EQUIVARIANT ALGEBRAIC GEOMETRY

TONY FENG
BASED ON LECTURES OF RAVI VAKIL

Contents

DISCLAIMER

This document originated from a set of lectures that I "live-TEXed" during a course offered by Ravi Vakil at Stanford University in the winter of 2015. The format of the course was somewhat unusual in that the first two weeks' worth of lectures were presented by Dan Edidin, as an overview of his theorem on "Riemann-Roch theorem for stacks" via equivariant algebraic geometry. Some background lectures were given outside of class as well, by both Dan and Ravi. Afterwards, Ravi took over the lectures and fleshed out the argument and examples (with a couple of guest lectures by Arnav Tripathy and Dan Litt sprinkled in).

While these unusual features of the course worked in the classroom, I felt upon looking back at my notes that I had failed to capture an account that would make any sense to an outside reader. Therefore, I have heavily modified the organization and content of the notes. The major addition has been of background material, which was mostly assumed (or black-boxed) during the course. My goal was to make everything accessible to a student familiar with algebraic geometry to the extent of a first year of scheme theory - in particular, the guiding principle has been to include background material that I wish I knew at the start of the class. Indeed, a major reason for my revamping of these notes was to solidify my comfort with that background.

Aside from that, the organization and presentation have also been significantly revised. This inevitably means that there will be many mistakes and typos, which of course should be blamed on me. On the other hand, the mathematical insights should be attributed elsewhere - mostly probably to Ravi, but various snippets are drawn from other sources or personal notes. I also apologize for inadequacies in the references.

1. Examples and motivation

The official objective of these notes is to give a proof of a "Riemann-Roch theorem for stacks" via equivariant algebraic geometry, following [Edi13]. However, our real agenda is to use this result as an excuse to meet some of the main characters of modern algebraic geometry - such as Chow groups, intersection theory, and stacks - which lie beyond a first course in algebraic geometry, and to see how they can be marshalled to answer very concrete questions.

The goal of this first section is to carefully study some simple examples, to see what such a Riemann-Roch theorem for stacks might say. In particular, we will examine the spaces $\mathbb{P}(a,b) =: \operatorname{Proj} k[x,y]$ with x in degree a and y in degree b. As schemes, these are of course all isomorphic to $\mathbb{P}^1$, but we will already see that for these simple examples something more subtle is going on.

1.1. Setup.

On $\mathbb{P}(a,b)$ we have an action of the group $\mathbb{G}_m$ (which we think of as just being $\mathbb{C}^*$) as follows: $\mathbb{G}_m$ acts on $\mathbb{C}[x,y]$ sending

$$x \mapsto \lambda^{-a} x$$
$$y \mapsto \lambda^{-b} y.$$

(There is no need to restrict our attention to $\mathbb{C}$, but on the other hand we'll see that there is not much to be gained in aiming for maximum generality.) We'll write down some explicit line bundles, count the dimension of the space of sections, and compare what we get with what Riemann-Roch tells us.

To construct a line bundle, we simply take a free module over $\mathbb{C}[x,y]$ with generator T, and extend the action of $\mathbb{C}^*$ to this bundle, say by by the action $\lambda \cdot T = \lambda^{\ell} T$. Let us call this bundle ξ^{ℓ}. The question that we are interested in, for this and the upcoming examples, is the following.

Question. What is the dimension of the space of $\mathbb{G}_m$-invariant sections of ξ^{ℓ} on $\mathbb{P}(a,b)$?

This dimension is denoted, as usual, by $h^0(\mathbb{P}(a,b), \xi^{\ell})$. Note that the $\mathbb{G}_m$-invariant sections will be precisely the $\mathbb{G}_m$-invariant elements of the module $\mathbb{C}[x,y,T]$.

1.2. Examples.

Let's consider first the simplest case: $\mathbb{P}(1,1)$.

- When $\ell = 0$, the only invariant sections are the constants, so we see that $h^0(\mathbb{P}(1,1), \xi^0) = 1$.
- When $\ell = 1$, the invariant sections are spanned by xT and yT, so we see that $h^0(\mathbb{P}(1,1), \xi^1) = 2$.
- When $\ell = 2$, the invariant sections are spanned by x^2T, xyT, y^2T (now you see why we chose negative weights in defining the original action!).
- It is now easy to see that $\xi^{\ell} \longleftrightarrow \mathscr{O}(\ell)$, so in particular $h^0(\mathbb{P}(1,1), \xi^{\ell}) = \ell + 1$.

ℓ	$h^0(\mathbb{P}(1,1),\xi^\ell)$
0	1
1	2
2	3
ℓ	$\ell+1$

Let's move on to the next simplest example: $\mathbb{P}(1,2)$.

- When $\ell=0$, the only invariant sections are again the constants, so $h^0(\mathbb{P}(1,2),\xi^0)=1$.
- When $\ell=1$, the invariant sections are spanned by xT this time, so $h^0(\mathbb{P}(1,2),\xi^1)=1$.
- When $\ell=2$, the invariant sections are spanned by x^2T, y^2T, so $h^0(\mathbb{P}(1,2),\xi^2)=2$.
- When $\ell=3$, the invariant sections are spanned by x^3T, xyT so $h^0(\mathbb{P}(1,2),\xi^3)=2$.
- In general, the dimension we seek is the number of non-negative integral solutions to $i+2j=\ell$, which is easily seen to be $\lfloor \ell/2\rfloor+1$.

ℓ	$h^0(\mathbb{P}(1,1),\xi^\ell)$	$h^0(\mathbb{P}(1,2),\xi^\ell)$
0	1	1
1	2	1
2	3	2
3	4	2
4	5	3
5	6	3

We see that for $\mathbb{P}(1,2)$ the dimension of the space of sections grows about half as fast as it did for $\mathbb{P}(1,1)$. *What's going on?*

Exercise 1.1. Compute the formula for $h^0(\mathbb{P}(a,b),\xi^\ell)$ for some other values of a,b. Do you have a guess for the general formula?

The Riemann-Roch theorem is precisely a statement about the dimension of the global sections of a line bundle. According to Riemann-Roch,

$$\begin{aligned}\chi(\mathscr{L}) = h^0(\mathscr{L}) - h^1(\mathscr{L}) &= d+1-g \\ &= d+\frac{1}{2}\chi_{\text{top}}.\end{aligned}$$

With $d=\ell$, this agrees well with the column describing the global sections of ξ^ℓ on $h^0(\mathbb{P}^1(1,1))$, as we even noted in that calculation. However, it seems to disagree with the calculation for $\mathbb{P}(1,2)$, which grows at about half the rate. In fact, we can easily compute that

$$h^0(\mathbb{P}(1,2),\xi^\ell) = \frac{\ell}{2}+\frac{3}{4}+(-1)^\ell\frac{1}{4}. \tag{1.1}$$

In general, we predict something like

$$\boxed{h^0(\mathbb{P}(a,b),\xi^\ell) = \frac{\ell}{ab} + (\text{something depending only on } \ell \text{ modulo } ab).}$$

Let's see if we can at least heuristically figure out why this might be. What is the Euler characteristic of $\mathbb{P}(1,2)$? Since this is isomorphic to $\mathbb{P}^1$ the first guess would be that it should be 2.

However, we can think about this another way. The point $[s:t] \in \mathbb{P}(a,b)$ corresponds to the $\mathbb{G}_m$-orbit of (s,t) in $\mathbb{A}^2 = \operatorname{Spec} k[x,y]$. Most of the orbits are acted on freely by $\mathbb{G}_m$, but *one* is not: the point $(0,1)$ has stabilizer $\{-1,1\} \cong \mathbb{Z}/2$ because -1 acts on the t coordinate by its *square*, which is 1. Therefore, this orbit is only "half" as large as the others, and we might imagine it as being only "half" a point in $\mathbb{P}(a,b)$. If we count in this way, then $\mathbb{P}(1,2)$ only has Euler characteristic $3/2$ (we traded a full point for "half a point").

Assuming for simplicity that $\gcd(a,b) = 1$, the same reasoning "shows" that $\mathbb{P}(a,b)$ should have Euler characteristic $1 + \frac{1}{a} + \frac{1}{b}$. Plugging this refined Euler characteristic into Riemann-Roch, we guess that

$$h^0(\mathbb{P}(a,b), \xi^\ell) = \frac{\ell}{ab} + \frac{1/a + 1/b}{2}.$$

For $a = b = 1$, we recover $\ell + 1$ on the right hand side as before. For $a = 1, b = 2$ we obtain $\frac{\ell}{2} + \frac{3}{4}$. This is obviously not correct, since we must obtain an integer, but it captures everything in (1.1) except for the term $(-1)^\ell \frac{1}{4}$. This is clearly a sign that something is interesting going on! We refine our prediction to

$$h^0(\mathbb{P}(a,b), \xi^\ell) = \frac{\ell}{ab} + \frac{1/a + 1/b}{2} + (\text{oscillation term}).$$

Exercise 1.2. Work out more examples, and identify the oscillation term.

What is this oscillation term? One might be tempted to guess at first that it has something to do with a higher cohomology group, but this doesn't seem consistent with the fact that the error term even oscillates in *sign*. Clearly what we need is a version of the Riemann-Roch theorem which plays well with group actions. We now embark on a journey that will clarify the existence and nature of the examples we've encountered here.

Part 1. Some non-equivariant ingredients

2. Chow groups

2.1. **The definition.** You can think of Chow groups as being something like a "homology theory" for algebraic varieties. In particular, you can think of elements of $CH_*(X)$ as representing subvarieties of X, just as a map of closed, oriented, connected manifolds $Y \to X$ induces an element of $H_*(X;\mathbb{Z})$, namely the image of the fundamental class of Y.

Definition 2.1. Formally, we define the group of *cycles* $Z_k(X)$ to be the free abelian group on k-dimensional subvarieties (which we take by definition to be closed, irreducible, reduced). Then $Z_*(X) = \bigoplus_k Z_k(X)$.

We say that two cycles in $Z_k(X)$ are *rationally equivalent* if there exists a cycle on $\mathbb{P}^1 \times X$ whose restrictions to the fibers $\{t_0\} \times X$ and $\{t_1\} \times X$ are A_0 and A_1.

The subgroup $B_k(X)$ is generated by differences of rationally equivalent varieties. (Warning: this is non-standard notation!) We set $B_*(X) := \bigoplus_k B_k(X)$.

The (graded) *Chow group* $CH_*(X)$ is the quotient $Z_*(X)/B_*(X)$, and we have the natural quotient grading

$$CH_*(X) = \bigoplus_k CH_k(X) := Z_k(X)/B_k(X).$$

One can think of rational equivalence as stating that there is a "family" parametrized by a segment in $\mathbb{P}^1$ whose boundary is $A_0 - A_1$. This is reminiscent of cobordism.

Example 2.2. Let's play around with a couple of special cases.

Any two points in $\mathbb{A}^n$ are rationally equivalent, because we can pick a line between them and "move" one point to the other. In fact, any point is rationally equivalent to the empty set because we can "push it off" to ∞, so $CH_0(\mathbb{A}^n) = 0$.

Any hypersurface $\{f = 0\}$ in $\mathbb{A}^n$ is also rationally equivalent to the empty set, as the graph of $f\colon \mathbb{A}^n \to \mathbb{P}^1$ is a cycle in $\mathbb{A}^n \times \mathbb{P}^1$ whose fiber over ∞ is empty.

In fact, we claim that

$$CH^*(\mathbb{A}^n) \cong \begin{cases} \mathbb{Z} & * = n, \\ 0 & * \neq 0 \end{cases}$$

It suffices to show that any proper subvariety $W \subset \mathbb{A}^n$ is rationally equivalent to the empty set. We will try the same pushing off trick. As W is a proper subvariety, we may assume that $O \notin Y$. Consider the subvariety $\widetilde{W} \subset \mathbb{A}^n \times (\mathbb{A}^1 - \{O\})$ defined by

$$\widetilde{W} = \{(z,t) \mid \frac{z}{t} \in W\}.$$

Geometrically, $\widetilde{W}$ is the family whose fiber over t is the dilation of W by t. The fiber over ∞ should morally be $\emptyset$, as we have "pushed away" all the points. However, let's see this explicitly.

In terms of equations, $\widetilde{W}$ is cut cut out by the ideal $\{f(z/t) \colon f(z) \in I(Y)\}$. Thus the closure of $\widetilde{W}$ is a family in $\mathbb{A}^n \times \mathbb{P}^1$ whose fiber over $t = 1$ is precisely Y. As $O \notin W$, there exists a polynomial $g(z)$ vanishing on Y and having non-zero constant term: $g(z) = c + \ldots$. Then $g(z/t) = c + t^{-1} \ldots$ has the value c on the fiber $\mathbb{A}^n \times \{\infty\}$, so the ideal of $\widetilde{W}$ restricts to the unit ideal over ∞.

Example 2.3. Any two points on a genus $g > 0$ projective curve are *not* rationally equivalent - if they were, then the corresponding cycle in $X \times \mathbb{P}^1$ would give a birational map $X \to \mathbb{P}^1$.

Here is another characterization of $B(X)$. For any rational function f on a subvariety $Y \subset X$, we can associate a divisor

$$\operatorname{Div}(f) = \sum_{\substack{W \subset Y \\ \text{codim } 1}} \operatorname{ord}_W(f)[W].$$

Proposition 2.4. *The group of boundaries $B(X)$ is coincides with the group generated by* $\operatorname{Div}(f)$ *as f and Y vary.*

Proof sketch. One inclusion is quite easy: given a rational function f, we get a rational map $f \colon Y \to \mathbb{P}^1$. The graph of f (the closure of the usual graph on an open subset) is a cycle in $Y \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$, the difference of whose fibers over 0 and ∞ is precisely $\operatorname{Div}(f)$. This shows that $\operatorname{Div}(f) \subset B(X)$.

The other direction is a bit more subtle. Given a cycle $\widetilde{V} \subset X \times \mathbb{P}^1$ with boundary cycle $A_0 - A_\infty$, the projection map determines a rational function $\widetilde{f}$ on V such that $\operatorname{Div}(\widetilde{f}) = [A_0] - [A_\infty]$. The map $V \to X$ is generically finite over its image (unless $\widetilde{V} = X \times \mathbb{P}^1$), and we want to present $[A_0 - A_\infty]$ as the divisor of some function on X. This is a very general situation, and it turns out that the function $\operatorname{Nm}_{V/X}(\widetilde{f})$ does the trick. $\square$

Example 2.5. $CH_n(X) = Z_n(X)$ is the free abelian group on the irreducible (connected) components of X. $Z_{n-1}(X)$ is just the group of divisors of X. $B_{n-1}(X)$

If X has pure dimension n, then $CH_{n-1}(X) \cong \operatorname{Cl}(X)$, i.e. divisors modulo principal divisors.

There is a map $CH_*(X) \to H_*(X)$, essentially by inclusion of the fundamental class (as we discussed previously). This gets a little messy because you have to define the fundamental class of a singular variety, but it works out. The map is actually more like a cobordism theory than a homology theory, and Totaro showed that it factors through the complex cobordism ring MU^*.

Remark 2.6. A very naïve conjecture would be that if $X/\mathbb{C}$ is a smooth projective variety, then the map $CH_*(X) \to H_*(X;\mathbb{Z})$ is *surjective*, i.e. any element of $H_*(X;\mathbb{Z})$ is obtained as the fundamental class of some algebraic subvariety. This fails for at least two reasons: first, one cannot expect this to be true integrally, but only rationally. Second, there are some constraints from Hodge theory. If one refines the conjecture appropriately to account for these obstructions, then one arrives at the *Hodge conjecture*.

Definition 2.7. If X is a compact complex manifold of dimension n, then Poincaré duality "identifies" $H^i(X)$ and $H_{n-i}(X)$. Motivated by this, we define $CH^i(X) := CH_{n-i}(X)$.

2.2. **Functoriality.** As the homology and cohomology are functorial, one might expect functoriality properties for Chow groups. These are a little subtle, but they do exist.

Proper pushforward. If $f \colon Y \to X$ is a proper map, then we can "push forward" subvarieties to subvarieties. However, one has to take care that this map preserve rational equivalences. We define $f_*\colon CH_*Y \to CH_*X$ by

$$f_*([A]) = \begin{cases} 0 & \dim f(A) < \dim A, \\ n[f(A)] & [K(A):K(f(A))] = n. \end{cases}$$

We will mostly just be thinking of the case where f is a closed embedding, in which case $f_*([A]) = [f(A)]$ on the nose.

Flat pullback. If $f\colon X \to Z$ is flat, then we may define a pullback map $f^*\colon CH^*Y \to CH^*X$ which is determined by

$$f^*([A]) = [f^{-1}(A)]$$

when $f^{-1}(A)$ is reduced.

2.3. **Properties.** We now discuss some features of the Chow groups that will be useful for computations.

Excision. If $Y \subset X$ is a closed subscheme and $U = Y \setminus X$ is its complement, then the inclusion and restriction maps of cycles give a right exact sequence

$$CH_*(Y) \xrightarrow{j_*} CH_*(X) \xrightarrow{\iota^*} CH_*(U) \to 0. \tag{2.1}$$

This is analogous to the excision axiom in algebraic topology.

Homotopy invariance. If $\pi\colon V \to X$ is an affine space bundle (i.e. a fiber bundle whose fibers are affine space), then the induced map $\pi^*\colon CH^*(X) \to CH^*(V)$ is a surjection. If V is actually a *vector* bundle (i.e. there exists a section), then π^* is an isomorphism. This is analogous to the fact that a vector bundle is homotopy equivalent to its base.

Mayer-Vietoris. If Y_1 and Y_2 are two subvarieties of X, then we have a right exact sequence

$$CH_*(Y_1 \cap Y_2) \to CH_*(Y_1) \oplus CH_*(Y_2) \to CH_*(Y_1 \cup Y_2) \to 0$$

induced by the usual maps.

2.4. **Ring structure.** In fact, $CH_*(X)$ has a *ring structure.* This might seem weird at first, if we're thinking of CH_* as some algebraic analogue of homology, but recall that compact complex manifolds *also* have a ring structure coming from the intersection product (dual to the cup product via Poincaré duality), which has the property that the intersection of the homology classes represented by two transversely intersecting, complementary-dimensional submanifolds is precisely the number of intersection points.

Definition 2.8. We say that subvarieties $A, B \subset X$ *intersect generically transversely* if they intersect transversely at a generic point of each component of $A \cap B$.

Theorem 2.9. *There exists a unique product structure on $CH^*(X)$ satisfying the condition that if A, B are generically transverse then $[A] \cdot [B] = [A \cap B]$. This product structure makes $CH^*(X)$ a commutative graded ring.*

Projection formula. The compatibility with the ring structure and the push/pull operations is described by the *projection formula*

$$f_*(\alpha \cdot f^*\beta) = f_*\alpha \cdot \beta.$$

Example 2.10. If X is a quasiprojective surface and D is an ample line bundle on X, then $A + nD$ and $B + nD$ will be very ample for $n \gg 0$. By Bertini's Theorem, we can find representatives in the class of $[A + nD]$ and $[B + nD]$ that intersect generically transversely. Then linearity forces the value of $[A] \cdot [B]$. (This is the approach to intersection theory on surfaces taken by Hartshorne.)

Example 2.11. We saw earlier that

$$CH^*(\mathbb{A}^n) \cong \begin{cases} \mathbb{Z} & * = 0, \\ 0 & * > 0. \end{cases}$$

We claim that the same holds for any open subset $U \subset \mathbb{A}^n$. Indeed, $Y := \mathbb{A}^n \setminus U$ is a closed subset of dimension at most $n-1$, so the excision exact sequence gives a surjection $CH_*(\mathbb{A}^n) \to CH_*(U)$.

Example 2.12. Let's compute $CH^*(\mathbb{P}^n)$. We have an inclusion of $\mathbb{P}^{n-1}$ as a closed subscheme, with the complement being $\mathbb{A}^n$. Therefore, the excision exact sequence is

$$CH_*(\mathbb{P}^{n-1}) \to CH_*(\mathbb{P}^n) \to CH_*(\mathbb{A}^n) \to 0$$

but we know that $CH^*(\mathbb{A}^n) = \mathbb{Z}$ (generated by the fundamental class) if $* = 0$ and 0 otherwise. Therefore, $CH^*(\mathbb{P}^n)$ is generated by the fundamental class and $CH_*(\mathbb{P}^{n-1})$.

We claim that $CH^*(\mathbb{P}^n) = \mathbb{Z}[h]/h^{n+1}$, where h represents the class of a hyperplane, i.e. the image of the fundamental class of $\mathbb{P}^{n-1}$. Let h' be the hyperplane class of $\mathbb{P}^{n-1}$, which maps to h^2 in $\mathbb{P}^n$. Then if $a(h')^k = 0$, we would have $ah^{2k} = 0$. Since the intersection product is well-defined, we could intersect with an $n-2k$-plane to find that $a = 0$ by Bezout's theorem.

In fact, an easy generalization of this argument shows that whenever X has an affine stratification, i.e. a partition into affine spaces $\{U_i\}$ such that if U_i intersects $\overline{U_j}$, then $U_i \supset U_j$, then CH^*X is generated by the closed strata, i.e. the classes of the $\overline{U_i}$.

Corollary 2.13. *If X has a stratification by open subsets of affine spaces, then $CH_*(X)$ is generated by the classes of its closed strata.*

3. K-THEORY

3.1. **Ordinary K-theory.** Let X be a scheme. There are two possible definitions of a Grothendieck K-group associated to X, using vector bundles or coherent sheaves

Definition 3.1. We define the *Grothendieck group of vector bundles* $K_0(X)$ to be the free abelian group generated by $\{\mathscr{E} = \text{ vector bundle}/X\}$ modulo the equivalence relation $[\mathscr{E}] = [\mathscr{E}'] + [\mathscr{E}'']$ if there exists a short exact sequence

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0.$$

We define the *Grothendieck group of coherent sheaves* $G_0(X)$ to be the free abelian group generated by $\{\mathscr{F} = \text{ coherent sheaf}/X\}$ modulo the equivalence relation $[\mathscr{F}] = [\mathscr{F}'] + [\mathscr{F}'']$ if there exists a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0.$$

Remark 3.2. If $j: Z \hookrightarrow X$ is a closed embedding and $i: U = X \setminus Z \hookrightarrow X$ is open embedding of the complement, then there is a short exact sequence

$$G_0(Z) \xrightarrow{j_*} G_0(X) \xrightarrow{i^*} G_0(U) \to 0.$$

We will not discuss higher K-theory, but we remark that it completes this short exact sequence to a long exact sequence.

There is a forgetful map $K_0(X) \to G_0(X)$, since a vector bundle is a coherent sheaf.

Theorem 3.3. *If X is smooth and projective, then this map is in fact an isomorphism (for instance, there is a finite resolution of any coherent sheaf by locally free sheaves).*

Proof. The key point is that every coherent sheaf has a finite resolution by locally free sheaves.

The easy direction is to show that this is a surjection: there is a map $K_0(X) \to G_0(X)$. If $\mathscr{F}$ is coherent, then $\mathscr{F}$ has a finite resolution by locally free sheaves:

$$0 \to P_n \to \ldots \to P_0 \to \mathscr{F} \to 0.$$

Therefore, $[\mathscr{F}]$ is the image of $\sum(-1)^i[P_i]$.

The harder direction is to define the inverse $G_0(X) \to K_0(X)$. It is not clear that the map $[\mathscr{F}] \mapsto \sum(-1)^i[P_i]$ is well-defined, i.e. this is independent of the resolution and additive, in that if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ then $\chi(P_0) = \chi(P_0'') + \chi(P')$.

This is totally categorical. If $\mathscr{A}$ is an abelian category (in our case, coherent sheaves) and $\mathscr{P} \subset \mathscr{A}$ is an exact subcategory (in our case, locally free sheaves) satisfying the axioms:

(1) If $P_1 \twoheadrightarrow P_0$ is an epimorphism, then its kernel is in $\mathscr{P}$,
(2) Every object $A \in \mathscr{A}$ has a finite resolution

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to A \to 0 \quad P_i \in \mathscr{P}$$

then $K_0(P) \cong K_0(G)$.

In Fulton's *Intersection Theory*, Appendix B.8.3 (440) the proof is outlined. There is a reference to a paper of Borel-Serre (1958).

Let's give the proof. There are two key facts concerning a surjection $\mathscr{F}' \twoheadrightarrow \mathscr{F}$:

(1) If $P_\bullet$ is a finite resolution, then there exists a finite resolution $P'_\bullet \to \mathscr{F}'$ with epimorphism $P'_\bullet \twoheadrightarrow P$ (whose kernel is then in $\mathscr{P}$ by the first axiom).
(2) If $P_\bullet \to \mathscr{F}$ and $P'_\bullet \to \mathscr{F}$ are two resolutions, then there exists a resolution $P'' \to \mathscr{F}$ with $P''_\bullet \twoheadrightarrow P'_\bullet$ and $P''_\bullet \twoheadrightarrow P_\bullet$.

Lemma 3.4 ((Lemma 14 in Borel-Serre, but less general))**.** *If $\mathscr{F} \in \mathscr{A}$ and we have two short exact sequence*

$$0 \to K \to P \to \mathscr{F} \to 0$$

and

$$0 \to K' \to P' \to \mathscr{F} \to 0$$

then there exists a short exact sequence

$$0 \to K'' \to P'' \to \mathscr{F} \to 0$$

surjecting onto the previous two.

