Qualifying Exam Practice Test

Generated on January 21, 2019

1: (Fall 2010 Algebra Qual, Morning #5)

(5) Let $G = GL(2, \mathbb{F}_q)$, where $q = p^n$, p prime. Let Π be the set of one-dimensional subspaces in $V = \mathbb{F}_q^2$. Since G acts on V by matrix multiplication, it acts on Π.

(i) Show that if $\ell \in \Pi$ then the stabilizer of ℓ in G contains a unique p-Sylow subgroup of G. How many p-Sylow subgroups does G have, and what is their order?

(ii) Prove that if ℓ_1, ℓ_2 and ℓ_3 are three distinct one-dimensional subspaces of V, then there is an element g of G such that $g\ell_1 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $g\ell_2 = \mathbb{F}_q \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g\ell_3 = \mathbb{F}_q \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Show that if P_1, P_2 and P_3 are three distinct p-Sylow subgroups of G, and if Q_1, Q_2 and Q_3 are another three distinct p-Sylow subgroups of G, then there exists a $g \in G$ such that

$$gP_1g^{-1} = Q_1, \qquad gP_2g^{-1} = Q_2, \qquad gP_3g^{-1} = Q_3.$$

2: (Spring 2011 Algebra Qual, Morning #1)

1. (a) Prove that if G is a finite group and H is a proper subgroup, then G is not a union of conjugates of H. (Hint: the conjugates all contain the identity.)

(b) Suppose G is a (finite) transitive group of permutations of a finite set X of n objects, $n > 1$. Prove that there exists $g \in G$ with no fixed points in X. (Hint: use part (a).)

3: (Fall 2013 Algebra Qual, Afternoon #1)

1. Let us say that a subgroup H of a group G is a *malnormal* subgroup if $gHg^{-1} \cap H = \{1\}$ for all $g \in G - H$.

Let G be a finite group acting transitively on a set S. We call G a *Frobenius group* if no nontrivial element $g \neq 1$ of G fixes more than one element of S.

(a) (5 points) Choose $x \in S$ and set $H = \mathrm{Stab}_G(x) = \{g \in G | gx = x\}$. Prove that G is a Frobenius group if and only if H is a malnormal subgroup of G.

(b) (5 points) Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q) \right\}$ and $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_q) \right\}$. Prove that H is a malnormal subgroup of G. (**Hint:** Let $S = \mathbb{F}_q$.)

4: (Fall 2012 Algebra Qual, Afternoon #5)

(10) Let $G = SL_3(\mathbf{F}_p)$, where p is an odd prime. Let ℓ be a prime divisor of $p^2 + p + 1$.

(i) (5 pts) Suppose $\ell > 3$. Prove that the ℓ-Sylow subgroups of G are cyclic.

(ii) (5 pts) Suppose that $\ell = 3$. Prove that the ℓ-Sylow subgroups of G are *not* cyclic.

5: (Spring 2014 Algebra Qual, Morning #3)

PROBLEM M3. Let G be a group of order $56 = 7 \cdot 8$. Let P_2 and P_7 be 2-Sylow and 7-Sylow subgroups, respectively.

(i) (3 pts) Show that if P_2 and P_7 are both normal then $xy = yx$ for $x \in P_2$ and $y \in P_7$.

(ii) (3 pts) Show that either P_2 or P_7 is normal.

(iii) (4 pts) Give an example with non-normal P_2, and another with non-normal P_7.

6: (Spring 2015 Algebra Qual, Afternoon #2)

Question 2. Let $p > 2$ be an odd prime number.

(i) (5 pts) Let G be a finite group and P a p-Sylow subgroup of G. Let N be the normalizer of P. Let x and y be elements of the centralizer $C(P)$ of P. Show that if x and y are conjugate in G then they are conjugate in N. **Hint:** Apply the Sylow theorems in the centralizer $C(y)$ of y.

(ii) (5 pts) Let $G = GL_2(\mathbf{F}_p)$. Find a Sylow p-subgroup and let N be its normalizer. Show that there may be elements of N that are conjugate in G but not conjugate in N.

7: (Fall 2011 Algebra Qual, Morning #1)

(1) Let p be a prime, $G = GL_3(\mathbb{Z}/p^5\mathbb{Z})$.

(i) Show that the natural map $G \to GL_3(\mathbb{Z}/p\mathbb{Z})$ is surjective, and compute the order of the kernel.

(ii) Compute the size of G, and describe an explicit p-Sylow subgroup of G.

8: (Fall 2012 Algebra Qual, Morning #2)

(2) Let $G = SL_n(\mathbf{F}_p)$ for a prime p and an integer $n > 1$.

(i) Find a Sylow p-subgroup P of G and compute its order.

(ii) Give an *explicit* sequence of subgroups $1 = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_m = P$ such that for all $0 \leq i < m$, P_i is normal in P_{i+1} and the quotient P_{i+1}/P_i is abelian.

9: (Spring 2013 Algebra Qual, Morning #3)

PROBLEM M3. Let $G = GL_2(\mathbf{Z}/9\mathbf{Z})$.

(i) (3 points) Compute the order of G.

(ii) (3 points) Prove $g \in G$ has 3-power order if and only if its image in $GL_2(\mathbf{Z}/3\mathbf{Z})$ does.

(iii) (4 points) Show that a Sylow 2-subgroup of G is isomorphic to $\mathbf{F}_9^\times \rtimes (\mathbf{Z}/2\mathbf{Z})$ where the nontrivial element $1 \in \mathbf{Z}/2\mathbf{Z}$ acts on $\mathbf{F}_9^\times$ via $x \mapsto x^3$.

10: (Fall 2010 Algebra Qual, Afternoon #1)

(1) Let G be a subgroup of a finite p-group H (p a prime) such that the natural homomorphism $G \to H/[H,H]$ is surjective. Prove that $G = H$ by induction on $|H|$ as follows:

  (i) Suppose N is any nontrivial normal subgroup of H; show (using the inductive assumption) that $G \cdot N = H$.

  (ii) Let Z be the center of H. Using (i) we have $G \cdot Z = H$; explain why $G \cap Z$ cannot be trivial. Now set $N = G \cap Z$ in (i).