}^1(W_E,\widehat{G})/\widehat{G})$ *on* $D_{\mathrm{lis}}(\mathrm{Bun}_G,\mathbf{Z}_\ell)$.

Remark 27.14. There is related work of Nadler-Yun, and Gaitsgory-Kazhdan-Rozenblyum-Varshavsky. One novelty here is that we work integrally.

Example 27.15. Let us explain how this recovers the map of Bernstein centers. There is a functor $\mathrm{Rep}(\widehat{G}\rtimes Q)^I \to \mathrm{Perf}(\mathcal{Z}^1(W_E,\widehat{G})/\widehat{G})^{W_E^I}$ by pullback and tensor. Theorem 27.13 then gives a functor to $\mathrm{End}(\mathcal{C})^{W_E^I}$.

27.7. **Elliptic L-parameters.** What does this mean for "elliptic" L-parameters?

Assume for simplicity that G is semisimple, with coefficients $\overline{\mathbf{Q}}_\ell$. Say φ is *elliptic* if it defines an isolated component of $\mathcal{Z}^1(W_E,\widehat{G})_{\overline{\mathbf{Q}}_\ell}/\widehat{G}$, i.e. $S_\varphi \subset \widehat{G}$ its centralizer, the inclusion

$$[*/S_\varphi] \subset [\mathcal{Z}^1(W_E,\widehat{G})/\widehat{G}]$$

is open and closed. In particular it is cut out by an idempotent. Hence from the spectral action, we get a corresponding component

$$D^{\varphi}_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbf{Q}}_\ell) \overset{\oplus}{\subset} D_{\mathrm{lis}}(\mathrm{Bun}_G,\overline{\mathbf{Q}}_\ell).$$

Now we have the spectral action on $D_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell)$, whereas what acts on $D^\varphi_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell)$ is $\text{Rep}(S_\varphi)$. The compatibility of spectral action and Hecke action says: given $V \in \text{Rep}(\widehat{G} \rtimes Q)^I$, if we decompose

$$V|_{S_\varphi \times W_E} = \bigoplus_{i=1}^r W_i \boxtimes r_i \quad \text{where } W_i \in \text{Irr}\,\text{Rep}(S_\varphi) \text{ and } r_i \in \text{Rep}_{\overline{\mathbf{Q}}_\ell}(W_E),$$

then for $A \in D^\varphi_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell)$,

$$T_V(A) = \bigoplus_{i=1}^m \text{Act}_{W_i}(A) \otimes r_i \in (D^\varphi_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell))^{W_E}$$

where here $\text{Act}_{W_i}(A)$ refers to the action of $\text{Rep}(S_\varphi)$ on $D^\varphi_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell)$.

In some sense this "is" the Kottwitz Conjecture.

Proposition 27.16. *All $A \in D^\varphi_{\text{lis}}(\text{Bun}_G, \overline{\mathbf{Q}}_\ell)$ are concentrated on the semistable locus, and correspond to supercuspidal representations.*

Proof. This is implied by compatibility with parabolic induction, as the non-semistable strata are related to Levi's, whose parameters must come from parabolic induction. □

We have a decomposition

$$D^\varphi_{\text{lis}} = \bigoplus_{b \in B(G)_{\text{basic}}} \bigoplus_{\substack{\pi \text{ irr. supercusp.} \\ \text{rep'n of } G_b(E) \\ \varphi_\pi = \varphi}} D(\overline{\mathbf{Q}}_\ell) \cdot [\pi].$$

This has an action of $\text{Rep}(S_\varphi)$. If S_φ is abelian (as happens in classical types), the characters of S_φ will permute these π's. This is a "Jacquet-Langlands correspondence".

Also, $T_V([\pi])$ is the π-isotypic component of the cohomology of some moduli spaces of local shtukas, and we have

$$T_V([\pi]) = \bigoplus_{i=1}^m \text{Act}_{W_i}([\pi]) \otimes r_i,$$

where the $\text{Act}_{W_i}([\pi])$ are Jacquet-Langlands transfers. This matches the prediction of the Kottwitz Conjecture. However, we are not proving the Kottwitz Conjecture, because we are missing the parametrizaton of all π with $\varphi_\pi = \varphi$.

Conjecture 27.17. *Assume G is quasisplit. Fix Whittaker data. Then there is a unique generic π_0 with $\varphi_{\pi_0} = \varphi$, and the functor*

$$\text{Perf}([*/S_\varphi]) \to (D^\varphi_{\text{lis}})^\omega = \bigoplus_b \bigoplus_\pi \text{Perf}(\overline{\mathbf{Q}}_\ell)[\pi]$$

sending $W \mapsto \text{Act}_W([\pi_0])$ is an equivalence of categories. In particular, this induces a bijection $\text{IrrRep}(S_\varphi) \xrightarrow{\sim} \{\pi\}$.

Remark 27.18. This is Kaletha's formulation of LLC using $B(G)_{\text{basic}}$.

27.8. **Back to D_{lis}.** We now return to a conjecture of the whole category of sheaves on Bun_G.

Fix Whittaker data, i.e. $U \subset B \subset G$, and a non-degenerate character $\psi\colon U(E) \to \breve{\mathbf{Z}}_\ell^\times$. Then we have the representation $\text{c-Ind}_{U(E)}^{G(E)}(\psi)$.

Conjecture 27.19. *There is an equivalence*

$$D^b_{\mathrm{coh}}(\mathcal{Z}^1(W_E, \widehat{G})_{\mathbf{Q}_\ell}/\widehat{G}) \cong D_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbf{Q}}_\ell)^\omega$$

which is linear over $\mathrm{Perf}(\mathcal{Z}^1(W_E, \widehat{G})_{\overline{\mathbf{Q}}_\ell}/\widehat{G})$*, taking* $\mathcal{O}_{\mathcal{Z}^1(W_E,\widehat{G})_{\overline{\mathbf{Q}}_\ell}/\widehat{G}}$ *to* $[\text{c-Ind}_{U(E)}^{G(E)}(\psi)]$.

Remark 27.20. The spectral action determines the functor of Conjecture 27.19 on the subcategory $\mathrm{Perf}(\mathcal{Z}^1(W_E, \widehat{G})_{\overline{\mathbf{Q}}_\ell}/\widehat{G}) \subset D^b_{\mathrm{coh}}(\mathcal{Z}^1(W_E, \widehat{G})_{\mathbf{Q}_\ell}/\widehat{G})$. The difference between $D^b_{\mathrm{coh}}(\mathcal{Z}^1(W_E, \widehat{G})_{\mathbf{Q}_\ell}/\widehat{G})$ and $\mathrm{Perf}(\mathcal{Z}^1(W_E, \widehat{G})_{\overline{\mathbf{Q}}_\ell}/\widehat{G})$ comes from the singularities of $\mathcal{Z}^1(W_E, \widehat{G})_{\mathbf{Q}_\ell}$.

Remark 27.21. Integrally, we expect that there is a categorical equivalence but we need to impose a "nilpotent singular support" condition on the stack of Langlands parameters. We expect the equivalence to be

$$D^b_{\mathrm{coh,Nilp}}(\mathcal{Z}^1(W_E, \widehat{G})_{\breve{\mathbf{Z}}_\ell}/\widehat{G}) \cong D_{\mathrm{lis}}(\mathrm{Bun}_G, \breve{\mathbf{Z}}_\ell)^\omega.$$

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