Proof. Given P and P', we take the "fibered product"

$$\begin{array}{ccc} P \times_{\mathscr{F}} P' & \longrightarrow & P' \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathscr{F} \end{array}$$

The fibered product surjects to P and P', as the original maps were. Now it may not lie in $\mathscr{P}$, but we let P_2 be any object of $\mathscr{P}$ surjecting to $P \times_{\mathscr{F}} P'$. Let P_3 surject to K, P'_3 surject to K', then $P'' = P_2 + P_3 + P'_3$. Then define the correct map $P'' \to \mathscr{F}$. ☐

Another trick? Given $\mathscr{F}' \to \mathscr{F}$, and a resolution $P_\bullet \to \mathscr{F}$, we can find a resolution $P'_\bullet \to \mathscr{F}'$ such that

$$\begin{array}{ccc} P'_\bullet & \longrightarrow & P_\bullet \\ \downarrow & & \downarrow \\ \mathscr{F}' & \longrightarrow & \mathscr{F}. \end{array}$$

Then, if $P_\bullet$ and $P'_\bullet$ are two resolutions of $\mathscr{F}$, then we form

$$\begin{array}{ccccc} P''_0 & \longrightarrow & P'_0 \oplus P'_1 & & \\ \downarrow & & \downarrow & & \\ \mathscr{F} & \xrightarrow{\Delta} & \mathscr{F} \oplus \mathscr{F} & \longrightarrow & F \end{array}$$

☐

3.2. **Properties of K-theory.** The group $K_0(X)$ is in some sense the more natural object - it was first defined by topologists, in a topological setting. However, for various purposes $G_0(X)$ is actually easier to work with (even though they are the same in many of settings). This is exhibited in the functoriality properties discussed below: many of them are defined on one of the K-groups but not clearl on the other.

Product structure. The groups $K_0(X)$ has a *product structure* induced by the tensor product of vector bundles or coherent sheaves. The tensor product also defines an *action* of $K_0(X)$ on $G_0(X)$, giving the latter the structure of a $K_0(X)$-module.

Pullback. If $f: X \to Y$ is any morphism, then there is a map induced by pullback

$$f^*: K_0(Y) \to K_0(X)$$

taking $[\mathscr{E}] \mapsto [f^*\mathscr{E}]$. This is well-defined because pullback is exact on vector bundles.

Proper pushforward. If $f: X \to Y$ is *proper*, then there is a pushforward $G_0(X) \to G_0(Y)$ related to the Euler characteristic:

$$f_*[\mathscr{F}] = \sum_i (-1)^i [R^i f_* \mathscr{F}].$$

The properness is needed, of course, to ensure that the pushforward is still coherent, and also to ensure that the pushforward is exact.

Example 3.5. If X is proper over Spec k (e.g. X is projective), then

$$f_*[\mathscr{F}] = \sum_i (-1)^i [H^i(X, \mathscr{F})] = \chi(\mathscr{F}) \in G_0(\mathrm{pt}) \cong \mathbb{Z}.$$

Flat pullback. If $f: X \to Y$ is *flat*, then there is a pullback map

$$f^*: G_0(Y) \to G_0(X)$$

sending $[\mathscr{F}] \mapsto [f^*\mathscr{F}]$. The flatness is needed to ensure that the relations are preserved (i.e. pulling back preserves exactness).

Projection formula. If $f: X \to Y$ is proper, then the *projection formula* says that for $\alpha \in K_0(Y)$ and $\beta \in G_0(X)$,

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta.$$

In other words, f_* is a $K_0(Y)$-module homomorphism. This is a *very important* fact!s

4. Intersection theory and Chern classes

4.1. **Chern classes.** The reader may be familiar with the construction of Chern classes from algebraic topology. In our algebraic setting, the Chern classes of K-theory classes on X will be valued in the *Chow ring* of X. The usual "topological Chern class" is then obtained by the cycle class map from the Chow ring to cohomology.

Let's first discuss the case of vector bundles. The Chern class of a vector bundle is a class in the Chow ring, satisfying some properties. First, it is specified in the elemental case of line bundles. Next, for a vector bundle built out of smaller vector bundles, there is an expression for its Chern class in terms of the Chern classes of the smaller bundles. Finally, the Chern class is required to satisfy a certain functoriality property.

Definition 4.1. The *Chern class* is an assigment from vector bundles on smooth varieties X to elements of $CH^*(X)$, sending a rank r vector bundle E to

$$E \rightsquigarrow c(E) = 1 + c_1(E) + c_2(E) + \ldots + c_r(E) \in CH^*(X),$$

with the following properties.

(1) If $\mathscr{L}$ is a line bundle on X, and $\mathscr{L} \cong \mathscr{O}(D)$ for a Weil divisor D, then $c_1(\mathscr{L}) = [D] \in CH^1(X)$.

(2) If we have a short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

then we demand that

$$c(E) = c(E')c(E'').$$

(3) If $f : X \to Y$ is a morphism of smooth varieties and E is a vector bundle on Y, then

$$c(f^*E) = f^*c(E).$$

Although this definition is not constructive, it is robust in practice for computing Chern classes.

Example 4.2. If L_1 and L_2 are two line bundles, then

$$c(L_1 \otimes L_2) = 1 + c_1(L_1) + c_2(L_2)$$

because tensor product corresponds to addition of Weil divisors.

Example 4.3. Let's compute $c(T\mathbb{P}^n)$. First, we have an embedding of the tautological bundle on $\mathbb{P}^n$ into $\mathbb{P}^n \times \mathbb{C}^{n+1}$:

$$0 \to \mathscr{O}(-1) \to \mathscr{O}^{n+1} \to Q \to 0.$$

The fiber of the quotient Q at $[L]$ is canonically $\mathbb{C}^{n+1}/L$. Tensoring with $\mathscr{O}(1)$, we obtain

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{n+1} \to Q(1) \to 0. \tag{4.1}$$

Thinking of $\mathscr{O}(1)$ as the dual of the tautological bundle, its fiber at $[L]$ is canonically $\mathrm{Hom}([L], \mathbb{C})$. Therefore, the fiber of $Q(1)$ at $[L]$ is canonically $\mathrm{Hom}([L], \mathbb{C}^{n+1}/[L])$, which is the tangent space at $[L]$. Thus (4.1) becomes

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{n+1} \to T\mathbb{P}^n \to 0. \tag{4.2}$$

Exercise 4.4. Prove (4.2) rigorously to your satisfaction.

We conclude that

$$c(T\mathbb{P}^n) = c(T\mathbb{P}^n)c(\mathcal{O}) = c(\mathcal{O}(1)^{n+1}) = (1+h)^{n+1}.$$

Grothendieck proved that every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. For $\mathbb{P}^2$, we see that $c(T\mathbb{P}^2) = 1+3h+3h^2$. This doesn't factor into a product of two linear polynomials, so $T\mathbb{P}^2$ cannot be an extension of two line bundles.

Extensions. Now we can extend the definition of the Chern class to $K_0(X)$ by demanding it to be a group homomorphism from $K_0(X)$ to the multiplicative group of $CH^*(X)$. Concretely, this is determined by $c(-[E]) = c(E)^{-1} \in CH^*(X)$ - this makes sense because the form $c(E) = 1+\dots$ implies that $c(E)$ is invertible in $CH^*(X)$.

Since $K_0(X) \cong G_0(X)$ if X is a smooth variety, this also defines a notion of Chern class on coherent sheaves. Concretely, if $\mathscr{F}$ is a coherent sheaf on X then we have a finite resolution

$$0 \to \mathscr{E}_n \to \mathscr{E}_{n-1} \to \dots \to \mathscr{E}_1 \to \mathscr{F} \to 0$$

so that $[\mathscr{F}] = -\sum_{i=1}^n (-1)^i [\mathscr{E}_i]$, and

$$c(\mathscr{F}) = \frac{\prod_{i \text{ odd}} c(\mathscr{E}_i)}{\prod_{i \text{ even}} c(\mathscr{E}_i)}.$$

4.2. **The splitting principle.** The *splitting principle* says that

> For the purpose of proving identities between Chern classes of vector bundles, one can assume that the vector bundles are sums of line bundles.

This makes computations easier because we know very explicitly the Chern classes of line bundles. If $E = L_1 \oplus \dots \oplus L_r$, then

$$c(E) = \prod_{i=1}^r (1 + c_1(L_i)).$$

Thus, $c_d(E)$ is the dth elementary symmetric polynomial in the $c_1(L_i)$.

Let's illustrate with some examples, and then revisit the question of why this works.

Example 4.5. Let E and E' be two vector bundles. Then what is the Chern classes of $E \otimes E'$? The splitting principle tells us to *pretend* that $E = L_1 \oplus \dots \oplus L_m$ and $E' = L'_1 \oplus \dots \oplus L'_n$. Then

$$E \otimes E' = \bigoplus_{i,j} L_i \otimes L'_j$$

so

$$c(E \otimes E') = \prod_{i,j} (1 + c_1(L_i))(1 + c_1(L'_j)).$$

For instance, we find that

$$\begin{aligned} c_1(E\otimes E') &= \sum_{i,j}(c_1(L_i)+c_1(L'_j)) \\ &= n\sum_i c_1(L_i)+m\sum_j c_1(L'_j) \\ &= nc_1(E)+mc_1(E'). \end{aligned}$$

Although we started the calculation under the assumption that E and E' split, the final answer $c_1(E\otimes E') = nc_1(E)+mc_1(E')$ does not reference this assumption. The splitting principle tells us that this is true even if the bundles do not split.

Example 4.6. If $E\cong\bigoplus L_i$, then $\det E = \bigotimes L_i$. Therefore,

$$c_1(E) = c_1(\det E).$$

The splitting principle tells us that this is true for general E.

The justification for the splitting principle is that for any vector bundle E on X, one can find a map $f\colon X'\to X$ such f^*E has a filtration by line bundles, and also the induced map on Chow rings $f^*\colon CH^*(X)\to CH^*(X')$ is injective. The existence of the filtration implies that $c(f^*E)$ is the product of the Chern classes of its line bundle factors. The injectivity implies that to prove an identity for $c(E)$ on X, it suffices to prove the pulled-back identity on X', when one has $c(E)$ splits.

Let $V\to X$ be a vector bundle. We "wish" that $V = L_1\oplus L_2\oplus\ldots\oplus L_n$. One can always "achieve" this in topology by an appropriate pullback, but it is too ambitious to demand in algebraic geometry. Instead, we "wish" that there were a *filtration*

$$V\supset V_{n-1}\supset V_{n-2}\supset\ldots\supset 0$$

with each successive quotient $V_i/V_{i-1}\cong L_i$ a line bundle.

The space X' can be constructed as follows. For a vector bundle $V\to X$, we create a *flag bundle* $\mathrm{Fl}(V)$ equipped with a map $\pi\colon \mathrm{Fl}(V)\to X$. The fiber of $\mathrm{Fl}(V)\to X$ over $x\in X$ is the (full) flag variety the fiber V_x, which is a projective variety. Tautologically the pullback of V to $\mathrm{Fl}(V)$ has a filtration by line bundles.

So why is the pullback $CH^*(X)\to CH^*(\mathrm{Fl}(V))$ injective? The map $\mathrm{Fl}(V)\to X$ can be realized as a sequence of projective bundles (choosing one step of the filtration at a time). Now the result follows from a general property of projective bundles, as you can create a rational section by pulling back a class and intersecting with an appropriate power of the hyperplane class coming from the tautological bundle.

To elaborate, suppose $\pi\colon P\to X$ is a projective bundle of relative dimension r (i.e. the fibers are $\mathbb{P}^r$). If $\alpha\in CH^d(X)$, then $\pi^*\alpha$ lives in $CH^{d+r}(X)$ and is represented fiberwise by a dimension r variety. There is a tautological line bundle $\mathscr{L}$ on P, whose dual has first Chern class represented by a hyperplane in each fiber $\mathbb{P}^r$. Then $\pi_*(\pi^*\alpha\cap c_1(\mathscr{L}^*)^r)$ is a multiple of α.

4.3. **Some characteristic classes.** Let $E\to X$ be a vector bundle of rank r. We define the *Chern roots* $\alpha_1,\ldots,\alpha_r$ of E to be the formal roots of

$$c(E) = 1+c_1(E)t+c_2(E)t^2+\ldots+c_r(E)t^r,$$

i.e.

$$\prod_{i=1}^{r}(1+a_i t) = 1 + c_1(E)t + c_2(E)t^2 + \ldots + c_r(E)t^r$$

so $c_d(E)$ is the dth elementary symmetric polynomial in the α_i. If $E = \bigoplus L_i$ then $\alpha_i = c_1(L_i)$, but in general the $\alpha_i i$ may not exist in $CH^*(X)$. They play the role of hypothetical Chern classes in the pretend world generated by the splitting principle. In computations, the roles of the L_i are symmetric and so any expressions obtained in the a_i can be rephrased in terms of the honest Chern classes of E.

Definition 4.7. If E is a vector bundle on X, then the *Chern character* is

$$\operatorname{ch}(E) = \sum \exp(\alpha_i)$$

where the α_i are the Chern roots of E.

By an application of the splitting principle,

$$\operatorname{ch}(E \oplus E') = \operatorname{ch}(E) \oplus \operatorname{ch}(E')$$

and

$$\operatorname{ch}(E \otimes E') = \operatorname{ch}(E) \cdot \operatorname{ch}(E').$$

Therefore, the Chern character defines a *ring homomorphism* $K_0(X) \to CH^*(X)$.

Definition 4.8. The *Todd class* of E is

$$\operatorname{Td}(E) = \prod_i \frac{\alpha_i}{1-\exp(-\alpha_i)} = 1 + \frac{1}{12}c_1 + \frac{1}{12}(c_1^2 + c_2) + \ldots.$$

Note that this is invertible.

Example 4.9. In Example 4.3 we computed that $c(T\mathbb{P}^n) = (1+h)^{n+1}$. This is the same as the Chern class of $\mathcal{O}(1)^{n+1}$, so the Todd class of $T\mathbb{P}^n$ is the same as the Todd class of $\mathcal{O}(1)^{n+1}$. But the latter bundle clearly has Chern roots h with multiplicity $n+1$, so we conclude that

$$\operatorname{Td}(T_X) = \left(\frac{h}{1-e^{-h}}\right)^{n+1}$$

4.4. Interpretation as degeneracy loci. ♠♠♠ TONY: [should I say anything?]

5. THE RIEMANN-ROCH THEOREM

5.1. **An overview.** To motivate the equivariant Riemann-Roch Theorem and the objects that need to be introduced, we will give a very brief overview of the classical Riemann-Roch Theorem and its generalizations.

One perspective on the Riemann-Roch Theorem is that it is a tool to compute $h^0(X,\mathscr{L})$ for a line bundle $\mathscr{L}$ on a variety X. You can view this as a problem in *analysis*, since for curves it boils down to a question about the existence of meromorphic functions with prescribed zeros and poles.

It turns out that the quantity $h^0(X,\mathscr{L})$ is not so well-behaved, and a much more tractable quantity is the *Euler characteristic*:

$$\chi(\mathscr{L}) = \sum_i (-1)^i h^i(X,\mathscr{L}).$$

Theorem 5.1 (Riemann-Roch Theorem)**.** *If X be a smooth projective curve and D is a divisor on X, then*

$$\chi(X,D) = \deg D + 1 - g.$$

If X is a smooth projective curve, then *Serre duality* asserts that

$$h^1(X,\mathscr{L}) \cong h^0(X, K_X \otimes \mathscr{L}^\vee).$$

This gives the usual reformulation left hand side as

$$h^0(X,D) - h^0(X,K-D).$$

The Riemann-Roch theorem may be re-interpreted in terms of intersection theory. If you are familiar with Atiyah-Singer index theory, then you know the philosophy that "topological information gives analytic information." The Riemann-Roch theorem is a result in this spirit (in fact, over $\mathbb{C}$ it is a special case of the Atiyah-Singer index theorem). This is the formulation generalized by Hirzebruch to higher dimensional varieties.

Theorem 5.2 (Hirzebruch-Riemann-Roch)**.** *Let X be a smooth projective variety and $\mathscr{E}$ a vector bundle on X. Then*

$$\chi(\mathscr{E}) = \deg(\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(T_X))$$

where $\mathrm{ch}(\mathscr{E})$ *is the Chern character of* $\mathscr{E}$ *and* Td *is the Todd class.*

Grothendieck recognized that this should be an instance of a more general assertion about a *morphism* of schemes, in the special case where the target is a point.

Theorem 5.3 (Grothendieck-Riemann-Roch)**.** *If $f: X \to Y$ is a proper morphism of smooth projective varieties, then*

$$\mathrm{ch}(f_*\mathscr{E}) = f_*(\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(T_f)).$$

One can view this as relating two kinds of pushforwards in some sense: the left hand side is a pushforward in K-theory, and the right hand side is an intersecton theoretic poushforward.

5.2. **Curves and surfaces.** We review the easy versions of Riemann-Roch theorems in the familiar example of curves, and the perhaps less-familiar example of surfaces.

Let X be a smooth projective curve. The key to the Riemann-Roch theorem is that the Euler characteristic is additive in short exact sequences. Let D be a divisor on X, and $p \in X$. Then we have a short exact sequence

$$0 \to \mathscr{O}(D-p) \to \mathscr{O}(D) \to \mathscr{O}(D)|_p \to 0.$$

Thus, by the additivity of χ,

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D-p)) + \chi(\mathscr{O}_p) = 1 + \chi(\mathscr{O}(D-p)).$$

Now if we define the *genus* of X by

$$\chi(\mathscr{O}_X) = 1 - g$$

it follows from an easy induction that

$$\chi(\mathscr{O}_D) = \deg D + 1 - g.$$

This is the Riemann-Roch theorem for curves, without the "hard" input of Serre dualitly.

Next let's turn our attention to surfaces. If X is a surface and D is a divisor on X, and C is an irreducible closed curve in X, then by the same reasoning as before we have

$$0 \to \mathscr{O}(D-C) \to \mathscr{O}(D) \to \mathscr{O}(D)|_C \to 0.$$

Thus,

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D-C)) + \chi(\mathscr{O}(D)|_C).$$

Now we can apply the Riemann-Roch theorem for curves to deduce that

$$\chi(\mathscr{O}(D)|_C) = \deg \mathscr{O}(D)|_C + 1 - g(C).$$

It follows easily from the definition of the intersection product that $\deg \mathscr{O}(D)|_C = D \cdot C$, and $g(C)$ is determined by the adjunction formula

$$2g(C) - 2 = (C + K_X) \cdot C = C \cdot C + K_X \cdot C.$$

To summarize, we have found that

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}(D-C)) + \frac{1}{2}(2D - C - K_X) \cdot C$$

and an easy induction proves:

Theorem 5.4 (Riemann-Roch for surfaces)**.** *Let X be a smooth projective surface and D a divisor on X. Then*

$$\chi(\mathscr{O}(D)) = \chi(\mathscr{O}) + \frac{1}{2}(D - K_X) \cdot D.$$

5.3. **Hirzebruch-Riemann-Roch.** We recall the statement of the Hirzebruch-Riemann-Roch theorem, stated in a slightly different way.♠♠♠ TONY: [I do not know if this is how Hirzebruch thought about it... probably not] Let X be a smooth proper variety over a field k and $\mathscr{E}$ a vector bundle on X. The Riemann-Roch theorem is concerned with a formula for the Euler characteristic of $\mathscr{E}$. Another way to think about this is that $\mathscr{E}$ represents a K-theory class $[\mathscr{E}]$, and the Euler characteristic is precisely the image of $[\mathscr{E}]$ under pushforward to $K_0(\text{pt} = \operatorname{Spec} k) \cong \mathbb{Z}$, where the latter identification is by the (virtual) dimension.

Now, one way to "access" K-theory is by mapping it to cohomology, or in our case the Chow ring, via the Chern class. We just discussed that there is a *ring homomorphism* $\text{ch}\colon K_0(X) \to CH^*(X)$ given by the Chern character, so it is natural to ask what the pushforward on K-theory corresponds to at the level of Chow rings.

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\text{ch}} & CH^*(X) \\ \downarrow{\scriptstyle \pi_*} & & \vdots \\ K_0(\text{pt}) & \xrightarrow{\text{ch}} & CH^*(\text{pt}). \end{array}$$

The obvious guess would be that the right hand side is also given the pushforward on Chow groups. However, this is not quite right. It was Hirzebruch who realized that to get commutativity, one has to "twist" by the Todd class. To make this well-defined, we must pass to rational coefficients (i.e. tensor everything with $\mathbb{Q}$).

Theorem 5.5 (Hirzebruch-Riemann-Roch)**.** *Let X be a smooth projective variety and $\mathscr{E}$ a vector bundle on X. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}\cdot\text{Td}(X)} & CH^*_{\mathbb{Q}}(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(\text{pt})_{\mathbb{Q}} & \xrightarrow{\text{ch}\cdot\text{Td}(X)} & CH^*_{\mathbb{Q}}(\text{pt}) \end{array}$$

Example 5.6. Let's apply the Hirzebruch-Riemann-Roch Theorem to a curve and see what we get. Let X be a curve and $\mathscr{E}$ a vector bundle of rank r on X. Then the HRR tells us that the following diagram commutes:

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}\cdot\text{Td}(X)} & CH^*_{\mathbb{Q}}(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(\text{pt})_{\mathbb{Q}} & \xrightarrow{\text{ch}\cdot\text{Td}(X)} & CH^*_{\mathbb{Q}}(\text{pt}) \end{array}$$

As discussed above, under the identification $K_0(\text{pt})_{\mathbb{Q}} \cong CH^*_{\mathbb{Q}}(\text{pt}) \cong \mathbb{Q}$ we have

$$\pi_*([\mathscr{E}]) = \chi(\mathscr{E}) = h^0(X, \mathscr{E}) - h^1(X, \mathscr{E}).$$

To compute what happens on the other side, we have to figure out the Chern character of $\mathscr{E}$ and the Todd class of X. If E has Chern roots α_i, then

$$\mathrm{ch}(E) = \sum_i \exp(\alpha_i) = r + \sum_i \alpha_i = r + c_1(\mathscr{E}).$$

(There are no higher terms because we are working on a curve.)

Next, if TX has Chern roots β_j then the Todd class is

$$\begin{aligned}\prod_j \frac{\beta_j}{1-e^{-\beta_j}} &= \prod_j \frac{\beta_j}{\beta_j - \frac{1}{2}\beta_j^2} \\ &= \prod_j \frac{1}{1-\frac{1}{2}\beta_j} \\ &= 1 + \frac{1}{2}c_1(TX)\end{aligned}$$

On the curve X we have $TX = (K_X)^*$, so $c_1(TX) = -c_1(K_X)^*$.

When pushing forward to $CH_*(\mathrm{pt})$, the classes in $CH_1(X)$ die. Therefore, HRR tells us that

$$\begin{aligned}\chi(\mathscr{E}) &= (r + c_1(\mathscr{E}))(1 - \frac{1}{2}c_1(K_X)) \\ &= \deg c_1(\mathscr{E}) + r(1-g).\end{aligned}$$

This is Riemann-Roch for curves!

Example 5.7. Let's apply HRR to surfaces. If $\mathscr{E}$ is a rank r vector bundle on X, then we have as before

$$\chi(\mathscr{E}) = \pi_*(c_1(\mathscr{E}) \cdot \mathrm{Td}(X)).$$

This time, if $\mathscr{E}$ has Chern roots α_i then

$$\begin{aligned}c_1(\mathscr{E}) &= \sum \exp(\alpha_i) \\ &= r + \sum \alpha_i + \frac{1}{2}\alpha_i^2 \\ &= r + c_1(\mathscr{E}) + \frac{1}{2}(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E}))\end{aligned}$$

If TX has Chern roots β_j then

$$\begin{aligned}\mathrm{Td}(TX) &= \sum_j \frac{\beta_j}{\beta_j - \frac{1}{2}\beta_j^2 + \frac{1}{6}\beta_j^3} \\ &= \sum_j \frac{1}{1 - \frac{1}{2}\beta_j + \frac{1}{6}\beta_j^2} \\ &= r + \frac{1}{2}\sum \beta_j - \frac{1}{6}\sum \beta_j^2 + \frac{1}{4}\sum \beta_j^2.\end{aligned}$$

Again, when taking the pushforward only the degree two terms, corresponding to $CH_0(X)$, survive. Separating out the terms which are already present in the trivial bundle, we obtain

$$\chi(\mathscr{E}) - r\chi(\mathscr{O}) = c_1(\mathscr{E}) \cdot \frac{1}{2} c_1(TX) + \frac{1}{2} c_1(\mathscr{E})^2.$$

If $\mathscr{E} \cong \mathscr{O}(D)$ is a line bundle, then $\frac{1}{2}c_1(\mathscr{E})^2 = \frac{1}{2} D \cdot D$ and $c_1(\mathscr{E}) \cdot \frac{1}{2} c_1(TX) = D \cdot \frac{1}{2} K_X$.

Remark 5.8. The formula $\chi(\mathscr{E}) = \deg(\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(T_X))$ of Hirzebruch-Riemann-Roch can be rephrased suggestively in terms of Chern-Weil theory as

$$\chi(\mathscr{E}) = \int \mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(T_X).$$

Example 5.9. Let Y be a smooth projective varieties with (étale) fundamental group finite of order d, and $\pi \colon X \to Y$ the "universal cover" of Y. Intuitively, we should have

$$\chi(\mathscr{O}_X) = d\,\chi(\mathscr{O}_Y).$$

Let's prove this as an application of Hirzebruch-Riemann-Roch. HRR tells us that

$$\chi(\mathscr{O}_X) = \deg(\mathrm{ch}(\mathscr{O}_{\mathscr{X}} \cdot \mathrm{Td}(T_X)).$$

Since $X \to Y$ is étale, we have $TY = \pi^* TX$. Therefore,

$$\chi(\mathscr{O}_X) = \deg(\mathrm{ch}(\mathscr{O}_{\mathscr{X}} \cdot \mathrm{Td}(T_X)) = \deg(\mathrm{ch}(\pi^*\mathscr{O}_Y) \cdot \mathrm{Td}(\pi^* TY)).$$

Now, taking the degree is the same as cupping with the fundamental class, so by the projection formula

$$\chi(\mathscr{O}_X) = d \deg(\mathscr{O}_Y \cdot \mathrm{Td}(TY)).$$

The right hand side is $\chi(\mathscr{O}_Y)$ by another application of Hirzebruch-Riemann-Roch.

5.4. **Grothendieck-Riemann-Roch.** At least in the form in which we stated the Hirzebruch-Riemann-Roch Theorem, Grothendieck's generalization contains few surprises. Grothendieck recognized that the HRR theorem was an instance of a more general relation on *families* over some base, in the special case where the base is a point.

Let $f \colon X \to Y$ be a proper morphism, and think of X as a "family" over Y. Grothendieck had the insight that given a vector bundle on X (i.e. a *family* of vector bundles over Y) a vector bundle on Y. The most naïve attempt would be to push forward, but one knows that this doesn't necessarily produce a vector bundle. However, at the level of K-theory everything does make sense - the constant of the Euler characteristic reflects the fact that the pushforward makese sense as a "virtual vector bundle." In the spirit of Hirzebruch, one can again ask what the picture looks like at the level of Chow rings.

Theorem 5.10 (Grothendieck-Riemann-Roch)**.** *Let $f \colon X \to Y$ be a proper morphism of nonsingular varieties. Then the following diagram commutes:*

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(X) \\ \downarrow f_* & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(Y) \end{array}$$

We will now embark on a proof of the Grothendieck-Riemann-Roch Theorem in the special case of *projective* morphisms. First we make some preliminary redictions. From the form of the theorem, it is immediate that it is "compatible under composition." In other words, if $g\colon Y \to Z$ is another proper morphism of smooth schemes, then we can concatenate the two commutative diagrams and deduce the result for the composite map.

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot \mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(X) \\ \downarrow f_* & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot \mathrm{Td}(Y)} & CH^*_{\mathbb{Q}}(Y) \\ \downarrow g_* & & \downarrow g_* \\ K_0(Z) & \xrightarrow{\mathrm{ch}\cdot \mathrm{Td}(Z)} & CH^*_{\mathbb{Q}}(Z) \end{array}$$

Therefore, to prove the theorem for projective morphisms it suffices to establish it in two special cases:

(1) The projection map $\pi\colon \mathbb{P}^n \times X \to X$, and
(2) A regular embedding $X \hookrightarrow Y$.

5.5. **GRR for projective spaces.** Let's first study the structure morphism over $Y = \operatorname{Spec} k$ and $X = \mathbb{P}^n_k$. Then for $\mathscr{F} \in K^*(X)$, the pushforward $\pi_*\mathscr{F}$ is just the Euler characteristic under the natural identification $K_0(Y) \cong \mathbb{Z}$ via the dimension.

Lemma 5.11. *The group $K(X)$ is generated by the line bundles $\mathscr{O}(n)$.*

Proof. The Hilbert Syzygy Theorem shows that every coherent sheave can be resolved by (direct sums of) line bundles, and every line bundle on $\mathbb{P}^n$ is $\mathscr{O}(n)$ for some n. ☐

Alternative proof. Induct using the excision sequence for K-theory:

$$K_0(\mathbb{P}^{m-1}) \to K_0(\mathbb{P}^m) \to K_0(\mathbb{A}^n) \to 0.$$

By homotopy invariance, $K_0(\mathbb{A}^m)_{\mathbb{Q}} \cong K_0(\text{point})_{\mathbb{Q}} = \mathbb{Q}$. By induction, $G_0(\mathbb{P}^{m-1})$ is generated by line bundles. The image in $K_0(\mathbb{P}^m)$ is then generated by the classes of line bundles on hyperplanes, but those are just restrictions of line bundles on $\mathbb{P}^m$ via the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^m}(n-1) \to \mathscr{O}_{\mathbb{P}^m}(n) \to \mathscr{O}_{\mathbb{P}^{m-1}}(n) \to 0.$$

☐

Since the claimed commutative diagram of Theorem 5.10 is additive, it suffices to show the result for $\mathscr{O}(n)$. Now recall that for $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^{\geq 0}$,

$$\boxed{\chi_{\mathbb{P}^m}(\mathscr{O}(n)) = \binom{n+m}{n} = \frac{(n+m)\dots(n+1)}{n!}.}$$

We just have to check that this agrees with what we get from taking the Chern character, twisting by the Todd class, and then taking the pushforward.

- It is immediate from the definition that $\mathrm{ch}(\mathscr{O}(n)) = e^{nh}$ where h is the hyperplane class.
- It was discussed in Example 4.9 that

$$\mathrm{Td}(T_{\mathbb{P}^m}) = \left(\frac{h}{1-e^{-h}}\right)^{m+1}.$$

So tracing across the top and right in the GRR diagram gives

$$\begin{array}{ccc} [\mathscr{O}(n)] & \longmapsto & e^{nh}\left(\frac{h}{1-e^{-h}}\right)^{m+1} \\ & & \downarrow \pi_* \\ & & ? \end{array}$$

The pushforward on $CH^*(X)$ takes the degree of the component in $CH_0(X)$, which is the coefficient of h^m. Therefore, the Grothendieck-Riemann-Roch theorem in this case boils down to the fact that the coefficient of h^m in $e^{nh}\frac{h^{m+1}}{(1-e^{-h})^{m+1}}$ is $\binom{n+m}{m}$.

This is the residue of $\frac{e^{nh}}{(1-e^{-h})^{m+1}}\,dh$ at $h=0$. Setting $z = e^h$, we are looking for the residue of

$$\frac{z^n}{(1-z^{-1})^{m+1}}\,d(\log z) \text{ at } z=1.$$

Now,

$$\begin{aligned} \mathrm{Res}_{z=1}\frac{z^n}{(1-z^{-1})^{m+1}}\,d(\log z) &= \mathrm{Res}_{z=1}\frac{z^n z^{m+1}}{(z-1)^{m+1}}\frac{d(z-1)}{z} \\ &= \mathrm{Res}_{z=1}\frac{z^{n+m}}{(z-1)^{m+1}}\,dz \\ &= \mathrm{Res}_{z=1}\frac{(z-1+1)^{n+m}}{(z-1)^{m+1}}\,dz. \end{aligned}$$

Now the result is clearly the coefficient of $(z-1)^m$ in $(z-1+1)^{n+m}$, which is precisely $\binom{n+m}{m}$. That completes the proof of GRR for projective space *over a field*.

Now we tackle the more general case of projective space over an arbitrary base: $X = \mathbb{P}^m \times Y \xrightarrow{\pi} Y$. We want to show that the diagram commutes:

$$\begin{array}{ccc} K_0(\mathbb{P}^m\times Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(\mathbb{P}^m\times Y) \\ \downarrow \pi_* & & \downarrow \pi_* \\ K_0(Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(Y)} & CH^*_{\mathbb{Q}}(Y) \end{array}$$

The strategy will be to show that the diagram is "base changed" from $Y = \mathrm{Spec}\, k$ in an appropriate sense.

Lemma 5.12. *The natural maps*

$$K_0(Y)\otimes K_0(\mathbb{P}^m) \to K_0(\mathbb{P}^m\times Y)$$

induced by the exterior (tensor) product, and

$$CH^*(Y) \otimes CH^*(\mathbb{P}^m) \to CH^*(\mathbb{P}^m \times Y)$$

induced by pullback and intersection product are both surjective.

Proof. We argue by induction. The base case $m = 0$ is obvious. We can cutting $Y \times \mathbb{P}^m$ along $Y \times \mathbb{P}^{m-1}$ (a fiberwise hyperplane section), and apply the excision exact sequence

$$CH_*(Y \times \mathbb{P}^{m-1}) \to CH_*(Y \times \mathbb{P}^m) \to CH_*(Y \times \mathbb{A}^m) \to 0.$$

By the homotopy axiom, we have $K_0(Y \times \mathbb{A}^m) \cong K_0(Y)$. This, together with some accounting of the ring map $CH^*(\mathbb{P}^m) \to CH^*(\mathbb{P}^{m-1})$, establishes the result. $\square$

Now, the splitting $T_X = T_{\mathbb{P}^m} \oplus T_Y$ induces a factorization $\mathrm{Td}(X) = \mathrm{Td}(Y)\mathrm{Td}(\mathbb{P}^m)$. The exterior product induces a factorization of diagrams

$$\begin{array}{ccc} K_0(Y)_{\mathbb{Q}} \times K_0(\mathbb{P}^m)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(Y),\mathrm{ch}\cdot\mathrm{Td}(\mathbb{P}^n)} & CH^*_{\mathbb{Q}}(Y) \times CH^*_{\mathbb{Q}}(\mathbb{P}^m) \\ \downarrow & & \downarrow \\ K_0(\mathbb{P}^m \times Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(\mathbb{P}^m \times Y) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\ K_0(Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(Y)} & CH^*_{\mathbb{Q}}(Y) \end{array}$$

The outer diagram commmutes because it is the product of $K_0(Y)$ with the GRR theorem for Spec k. Then surjectivity implies that the lower diagram commutes, which establishes the theorem.

5.6. **Why the Todd class?** Let's think about reversing-engineering this formula. We seek some power series $f(x)$ such that the coefficient of x^{-1} in $e^{nx} f(x)^{m+1}$ is $\binom{n+m}{m}$.

5.7. **GRR for regular embeddings.** We now study the case of a regular embedding $X \hookrightarrow Y$. Topologically, we should be able to factor a regular embedding $X \hookrightarrow Y$ through the *tubular neighborhood* of Y. An algebro-geometric linearization of this situation is to think of X as sitting in its normal bundle, so let's tackle that case first.

Embedding in normal bundle. Let $p \colon N \to X$ be a vector bundle, with the zero section $\sigma \colon X \to Y$. Let's replace V with its projective closure Y, but (abusing notation) retain the notation p and σ.

$$\begin{array}{ccc} N & \hookrightarrow & Y = \mathbb{P}(N \oplus \mathcal{O}_X) \\ & {\scriptstyle \sigma} \nwarrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

Now Y has a tautological line bundle $\mathcal{O}(-1)$, being a projective bundle, which sits in an exact sequence

$$0 \to \mathcal{O}(-1) \to p^*(N \oplus \mathcal{O}_X) \to Q \to 0.$$

We claim that $\sigma^* Q = N$. Indeed, in terms of coordinates $(\vec{x}, \vec{n}, z)$ on Y, the coordinates on $p^*(N \oplus \mathcal{O}_X)$ can also be thought of as $\vec{n}, z$ and the map $\mathcal{O}(-1) \to p^*(N \oplus \mathcal{O}_X)$ is the inclusion of "$(\vec{x}, \vec{n}, z, \vec{n}, z) \hookrightarrow (\vec{x}, \vec{n}, z, \vec{n}', z')$." Now it is clear that when restricted to the subset $\sigma(X) \subset Y$, which is defined by the coordinates $\vec{n} = 0$, the map $\mathcal{O}(-1)|_{\sigma(X)} \to p^*(N \oplus \mathcal{O}_X)|_{\sigma(X)}$ maps isomorphically to the summan $p^*\mathcal{O}_X$, and the quotient is isomorphic to N.

There is a global section s of Q, obtained from the map $p^*\mathcal{O}_X \cong \mathcal{O}_Y \to Q$. This vanishes precisely on the subset where the map $\mathcal{O}(-1) \to p^*(N \oplus \mathcal{O}_X)$ coincides with the inclusion of the summan $p^*\mathcal{O}_X$, which from the above discussion is precisely $\sigma(X)$.

Exercise 5.13. Go through these calculations rigorously if you are not convinced.

We want to prove the commutativity of the diagram

$$\begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH^*_{\mathbb{Q}}(X) \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi^*} \\ K_0(Y)_{\mathbb{Q}} & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(Y)} & CH^*_{\mathbb{Q}}(Y) \end{array}$$

It suffices to check it on vector bundles, since they generate K-theory. Let $\mathcal{E}$ be a vector bundle on X, and consider $[\pi_*\mathcal{E}] \in K(Y)$ (here $\pi_*\mathcal{E}$ is an honest pushforward since the morphism is affine, being a closed embedding). Unfortunately $\pi_*\mathcal{E}$ isn't a vector bundle, and we only really know how to compute Chern classes of vector bundles, so we have to resolve this by vector bundles.

Example 5.14. (The Koszul complex) Let's warm up on a special case, by resolving the structure sheaf of the origin in $\mathbb{A}^n$, which is $k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$. This is done by the *Koszul complex*. Let $R = k[x_1, \ldots, x_n]$. We can start off our resolution with the surjection $R \twoheadrightarrow k[x_1, \ldots, x_n]/(x_1, \ldots, x_n)$. The kernel is $(x_1, \ldots, x_n)$, and the natural next step is to take the free module on these generators.

$$Rx_1 \oplus \ldots \oplus Rx_n \to R \to k[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \to 0.$$

But there are obvious relations here, as $Rx_1 \oplus Rx_2$ contains $(-x_2)x_1 - (x_1)x_2$, which is killed in the map to R. Let M be the free R-module on symbols $X_1, \ldots, X_n$. You can see that each term is the next exterior power of M, so the final resolution we obtain is

$$0 \to \wedge^n M \to \wedge^{n-1} M \to \ldots \to \wedge^2 M \to M \to R \to k[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \to 0.$$

Now let's globalize. The regular embedding $X \hookrightarrow Y$ is locally cut out by r equations, which can be thought of as the coordinates of the global section $p^*\mathcal{O}_X \to Q$. Therefore, we have a Koszul resolution

$$\ldots \to \bigwedge^2 Q^\vee \to Q^\vee \to \mathcal{O}_Y \to \sigma_*\mathcal{O}_X \to 0.$$

(We dualize because because the equations cutting out $\sigma(X)$ are considered as *functions on* $\mathbb{Q}$.) We can then tensor this with $p^*\mathcal{E}$ to get a resolution of $\sigma_*\mathcal{E}$.

Let $\alpha_1, \dots, \alpha_r$ be the Chern roots of Q. Since the Chern character is multiplicative and additive, we have

$$\mathrm{ch}(\sigma_* \mathscr{E}) = \mathrm{ch}(\wedge^\bullet Q^\vee \otimes p^* \mathscr{E}) = \mathrm{ch}(\wedge^\bullet Q^\vee) \cdot \mathrm{ch}(p^* \mathscr{E}).$$

The first factor is (by additivity)

$$\mathrm{ch}(\wedge^\bullet Q^\vee) = \sum_p (-1)^p \,\mathrm{ch}(\bigwedge^p Q^\vee).$$

The Chern roots of $Q^\vee$ are $-\alpha_1, \dots, -\alpha_r$, so the Chern roots of $\bigwedge^p Q^\vee$ are p-fold products of the Chern roots of $Q^\vee$, and in the end one gets

$$\mathrm{ch}(\wedge^\bullet Q^\vee) = \sum_p (-1)^p \sum e^{-\alpha_{i_1}} \dots e^{-\alpha_{i_p}} = \prod_{i=1}^r (1 - e^{-\alpha_i}).$$

We can rewrite this in a more convenient form, noting that $c_r(Q) = \prod \alpha_i$:

$$\begin{aligned}
\mathrm{ch}(\wedge^\bullet Q^\vee) &= \prod_{i=1}^r (1 - e^{-\alpha_i}) \\
&= \prod_{i=1}^r \frac{(1 - e^{-\alpha_i})}{\alpha_i} \prod_{i=1}^r \alpha_i \\
&= \mathrm{Td}(Q)^{-1} \cdot c_r(Q).
\end{aligned}$$

Therefore, tracing through the lower path of the commutative diagram gives

$$\frac{c_r(Q)}{\mathrm{Td}(Q)} \cdot \mathrm{ch}(p^* \mathscr{E}) \mathrm{Td}(Y)$$

We want to match this up with what we get by tracing though the other path.

$$\begin{array}{ccc}
[\mathscr{E}] & \longmapsto & \mathrm{Td}(X) \cdot \mathrm{ch}(\mathscr{E}) \\
\big\downarrow & & \big\downarrow \\
\sigma_*[\mathscr{E}] \longmapsto & \frac{c_r(Q)}{\mathrm{Td}(Q)} \cdot \mathrm{ch}(p^* \mathscr{E}) \mathrm{Td}(Y) \overset{?}{=\!=\!=} & \sigma_* \mathrm{Td}(X) \mathrm{ch}(\mathscr{E}).
\end{array}$$

Lemma 5.15. *If $\sigma \colon X \hookrightarrow Y$ is a regular embedding with normal bundle N of rank r, then $\sigma_* \sigma^* \beta = c_r(N) \cdot \beta$, which you can think of geometrically as "intersecting β with X."*

The proof is immediate from the definition of the pullback. For a sanity check, note that if $\alpha \in CH^k(Y)$, then $\sigma^* \alpha \in CH^k(X)$ and $\sigma_* \sigma^* \alpha \in CH^{k+r}(Y)$. That is, the map $\alpha \mapsto \sigma_* \sigma^* \alpha$ increases the codimension by r.

Using this, we can complete our computation. By the Lemma,

$$\frac{c_d(Q)}{\mathrm{Td}(Q)} \mathrm{ch}(p^* \mathscr{E}) \cdot \mathrm{Td}(Y) = \sigma_* \sigma^* (\mathrm{ch}(p^* \mathscr{E})) \cdot \mathrm{Td}(Y) / \mathrm{Td}(Q).$$

By the projection formula,

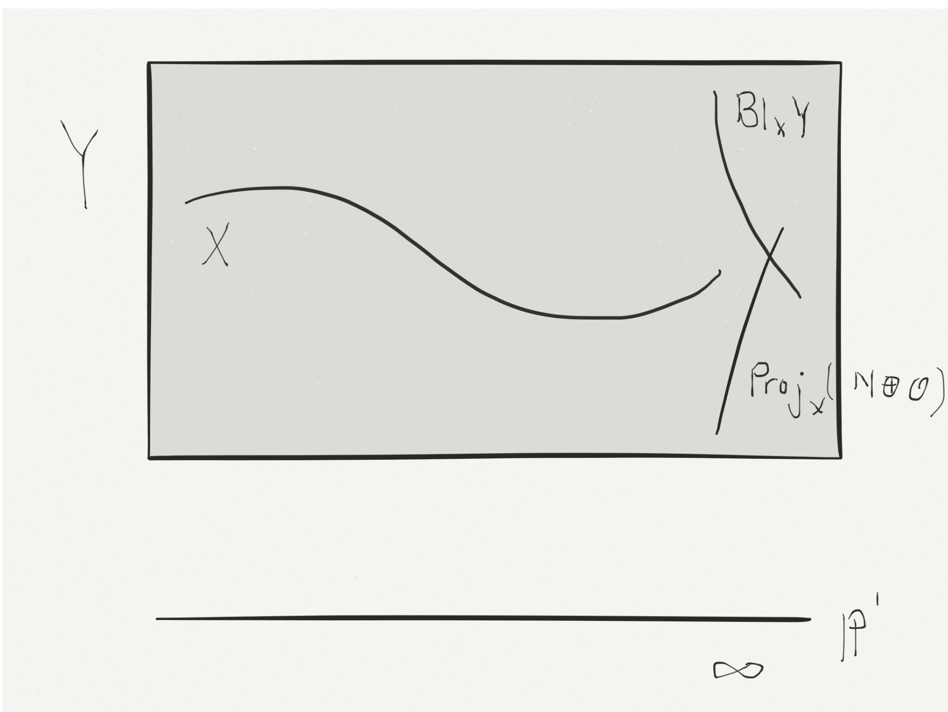
$$\sigma_* \sigma^* (\mathrm{ch}(p^* \mathscr{E})) \cdot \mathrm{Td}(Y) / \mathrm{Td}(Q) = \sigma_* (\sigma^* (\mathrm{ch}(p^* \mathscr{E})) \cdot \sigma^* \mathrm{Td}(Y) / \mathrm{Td}(Q))$$

But by the multiplicativity of the Todd class, the functoriality of Chern classes, and the fact that $\sigma^* Q = N_{X/Y}$, we see that the right hand side is $\sigma_*(\mathscr{E} \cdot \mathrm{Td}(X))$, as desired.

The general case. Now we do the general case of closed embeddings. We use the algebro-geometric analogue of the "tubular neighborhood," which is a construction called *deformation to the normal cone*. Let X be a closed subscheme of Y (in our case, even a regular embedding). To "get at" the tubular neighborhood, we form $Y \times \mathbb{P}^1$, which has the subscheme $X \times \mathbb{P}^1$. This maps to $\mathbb{P}^1$ by the obvious projection. We then blow up $X \times \{\infty\}$.

What does this look like? Over ∞, we get two components: $\mathrm{Bl}_X Y$ and $\mathrm{Proj}_X(N_{X/Y} \oplus \mathscr{O})$, which is the projective closure of the normal cone to X in Y. These intersect along $\mathrm{Proj}_X(N_{X/Y})$, which is the projectivization of the normal cone. Therefore, in the fiber over ∞ we have X sitting inside $\mathrm{Proj}_X(N_{X/Y})$ as the zero section, which in turn sits inside $\mathrm{Proj}_X(N_{X/Y} \oplus \mathscr{O}_X)$. This all follows from the general fact is that the exceptional divisor is always the projectivization of the normal bundle, and the normal bundle of X in $Y \times \mathbb{P}^1$ is $N \oplus \mathscr{O}$.

You can visualize the inclusion $X \hookrightarrow N_{X/Y}$ as "analytically zooming in" on X.



We claim that $\mathrm{Bl}_{X\times\infty}(Y \times \mathbb{P}^1)$ is still flat over $\mathbb{P}^1$. As the base is a smooth proper curve, we just have to see that blowing up introduces no associated points, but that is a general property of blowups.

So we have a diagram

$$\begin{array}{ccccc}
X & \hookrightarrow & X \times \mathbb{P}^1 & \hookleftarrow & X \\
\downarrow \sigma & & \downarrow \pi & & \downarrow \sigma' \\
Y & \xrightarrow{i} & \mathrm{Bl}_{X\times\infty}(Y \times \mathbb{P}^1) & \xleftarrow{j} & \mathbb{P}(N \oplus \mathscr{O}) \cup \mathrm{Bl}_X Y \\
\downarrow & & \downarrow & & \downarrow \\
0 & \hookrightarrow & \mathbb{P}^1 & \hookleftarrow & \infty
\end{array}$$

Denote $M = \mathrm{Bl}_{X\times\infty} Y$.

Let $\mathscr{E}$ be a vector bundle on X. We want to compute $[\sigma_*\mathscr{E}] \in K_0(Y)$. We'll take an indirect route, by first pulling back to $X \times \mathbb{P}^1$ and then resolving the pushforward to M:

$$G_\bullet \to \pi_* p_1^* \mathscr{E} \to 0.$$

Since everything is flat over $\mathbb{P}^1$, this sequence *remains exact upon restricting to a fiber*. The idea is that since 0 and ∞ are "homologous" in $\mathbb{P}^1$, we get the same class in K-theory upon restriction to the fibers over 0 and ∞.

Let's now argue formally. When we restrict $\pi_* p_1^* \mathscr{E}$ to the fiber over 0, we recover $\sigma_*\mathscr{E}$. Therefore $i^*\mathscr{G}_\bullet$ is a resolution of $\sigma_*\mathscr{E}$, so

$$\mathrm{ch}(\sigma_*\mathscr{E}) = \mathrm{ch}(i^*\mathscr{G}_\bullet) = i^* \mathrm{ch}(\mathscr{G}_\bullet).$$

This computation lives on Y, but since we want to transfer things along M we push it forward via i. Letting $M_0 \cong Y$ be the fiber over 0, we have

$$i_* \mathrm{ch}(\sigma_*\mathscr{E}) = i_* i^* \mathrm{ch}(\mathscr{G}_\bullet) = \mathrm{ch}(\mathscr{G}_\bullet) \cdot [M_0].$$

Since 0 is linearly equivalent to ∞ on $\mathbb{P}^1$, M_0 is linearly equivalent to $M_\infty \cong \mathbb{P}(N \oplus \mathscr{O}) \cup \mathrm{Bl}_X Y$ on M, so

$$\begin{aligned} i_* \mathrm{ch}(\sigma_*\mathscr{E}) &= \mathrm{ch}(\mathscr{G}_\bullet) \cdot [M_0] = \mathrm{ch}(\mathscr{G}_\bullet) \cdot [M_\infty] \\ &= \mathrm{ch}(\mathscr{G}_\bullet) \cdot [\mathrm{Proj}_X(N \oplus \mathscr{O})] + \mathrm{ch}(\mathscr{G}_\bullet) \cdot [\mathrm{Bl}_X Y]. \end{aligned}$$

But $\mathscr{G}_\bullet$ was a resolution for a sheaf supported on $X \times \mathbb{P}^1$, and in particular not on $[\mathrm{Bl}_X Y]$. Therefore, $\mathscr{G}_\bullet$ restricts to an exact resolution on $[\mathrm{Bl}_X Y]$, so $\mathrm{ch}(\mathscr{G}_\bullet) \cdot [\mathrm{Bl}_X Y] = 0$. Then by the same argument applied to the inclusion of X in the fiber over infinity, we have

$$i_* \mathrm{ch}(\sigma_*\mathscr{E}) = j_* \mathrm{ch}(\sigma'_*\mathscr{E}).$$

Since $\sigma' \colon X \to \mathrm{Proj}_X(N \oplus \mathscr{O})$ factors through the zero section $X \hookrightarrow N_{X/Y}$, our computation in that special case implies that

$$j_* \mathrm{ch}(\sigma'_*\mathscr{E}) = j_* \sigma'_* (\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(N_{X/Y}^\vee)).$$

In summary, we have shown that

$$i_* \mathrm{ch}(\sigma_*\mathscr{E}) = j_* \sigma'_* (\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(N_{X/Y}^\vee)). \tag{5.1}$$

To obtain an identity in the Chow ring of Y, we push down via the blowdown and projection $pr \colon \mathrm{Bl}_{X\times\infty}(Y \times \mathbb{P}^1) \to Y \times \mathbb{P}^1 \to Y$. Noting that $pr \colon i = \mathrm{Id}_Y$ and $pr \colon j \colon \sigma' = \sigma$, applying pr_* to (5.1) gives

$$\mathrm{ch}(\sigma_*\mathscr{E}) = \sigma_* (\mathrm{ch}(\mathscr{E}) \cdot \mathrm{Td}(N_{X/Y}^\vee)).$$

It only remains to unwind the right hand side to obtain the desired form:

$$\begin{aligned}\sigma_*(\mathrm{ch}(\mathscr{E})\cdot \mathrm{Td}(N_{X/Y}^\vee)) &= \sigma_*(\mathrm{ch}(\mathscr{E})\cdot \mathrm{Td}(X)\cdot \sigma^*\mathrm{Td}(Y)^{-1}) \\ &= \sigma_*(\mathrm{ch}(\mathscr{E})\cdot \mathrm{Td}(X))\cdot \mathrm{Td}(Y)^{-1}.\end{aligned}$$

This (finally!) completes the proof of the Grothendieck-Riemann-Roch theorem.

5.8. **Applications of GRR.**

Theorem 5.16. *Let X be smooth. Then* $\tau\colon K(X)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}\cdot \mathrm{Td}(T_X)} CH(X)_{\mathbb{Q}}$ *is an isomorphism.*

We begin with a technical result.

Lemma 5.17. *Let* Y *be smooth and* $f\colon X\hookrightarrow Y$ *the inclusion of an irreducible subvariety. Then* $\mathrm{ch}([\mathscr{O}_X])\in CH_*(Y)$ *is of the form* $[X]$ *plus lower dimension terms.*

Proof. If X is smooth, then we have a regular embedding and we can directly apply Grothendieck-Riemann-Roch:

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(X)} & CH(X) \\ \downarrow f_* & & \downarrow f_* \\ K(Y) & \xrightarrow[\mathrm{ch}\cdot\mathrm{Td}(Y)]{} & CH(Y) \end{array}$$

Then GRR tells us that $\mathrm{ch}([\mathscr{O}_X]) = f_*(\mathrm{Td}(N))^{-1}[X]$. As the Todd class is invertible, of the form 1 plus higher codimension terms, we get

$$\mathrm{ch}([\mathscr{O}_X] = f_*[X] + (\text{higher codimension terms}).$$

If X is not smooth, then at least it has a dense open subset which is smooth. Suppose that the complement is Z. We have a sequence

$$CH(Z)\to CH(X)\to CH(X\setminus Z)\to 0.$$

The first (smooth) case covers the assertion for $CH(X\backslash Z)$, and the rest is lower-dimensional terms, so we are done. ☐

Proof Sketch of Theorem 5.16. We claim that we have an isomorphism $CH^*(X) = Gr(G_0(X))$ sending $Z\mapsto \mathscr{O}_Z$ (the grading is by the dimension of the support of the coherent sheaf). Since X is smooth, this in turn is isomorphic to $Gr(K_0(X))$.

To check that the group homomorphism $CH(X)\to Gr(K(X))$ sending $[Z]\mapsto[\mathscr{O}_Z]$ is well-defied, one just has to check that two linearly equivalent points in $\mathbb{P}^1$ gives the same class in $Gr(K(X))$, because all relationsin $CH_*(X)$ are pullbacks from this situation. Note that we really do need to pass to the associated graded in order to obtain a homomorphism, because $[V+W]$ should go to $[\mathscr{O}_V]+[\mathscr{O}_W]-[\mathscr{O}_{V\cap W}]$.

By the preceding lemma, the composition

$$K(X)\xrightarrow{\mathrm{ch}} CH(X)\xrightarrow{\sim} Gr(K(X))\to K(X)$$

is the identity modulo lower order terms. It thus suffices to show that $K(X)\xrightarrow{\mathrm{ch}} CH(X)$ is surjective.

Well, if you start with a coherent sheaf $\mathscr{F}$ on X, with rank r, then it is "mostly a vector bundle" in the sense that there exists a dense open subset $U \subset X$ such that $\mathscr{F}|_U$ is a vector bundle over U. Why is this true? After twisting by a sufficiently ample divisor, one can produce a map $\mathscr{O}_X^r \to \mathscr{F}(n \gg 0)$ inducing an isomorphism on the generic point, and then the kernel and cokernel are supported on things on higher codimension. Then one untwists and inducts. □

Applications to moduli spaces. Let $\mathscr{M}_g$ be the moduli space (stack) of genus g curves. Whenever you have a moduli space it's useful to understand its cohomology and Chow ring, as that gives you "characteristic classes" for the relevant structure.

So how can we get a handle on $H^*(\mathscr{M}_g)$? Well, one explicit way is to study subvarieties on $\mathscr{M}_g$ (e.g. the locus of hyperelliptic curves), as these should be "Poincaré dual" to a cohomology classes.

Another way is to use our knowledge of vector bundles $\mathscr{M}_g$ and compute characteristic classes of them. For instance, $\mathscr{M}_g$ has a universal curve $\mathscr{C}_g \xrightarrow{\pi} \mathscr{M}_g$, whose fiber over a point is the corresponding curve. Let ω be the relative dualizing sheaf of $\mathscr{C}_g/\mathscr{M}_g$, i.e. and $\Omega = \pi_*\omega$. This has rank g, as the space of holomorphic differentials on a genus g curve is g-dimensional. Therefore, we obtain characteristic classes $\lambda_1, \dots, \lambda_g \in H^*(\mathscr{M}_g)$, namely $\lambda_i := c_i(\Omega)$.

On the other hand, we have cohomology classes $c_i(\omega) \in H^*(\mathscr{C}_g)$ and from them we can obtain cohomology classes $\pi_*(c_i(\omega)) \in H^*(\mathscr{M}_g)$. Grothendieck-Riemann-Roch says that these two sets of cohomology classes should be related.

This is a prototypical example of GRR is used. You want to study cohomology classes on $\mathscr{M}_g$, and you obtain two collections via characteristic classes on some family, and you want to compare them.

Part 2. **Some equivariant ingredients**

6. Topological classifying spaces

The purpose of this section is entirely to motivate issues that will arise in attempting to define "quotient stacks" and their Chow groups.

6.1. **Principal G-bundles.** A central theme in algebraic topology is to attach *algebraic invariants* to topological spaces, such as homotopy groups, homology groups, cohomoloy groups, etc. We will be discussing a very specific kind of topological object, which is nonetheless ubiquitous: the *principal G-bundle*. You may be familiar with *characteristic classes* of vector bundles (which arose even in the statement of the Riemann-Roch theorem), which are algebraic invariants attached to a special kind of principal G-bundle (for $G = \mathrm{GL}_n$).

Definition 6.1. Let G be a topological group. A *principal G-bundle over X* is a topological space P equipped with a continuous, free action of G and a map

$$\pi\colon P \to X$$

such that

(1) π identifies the quotient space $G\backslash P$ with X, and
(2) π is locally trivial, i.e. for all $x \in X$ there is an open neighborhood $x \in U \subset X$ such that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\ \cong\ } & U \times G \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Remark 6.2. We say that G acts *freely* on Y if the map $G \times Y \to Y \times Y$ sending $(g, y) \mapsto (y, gy)$ is a homeomorphism onto its image. The bijectivity is equivalent to all stabilizers being trivial, which is the familiar notion of free action for discrete groups.

Example 6.3. The *trivial G-bundle* on X is the product space $G \times X$ with the obvious projection map.

Any G-bundle $\pi\colon P \to X$ admitting a global section $s\colon X \to P$ is trivial, as we can view s as giving a coherent choice of identity element in each fiber. Concretely, we have a map $G \times X \to P$ sending $(g, x) \mapsto gs(x)$, which is necessarily an isomorphism.

Example 6.4. As with vector bundles, one can think of principal G-bundles in terms of gluing. Explicitly, if $\{(U_\alpha, \phi_\alpha)\}$is a trivialization of $\pi\colon P \to X$, then P is determined by the transition functions $\tau_{\beta\alpha}\colon U \to G$:

$$G \times U_\alpha \xrightarrow{\phi_\alpha^{-1}} p^{-1}(U_\alpha) \xrightarrow{\phi_\beta} G \times U_\beta, \qquad \tau_{\beta\alpha}\colon G \times U_\alpha \to G \times U_\beta$$

These transition functions must satisfy the *cocycle conditions* to be consistent. Thus G is called the "structure group."

One immediate consequence of this definition is that if $G = \mathrm{GL}_n$, then our transition functions also define a vector bundle. So we see that *there is an equivalence between principal* GL_n*-bundles and vector bundles*! In fact whenever G is the automorphism group of a certain structure, a principal G-bundle will have an alternate interpretation in terms of that structure.

6.2. **Characteristic classes.** Even if you are not familiar with principal G-bundles, you have probably encountered plenty of vector bundles and appreciate their importance. Vector bundles can be hard to "classify," but a first step is to attach algebraic invariants to vector bundles. Namely, to any vector bundle $V \to X$ one can associate elements of $H^*(X)$ *in a functorial way* (meaning compatibly with pullbacks).

Example 6.5. If $V \to X$ is a complex vector bundle of (complex) rank n, then there are *Chern classes* $c_1(V), c_2(V), \ldots, c_n(V) \in H^*(X; \mathbb{Z})$.

Example 6.6. If $V \to X$ is a real vector bundle of rank n, then there are *Stiefel-Whitney classes* $w_1(V), w_2(V), \ldots, w_n(V) \in H^*(X; \mathbb{Z}/2)$.

One might ask *why* these characteristic classes exist, and why there aren't any more out there waiting to be discovered. Classifying spaces answer these questions in a very elegant way.

Definition 6.7. The *classifying space* BG (well-defined up to homotopy) is a space representing the functor **Top** $\to$ **Set** sending

$$X \mapsto \{\text{principal } G\text{-bundles on } X\}/\text{isom}.$$

on a "nice enough" subcategory of spaces (e.g. CW complexes). In other words, there is a natural bijection

$$\mathrm{Hom}(X, BG)/\text{homotopy} \longleftrightarrow \{\text{principal } G\text{-bundles on } X\}/\text{isom}$$

It is a theorem that such a space always exists. In fact, here is a "concrete" construction. Take a contractible space EG on which G acts freely. [Why does such a thing always exist?] Then $BG = EG/G$.

The map $EG \to BG$ is the "universal principal G-bundle," and it corresponds to the identity map $BG \to BG$. Given a map $f : X \to BG$, the corresponding principal G-bundle is the pullback

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

A consequence of the definition is that $H^*(BG)$ *parametrizes all functorial assignments of cohomology classes to principal* G*-bundles.*

Proof. Indeed, given a vector bundle $P \to X$, we get a map $X \to BG$ pulling back the universal bundle to P.

$$\begin{array}{ccc} P & \dashrightarrow & EG \\ \downarrow & & \downarrow \\ X & \dashrightarrow & BG \end{array}$$

We can then assign to P the pullback of a cohomology class of BG along this map. Conversely, suppose we have such an assignment. Then it is completely determined by its value on the universal bundle, as any other bundle is a pullback of this one. □

We claim that if we have a group homomorphism $H \to G$, then we get a map $BH \to BG$. Indeed, by definition giving a map $BH \to BG$ is the same as giving a functorial recipe for turning a principal H bundle into a principal G-bundle. One perspective on a principle H-bundle is in terms of transition functions with values in H satisfying the cocycle conditions. But if we compose that with the homomorphism to G, then we get transition functions valued in G satisfying the cocycle conditions, hence a principal G-bundle.

If $H \subset G$ is a subgroup, we can choose the map $BH \to BG$ to be a fibration with fiber G/H. To see this, note that H acts freely on EG a fortiori, and the fibers of $EG/H \to EG/G$ are evidently G/H.

The result we state now is probably not the optimal one, but it suffices for our purposes.

Proposition 6.8. *If $H \hookrightarrow G$ is a weak homotopy equivalence, then $BH \to BG$ is a weak homotopy equivalence.*

Proof. Recall that $H \hookrightarrow G$ is a *weak homotopy equivalence* if it induces isomorphisms on all homotopy groups, which implies (Hurewicz's Theorem) that it induces isomorphisms on all (co)homology groups.

By the long exact sequence of homotopy groups for the fibration $H \to G \to G/H$, we see that $\pi_i(G/H) = 0$ for $i > 0$. Next applying the long exact sequence of homotopy groups for the fibration $G/H \to BH \to BG$ shows that BH and BG are weakly homotopy equivalent. □

Example 6.9. By Proposition 6.8, $B\,\mathrm{GL}(1,\mathbb{R}) \cong B\mathbb{Z}/2$. What is $B\mathbb{Z}/2$? Well, $\mathbb{Z}/2$ acts freely on S^∞, which is contractible. So $B\mathbb{Z}/2 \cong \mathbb{R}P^\infty$. This has a cell structure, with one cell of each dimension and in $\mathbb{Z}/2$-(co)homology, the boundary maps are 0 (that's what makes it easy to calculate!). In fact, $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_1]$ where $|w_1| = 1$.

Given any real line bundle $L \to X$, we get a map $f : X \to \mathbb{R}P^\infty$ such that the pullback of the tautological bundle is L. The *first Stiefel-Whitney class* $w_1(L)$ is precisely $f^*[w_1]$.

Example 6.10. By Proposition 6.8, $B\,\mathrm{GL}(1,\mathbb{C}) \cong BS^1$. Again, S^1 acts on $S^\infty \subset \mathbb{C}^\infty$ by multiplication, and the quotient is $\mathbb{C}P^\infty$. The cohomology ring is $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[c_1]$.

Given any *complex* line bundle $L \to X$, we get a map $f : X \to \mathbb{C}P^\infty$ such that the pullback of the tautological bundle is L. The *first Chern class* $c_1(L)$ is precisely $f^*[c_1]$.

7. STACKS

In this and the next section, we give a quick crash course on *stacks* (borrowing heavily from [Hei10]). Our goal is not to cover, in any way, a comprehensive treatment of the theory, and indeed not even the small corner of it that we need later. Even for the small slice of material that we treat, we do not claim to present the "optimal" definitions or resuts.

Our aim is to modest: to motivate the rather abstract definition, highlight the main examples which will be relevant to us, and dip just deep enough into the theory so that the reader will comfortable with the basic language of stacks. While the precise definitions can often be unwieldy, a familiarity with the core ideas can go a long way towards guessing or "black boxing" technical issues whenever they arise.

7.1. **Motivation: quotient stacks.** The idea of "quotient stack" in algebraic geometry is modeled on that of classifying spaces. Some of the topological notions - homotopy, contractibility, quotient topology - do not translate so easily to the realm of algebraic geometry, but the formal properties do.

What would it *mean* to have "classifying spaces" in algebraic geometry? Ideally, we could find a *scheme* BG such that

$$\mathrm{Hom}_{\mathbf{Sch}}(X, BG) \longleftrightarrow \{\text{principal } G\text{-bundles on } X\}.$$

In other words, we want to represent the functor taking a scheme X to *algebraic* principal G-bundles over X.

Remark 7.1. Here I am brushing an important technical point under the rug: the "local triviality" should not be with respect to the standard Zariski topology (except in lucky cases), but some finer Grothendieck topology. We'll discuss this in greater detail later.

Unfortunately, no such space BG exists in the category of schemes. With a bit of familiarity with the theory of moduli spaces, it is easy to see why: for instance, as long as G has non-trivial center, a principal G-bundle will have non-trivial automorphisms given by multiplication by non-trivial central elements. To obtain a "space" with the right properties, we will have to enlarge our concept of algebro-geometric space, which is the realm of *stacks.*

More generally, if X is a scheme and G is any group acting on X, then we want to be able to produce a space $[X/G]$ such that sheaves on $[X/G]$ are "the same" as G-equivariant sheaves on X. This sort of object $[X/G]$ is called a *quotient stack.*

Example 7.2. For a group G acting on a point pt, the stack [point/G] is called *the classifying stack BG*. Then we want sheaves on BG to be equivalent to representations of G.

Topologically, taking the quotient of a point by G is just a point, so the stack must remember additional data. A first (not very accurate) approximation is to think of the stack $[X/G]$ as the quotient space together with the information of the stabilizer groups of G at corresponding points of X. However, this is a dangerously inaccurate slogan in general.

7.2. **Prototypical examples.** Perhaps the first definition of schemes that everybody learns is in terms of concrete "geometric" data: a topological space and a sheaf of functions, satisfying some local conditions... However, an alternate perspective, which proves very fruitful, view schemes as *functors of points* from schemes (or just rings) to sets. This second approach is the one that we take en route to stacks.

Formally, a stack will be defined as a *functor on the category of schemes* satisfying certain conditions. There will be various levels of "geometric" conditions that we can impose, which will force the stacks to conform more closely to our traditional conception of spaces in algebraic geometry (but at a price of generality). Thus, stacks are almost tautologically introduced as representing "spaces" for functors that cannot be represented in the traditional world of schemes. The general conditions are modelled on some particular examples that we now discuss.

Example 7.3. Consider the functor $\varphi_g\colon \mathbf{Sch} \to \mathbf{Sets}$ sending X to the set of isomorphism classes of flat families $f\colon Y \to X$ with fibers being smooth projective curves of genus g.

If this functor were representable by some space $\mathscr{M}_g$ (the "fine moduli space of curves of genus g") then, analogously to the existence of the universal bundle on BG, the identity map $\mathscr{M}_g \to \mathscr{M}_g$ would correspond to a *universal family of genus g curves over M_g* from which all other families would be pulled back. That is, the identification

$$\mathrm{Hom}_{\mathrm{Sch}}(X, \mathscr{M}_g) = \varphi_g(X)$$

would send a map $X \to \mathscr{M}_g$ to the pullback of the universal family, and all families would be obatined in this way.

This is not true as stated: there is no "fine moduli space of curves of genus g" (which is a scheme). The problem is that some curves have nontrivial automorphisms.

Why is the presence of automorphisms problematic for the existence of moduli spaces? Essentially, automorphisms allow us to construct "families" which are locally trivial but not globally trivial. This means that any representing morphism must be locally constant, but not globally constant. That of course is impossible in general.

Example 7.4. To illustrate, let's consider a concrete example. If $G = \mathbb{Z}/2$ acts on a curve D of genus g, then we can produce a nontrivial family of curves over X, all of whose fibers are isomorphic to D.

Indeed, let $E \to X$ be any principal $\mathbb{Z}/2$-bundle of schemes. Then consider the family of curves over X, $(E \times D)/(\mathbb{Z}/2)$. It has fibers all isomorphic to D, but it is typically not a trivial bundle.

If $\mathscr{M}_g$ existed, then the family would corresponding to a morphism $X \to \mathscr{M}_g$ which would necessarily be constant (since all the fibers are isomorphic). However, that would imply that the pullback is the trivial bundle.

Example 7.5. Consider the functor $\mathbf{Sch} \to \mathbf{Sets}$ sending X to the set of isomorphism classes of vector bundles of rank n on X (or more generally, principal G-bundles on X). We will shortly see that the right target category to consider is not the category of sets but the category of *groupoids*; we can ignore this distinction for now. This should be represented by a stack BG.

7.3. **Definition of stacks.** We have just remarked on the limitations of the notion of "set up to isomorphism" in dealing with moduli problems where the objects have *automorphisms*. A more uniform way to deal with this is to *keep track of automorphisms*. The naturual apparatus to do this is a categorical generalization of sets called *groupoids*. A "set up to isomorphism" may be viewed as a category in which all morphisms are the identity map.

Definition 7.6. A *groupoid* is a category in which all morphisms are isomoprhism.

A stack will be defined as a particular kind of contravariant functor **Sch** → **Grpoid**, generalizing the the Yoneda embedding of schemes into contravariant functors **Sch** → **Set**. To see what axioms we should impose in order to define a "reasonable" category, keep in mind the two prototypical examples we discussed in the previous section:

- $\mathcal{M}_g$ associates to a scheme X the groupoid of flat families of smooth, projective, genus g curves over X.
- BG associates to a scheme X the groupoid of principal G-bundles on X.

We reiterate for emphasis that since we are considering groupoids instead of sets, many families or bundles which would ordinarily be identified will be considered as distinct objects, "connected" by isomorphisms.

Now, what are the features of algebro-geometric families over a space that we would like to codify?

- The functoriality via pullback of families: if $X \to X'$ is a morphism of schemes and F' is a family over X', then one has a pullback family over X:

$$\begin{array}{ccc} F & \longrightarrow & F \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

- The fact that families can be glued together from loca; data, and similarly for morphisms.

There are some subtleties in formulating these issues properly. For example, the usual pullback is constructed "up to isomorphism." This means that there is no natural "single-valued" notion of pullback, so that we can only define *a* (not necessarily unique) pullback object for a given family, obviously all of which are isomorphic. But if one is prepared to ignore set-theoretic issues, then one can make a coherent choice of distinguished pullback object (a "cleavage") and assume the existence of pullback functors.

Next, it turns out that stacks are naturally regarded as 2-*categories*, meaning that there are *morphisms between morphisms*. Thus it turns out not to be natural to view stacks as literal functors in the usual sense, but functors "up to homotopy." Concretely, this means that the composition of pullbacks is not equal on the nose to the pullback of the composition, but that there exists a natural transformation between them. (Invoking again the analogy with the homotopy category, it is more natural to ask that two maps be homotopic rather than equal on the nose.)

To glue families and morphisms, we need a notion of "open covering" of a scheme. The Zariski topology is an option, but it is decidedly inadequate terms. In technical terms, what we need is a *Grothendieck topology* on **Sch**. Concretely, this means that we specify for each scheme a family of morphisms which are to be considered "coverings" in our "topology," which satisfy certain coherence axioms modelled on the properties of topological coverings. We could almost proceed directly to the definition of a stack, but it seems worthwhile to embark on a brief digression about Grothendieck topologies.

Definition 7.7. Let $\mathscr{C}$ be a category. A *Grothendieck topology* on $\mathscr{C}$ is a collection of sets of morphisms $\{U_i \to U\}$ called *coverings*, such that

(1) for any isomorphism $U \to V$, $\{U \to V\}$ is a covering of V.
(2) If $\{U_i \to U\}$ is a covering of U and $V \to U$ is a morphism in C, then the fiber products $V \times_U U_i$ exist in C and $\{V \times_U U_i \to V\}$ is a covering of V.
(3) If $\{U_i \to U\}$ is a covering, and for each i we have a covering $\{V_{ij} \to U_i\}$, then $\{V_{ij} \to U\}$ is a covering.

Example 7.8. Let X be a topological space, $\mathscr{C}$ be the category of open subsets of X (morphisms are inclusions). Define a covering $\{U_i \to U\}$ to be a collection of open sets whose union is U. This is a Grothendieck topology, and the traditional notion of covering.

Example 7.9. On the category **Sch** there are four commonly used Zariski topologies, listed from coarsest to finest.

- The *Zariski topology*: a collection of arrows $\{U_i \to U\}$ is a covering if the U_i form an open cover of U in the Zariski topology.
- The *étale topology*: a collection of arrows $\{U_i \to U\}$ is a covering if the morphism $\coprod U_i \to S$ is étale and surjective.
- The *fppf topology*: a collection of arrows $\{U_i \to U\}$ is a covering if the morphism $\coprod U_i \to S$ is faithfully flat and locally of finite presentation.
- The *fpqc (flat) topology*: a collection of arrows $\{U_i \to U\}$ is a covering if the morphism $\coprod U_i \to S$ is faithfully flat, and every affine subset of U is the image of some quasicompact open subset in $\coprod U_i$.

Definition 7.10. A *site* is a category $\mathscr{C}$ equipped with a Grothendieck topology.

For a site $\mathscr{C}$, a *sheaf* $\mathscr{M}$ is a contravariant functor $\mathscr{C} \to$ **Set** such that

(1) (Identity axiom) For every covering $\{U_i \to U\}$, if two sections $a, b \in \mathscr{M}(U)$ agree when pulled back to $\mathscr{M}(U_i)$ for all i, then $a = b$.
(2) (Sheaf axiom) For any covering $\{U_i \to U\}$ and any $a_i \in \mathscr{M}(U_i)$ agreeing in $\mathscr{M}(U_i \times_U U_j)$ for all i, j, there is a section $a \in \mathscr{M}(U)$ that restricts to all a_i.

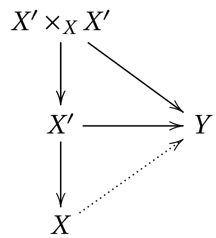
Thus we see that the notion of a sheaf on a site captures the notion of "gluing objects" that we wanted stacks to have.

It turns out that we could take either the étale, fppf, or fpqc topologies for our purposes, but the Zariski topology is definitely too coarse. However, as a sanity check we had better make sure that schemes, viewed through their functors of points, actually form sheaves in these topologies (it is obvious that they are sheaves for the Zariski topology).

Theorem 7.11 (Grothendieck)**.** *Let S be a scheme. A representable functor on* ***Sch****/S is a sheaf in the flat topology.*

Proof sketch. Let's first just digest the assertion. Given a scheme $Y \to S$ we consider the contravariant functor $X \mapsto \mathrm{Hom}_S(X, Y)$. This is a functor from schemes over S to sets, and the claim is that this is a sheaf in the flat topology.

For simplicity, we just consider a flat covering which consists of a single morphism, and we work in the category of k-schemes for some field k. The theorem asserts that for any surjective flat quasicompact morphism $X' \to X$ over S, and any morphism $X' \to Y$ over S such that the two compositions $X' \times_X X' \to X' \to Y$ are equal, there is a map from $X \to Y$ inducing $X' \to Y$.



Since morphisms are local in the Zariski topology, we may reduce to the case where Y is affine. Then a map to Y is a map to affine space with some algebraic conditions, so we reduces to the case where $Y = \mathbb{A}^n$. But then a map to Y is the same as a bunch of maps to $\mathbb{A}^1$, so we may reduce to the case $Y = \mathbb{A}^1$.

That is, we're given a regular function $f \in \mathscr{O}(X')$ such that the two pullbacks to $X' \times_X X'$ are equal, and we want to deduce that f is pulled back from X.

Let's think about what this means algebraically. Reducing to the case where B is affine, we have a faithfully flat A-algebra B, and we want to know that if $f \in B$ satisfies $1 \otimes f = f \otimes 1$ in $B \otimes_A B$, then f comes from A.

This is a special case of the exactness of the *Amitsur complex*

$$0 \to A \to B \to B \otimes_A B \tag{7.1}$$

(the last map sends $b \mapsto b \otimes 1 - 1 \otimes b$) which is usually proved early on in descent theory.

Exercise 7.12. Prove that (7.1) is exact. (Hint: it suffices to show exactness after tensoring with B. Then consider the section $B \otimes_A B \otimes_A B \to B \otimes_A B$ obtained by multiplying the first two coordinates.

□

Now we have all the notions that we need in order to define stacks. We adopt as our site **Sch** with the étale, fppf, or fpqc topology.

Definition 7.13. A *stack* $\mathscr{M}$ is sheaf of groupoids

$$\mathscr{M} : \mathbf{Sch} \to \mathbf{Grpoid}.$$

Explicitly, $\mathscr{M}$ is the data of

(1) for each $X \in Ob(\mathbf{Sch})$ a groupoid $\mathscr{M}(X)$,
(2) for each morphism $f : X \to Y \in Mor(\mathbf{Sch})$ a functor $f^* : \mathscr{M}(Y) \to \mathscr{M}(X)$,
(3) for any pair of morphisms $f : X \to Y$ and $g : Y \to Z$ a natural transformation

$$(f \circ g)^* \implies g^* \circ f^*$$

satisfying the gluing conditions

- (Objects glue) Given a covering $\{U_i \to X\}$, and $\{F_i \in \mathscr{M}(U_i)\}$ a collection of objects, and for each pair i, j an isomorphism $\tau_{ij}\colon F_i|_{U_i \times_U U_j} \to F_j|_{U_i \times_U U_j}$ satisfying a cocycle condition on triple "overlaps," *then* there exists $F \in \mathscr{M}(U)$, unique up to unique isomorphism, together with isomorphisms $\phi_i\colon F|_{U_i} \cong F_i$ such that $\tau_{ij} = \phi_j \circ \phi_i^{-1}$.
- (Morphisms glue) Given $F, F' \in \mathscr{M}(X)$, and a covering $\{U_i \to X\}$ together with a morphism $\psi_i\colon F|_{U_i} \to\to F'|_{U_i}$ for each i such that for each pair i, j we have $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$, *then* there exists a unique morphism $\psi\colon F \to F'$ such that $\psi|_{U_i} = \psi_i$.

It is basically immediate from the definition that $\mathscr{M}_g$ and BG are stacks. However, for the purposes of working geometrically we will want to restrict our attention to a smaller class of "algebraic stacks" (and even further within them) in which we can really do geometry. It is a non-trivial theorem that $\mathscr{M}_g$ and BG lie in those restricted classes of stacks.

Definition 7.14. Let X be a scheme and G an algebraic group acting on X. We define the *quotient stack* $[X/G]$ sending T to the groupoid of principal G-bundles $P \to T$ equipped with a G-equivariant map to X, i.e. diagrams

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equiv.}} & X \\ {\scriptstyle G-\text{bundle}}\downarrow & & \\ T & & \end{array}$$

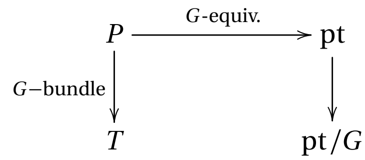
As a sanity check, let's make sure that if X/G exists as a scheme then we recover the same definition. In that case, the claim would be that there is a canonical isomoprhism between $\mathrm{Hom}(T, X/G)$ and the set

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equiv.}} & X \\ {\scriptstyle G-\text{bundle}}\downarrow & & \\ T & & \end{array}$$

Indeed, $X \to X/G$ is a principal G-bundle, so given a morphism $T \to X/G$ we can pull back X to obtain a principal G-bundle $P \to T$, and the map $P \to X$ is automatically G-equivariant. Conversely, given a diagram as above we get a map $T \to X/G$ by descent.

$$\begin{array}{ccc} P & \xrightarrow{G\text{-equiv.}} & X \\ {\scriptstyle G-\text{bundle}}\downarrow & & \downarrow \\ T & \dashrightarrow & X/G \end{array}$$

Example 7.15. Let G be an algebraic group over a field k. We claim that BG is the quotient stack $[\mathrm{pt}/G]$, where the G action is the only possible one on $\mathrm{pt} = \mathrm{Spec}\ k$. By definition, if T is a scheme then $\mathrm{Hom}(T, [\mathrm{pt}/G])$ is the groupoid of diagrams



where P is a principal G-bundle on T. But given any such P, the structure map to pt is G-equivariant and unique, so this is just the same as the data of P.

7.4. **Morphisms of stacks.** Morphisms of stacks are natural transformations, but since stacks are not quite functors in the traditional sense (they are functors "up to homotopy," e.g. they need not preserve compositions on the nose, but only up to natural transformations), there are a few minor novelties in formulating the correct definition.

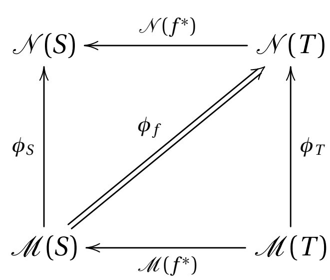
Definition 7.16. A *morphism of stacks* $\phi\colon \mathscr{M} \to \mathscr{N}$ consists of, for each $T \in Ob(\mathbf{Sch})$, a functor

$$\phi_T\colon \mathscr{M}(T) \to \mathscr{N}(T)$$

together with, for each morphism $f\colon S \to T \in Mor(\mathbf{Sch})$, a natural transformation

$$\phi_f\colon \phi_S \circ \mathscr{M}(f)^* \implies \mathscr{N}(f^*) \circ \phi_T$$

We emphasize that this means that the usual diagram depicted below does not commute on the nose, but there is a natural transformation between the two compositions (which is necessarily an isomorphism on all objects because the categories are all groupoids).



Yoneda's Lemma. There is a version of Yoneda's Lemma for stacks. Let T be a scheme, and denote by $\underline{T}$ the stack $S \mapsto \mathrm{Hom}(S, T)$.

Theorem 7.17 (Yoneda)**.** *For any stack $\mathscr{F}$, there is a natural equivalence of categories*

$$\mathrm{Hom}(\underline{T}, \mathscr{M}) \cong \mathscr{M}(T).$$

Exercise 7.18. Prove this. The argument is *almost* identical to the usual one, except that one has to take some care since for example morphisms of stacks only preserve pullbacks up to natural equivalence.

Example 7.19. We have a morphism $B\mathrm{GL}_n \times B\mathrm{GL}_n \to B\mathrm{GL}_{mn}$ induced by the tensor product of the associated vector bundles.

We mentioned that stacks form a 2-category, i.e. there is a natural notion of morphism between morphisms of stacks. Namely, if ϕ and ψ are two morphisms $\mathcal{M} \to \mathcal{N}$, then a 2-morphism between ϕ and ψ is the data of a natural transformation $\phi_T \to \psi_T$ for each $T \in Ob(\mathbf{Sch})$, compatible with the pull-back functors in the sense that for any $f: S \to T \in Mor(\mathbf{Sch})$, the following diagram commutes (on the nose!)

$$\begin{array}{ccc} \phi_S \circ \mathcal{M}(f)^* & \xrightarrow{\phi_f} & \mathcal{N}(f)^* \circ \phi_T \\ \downarrow & & \downarrow \\ \psi_S \circ \mathcal{M}(f)^* & \xrightarrow{\psi_f} & \mathcal{N}(f)^* \circ \psi_T \end{array}$$

Example 7.20. Consider the stack BG and $\mathrm{Id}: BG \to BG$ as morphism of stacks. If $z \in Z(G)$, then "multiplication by z" is a 2-morphism $\mathrm{Id} \to \mathrm{Id}$.

7.5. **Fibered products of stacks.** The definition of the fibered product is also a little different from usual, because it is not natural to demand equality of morphisms (along two compositions), but only a natural transformation between them.

Definition 7.21. If $\phi: \mathcal{M} \to \mathcal{X}$ and $\psi: \mathcal{N} \to \mathcal{X}$ are two maps of stacks, then the *fibered product*

$$\begin{array}{ccc} \mathcal{N} \times_{\mathcal{X}} \mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \phi \\ \mathcal{N} & \xrightarrow[\psi]{} & \mathcal{X} \end{array}$$

is the functor sending T to the groupoid

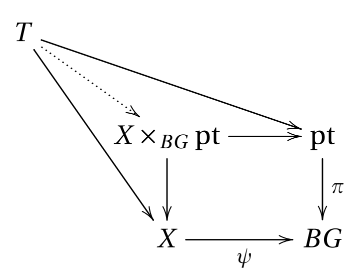
$$\{(m, n, \alpha) \mid m \in \mathcal{M}(T), n \in \mathcal{N}(T), \alpha: \phi(m) \to \psi(n)\}.$$

Example 7.22. Let X be a scheme and G an algebraic group. We want to digest the meaning of the fibered product

$$\begin{array}{ccc} X \times_{BG} \mathrm{pt} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow[\psi]{} & BG \end{array}$$

Part of the structure data is the morphism $X \to BG = [\mathrm{pt}/G]$, which is the same as specifying a principal G-bundle $P \to X$.

Let's try to understand $X \times_{BG} \mathrm{pt}$ through its functor of points.



By definition, a T-point of $X \times_{BG} \mathrm{pt}$ is the same as maps $f\colon T \to X$ and $g\colon T \to \mathrm{pt}$, together with a map (necessarily an isomorphism)

$$\alpha\colon \psi \circ f(T) \to \pi \circ g(T).$$

What does this really mean? Again, $\psi \circ f(T)$ is equivalent to the data of a principal G-bundle on T, namely f^*P. Similarly, $\pi \circ g(T)$ is equivalent to the data of another principal G-bundle on T, namely $g^* \mathrm{pt}$, which is the trivial bundle. So that means α is a trivialization of f^*P (i.e. an isomorphism with the trivial bundle), which is the same as a section $T \to f^*P$.

In conclusion, we have seen that a map $T \to X \times_{BG} \mathrm{pt}$ is the same as a map $f\colon T \to X$ and a section $s\colon T \to f^*P$

$$\begin{array}{ccc} f^*P & \longrightarrow & P \\ \downarrow \uparrow s & & \downarrow \\ T & \xrightarrow{f} & X \end{array}$$

This is the same as specifying a map $T \to P$. So we see that $X \times_{BG} \mathrm{pt}(T) \cong P(T)$, or $X \times_{BG} \mathrm{pt} \cong P$. Of course, this is what we expected: $\mathrm{pt} \to BG$ is the "universal" G-bundle. Note that the presence of the map α, measuring the difference along the two pullbacks, is crucial to achieving the the right characterization!

Exercise 7.23. Let G be an algebraic group and $H \subset G$ a closed subgroup.

- Show that there is a canonical map $BH \to BG$.
- Show that there is a (2-)cartesian diagram

$$\begin{array}{ccc} G/H & \longrightarrow & BH \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & BG \end{array}$$

Exercise 7.24. Let G be an algebraic group and $H \triangleright G$ a closed normal subgroup.

- Show that there is a canonical map $BG \to B(G/H)$.
- Show that there is a (2-)cartesian diagram

$$\begin{array}{ccc} BH & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & B(G/H) \end{array}$$

Exercise 7.25. If $\mathscr{M}$ is a stack, then we define its *inertia stack* $I(\mathscr{M})$ is the fibered product $\mathscr{M} \times_{(\mathscr{M} \times \mathscr{M})} \mathscr{M}$. Show that

$$I(\mathscr{M})(T) = \{(m, \alpha) \mid m \in \mathscr{M}(T), \alpha \in \mathrm{Aut}(m)\}.$$

In particular, if $\mathscr{M} = [X/G]$ then

$$I\mathscr{M} = \{(g, x) \mid x \in X, g \in \mathrm{Aut}(X)\}.$$

8. Algebraic stacks

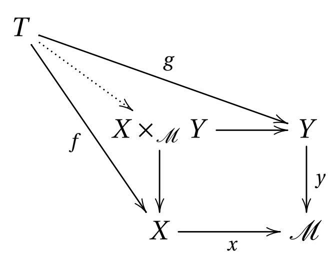
8.1. **Artin stacks.** The category of stacks that we have just defined is too general to work with in algebraic geometry. We would like to upgrade notions of smoothness, dimension, sheaves, etc. from schemes to stacks. For instance, we would like to say in some meaninful way that "$\mathscr{M}_g$ has dimension $3g-3$." To make these notions precise, we need to demand a little extra structure. This comes in the form of an "atlas" covering the stalk by a scheme.

For motivation, look back at Example 7.22. Here the stack BG admits a "uniformization" from a *scheme* pt. If we base change along any map from a scheme $X \to BG$ then the fibered product is the total space of a principal G-bundle over X, and in particular a smooth, surjective morphism. Thus, the map pt $\to BG$ should morally be considered as "smooth covering." Thus, we should able to use it to descend the usual geometric properties of schemes to BG.

This is the prototype for the notion of algebraic stack, which is a stack equipped with a covering by a scheme (an "atlas") satisfying some conditions. This will allow us to transfer geometric notions from schemes to stacks, provided that they satisfy good descent properties. Before we can give a complete definition, we need to highlight some technical considerations.

Definition 8.1. Let $\mathscr{M}$ be a stack, and let $x: X \to \mathscr{M}$ and $y: Y \to \mathscr{M}$ be two maps from *schemes*. Then the fiber product $X \times_{\mathscr{M}} Y$ (along x and y) is called the *isomorphism stack* $\mathrm{Isom}(x,y)$.

To see why this is a reasonable name, consider its T-points:

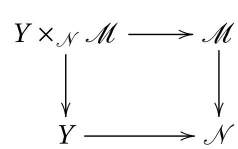


We have

$$\mathrm{Isom}(x,y)(T) = \{(f,g,\alpha) \mid f: T \to X, g: T \to X, \alpha: f^*x \cong g^*y\}.$$

This is not only a stack but a sheaf (it lands in the subcategory **Set** $\subset$ **Grpoid** consisting of groupoids with trivial automorphism groups). In fact, in many natural examples it is actually a scheme (e.g. Example 7.22). This general phenomenon has a name:

Definition 8.2. We say that a morphism of stacks $\mathscr{M} \to \mathscr{N}$ is *representable* if the base change along any map from a scheme Y to $\mathscr{N}$ is again a scheme:



In the future, we may use the convention that stacks denoted by ordinary Roman letters are schemes.

Definition 8.3. We say that a stack $\mathscr{M}$ is an *algebraic stack* (or *Artin stack*) if the following conditions hold.

(1) For any schemes X, Y and maps $X \to \mathscr{M}$, $Y \to \mathscr{M}$ the fibered product $X \times_{\mathscr{M}} Y$ is a scheme.
(2) There exists a scheme U (which we may refer to as an *atlas*) with a map $u : U \to \mathscr{M}$ such that for all maps from a scheme $X \to \mathscr{M}$, the base change of u is smooth and surjective:

$$\begin{array}{ccc} X \times_{\mathscr{M}} U & \longrightarrow & U \\ \downarrow & & \downarrow u \\ X & \longrightarrow & \mathscr{M} \end{array}$$

(3) The forgetful map $\mathrm{Isom}(u,u) = U \times_{\mathscr{M}} U \to U \times U$ is quasicompact and separated.

Remark 8.4. The third condition is a technical condition that you can ignore for our purposes.

Another unimportant technical remark is that it can be better to relax (1) by allowing $X \times_{\mathscr{M}} Y$ to be an "algebraic space" instead of a scheme. (Algebraic spaces sit between schemes and algebraic stacks.)

Exercise 8.5. Show that condition (1) is equivalent to the diagonal map $\mathscr{M} \to \mathscr{M} \times \mathscr{M}$ being representable.

8.2. **Deligne-Mumford stacks.** There is a subclass of Artin stacks which are "nicer" in many ways, and often come up in practice.

Definition 8.6. We say that an Artin stack $\mathscr{M}$ is a *Deligne-Mumford stack* (or *DM stack* for short) if the atlas U can be taken such that any base change $X \times_{\mathscr{M}} U \to X$ is étale and surjective:

$$\begin{array}{ccc} X \times_{\mathscr{M}} U & \longrightarrow & U \\ \text{étale} \downarrow & & \downarrow u \\ X & \longrightarrow & \mathscr{M} \end{array}$$

Remark 8.7. It is usually also required that the diagonal morphism $\mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is quasicompact. We shall ignore this issue.

Example 8.8. The computation in Example 7.22 shows that BG is an Artin stack (but not a Deligne-Mumford stack if G has positive dimension).

Exercise 8.9. Show that $[X/G]$ is an Artin stack.

Facts about DM stacks.

Theorem 8.10. *If $\mathscr{M}$ is a Deligne-Mumford stack and T is a quasicompact scheme, then any object of $\mathscr{M}(T)$ has only finitely many automorphisms.*

Theorem 8.11. *If G acts on X with finite reduced stabilizers, then $[X/G]$ is a Deligne-Mumford stack.*

Proof. See [Edi00] for a discussion of this proof and many others. □

Theorem 8.12 (Rydh)**.** *If* Spec $k \xrightarrow{x} [X/G]$ *is a point and $[X/G]$ is a DM stack, then there exists a stabilizer-preserving morphism $[U/G_x] \to [X/G]$ where U is affine.*

This gives a local picture of a DM stack as a quotient of a scheme by a *finite* group, and Artin stacks as being locally quotients by positive-dimensional groups. However, we caution against taking this picture too literally: for instance, there are Artin stacks all of whose objects have finite stabilizers.

8.3. **Algebraic spaces.**

Definition 8.13. An *algebraic space* is a Deligne-Mumford stack which is a sheaf of *sets* (rather than groupoids). In other words, it is a functor on **Sch** with no non-trivial automorphisms.

Example 8.14. For a scheme X and a finite group G acting *freely* on X, the quotient stack $[X/G]$ is an algebraic space. There is an étale atlas given by $X \to [X/G]$.

If X is quasiprojective, then X/G is a quasiprojective scheme. If X is not quasiprojective, then Hironaka showed that X/G need not be a scheme.

Example 8.15. We give an example of an algebraic space that is not a scheme. Let Y be the quotient of $\mathbb{A}^1$ modulo the étale equivalence relation $x \sim -x$ if $x \neq 0$.

This algebraic space comes with a morphism $Y \to \mathbb{A}^1$, sending $x \mapsto x^2$, which is an isomorphism over $\mathbb{A}^1 - \{0\}$. Moreover, this is bijective, and both Y and $\mathbb{A}^1$ are smooth over k. But f is not étale at 0 (so f is not an isomorphism). Indeed, in a neighborhood of 0 the map f "looks like" $x \mapsto x^2$.

8.4. **Geometric properties.** Any property which is "local" in the smooth topology can be immediately generalized from schemes to Artin stacks using an atlas. Though the reader can probably guess how this works, we spell it out explicitly.

Definition 8.16. Let P be a property of schemes that "can be checked" on coverings in the sense that if X is a scheme and $X' \to X$ is a covering, then X has P if and only if X' does.

Let $\mathscr{M}$ be an Artin stack. Then we say that *$\mathscr{M}$ has property P* if for some (equivalently, any) atlas $u: U \to \mathscr{M}$, U has property P.

Example 8.17. This applies with $P =$ smooth, normal, reduced, locally of finite presentation, locally noetherian, regular...

More generally, it is straightforward to generalize this to a relative version when the morphism is representable.

Definition 8.18. Let P be a property of morphisms of schemes $X \to Y$ that "can be checked" on coverings in the sense that if $Y' \to Y$ is a covering, then $X \to Y$ has property P if and only if $X \times_Y Y' \to Y$ has property P.

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

Let $\mathcal{M} \to \mathcal{N}$ be a **representable** morphism of Artin stacks. Then we say that $\mathcal{M} \to \mathcal{N}$ *has property P* if for some (or equivalently, any) atlas $u\colon U \to \mathcal{N}$ an atlas, the base-change $U \times_{\mathcal{N}} \mathcal{M} \to U$ has property P.

$$\begin{array}{ccc} U \times_{\mathcal{N}} \mathcal{M} & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow u \\ U & \longrightarrow & \mathcal{N} \end{array}$$

Example 8.19. This applies with $P =$ closed immersion, open immersion, affine, proper, finite...

Example 8.20. This can be used to define the notion of "closed substack" and "open substack." In particular, using the atlas $X \to [X/G]$ we see that open substacks of $[X/G]$ are of the form $[V/G]$ for an open subscheme $V \subset X$, and likewise for closed substacks.

Finally, when a morphism of stacks is *not* representable we can still transfer properties that are local on the source and target for the given topology.

Definition 8.21. Let P be a property of morphisms of schemes $X \to Y$ that can be "checked locally on source and target" in the sense that if $Y' \to Y$ and $X' \to X$ are coverings fitting into a commutative (but not necessarily fiber!) square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

then $X \to Y$ has property P if and only if $X' \to Y'$ has property P.

Let $\mathcal{M} \to \mathcal{N}$ be an arbitrary morphism of Artin stacks. Then we say that $\mathcal{M} \to \mathcal{N}$ *has property P* if there exist atlases $u\colon U \to \mathcal{M}$ and $v\colon V \to \mathcal{N}$ fitting into a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow u \\ V & \longrightarrow & \mathcal{N} \end{array}$$

such that $U \to V$ has property P.

Example 8.22. This applies with $P =$ smooth, flat, locally of finite presentation...

Example 8.23. The map $BG \to \mathrm{pt}$ is not representable. This is a consequence of the following exercise:

Exercise 8.24. Show that a representable morphism induces an injection on automorphisms groups of objects.

This is actually an isomorphism if one replaces schemes with algebraic spaces in all the definitions.

Example 8.25. A situation that arises frequently in algebraic geometry is that one has an action of $G = \mathrm{GL}_n$ on a space X, but the center acts trivially. Then there is a morphism of stack quotients $[X/\mathrm{GL}_n] \to [X/\mathrm{PGL}_n]$ which is smooth and surjective but *not* representable.

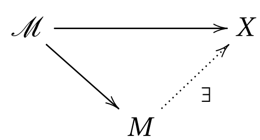
8.5. **Sheaves on stacks.** We can use the same idea to "descend' the notion of sheaf on a stack. As before, the way to proceed is to find a "descent" formulation of the notion on schemes (in terms of the appropriate topology), and apply this to an atlas $u: U \to \mathscr{M}$.

In the case of schemes, if $U \to X$ is covering then a quasicoherent sheaf on X is the data of a quasicoherent sheaf on U, equipped with an isomorphism $p_1^* \mathscr{F} \cong p_2^* \mathscr{F}$ on $U \times_X U$ satisfying cocycle conditions on $U \times_X U \times_X U$.

Definition 8.26. Let $\mathscr{M}$ be a stack and $u: U \to X$ an atlas. A *quasicoherent sheaf on* $\mathscr{M}$ is the data of a sheaf $\mathscr{F}_U$ on U, together with an isomorphism $p_1^* \mathscr{F} \cong p_2^* \mathscr{F}$ on $U \times_{\mathscr{M}} U$ satisfying cocycle conditions on $U \times_{\mathscr{M}} U \times_{\mathscr{M}} U$.

8.6. **Coarse moduli spaces.** Classically, a *coarse moduli space* M_g of genus g curves was constructed using Geometric Invariant Theory. One might ask what the relation is between M_g and the moduli stack $\mathscr{M}_g$. This relationship fits into a general situation.

Definition 8.27. For a stack $\mathscr{M}$, a *categorical quotient* is a scheme M such that any map $\mathscr{M} \to X$ to a scheme (algebraic space) factors uniquely through M:



A *coarse moduli space* for $\mathscr{M}$ is a categorical quotient M which induces a bijection on geometric points: $\mathscr{M}(\overline{k}) = M(\overline{k})$. (This is obviously unique if it exists.)

Theorem 8.28 (Keel-Mori)**.** *Every separated Deligne-Mumford stack has a coarse moduli space.*

Example 8.29. Categorical quotients can be smaller than expected. For example, the categorical quotient of $[\mathbb{A}^n/\mathbb{G}_m]$ is just a point because any $\mathbb{G}_m$-equivariant map to $\mathbb{A}^n$ must be constant.

Example 8.30. If X is a scheme (algebraic space) and G acts on X via *finite* stabilizers, then $[X/G]$ has a coarse moduli space M. Moreover, $[X/G] \to M$ is proper and quasi-finite.

Example 8.31. The moduli stack of elliptic curves over $\mathbb{C}$ is the quotient stack $\mathbb{H}/SL(2,\mathbb{Z})$. The corresponding coarse moduli space is just $\mathbb{A}^1_{\mathbb{C}}$. This isomorphism is called the j-invariant.

In this case, the difference between the stack and the coarse moduli space has to do with the extra data of the automorphisms, which is captured in the stack. Most elliptic curves have automorphism group $\mathbb{Z}/2$ (with the non-trivial element being the inverse in the group law), but one (up to isomorphism over $\mathbb{C}$) has automorphism group $\mathbb{Z}/4$ and one has automorphism group $\mathbb{Z}/6$.

The fact that the moduli stack of elliptic curves is a quotient $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ means that this moduli stack has a Kähler metric with curvature -1. This has some interesting geometric consequences: for instance, it implies that any family of elliptic curves over $\mathbb{P}^1$, or $\mathbb{A}^1$, must have all fibers isomorphic. Indeed, such a family would give a morphism $\mathbb{A}^1 \to [\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})]$. The universal cover of this latter space is $\mathbb{H}$, and since $\mathbb{A}^1$ is simply-connected, this lifts to a map $\mathbb{A}^1 \to \mathbb{H}$. Since $\mathbb{H}$ is isomorphic to the unit disc as a complex manifold, Liouville's theorem implies that any such analytic map is constant.

On the other hand, the fact that the coarse moduli space of elliptic curves is isomorphic to the affine line (where the natural Kähler metric has curvature 0 rather than -1) also has geometric consequences. For instance, the fact that there are infinitely many distinct elliptic curves over $\mathbb{Q}$ follows immediately from the fact that the affine line has infinitely many rational points, and one can check that there is an elliptic curve over $\mathbb{Q}$ with any given j-invariant.

By contrast, for big enough positive integers n the coarse moduli space of elliptic curves with a point of order n has genus at least 2. Faltings showed that every curve of genus at least 2 over $\mathbb{Q}$ has only finitely many rational points. (The special case of modular curves which is relevant here was established earlier by Mazur.) So for all sufficiently large n, there are only finitely many elliptic curves over Q with a point of order n, up to isomorphism over $\overline{\mathbb{Q}}$.

9. Chow groups of quotient stacks

9.1. **Construction.** Although stacks might seem too far from "genuine" geometric spaces to admit a Chow ring, Totaro [Tot99] gave a definition of the "Chow ring of BG." Edidin-Graham subsequently generalized [EG00] this definition to quotient stacks. The reader may be disappointed or pleased to learn that it does not involve any of the formalism of stacks that we have just belabored!

The motivation goes back to the topological construction of BG as EG/G, where EG is a contractible space with a free G-action. We don't have the luxury of an algebro-geometric version of EG, but if we can pick an "almost contractible" space with a free G-action then we might expect to get a reasonable definition of (at least some) Chow groups.

Consider the topological story. If V is contractible and $S \subset V$ has high codimension, say real codimension at least $i+1$, then the inclusion $V-S \hookrightarrow V$ will induce an *isomorphism* on homotopy groups the $\pi_0, \ldots, \pi_i$. In particular, $V-S$ will be i-connected. Then the map $V-S \to EG$ will induce an isomorphism on homotopy groups in dimension up to i, hence also an isomorphism on homology groups up to dimension $i+1$ by Hurewicz's theorem. Motivated by this, Totaro defined $CH^i(BG) = CH^i(V-S)/G$ for any pair (V,S) such that G acts freely on $V-S$ and $\operatorname{codim}_V S > i$. The more general definition of Edidin-Graham is:

Definition 9.1. Let X be a smooth quasiprojective variety with an action of an algebraic group G. If V is a faithful G-representation and $S \subset V$ is a subset with $\operatorname{codim}_V S > i$, then we define the *$G$-equivariant Chow ring*

$$CH^i_G([X/G]) := CH^i_G(X) := CH^i(X \times (V-S))/G$$

where the G acts diagonally on $X \times (V-S)$.

In order for this to really be well-defined, we have to check that it is independent of the choice of (V,S).

Proof. We want to show that if (V,S) are (V',S') are two pairs such that G acts freely on both $V-S$ and $V'-S'$ and $\operatorname{codim}_V S, \operatorname{codim}_{V'} S' > i$ then

$$CH^*(X \times (V-S))/G \cong CH^*(X \times (V'-S'))/G \text{ for } * < i.$$

We use the "double fibration trick" due to Bogomolov in order to reduce to the special case where one pair "dominates" the other, in the sense that $V' = V$ and $S' \supset S$.

We first reduce to the case where the representations are equal by considering a common domination by $V \times V'$. Then $X \times (V-S) \times V'$ is a vector bundle over $X \times (V-S)$, so $(X \times (V-S) \times V')/G$ exists and $X \times S \times V'$ has codimension at least i in $X \times V \times V'$. Similarly, $X \times V \times (V'-S')$ is a vector bundle over $X \times (V'-S')$, satisfying the right conditions. By the homotopy axiom, a space as the same Chow groups as any vector bundle over it. This reduces to the case $V = V'$.

Next, replacing S' with $S \cup S'$ allows us to assume that $S' \supset S$. Then we apply the excision axiom (2.1):

$$CH_*(X \times (S'-S))/G \to CH_*(X \times (V-S))/G \to CH_*(X \times (V-S'))/G \to 0$$

is exact. But since S' has codimension greater than i, $CH_*(X \times (S' - S))/G$ vanishes up to codimension i, so the map $CH_*(X \times (V - S))/G \to CH_*(X \times (V - S'))/G$ must be an isomorphism in codimension up to i. □

We can informally think of $[X/G]$ as $\varinjlim_i (X \times (V_i - S_i)/G$. As a limit of spaces this may not really make sense, but we have shown that the limit of *Chow groups* $\varinjlim_i CH^i(V_i - S_i/G)$ really does make sense, so we can just define it to be $CH^i(BG)$.

We can make all the definitions we want, but why is this a *good* definition?

Theorem 9.2. *Let G be a reductive group over a field k. Then the above group $CH^i(BG)$ is naturally identified with the set of (pullback) functorial assignments for every smooth quasiprojective variety X,*

$$\{\textit{principal } G\textit{-bundle over } X\} \to CH^i X.$$

Proof. See [Tot99]. □

Remark 9.3. This gives a natural ring structure on $CH^i X$, which agrees with what you think it is (namely the inverse limit of the ring structures on the finite approximations).

9.2. **Examples.** Our strategy for computing BG will be to find a representation of G such that the action is free on the complement of a high codimension subset S. We then need to compute the quotient variety $(V - S)/G$. When G is a finite group we can just take the ring of invariants in $V - S$; when G is a linear algebraic group, the quotient exists as a quasi-projective variety by general theory.

9.2.1. *Stratifications.*

Example 9.4. Let $G = \mathbb{G}_m$. Then G acts on $\mathbb{A}^{n+1}$ by scalar multiplication, and the action is free on $\mathbb{A}^{n+1} - \{O\}$. The quotient space is one that we know and love: $\mathbb{P}^n$. We computed earlier that $CH^*(\mathbb{P}^n) \cong \mathbb{Z}[c_1]$.

Let L_n be the tautological line bundle on $\mathbb{P}^n$. The inclusion via 0-section $\mathbb{P}^n \to L_n$ sends the class of the hyperplane to the class of a codimension 2 plane. On the one hand, we know that $CH^*(L_n) \to CH^*(\mathbb{P}^n)$ is an isomorphism by homotopy invariance. On the other hand, we have a pushforward map

$$CH^{n-k}(\mathbb{P}^n) = CH_k(\mathbb{P}^n) \to CH_k(L_n) \xrightarrow{\sim} CH_{k-1}(\mathbb{P}^n) = CH^{n-k+1}(\mathbb{P}^n).$$

For $k = n$, this composition takes the fundamental class of $\mathbb{P}^n$ to the class of the zero section in $CH_k(L)$, and then to the hyperplane class in $\mathbb{P}^n$. The upshot is that the inclusion of $\mathbb{P}^n$ as the zero section of L_n *induces multiplication by* $c_1(L_n)$ at the level of the Chow ring.

Taking the "limit" $n \to \infty$, where we get $B\mathbb{G}_m = \mathbb{P}^\infty$ and L_∞ is the "universal" line bundle, we obtain the general statement:

Corollary 9.5. *If $L \to X$ is any line bundle, then the map*

$$CH_k(X) \to CH_k(L) \cong CH_{k-1}(X)$$

induced by the inclusion $X \hookrightarrow L$ as the zero section corresponds to multiplication by $c_1(L) \in CH^*(X)$ in $CH^*(X)$.

This result will prove surprisingly useful!

Example 9.6. What's $B\,\mathrm{GL}(n)$? Let V be the standard representation of $\mathrm{GL}(n)$. Let $W = \mathrm{Hom}(\mathbb{A}^N, V) \cong V^N$ for $N \gg 0$. Then $\mathrm{GL}(V)$ acts freely on the open subset of *surjective* linear maps $\mathrm{Surj}(\mathbb{A}^n, V)$. The quotient space $\mathrm{Surj}(\mathbb{A}^n, V)/\mathrm{GL}(V)$ is isomorphic to $\mathrm{Gr}(N-n, N)$, by associating to a surjective linear map its kernel.

The codimension of the complement goes to ∞ with N, so we get that

$$CH^* B\,\mathrm{GL}(n) = \varinjlim CH^* \mathrm{Gr}(N-n, N)\text{“} = CH^* \mathrm{Gr}(\infty - n, \infty).\text{”}$$

As the Grassmannian also admits an *algebraic* affine stratification (via Schubert cells), its Chow groups are the free abelian group on the set of cells, like the ordinary cohomology ring. Therefore,

$$CH^* B\,\mathrm{GL}(n) \cong \mathbb{Z}[c_1, \dots, c_n] \quad |c_i| = i.$$

By the earlier theorem, each c_i furnishes a functorial assignment from rank n vector bundles $V \to X$ to $CH^i(X)$, which is called the *Chern class*.

9.2.2. *Finite groups.*

Example 9.7. Let's try to compute $CH^* B(\mathbb{Z}/2)$.

To compute $CH^{i=0}$, we can take the reflection representation on $\mathbb{A}^1$. Then $\mathbb{Z}/2$ acts freely on $A^1 - \{O\}$, with quotient again $\mathbb{A}^1 - \{O\}$, so

$$CH^0(B\mathbb{Z}/2) \cong CH^0(\mathbb{A}^1 - \{O\}) \cong \mathbb{Z}.$$

To compute $CH^{i=1}$, we can take the reflection representation on $\mathbb{A}^2$. Then $\mathbb{Z}/2$ acts freely on $\mathbb{A}^2 - \{0\}$ by multiplication by ± 1. The ring of invariants its $k[x^2, xy, y^2] \subset k[x, y]$, which you might recognize as the (affine) quadric cone $Q := \mathrm{Spec}\, k[u, v, w]/(uw - v^2)$. Removing the origin corresponds to removing the cone point.

When calculating the first Chow group, we may as well throw the cone point back in since it has codimension 2 (hence doesn't affect CH^1). We claim that $CH^1\mathbb{Q} \cong \mathbb{Z}/2$, generated by the class of a line through the origin lying on the cone, e.g. $u = 0$.

First let's see why twice the line should be zero. A plane tangent to the line intersects the cone in the double line. As the plane is rationally equivalent to zero on $\mathbb{A}^3$, its intersection is rationally equivalent to zero on the cone.

According to the basic exact sequence, the quotient of CH^1 by the class of this line is just CH^1 of the cone minus the hyperplane section. That corresponds to inverting u, in which case we get $\mathrm{Spec}\, k[u^{\pm}, v]$, which is an open subset of affine space, and hence has trivial CH^1.

In general, $\mathbb{Z}/2$ will act freely on $\mathbb{A}^n - \{O\}$, and you can see that the quotient will be $\mathrm{Spec}(k[x_1, \dots, x_n])_{2\bullet}$ minus the origin. That's the affine cone over the Veronese embedding of the smooth quadric in $\mathbb{P}^{n-1}$ minus the cone point. For the purposes of calculating the Chow groups, we can always throw the cone point back in. The CH_k of this variety will have a class represented by a k-plane contained in the quadric, whose double is the intersection of an ambient plane with the quadric, hence rationally equivalent to 0.

Exercise 9.8. Show that these plane classes generate, so that the Chow ring is $\mathbb{Z}[h]/(2h)$.

Don't sweat to much on this exercise - we will shortly see a more efficient way to see the answer.

Example 9.9. Let $P \to X$ be a principal $\mathbb{G}_m$-bundle (with X smooth). Then P is the total space of the corresponding line bundle L minus the 0-section. The excision sequence (viewing $X \hookrightarrow L$ as the zero section) gives

$$CH_*(X) \to CH_*(L) \to CH_*(L - X = P) \to 0.$$

But by Corollary 9.5, the map $CH_*(X) \to CH_*(L)$ is multiplication by $c_1(L)$, so we get that $CH_*(P) = CH_*(X)/c_1(L)$.

Compare this with the Gysin sequence of a circle bundle in topology:

$$\ldots \to H^{i-2}X \xrightarrow{c_1(L)} H^iX \to H^iP \to H^{i-1}X \to \ldots$$

Example 9.10. We can use the result of Example 9.9 to give a slick computation of $CH^*(B\mathbb{Z}/p)$. Let W be a faithful 1-dimensional representation of $\mathbb{Z}/p$ (i.e. via a non-trivial character) and $V = W^{\oplus n}$. As $\mathbb{Z}/p$ acts freely on $V - \{O\}$, an nth level approximation to $B(\mathbb{Z}/p)$ is $(V - \{O\})/(\mathbb{Z}/p)$. Now, this action factors through a representation of $\mathbb{G}_m$, via $\mathbb{Z}/p \hookrightarrow \mathbb{G}_m \hookrightarrow \mathrm{GL}(V)$. Therefore, we should have a fiber bundle

$$G_m/(\mathbb{Z}/p) \to B\mathbb{Z}/p \to B\mathbb{G}_m.$$

This doesn't really make sense in the category of schemes, but concretely $\mathbb{A}^n - \{O\}$ can be used as an "approximation to EG" for both $\mathbb{G}_m$ and $\mathbb{Z}/p$, so we have a genuine fiber bundle

$$\mathbb{G}_m/(\mathbb{Z}/p) \to (\mathbb{A}^n - \{O\})/(\mathbb{Z}/p) \to (\mathbb{A}^n - \{O\})/\mathbb{G}_m.$$

Of course, we computed the latter objects as $\mathbb{P}^{n-1}$, and $\mathbb{G}_m/(\mathbb{Z}/p) \cong \mathbb{G}_m$. This realizes $(\mathbb{A}^n - \{O\})/(\mathbb{Z}/p)$ as a $\mathbb{G}_m$-bundle over $\mathbb{P}^{n-1}$, corresopnding to the line bundle $\mathscr{O}(-p)$ as it's evidently the pth power of the tautological bundle. That puts this example in the context of 9.9, so

$$CH^*B(\mathbb{Z}/p) \cong CH^*\mathbb{P}^\infty/pc_1 \cong \mathbb{Z}[c_1]/pc_1.$$

Observe that we recover the computation of $B(\mathbb{Z}/2)$, with much less fussing around!

9.2.3. *Classical groups.* We now develop the tools to calculate the Chow ring of some classical groups.

Theorem 9.11. *Let G be an affine group scheme over k and V a faithful representation of G. Under the induced map*

$$CH^*BGL(V) \cong \mathbb{Z}[c_1, \ldots, c_n] \to CH^*BG$$

let $c_i \mapsto c_iV$. Then

$$CH^*(\mathrm{GL}(V)/G) \cong CH^*BG/(c_1V, \ldots, c_nV).$$

Proof. We (morally) have a fibration

$$\mathrm{GL}(n)/G \to BG \to B\,\mathrm{GL}(n).$$

By "looping" this, we also get

$$\mathrm{GL}(n) \to \mathrm{GL}(n)/G \to BG.$$

(To make this mathematically sound, argue that if $V - S$ approximates $E\,\mathrm{GL}(n)$ then we get

$$\mathrm{GL}(n)/G \to (V-S)/G \to (V-S)/\,\mathrm{GL}(n)$$

and

$$\mathrm{GL}(n) \to [H \times (V - S)]/G \to (V - S)/G.$$

where the G-action on a product is always diagonal.)

This shows that $\mathrm{GL}(n)/G$ is a principal $\mathrm{GL}(n)$-bundle over BG. That's the same as a vector bundle, so it suffices to show that if $P \to X$ is principal $\mathrm{GL}(n)$-bundle, then

$$CH^*(P) = CH^*(X)/(c_1(P), \dots, c_n(P)).$$

We already saw this in the special case $n = 1$. That implies the result for a direct sum of line bundles, i.e. a $(\mathbb{G}_m)^n$-bundle. Then the result holds for a Borel by homotopy invariance, and the general case can be reduced to the Borel by considering a flag (this is a reflection of the splitting principle in algebraic geometry).

□

We can use this to gain information about $CH^* BG$ if we know $CH^*(\mathrm{GL}(V)/G)$ (or vice versa). For instance, if $CH^*(\mathrm{GL}(V)/G)$ is trivial, then $CH^* BG$ is generated by $c_1 V, \dots, c_n V$.

Example 9.12. What is $CH^* BO(n)_{\mathbb{C}}$? Let V be the standard representation of $O(n)$, inducing an embedding $O(n) \hookrightarrow \mathrm{GL}(n)$. To apply Theorem 9.11, we need to understand $\mathrm{GL}(n)/O(n)$.

Well, $\mathrm{GL}(n)$ acts on symmetric forms on V, i.e. $\mathrm{Sym}^2 V^*$, which is isomorphic to $\mathbb{A}^{n(n+1)/2}$. All *non-degenerate* symmetric bilinear forms are $\mathrm{GL}(n)$-equivalent, and the stabilizer of a non-degenerate form is $O(n)$. Therefore, $\mathrm{GL}(n)/O(n)$ can be realized as an open subset of $\mathbb{A}^{n(n+1)/2}$, so

$$CH^* \mathrm{GL}(n)/O(n) = \begin{cases} \mathbb{Z} & * = 0, \\ 0 & * > 0. \end{cases}$$

By the theorem, we may conclude that $CH^* BO(n)$ is generated by $c_1, \dots, c_n$. What's the kernel?

As the representation V of $O(n)$ is self-dual, we get $c_1 = -c_1$, and in general $c_j = (-1)^j c_j(V)$. Therefore, $2c_j = 0$ for all *odd* j. In fact, these are the only relations. One way to see this is that the map

$$\mathbb{Z}[c_1, \dots, c_n]/(2c_{2k+1} = 0) \hookrightarrow H^*(BO(n), \mathbb{Z})$$

is injective, but this factors through Chow. Therefore, we conclude that

$$CH^* BO(n) \cong \mathbb{Z}[c_1, \dots, c_n]/(2c_{2k+1} = 0).$$

Example 9.13. What is $CH^* B\,\mathrm{Sp}(2n)_{\mathbb{C}}$? Again, let V be the standard representation of $\mathrm{Sp}(2n)$. Then $\mathrm{GL}(2n)$ acts transitively on the space of symplectic forms on V, which the non-degenerate ones being isomorphic to an open subset of affine space. Therefore, $CH^* BG$ will be generated by $CH^* \mathbb{Z}[c_1, \dots, c_{2n}]$. But what are the relations?

Again, the natural symplectic form makes V self-dual, so by the same reasoning we get $2c_i = 0$ for i odd. In fact, we claim that $c_i = 0$ for i odd. It suffices to show that $CH^* B\,\mathrm{Sp}(2n) \hookrightarrow CH^* BT = CH^* B\mathbb{G}_m$ (the maximal torus), as we checked that the latter is torsion-free. Since BT is an iterated affine space bundle over BB (the classifying space of the Borel), it suffices to show that $CH^* B\,\mathrm{Sp}(2n) \hookrightarrow CH^* BB$. This fits into a fiber bundle

$$\mathrm{Sp}(2n)/B \to BB \to B\,\mathrm{Sp}(2n).$$

Now, $\mathrm{Sp}(2n)$ is one of Grothendieck's list of "special groups," meaning that this bundle is actually *Zariski locally trivial*, so we can take a section over an open subset and then take its closure. (This is a version of Hilbert's Theorem 90 for the symplectic group, which is reflected in the linear algebra fact that there is only $2n$-dimensional symplectic vector space over any field.) This defines a cycle $\alpha \in CH^* BB$ pushing forward to the fundamental class $1 \in CH^0 B\,\mathrm{Sp}(2n)$. This gives a section for the induced map of Chow rings $CH^* B\,\mathrm{Sp}(2n) \to CH^* BB$, as the projection formula tells us that $f_*(\alpha \cdot f^* x) = x$ for all $x \in CH^* B\,\mathrm{Sp}(2n)$.

It is known that $H^*(B\,\mathrm{Sp}(2n), \mathbb{Z}) \cong \mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$, so by the same argument as above we must have found all relations in the Chow ring: we conclude that $CH^* B\,\mathrm{Sp}(2n) \cong \mathbb{Z}[c_2, c_4, \ldots, c_{2n}]$.

10. Equivariant K-theory

10.1. **Equivariant sheaves.** We now introduce a group action into the picture. Suppose that we have an algebraic group G acting on our space X. This means that we have an action morphism

$$G \times X \xrightarrow{a} X$$

satisfying the usual axioms, which can be expressed as certain commutative diagrams.

Definition 10.1. A morphism $f : X \to Y$ is G-*equivariant* with respect to given G-actions $G \times X \xrightarrow{a_X} X$ and $G \times Y \xrightarrow{a_Y} Y$ if the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{a_X} & X \\ \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{a_Y} & Y \end{array}$$

We want to have a notion of a "G-equivariant vector bundle" E on X. A natural definition would be to demand that that E admit a G-action such that the projection map $E \xrightarrow{\pi} X$ is G-equivariant. However, to better match the generalization to coherent sheaves we choose a different formulation:

Definition 10.2. Let $p_2 : G \times X \to X$ denote the second projection map. A G-*equivariant vector bundle* on X is a locally free sheaf $\mathscr{E}$ on X equipped with an isomorphism

$$\theta : a^*\mathscr{E} \cong p_2^*\mathscr{E}$$

on $G \times X$, satisfying the cocycle condition

$$p_{23}^*\theta \circ (1 \times a)^*\theta = (m \times 1)^*\theta$$

where $p_{23} : G \times G \times X \to G \times X$ is the projection and $m : G \times G \to G$ is the multiplication within G.

Exercise 10.3. Show that Definition 10.2 is equivalent to demanding that the projection map from the total space of $\mathscr{E}$ to X be G-equivariant.

Definition 10.4. Let $p_2 : G \times X \to X$ denote the second projection map. A G-*equivariant coherent sheaf* on X is a coherent sheaf $\mathscr{F}$ on X, equipped with an analogous isomorphism satisfying an analogous cocycle condition to that in Definition 10.2.

Definition 10.5. We denote by $G_0(G, X)$ the Grothendieck group of G-coherent sheaves and $K_0(X)$ the Grothendieck group of G-vector bundles.

Example 10.6. If $X = \operatorname{Spec} k$, then the notions of coherent sheaf and vector bundle on X coincide, both being a finite-dimensional k-vector space. A G-equivariant coherent sheaf on X is then simply a finite-dimensional representation of G. Thus, $G_0(G, X) = K_0(G, X) = R(G)$, the *representation ring* of G.

Example 10.7. If G is a finite abelian group over $\mathbb{C}$, then by the classical theory of finite group representations the representation ring is the free abelian group on the irreducible representations, which coincides with the usual group algebra $\mathbb{C}[G]$ (since all of the irreducibles are one-dimensional).

Example 10.8. Let $G = \mathbb{G}_m/\mathbb{C}$. Any representation of G is absolutely irreducible, and the irreducible representations are the characters. Therefore, the representation ring is $\mathbb{Z}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$. This coincides with the group algebra $\mathbb{C}[\mathbb{G}_m]$ again, but we emphasize that this is *not* true in general.

Proposition 10.9. *Let* $[X/G]$ *be a quotient stack. Then we have canonical isomorphisms*

$$K_0(G,X) \cong K_0([X/G]) \quad \textit{and} \quad G_0(G,X) \cong G_0([X/G])$$

Proof. Using the smooth atlas $X \to [X/G]$, we have by definition that a vector bundle (resp. coherent sheaf) on X/G is the same as a vector bundle (resp. coherent sheaf) on X satisfying descent data. We have to verify that this descent data corresponds precisely to the G-equivariance.

The definition of G acting freely on X is that the map

$$G \times X \xrightarrow{(a,\mathrm{Id})} X \times_{[X/G]} X$$

sendin $(g,x) \mapsto (gx,x)$ is an isomorphism, hence also that

$$G \times G \times X \to X \times_{[X/G]} X \times_{[X/G]} X$$

sending $(g,g',x) \mapsto (gg'x, g'x, x)$ is an isomorphism. We can use this to compare the definition of G-equivariance with the descent formalism from X to X/G. For instance, the map $m \times \mathrm{Id} \colon G \times G \times X \to G \times X$ translates into $p_{13} \colon X \times_{[X/G]} X \times_{[X/G]} X \to X \times_{[X/G]} X$.

Exercise 10.10. Complete the proof.

□

Example 10.11. By Example 10.6, we see that $K_0(BG) \cong G_0(BG) \cong R(G)$.

Example 10.12. Let $p \colon X \to B$ be a principal G-bundle. Then applying Proposition 10.9 with $X = X$, so that $B = [X/G]$, we obtain canonical isomorphisms

$$K_0(G,X) \cong K_0(B) \quad \text{and} \quad G_0(G,X) \cong G_0(B).$$

Example 10.13. Let G be a finite group. We have maps $\mathrm{pt} \to BG$ and $BG \to \mathrm{pt}$. The first map is the universal bundle over BG, which in terms of the functor of points sends a scheme T to the trivial G-bundle over T, and the second map sends any principal G-bundle over T to T.

We have $K(\mathrm{pt}) \cong \mathbb{Z}$ and $K(BG) \cong R(G)$. What are the pushforward maps $K(\mathrm{pt}) \to K(BG)$ and $K(BG) \to K(\mathrm{pt})$? Note that their composition must be the identity.

A sheaf on BG corresponds to a G-equivariant sheaf on the atlas pt; what sheaf is this? By definition, the map $\mathrm{pt} \to BG$ corresponds to a cartesian diagram

$$\begin{array}{ccc} G & \xrightarrow{f'} & \mathrm{pt} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathrm{pt} & \xrightarrow{f} & BG \end{array}$$

By base change, for a vector bundle V on pt we have $\pi^* f_* V \cong f'_*(\pi')^* V = V \otimes \operatorname{Spec} k[G]$. So we see that the map $K(\mathrm{pt}) \to K(BG)$ corresponds to *tensoring with the regular representation of* G.

Next, the map $K(BG) \to K(\mathrm{pt})$ corresponds to taking global sections. The global sections of a sheaf $\mathscr{F}$ on BG are the G-invariants of the global sections of the pullback sheaf on pt.

As a sanity check, we verify that the compositionis the identity: $(V \otimes k[G])^G \cong V$.

10.2. Atiyah-Bott localization. ♠♠♠ TONY: [TODO]

Part 3. **Equivariant Riemann-Roch**

11. Equivariant Riemann-Roch

11.1. **Baum-Fulton-MacPherson.** We are *finally* about to state the equivariant Riemann-Roch theorem of Edidin and Graham. First, we highlight a generalization of the Grothendieck-Riemann-Roch theorem to *singular* varieties, after which the equivariant Riemann-Roch theorem is patterned.

The *Baum-Fulton-MacPherson theorem* says that for all finite type schemes over k there exists an isomorphism $G(X) \xrightarrow{\tau_X} CH^*(X)$ which is covariant for *proper* morphisms $X \to Y$, which agrees with $\mathrm{ch}\cdot\mathrm{Td}(T_X)$ if X is smooth (and thus recovers Grothendieck-Riemann-Roch in that case.

$$\begin{array}{ccc} K(X)_{\mathbb{Q}} & \xrightarrow{\tau_X} & CH^*(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ K(Y)_{\mathbb{Q}} & \xrightarrow{\tau_Y} & CH^*(Y)_{\mathbb{Q}} \end{array}$$

In general, the map τ_X satisfies

$$\tau_X([\mathscr{O}_V]) = [V] + \text{lower order terms}$$

and indeed this underlies why τ_X is an isomorphism.

11.2. **Riemann-Roch for quotient stacks.** Let X be a scheme and G a linear algebraic group acting on X. We seek a Riemann-Roch theorem for the quotient stack $[X/G]$.

Remark 11.1. The quotient stack $[X/G]$ is Deligne-Mumford if all stabilizers are finite and geometrically reduced (the latter condition is automatic in characteristic 0) [EG00].

For notational purposes, from now on *we will use $K(\mathscr{X})$ to denote the rational Grothendieck group of coherent sheaves on $\mathscr{X}$).* We have a homomorphism $K([X/G]) \to CH([X/G])$, which may not be an isomorphism, but it factors through an isomorphism which we will shortly describe:

$$\begin{array}{ccc} K([X/G]) & \longrightarrow & CH([X/G]) \\ & \searrow \quad ? \quad \nearrow_{\cong} & \end{array}$$

The ring $K([X/G])$ has an *augmentation ideal*, i.e. the ideal of objects of rank 0. For example, $K(BG)$ is the representation ring of G, and has the augmentation ideal of rank 0 virtual representations. This is the "universal case." Any morphism $X \to \operatorname{Spec} k$, which is obviously flat, descends to a map $f\colon [X/G] \to BG$. This induces a pullback $f^*\colon K(BG) \to K([X/G])$, and the augmentation ideal for $K([X/G])$ is generated by the pullback of the augmentation ideal of $K(BG)$. Note that one can think of this again as the ideal of virtual vector bundles of rank 0.

Definition 11.2. We define $\widehat{K}([X/G])$ to be the completion of $K([X/G])$ at the augmentation ideal.

This is the missing ingredient ? in the diagram above. The map $K([X/G]) \to CH([X/G])$ should be something like $\mathrm{ch} \cdot \mathrm{Td}([X/G])$. Letting $\mathfrak{g}$ denote the Lie algebra of G equipped with the *adjoint action* of G, it is natural to think of $T_{[X/G]} = T_X/\mathfrak{g}$.

Theorem 11.3 (Edidin-Graham)**.** *The map* $\mathrm{ch} \cdot \mathrm{Td}(TX)/\mathrm{Td}(\mathfrak{g})$ *factors through an isomorphism*

$$\begin{array}{ccc} K([X/G]) & \xrightarrow{\mathrm{ch}\cdot\mathrm{Td}(TX)/\mathrm{Td}(\mathfrak{g})} & CH([X/G]) \\ & \searrow \quad \nearrow_{\cong} & \\ & \widehat{K}([X/G]) & \end{array}$$

Example 11.4. If $G = \mathbb{G}_m = \mathbb{C}^\times$, then (by Example 10.8) we have $R(G) = \mathbb{Z}[s, s^{-1}]$. The augmentation ideal is evidently generated by $s^a - s^b$, hence is the principal ideal $(s-1)$. So we see that the augmentation ideal cuts out the identity in Spec $R(G)$.

Let $X = \mathrm{Spec}\, k$, so $[X/G] = BG$. Then (by Example 10.11) $G([X/G]) \cong \mathbb{Z}[s, s^{-1}]$ and (by Example 9.4) $CH^*([X/G]) \cong \mathbb{Z}[[t]]$. Tracing through these identifications, the Chern root of the representation corresponding to s is t. Therefore, we can identify the map appearing in the equivariant Riemann-Roch theorem (which requires tensoring with $\mathbb{Q}$) with

$$\mathbb{Q}[s, s^{-1}] \to \mathbb{Q}[[t]]$$
$$s \mapsto e^t$$

This is not an isomorphism, as expected. However, the equivariant Riemann-Roch theorem predicts that it factors through the completion of $G(BG)$ at the augmentation ideal.

$$\begin{array}{ccc} \mathbb{Q}[s, s^{-1}] & \longrightarrow & \mathbb{Q}[[t]] \\ & \searrow \quad \nearrow & \\ & \widehat{\mathbb{Q}[s, s^{-1}]}_{(s-1)} & \end{array}$$

What is the completion $\widehat{\mathbb{Q}[s, s^{-1}]}_{(s-1)}$? If we set $u = s-1$, then we are taking the completion of $\mathbb{Q}[u, (1+u)^{-1}]$ at (u). But of course $(1+u)$ is already invertible in $\mathbb{Q}[[u]]$, so we simply obtain $\mathbb{Q}[[u]]$. Thus we indeed witness the isomorphism

$$\widehat{G}([X/G]) \cong \mathbb{Q}[[u]] \xrightarrow{\sim} \mathbb{Q}[[t]] \cong CH^*([X/G]).$$

However, this isomorphism sends $u \mapsto e^t - 1$, which is more "complicated" than the naïve map $u \mapsto t$.

Example 11.5. Let $X = \mathrm{Spec}\, k$ and $G = \mu_2$. Then we have

$$K(G, \mathrm{pt})_{\mathbb{Q}} = R(G)_{\mathbb{Q}} = \mathbb{Q}[x]/(x^2 - 1).$$

Note that this agrees with the coordinate ring, as expected from Example 10.7.

On the other side, the Chow ring of $[X/G]$ is $\mathbb{Q}[t]/2t \cong \mathbb{Q}$, by Example 9.7. This is obviously not isomorphic to the K-theory, but the equivariant Riemann-Roch theorem predicts an isomorphism after completing at the augmentation ideal.

Indeed, the augmentation ideal is $I = \langle \chi - 1 \rangle$, and completing $\mathbb{Q}[x]/(x^2-1)$ with respect to I gives$(\mathbb{Q}[x]/(x^2-1))_{(x-1)} \cong \mathbb{Q}[x]/(x-1) \cong \mathbb{Q}$.

Exercise 11.6. Check that the map $\tau = \mathrm{ch} \cdot \mathrm{Td}(TX/\mathfrak{g})$ sends $x \mapsto e^t = 1 \in K([X/G])$.

11.3. **Examples.**

Example 11.7. (Weighted projective line) In the introduction, we saw hints of a Riemann-Roch theorem for the weighted projective line $\mathbb{P}^1(a,b)$. Over $\mathbb{C}$, we have an action $\mathbb{C}^*$ on $\mathbb{A}^2$ sending $x \mapsto \lambda^{-a} x$ and $y \mapsto \lambda^{-b} y$. A line bundle on $\mathbb{P}(a,b)$ is a line bundle on $\mathbb{A}^2$ with a compatible action of $\mathbb{C}^*$.

Exercise 11.8. Show that the coarse moduli space of $\mathbb{P}^1(a,b)$ is the usual $\mathbb{P}^1$.

Any line bundle on $\mathbb{A}^2$ is trivial, so if we denote a generator by T then the action of $\mathbb{C}^*$ is necessarily of the form $T \mapsto \lambda^\ell T$. We denoted by ξ^ℓ the corresponding line bundle on $\mathbb{P}(a,b)$. Then we counted $\Gamma(\mathbb{P}(a,b)\xi^\ell)$. We saw a linear plus oscillatory behavior, and we wondered if there could be a Riemann-Roch formula underlying the results. The conjectured formula was

$$h^0(\mathbb{P}(a,b),\xi^\ell) = \frac{\ell}{ab} + \frac{1/a+1/b}{2} + (\text{oscillation term}). \tag{11.1}$$

Now let's analyze this using the equivariant Riemann-Roch theorem.

Computation of Chow ring. Let's warm up with the simplest case $a = 1, b = 1$. Then $\mathbb{P}^1(1,1) = (\mathbb{A}^2 \setminus 0)/\mathbb{G}_m$. We have an excision exact sequence for Chow rings

$$CH_*(\mathrm{pt}/\mathbb{G}_m) \to CH_*(\mathbb{A}^2/\mathbb{G}_m) \to CH_*(\mathbb{A}^2 - 0/\mathbb{G}_m) \to 0.$$

Now $CH^*(\mathrm{pt}/\mathbb{G}_m) = CH^*(B\mathbb{G}_m) = \mathbb{Z}[u]$. As $\mathbb{A}^2/\mathbb{G}_m$ is an $\mathbb{A}^2$-vector bundle, we also have $CH^*([\mathbb{A}^2/\mathbb{G}_m]) \cong \mathbb{Z}[t]$. What is the map $CH_*(\mathrm{pt}/\mathbb{G}_m) \to CH_*(\mathbb{A}^2/\mathbb{G}_m)$? The element $1 \in CH^*(\mathrm{pt}/\mathbb{G}_m)$ is always represented by the fundamental class, which is modelled at the finite approximations to the Chow ring by the fundamental class of $\mathbb{P}^n$. On the other hand, the finite approximations to the Chow ringof $[\mathbb{A}^2/\mathbb{G}_m]$ are $[\mathbb{A}^2 \times (\mathbb{A}^{n+1}-0)/\mathbb{G}_m] \cong \mathbb{P}^{n+2} - \mathbb{P}^1 \subset \mathbb{P}^{n+2}$, with t representing the hyperplane class. Since the intersection of two hyperplane classes in $\mathbb{P}^{n+2}$ is the class of $\mathbb{P}^n$, we may conclude that this map is sends $1 \mapsto t^2$. By the excision sequence, we deduce that $CH^*(\mathbb{A}^2 - 0/\mathbb{G}_m) \cong \mathbb{Z}[t]/t^2$.

This was the case $(a,b) = (1,1)$. In general, we see that in the map $CH_*(\mathrm{pt}/\mathbb{G}_m) \to CH_*(\mathbb{A}^2/\mathbb{G}_m)$ the fundamental class 1 is sent to the class of the "intersection of the x and y axes in $\mathbb{A}^2$." Looking in the approximations for $[\mathbb{A}^2 \times (\mathbb{A}^{n+1}-0)/\mathbb{G}_m]$, we see that these axes represent at and bt, since they are cut out by elements of degree a and b. The conclusionis that $1 \mapsto (at)\cdot(bt)$, hence

$$\boxed{CH^*(\mathbb{P}^1(a,b)) \cong \mathbb{Z}[t]/abt^2}.$$

Now the degree map on K-theory induces a map $\mathbb{Q}[t]/abt^2 \mapsto \mathbb{Q}$. We claim that under this map, $t \mapsto \frac{1}{ab}$. It has stuff going on at the $1/a$ and $1/b$ weighted points, which are the images of the line at and bt whose intersection cut out the pushforward of the class of the point.

Computation of K-theory. We also have an excision exact sequence

$$K(\mathrm{pt}/\mathbb{G}_m) \to K(\mathbb{A}^2/\mathbb{G}_m) \to K(\mathbb{A}^2 - 0/\mathbb{G}_m) \to 0.$$

We know that $K(\mathrm{pt}/\mathbb{G}_m)$ is the representation ring of $\mathbb{G}_m$, which is $\mathbb{Z}[s,s^{-1}]$, so similarly $K_0(\mathbb{A}^2/\mathbb{G}_m) \cong \mathbb{Z}[t,t^{-1}]$. What is the map $K_0(\mathrm{pt}/\mathbb{G}_m) \to K_0(\mathbb{A}^2/\mathbb{G}_m)$? It suffices to compute the image of 1, which is the class of $\mathscr{O}_0 \in G(\mathbb{A}^2/\mathbb{G}_m)$. To find this, we should resolve by vector bundles. We can use the Koszul complex:

$$0 \to \mathscr{O}_{\mathbb{A}^2} \xrightarrow{(x,y)^T} \mathscr{O}_{\mathbb{A}^2}^{\oplus 2} \xrightarrow{(-y,x)} \mathscr{O}_{\mathbb{A}^2} \to \mathscr{O}_{\mathrm{pt}} \to 0.$$

Now comes an important point. Here x and y represent *functions* on $\mathbb{A}^2$ rather than coordinates, so the action of $\mathbb{G}_m$ has weight *negative* a on x and *negative* b on y. Therefore, as *equivariant* bundles $\mathscr{O}_{\mathbb{A}^2}^{\oplus 2}$ has weight (a,b) (in order to make the map equivariant), etc. By passing to the alternating sum, we conclude that

$$[\mathscr{O}_{\mathrm{pt}}] = 1 - t^a - t^b + t^{a+b}.$$

Here's another way of going about this computation. We can first compute $[\mathscr{O}_{x-\mathrm{axis}}]$ by the resolution

$$0 \to \mathscr{O}_{\mathbb{A}^2} \xrightarrow{x} \mathscr{O}_{\mathbb{A}^2} \to \mathscr{O}_{x-\mathrm{axis}} \to 0.$$

This shows that $[\mathscr{O}_{x-\mathrm{axis}}] = (1-t^a)$. Then applying a similar argument on $\mathscr{O}_{x-\mathrm{axis}} \cong \mathbb{A}^1$, which now has an action of weight b, we obtain again $[\mathscr{O}_{\mathrm{pt}}] = (1-t^a)(1-t^b)$. So the conclusion is that

$$K(\mathbb{P}^1(a,b)) \cong \mathbb{Q}[t]/(1-t^a)(1-t^b).$$

The Todd class. We want to compute $T\mathbb{P}(a,b)$ because we eventually want to take its Todd class. For the normal $\mathbb{P}(1,1)$, you have the usual Euler exact sequence

$$0 \to \mathscr{O} \to \mathscr{O}(1) \oplus \mathscr{O}(1) \to T \to 0.$$

The same proof for $\mathbb{P}(a,b)$ gives a "weighted Euler exact sequence"

$$0 \to \mathscr{O} \to \xi^a \oplus \xi^b \to T \to 0.$$

By the multiplicativity of Todd class, we get

$$\begin{aligned} \mathrm{Td}(T\mathbb{P}(a,b)) &= \left(1 + \frac{1}{2}at + \dots\right)\left(1 + \frac{1}{2}bt + \dots\right) \\ &= 1 + \frac{1}{2}(a+b)t. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{ch}(\xi^\ell) \cdot \mathrm{Td}(T\mathbb{P}(a,b)) &= e^{\ell t}\left(1 + \frac{a+b}{2}t\right) \\ &= (1+\ell t)(1 + \frac{a+b}{2}t) \\ &= 1 + (\ell + \frac{a+b}{2})t \end{aligned}$$

which has degree $\frac{1}{ab}(\ell + \frac{a+b}{2})$. This agrees with what we guessed in (11.1), except for the error term. The error term is coming from the pieces supported away from $1 \in \operatorname{Spec} R(G)$, as we discussed before.

For simplicity, let's consider the case $a = 1, b = 2$. Then

$$K(B\mathbb{G}_m)_{\mathbb{Q}} \cong \mathbb{Q}[t]/(1-t^2)(1-t) \cong \mathbb{Q}[t]/(1-t)^2(1+t)$$

and what we computed above was the contribution from the $\mathbb{Q}[t]/(t-1)^2$ piece. The class of ξ^ℓ in K-theory is t^ℓ, which modulo $(t-1)^2$ is $\ell t + 1$, but modulo $t+1$ is $(-1)^\ell$. That is why we also get a contribution involving $(-1)^\ell$ to $h^0(\xi^\ell)$.

What about higher rank vector bundles? For a direct sum of line bundles, the naïve expected answer is

$$h^0(V) = \deg V + \operatorname{rank} V(\frac{1}{2}\chi_{\text{top}}).$$

Even though this is wrong wrong, we'll see that it is sometimes right for vector bundles. For instance, take $V = 1 + \xi$. Then we predict that

$$h^0(V) = \frac{1}{2} + 2(\frac{1}{2} \times \frac{3}{2}) = 2.$$

But $h^0(\mathcal{O}) = 1$ and $h^0(\xi) = 1$, so this is actually correct! You can check that our prediction for $\xi^k + \xi^{k+1}$ will be always be right. The reason comes from the fact that bundles of this form vanish at the ideal $(1+t)$, so there is no extra contribution.

More general weighted projective spaces. It is easy to see how this discussion generalizes. Given integers $a_0, \ldots, a_n$, we can consider

$$[\mathbb{A}^{n+1} - 0/\mathbb{C}^\times] =: \mathbb{P}^n(a_0, \ldots, a_n)$$

with the action $\lambda(x_0, \ldots, x_n) = (\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n)$.

The same argument shows that

$$K(\mathbb{P}^n(a_0, \ldots, a_n)) \cong \mathbb{Z}[\chi, \chi^{-1}]/(\chi^{a_0} - 1) \ldots (\chi^{a_n} - 1)$$

and the Chow ring is

$$CH^*(\mathbb{P}^n(a_0, \ldots, a_n)) = \mathbb{Z}[t]/(a_0 \ldots a_n)t^{n+1}.$$

Tensoring with $\mathbb{Q}$, we see that

$$K(\mathbb{P}(a_0, \ldots, a_n))_{\mathbb{Q}} \cong \mathbb{Q}[\chi, \chi^{-1}]/(\chi - 1)^{n+1}(\chi^{a_0 - 1} + \ldots + 1)(\chi^{a_n - 1} + \ldots + 1)$$

which can be decomposed as

$$\mathbb{Q}[\chi, \chi^{-1}]/(\chi - 1)^{n+1} \oplus \text{ stuff not supported at 1.}$$

The Riemann-Roch theorem identifies the first summand with $CH^*(\mathbb{P}^n(a_0, \ldots, a_n))_{\mathbb{Q}}$.

Now, the other summands depend very much on the specific choice of a_i.

12. The geometry of quotient stacks

We will work towards a proof of the Equivariant Riemann-Roch Theorem and other facts that we have mentioned, but not proved. However, we will do so in a more leisurely fashion, exploring as we go.

12.1. Four points on $\mathbb{P}^1$.

We have seen that basic questions about the simple example $\mathbb{P}^1$ has led to very deep and fascinating mathematics, and we return to it again. The goal is to explore how Deligne-Mumford (DM) stacks come up "in nature."

Consider the space "$\mathbb{P}^1$ with four unordered points" up to automorphisms of $\mathbb{P}^1$. This is a little vague so far, so let's try to digest what it means.

Given four *ordered* points on $\mathbb{P}^1$, there exists a unique automorphism sending the first three to $0,1,\infty$, and the fourth to λ. Therefore, "four ordered points" on $\mathbb{P}^1$ are parametrized by $\mathbb{P}^1$, via the *cross-ratio* λ (there are some automorphism issues at $\lambda = 0,1,\infty$ which we'll ignore).

We might say this in a slightly different way, that four unordered points are parametrized by "$\mathbb{P}^1/S_4$." This means that there is an S_4-action on "space of four ordered points" and we should quotient out by this group. But actually the Klein four subgroup $V_4 \subset S_4$ acts trivially, and we have an exact sequence

$$1 \to V_4 \to S_4 \to S_3 \to 1$$

so the space is really "$\mathbb{P}^1/S_3$." This is a projective curve admitting a degree 6 cover by $\mathbb{P}^1$, so it must itself be $\mathbb{P}^1$.

If we want to study "line bundles on the quotient," then we need to understand the group action on line bundles. For instance, we have the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$.

Problem 12.1. *When, and in how many ways, can one extend the S_4 action from $\mathbb{P}^1$ to $\mathcal{O}(n)$?*

In particular, such an extension would make $\Gamma(\mathcal{O}(n))$ into an S_4-representation, and then the graded ring $\bigoplus_{d\geq 0}\Gamma(\mathcal{O}(n))$ would be a representation of S_4.

Problem 12.2. *What does $\Gamma(\mathcal{O}(n))$ "look like" as n grows?*

An algebraic digression. Let's abstract this question a bit. Let G be a finite group (non-abelian) and let V be a finite-dimensional representation over $\mathbb{C}$. What representations occur in $\mathrm{Sym}^n V$? A related question: what representations occur in $V^{\otimes n}$?

Exercise 12.3. Find estimates on "how often" a given irreducible representation of G occurs in $\mathrm{Sym}^n V$ and $V^{\otimes n}$.

Example 12.4 (Finite cyclic groups). The representation ring of $G = \mathbb{Z}/n$ is $A = \mathbb{C}[t]/(t^n - 1)$. So we have "recovered" the group in this funny way from its representation ring, and the eigenspace decomposition corresponds to a decomposition as representations.

Example 12.5 (The multiplicative group). A finite-dimensional $\mathbb{G}_m$-representation is the same as a finite-dimensional module over $\mathbb{C}[t,t^{-1}]$. But $\mathrm{Spec}\,\mathbb{C}[t,t^{-1}] = \mathbb{G}_m$, so we again "recover" the group. However, things get subtler over a non-abelian group.

Now let's go back to our original example. We have an S_4 action on $\mathbb{P}^1$, which in stacky terms means that we have a diagram

$$\begin{array}{ccccc} \mathbb{P}^1 & \longrightarrow & [\mathbb{P}^1/S_4] & \longrightarrow & \mathbb{P}^1 \\ \downarrow & & \downarrow & & \\ \mathrm{pt} & \longrightarrow & BS_4 & & \end{array}$$

A line bundle on BS_4 is just a k-module with S_4 action, i.e. a representation of S_4, so the earlier question Problem 12.2 can be thought of as asking how to "push forward" a line bundle on $\mathbb{P}^1$ to a line bundle on BS_4.

Back to four points on $\mathbb{P}^1$. We saw that the action of S_4 on $\mathbb{P}^1$ factors through S_3, so let's study $[\mathbb{P}^1/S_3]$.

First of all, what is the action of S_3 on $\mathbb{P}^1$? Explicitly, it can be described as the effect on the cross-ratio induced by permuting $0, 1, \infty$. Let's calculate the effect in coordinates. Let x, y be coordinates on $\mathbb{P}^1$ and $u = x/y$. Then the elements of S_3 are

$$\begin{aligned} &u \mapsto u \\ &u \mapsto 1-u \\ &u \mapsto 1/u \\ &u \mapsto u/(u-1) \\ &u \mapsto 1/(u-1) \\ &u \mapsto (u-1)/u \end{aligned}$$

What are the points with non-trivial stabilizers? $\{0, 1, \infty\}$ and $\{1/2, 2, -1\}$ have stabilizer of size 2, and the roots of $u^2 - u + 1 = 0$, which are $\{\frac{1\pm\sqrt{3}}{2}\}$, form an orbit of size 2. We claim that this is everything. We could easily check this directly, but let's see it in a different way.

Intuitively, $\mathbb{P}^1$ should be a "covering space" of $[\mathbb{P}^1/S_3]$ (with deck transformations S_3). Since $\mathbb{P}^1$ has Euler characteristic 2, the space $[\mathbb{P}^1/S_3]$ should have Euler characteristic $2/6 = 1/3$. On the other hand, we are "removing" each point x that has a non-trivial stabilizer and pasting back in a point with "weight" $\frac{1}{\mathrm{Stab}(x)}$, for a total of

$$2 - 1 - 1 - 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = 1/3.$$

The reasoning here is that a "stacky point" with stabilizer of order 2 "is" $B(\mathbb{Z}/2)$, since it admits a double cover by a point.

Let's try to rigorously justify the claim that the inclusion of a point with $\mathbb{Z}/2$ stabilizer corresponds to a closed embedding $B(\mathbb{Z}/2) \hookrightarrow [\mathbb{P}^1/S_3]$. First of all, what *is* the map $B(\mathbb{Z}/2) \to [\mathbb{P}^1/S_3]$? We have defined this notion in terms of T-points.

A map from a test scheme T to $B(\mathbb{Z}/2)$ is a diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mathbb{Z}/2-\text{equiv.}} & \text{pt} \\ \downarrow{\scriptstyle \mathbb{Z}/2} & & \downarrow \\ T & \longrightarrow & B(\mathbb{Z}/2) \end{array}$$

From this we want to produce, in a functorial way, a diagram

$$\begin{array}{ccc} W & \xrightarrow{S_3\text{-equiv.}} & \mathbb{P}^1 \\ \downarrow{\scriptstyle S_3} & & \downarrow \\ T & \longrightarrow & [\mathbb{P}^1/S_3] \end{array}$$

which specifies a map from T to $[\mathbb{P}^1/S_3]$. To do so, we can extend the structure group on Y from $\mathbb{Z}/2$ to S_3 to produce a principal S_3-bundle $W \to T$. (Recall from Exercise 7.23 that given an inclusion $H \hookrightarrow G$, we have a map $BH \to BG$.)

Next, we need to define a map from W to $\mathbb{P}^1$. We begin by mapping Y to $\mathbb{P}^1$ by collapsing it to a point in $\mathbb{P}^1$ with stabilizer $\mathbb{Z}/2$. This is $\mathbb{Z}/2$-equivariant by definition. We extend the map to $W \to \mathbb{P}^1$ by sending $\sigma \cdot x \mapsto \sigma(x)$ for $\sigma \in S_3$. If this is well-defined, then it will be S_3-equivariant by contruction. But the only ambiguity comes from the $\mathbb{Z}/2$-action on Y, and the orbit of the point in $\mathbb{P}^1$ which is the image of Y is $\mathbb{Z}/2$-invariant.

The fact that $B(\mathbb{Z}/2) \hookrightarrow [\mathbb{P}^1/S_3]$ is a closed embedding follows from $Y \hookrightarrow W$ being a closed embedding.

Computation on $[\mathbb{P}^1/S_3]$**.** In order to get a computational handle on the stack $[\mathbb{P}^1/S_3]$, it is useful to produce an étale cover in terms of affines. If we choose an S_3-invariant open affine subset$U \subset \mathbb{P}^1$, then we obtain a diagram

$$\begin{array}{ccc} U & \overset{\text{open}}{\hookrightarrow} & \mathbb{P}^1 \\ {\scriptstyle S_3}\downarrow & & \downarrow{\scriptstyle S_3} \\ V & \overset{\text{open}}{\hookrightarrow} & [\mathbb{P}^1/S_3] \end{array}$$

We can take U to be $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which is Spec $\mathbb{C}[u]_{u(u-1)}$. A natural thing to do is to form $\text{Spec}(\mathbb{C}[u]_{u(u-1)})^{S_3}$, which will end up giving the *coarse* moduli space for V.

Exercise 12.6. Since we threw out one orbit, the coarse space should be $\mathbb{A}^1$. Show that $(\mathbb{C}[u]_{u(u-1)})^{S_3} \cong \mathbb{C}[t]$. What is t?

We have a cover Spec $\mathbb{C}[u]_{u(u-1)} \to [\text{Spec}\, \mathbb{C}[u]_{u(u-1)}/S_3]$. The latter has two "stacky" points, and the map should be an isomorphism away from the stacky points. How might one prove this? If you find the corresponding values of t, and poke them out, then you should find $\mathbb{A}^1 - \{\text{two points}\}$ with an honest S_3-cover.

Using the explicit data of this étale cover, we can push and pull sheaves to heart's content, and do all calculations on the honest scheme $\mathbb{P}^1$. For instance, what's the degree of a point?

$$\begin{array}{ccc} \mathbb{P}^1 & & \\ \big\downarrow{\scriptstyle \deg 6} & & \\ [\mathbb{P}^1/S_3] & \longrightarrow & \operatorname{Spec} k \end{array}$$

Let d be the degree of the point in $[\mathbb{P}^1/S_3]$ corresponding to the orbit $\{0,1,\infty\}$ consisting of 3 reduced points, which has deg 3. Then the degree of this point must be $\frac{3}{6} = 1/2$.

12.2. **Euler characteristics for stacks.** In the discussion of the previous sections, we exploited the idea of an "Euler characteristic" for stacks. Let's explore this. For a complex manifold, the *compactly supported Euler characteristic* is

$$\chi_e(X) = \sum_i (-1)^i h^i_c(X)$$

where $h^i_c(X) = \dim H^i_c(X)$ is the dimension of the compactly supported cohomology.

This is additive in nice situations, e.g. if $X = \underbrace{Y}_{\text{open}} \cup \underbrace{Z}_{\text{closed}}$ is a decomposition into open and closed subsets, then we have

$$\chi_e(X) = \chi_e(Y) + \chi_e(Z).$$

It is easy to check that under a finite étale map $f : X \to Y$ of degree d, we have

$$\chi_e(X) = d\chi_e(Y).$$

In order to extend this notion to stacks, we simply apply this formula to some étale cover. Of course, one has to check that it is well-defined (i.e. independent of cover). To do this, one takes the fibered product of any two étale covers by schemes (using that the diagonal is representable for Deligne-Mumford stacks):

$$\begin{array}{ccc} X \times_{\mathscr{X}} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ X' & \longrightarrow & \mathscr{X} \end{array}$$

In general one may not have a finite étale cover of $\mathscr{X}$, but it has some étale cover. If $\mathscr{X}$ is quasicompact, then one can choose an étale cover comprised of finitely many finite étale morphisms $\{U_i \to \mathscr{X}\}$, and one can stratify $\mathscr{X}$ by the number of pre-images, apply our previous definition for finite étale covers, and then add them up.

This "argument sketch" raises even more questions. What does it even mean to stratify a stack? We say that $\mathscr{X} = \mathscr{Y} \coprod \mathscr{Z}$ if any "point" $\operatorname{Spec} k \to \mathscr{X}$ factors through $\mathscr{Y}$ or $\mathscr{Z}$. We've basically just sketched an argument for the following result.

Proposition 12.7. *Suppose $\mathscr{X}$ is a finite type Deligne-Mumford stack with atlas $U \xrightarrow{\pi} \mathscr{X}$. Then there exists a finite collection of locally closed substacks $Z_1, \ldots, Z_n$ that are equidimensional, such that $\mathscr{X} = Z_1 \coprod \ldots \coprod Z_n$, and such that $\pi^{-1}(Z_i) \to Z_i$ is finite étale.*

This is a quasi-proposition in the sense that I am not sure that the proof is written down. The proof should just be straightforward from the following fact about schemes: given a finite type scheme X and a surjective étale map $U \to X$, the result is true, with the stratification being determined by fiber degrees.

The upshot is that the proposition allows us to define an Euler characteristic for finite-type DM stacks. .

Remark 12.8. It is a serious theorem that every Deligne-Mumford stack (finite type over a field k) has a *finite* cover by a scheme. This is non-trivial (the definition of an atlas doesn't involve properness) - see for instance [EHKV01].

This is a handry crutch to use. For instance, what does it mean for a morphism of stacks to be proper? We've defined this for *representable* morphisms, but there are settings of interest where we don't have this, e.g. $\overline{\mathcal{M}_g} \to \mathrm{pt}$. For testing properness, you can instead use a proper cover by a scheme.

We now restrict our attentiont to DM stacks, so we can talk about proper morphisms.

If $\mathcal{X} \to \mathcal{Y}$ is a proper morphism of (DM) stacks, then there is a pushforward map on Chow groups

$$\pi_* \colon CH_* \mathcal{X} \to CH_* \mathcal{Y}.$$

Here (unlike in the case of schemes), we crucially need rational coefficients. In particular, if $\mathcal{X}$ is proper over k then we get a "degree" map

$$CH_0(\mathcal{X}) \xrightarrow{\pi_*} CH_0(\mathrm{pt})$$

which is traditionally denoted by $\pi_* \beta = \int \beta$. The goal of an "equivariant Riemann-Roch Theorem" is to describe a "formula" for the Euler characteristic for DM stacks, generalizing Grothendieck-Riemann-Roch for schemes.

13. Computing Euler characteristics

13.1. **Definition of Euler characteristic.** Suppose G acts on X (tamely), and $[X/G]$ is a proper algebraic stack. Then the associated coarse moduli space $\mathscr{X} = [X/G]$ is a proper algebraic space. We can define an Euler characteristic

$$\chi\colon K_G(\mathscr{X}) \xrightarrow{\sim} K([X/G]) \to K(\mathrm{pt}) = \mathbb{Z}$$

by

$$\chi(\mathscr{F}) = \sum_i (-1)^i h^i(X, \mathscr{F})^G.$$

Here $h^i(X,\mathscr{F})^G$ denotes the dimension of the invariant subspace of cohomology, $\dim H^i(X,\mathscr{F})^G$.

Is this well-defined? One concern is that the cohomology groups may not be finite-dimensional if X is not proper. In general X need not be proper, but as long as $[X/G]$ is proper the vector spaces $H^i(X,\mathscr{F})^G$ will be finite-dimensional, because they agree with $H^i([X/G],\pi_*\mathscr{F}) \cong H^i(M, p_*\pi_*\mathscr{F})$ where $p\colon [X/G] \to M$ is a coarse moduli space for $[X/G]$ (Theorem 8.28).

13.2. **Locality of Equivariant Riemann-Roch.** We now attempt to give a general explanation for what's going on. Let $\mathscr{X} = [X/G]$ be a quotient stack which is smooth, proper, and Deligne-Mumford. Let $\mathscr{E}$ be a vector bundle on $\mathscr{X}$. We would like to be able to compute the holomorphic Euler characteristic. The first guess might be to write down the same expression as appears in Hirzebruch-Riemann-Roch:

$$\chi(\mathscr{X},\mathscr{E}) = \int_{\mathscr{X}} \mathrm{ch}(\mathscr{E})\mathrm{Td}(\mathscr{X}).$$

(Here the integral is shorthand for taking the pushforward to a point and then the degree.) However, this is not quite right, and here is where we see the main difference between shemes and stacks.

Let $\mathscr{E}_1 \in K(\mathscr{X}) = K_G(X)$ be the class constructed as follows. The space $K_G(X)$ is a module over the representation ring $R(G)$. The augmentation ideal corresponds to an element $1 \in \operatorname{Spec} R(G)$, and $\mathscr{E}_1$ is the component of $\mathscr{E}$ supported at 1.

Theorem 13.1. *We have*

$$\chi(\mathscr{X},\mathscr{E}_1) = \int_{\mathscr{X}} \mathrm{ch}(\mathscr{E})\mathrm{Td}(\mathscr{X}) \tag{13.1}$$

.

Definition 13.2. Some terminology on group actions:

- We say that the group action is *proper* if the action map

$$G \times X \xrightarrow{\pi} X \times X$$

  is a proper map. This is equivalent to the quotient stack $[X/G]$ being separated.
- We say G acts with *finite stabilizer* if π is finite. If G is affine, which is the case in all of our situations, then this is equivalent to π being proper.
- We say G acts with *finite stabilizers* if π is quasi-finite.
- We say that G acts *freely* if π is a closed embedding.

Remark 13.3. This admittedly terrible terminology is, unfortunately, well established in the literature.

Proof of Theorem 13.1. We now work towards the proof of Theorem 13.1, beginning with some general observation.

Lemma 13.4. *If $f\colon X' \to X$ is a finite map, then the pushforward map $f_*\colon CH_*(X') \to CH(X)$ is surjective.*

Proof. If $\deg f = d$, then $f^* f_*$ is multiplication by d. □

Next we use the fact that every DM stack $\mathscr{X}$ has a *finite* atlas. In our case $\mathscr{X} = [X/G]$, this means that there exists a finite G-equivariant map $X' \to X$ of schemes such that $\mathscr{X}' := [X'/G]$ is representable, i.e a scheme.

By the Lemma, $CH_*(X') \twoheadrightarrow CH_*(X)$, hence also

$$CH_*(G,X') = CH(\mathscr{X}') \twoheadrightarrow CH(\mathscr{X}) = CH_*(G,X).$$

This is still a surjection after completing after tensoring with $\mathbb{Q}$. Then by the Equivariant Riemann-Roch theorem, the corresponding map on the K-theory side is an isomorphism:

$$K(\mathscr{X}')_{\mathfrak{m}_1} \twoheadrightarrow K(\mathscr{X})_{\mathfrak{m}_1}.$$

But since $\mathscr{X}'$ is a scheme, its K-theory is supported at 1, i.e. $K(\mathscr{X}')_{\mathfrak{m}_1} = K(\mathscr{X}')$.

Now, to prove the theorem we may take $\alpha \in K(\mathscr{X})_{\mathfrak{m}_1}$ and show that

$$\int_{\mathscr{X}} \mathrm{ch}(\alpha)\mathrm{Td}(\mathscr{X}) = \chi(\alpha).$$

Write $\alpha = f_*\alpha'$ for some $\alpha' \in K(\mathscr{X}')$. Since f is a finite map, $\chi(\mathscr{X},\alpha) = \chi(\mathscr{X}',\alpha')$. We wish to apply the Hirzebruch-Riemann-Roch Theorem to $\mathscr{X}'$ with resect to α', although we meet the technical complication (which we shall ignore) that in general $\mathscr{X}'$ will be singular, so we need a generalization to singular schemes. In any case, that tells us that

$$\int_{\mathscr{X}} \chi(\alpha')\mathrm{Td}(\mathscr{X}') = \chi(\mathscr{X}',\alpha') = \chi(\mathscr{X},\alpha).$$

To finish off, we want to see that

$$\int_{\mathscr{X}} \chi(\alpha')\mathrm{Td}(\mathscr{X}') = \int_{\mathscr{X}} \chi(\alpha)\mathrm{Td}(\mathscr{X})$$

but this is precisely the statement that the equivariant Riemann-Roch map is covariant for proper morphisms. □

So the integral *only recovers the Euler characteristic over the identity component.* Why should we have expected this to be true?

(1) The Chern character can only possibly “see” the component at 1. Since the right hand side of (13.1) only knows $\mathscr{E}$ through the component at 1, we can’t expect to “see” more on the left hand side.

(2) There are various equivariant cohomology theories, and the naïve equivariant cohomology theory automatically involves a localization at 1. Namely, the naïve construction ("Borel-Moore") for equivariant cohomology of X under an action of G is

$$H^*_{BM,G}(X) := H^*(X \times EG/G).$$

You can define this for any kind of cohomology theory, and indeed this is how we defined the equivariant Chow groups. We claim that whenever we do this kind of construction, there is implicitly a localization at 1 involved. Since the Chern character maps to the Chow ring, which *is* constructed in this naïve way, it must also localize at 1.

Theorem 13.5 (Atiyah-Segal completion theorem)**.** *The map*

$$K_G(\mathrm{pt}) \to K(BG)$$

is an isomorphism after completing at the augmentation ideal.

Remark 13.6. The left hand side is $\mathrm{Rep}(G)$. The right hand side is the limit of K-theory in EG/G, i.e. the inverse limit of K-theory of $K((V_i - 0)/G)$ of a sequence of increasing, faithful G-representations

$$V_i \hookrightarrow V_{i+1} \hookrightarrow \dots$$

Example 13.7. For $G = \mathbb{G}_m$, we have $K_{\mathbb{G}_m}(\mathrm{pt}) \cong \mathrm{Rep}(\mathbb{G}_m) \cong \mathbb{Z}[t, t^{-1}]$. The natural action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} - 0$ is free, so $B\mathbb{G}_m = \lim_n \mathbb{P}^n$ and

$$K(B\mathbb{G}_m) = \varprojlim K(\mathbb{P}^n).$$

Now, $K(\mathbb{P}^n)$ is generated by line bundles, which are all of the form $\mathscr{O}(n)$. However, there are some relations. We claim that $K(\mathbb{P}^n)$ is generated by $1, t^{\pm 1}, t^{\pm 2}, \dots$. More precisely, there is an exact sequence

$$0 \to \mathscr{O} \to \mathscr{O}(1)^{n+1} \to \mathscr{O}(2)^{\binom{n+1}{2}} \to \dots \to \mathscr{O}(n+1) \to 0.$$

This is essentially the Koszul complex: the maps $\mathscr{O} \to \mathscr{O}(1)^{n+1}$ are multiplication by x_j, etc. This relation shows that

$$K(\mathbb{P}^n) \cong \mathbb{Z}[t]/(t-1)^{n+1}$$

so the inverse limit is the completion of $\mathbb{Z}[t]$ at $t-1$. But that's the same as the completion of the representation ring at $t-1$.

Upshot: the naïve formula only computes the Euler characteristic of the "part supported at 1." We need to perform some trickery in order to compute the other contributions.

Theorem 13.8 (Edidin-Graham)**.** *If G acts on X with finite stabilizers, then $K([X/G])_{\mathbb{Q}}$ is supported at a finite number of maximal ideals of* Spec $R(G)_{\mathbb{Q}}$*. In addition, we have $CH^i_G(X) = 0$ for $i > \dim X - \dim G$.*

You should think of $R(G)$ as some sort of model for the group G. The K-theory $K([X/G])$ is like a sheaf on Spec $R(G)$, and we are saying that it is supported at finitely many points. Notice how this is compatible with what we saw in Examples 11.4 and 11.5. So we have

$$K([X/G]) = \bigoplus_{\mathfrak{m} \in \operatorname{Spec} R(G)} K([X/G])_{\mathfrak{m}}$$

where we implicitly use rational coefficients here and henceforth.

Let $\mathfrak{m}_1$ be the augmentation ideal (corresponding to $1 \in G$). In these terms, the Equivariant Riemann-Roch theorem describes a diagram

$$\begin{array}{ccc} K([X/G])_{\mathbb{Q}} & \xrightarrow{\tau} & CH^*([X/G])_{\mathbb{Q}} \\ & \searrow \quad \nearrow_{\sim} & \\ & K([X/G])_{\mathfrak{m}_1} & \end{array}$$

To understand the other components of $G_0(G,X)$ we'll use the localization theorem in equivariant K-theory.

13.3. **Using the localization theorem.** We will focus our attention on the special case where $G = T$ is a *torus* over $\mathbb{C}$. We know (Theorem 13.8) that

$$K([G/X]) = \bigoplus_{h \in T} K([X/G])_{\mathfrak{m}_h}$$

Henceforth we will abbreviate $K(\cdot)_h := K(\cdot)_{\mathfrak{m}_h}$.

Remark 13.9. It turns out that h is *not* in the support of $K([G/X])$ if h acts without any fixed points. Therefore any h appearing in the support must fix a non-empty locus. Snce G is assumed to act with finite stabilizer, this implies that h must have finite order.

The equivariant Riemann-Roch theorem describes a map

$$K([X/G])_1 \xrightarrow{\tau_x} CH^*([X/G])_{\mathbb{Q}}.$$

Unfortunately, this map only gets at the part of the equivariant K-theory supported at $1 \in T$. We would like to "see" all of $K([X/G])$ in order to obtain a formula analogous to the classical Riemann-Roch Theorem for the Euler characteristic

Definition 13.10. If $\mathscr{V}$ is vector bundle on a scheme X, then we define

$$\lambda_{-1}(\mathscr{V}) = 1 - [\mathscr{V}^*] + [\bigwedge^2 \mathscr{V}^*] - \dots$$

A cute shorthand for this is $(1-t)^{\mathscr{V}^*}|_{t=1}$.

Theorem 13.11 (Localization Theorem)**.** *The inclusion $\iota_h \colon X^h \hookrightarrow X$ induces an isomorphism on (rational) K-theory:*

$$K([X/G])_h \cong K([X^h/G])_h.$$

Moreover, if $\lambda_{-1}(N_h) \in K(X^h)_h$ denotes the (equivariant) Euler class of N_h, then we have the explicit formula for $\alpha \in K([X/G])$:

$$\alpha = (\iota_h)_* \frac{\iota_h^* \alpha}{\lambda_{-1}(N_h^*)}.$$

This is helpful because it allows us to "twist" the information from the equivariant Riemann-Roch Theorem. Let us elaborate.

Recall from Example 10.8 that for $G = \mathbb{G}_m$, we have $R(G) \cong \mathbb{C}[G]$. It is easy to see that this will hold for tori as well. Now, for each $h \in G$ we have a "translation by h" map $G \xrightarrow{t_h} G$ sending $g \mapsto gh$, and this induces an automorphism of $\mathbb{C}[G] = R(G)_{\mathbb{C}} = K(BG)$ by $(t_h^\# f)(g) = f(hg)$. In particular, the map $t_h^\# \colon R(G)_{\mathbb{C}} \to R(G)_{\mathbb{C}}$ sends $\mathfrak{m}_h \mapsto \mathfrak{m}_1$.

If Y is a G-space, and $h \in G$ has finite order and acts trivially on Y (to see why this seemingly silly situation might arise, we will eventually apply this with $Y = X^h$), then $t_h^\#$ lifts to an automorphism of $K([Y/G])$ as follows. If $\mathscr{F}$ is a G-coherent sheaf on Y, then we have a decomposition

$$\mathscr{F} = \bigoplus_{\chi \in X^\bullet(\langle h \rangle)} \mathscr{F}(\chi)$$

(here $X^\bullet$ is the character group of the subgroup generated by h) and we define

$$\boxed{t_h^\#([\mathscr{F}]) := \bigoplus_{\chi} \chi(h) \mathscr{F}(\chi).}$$

Remark 13.12. You can think of this as being analogous to a (finite) Fourier transform.

The crucial point is the following.

Proposition 13.13. *For any $\mathscr{F} \in K([X/G])$, we have*

$$\chi([X/G], \mathscr{F}) = \chi([X/G], t_h(\mathscr{F})).$$

Proof. The Euler characteristic passes through the invariant subsheaf $\mathscr{F}^G$, which corresponds to the trivial character $\chi = 1$ in the decomposition

$$\mathscr{F} = \bigoplus_{\chi} \mathscr{F}(\chi).$$

Since t_h fixes the $\mathscr{F}(\chi)$, it doesn't affect the Euler characteristic. □

Now we can apply the Equivariant Riemann-Roch Theorem *after* twisting by h to detect the part supported at h:

$$\begin{array}{ccccc} K([X/G])_{\mathfrak{m}_h} & \xrightarrow{\sim} & K([X^h/G])_{\mathfrak{m}_h} & \xrightarrow{t_h^\#} & K([X^h/G])_{\mathfrak{m}_1} \\ & & & & \downarrow{\scriptstyle \sim\ \mathrm{ERR}} \\ & & & & CH^*([X^h/G])_{\mathbb{C}} \end{array}$$

Putting all of these together gives an isomorphism

$$K([X/G]) = \bigoplus_{h \in G} K([X/G])_{\mathfrak{m}_h} \xrightarrow{\sim} \bigoplus_{h \in G} CH^*([X^h/G]).$$

Using this we deduce:

Theorem 13.14. *Let $\mathscr{F}$ be an equivariant vector bundle on $\mathscr{X} = [X/G]$. Then we have*

$$\chi(\mathscr{X},\mathscr{F}) = \sum_{h\in G}\int_{[X^h/G]} \operatorname{ch}\left(t_h \frac{\iota_h^*\mathscr{F}}{\lambda_{-1}(N_h^*)}\right)\cdot \operatorname{Td}([X^h/G]).$$

13.4. **Localization in the Chow ring.** As somewhat of an aside, we describe a localization formula at the level of Chow rings (as opposed to K-theory). Let $\alpha \in CH_*^G(X)$ (recall that we are taking $\mathbb{Q}$-coefficients). Then we have an inclusion

$$\iota_G : X_G \hookrightarrow X.$$

Theorem 13.15. *We have*

$$\alpha = \iota_{G_*}\frac{\iota_G^*\alpha}{c_{top}(N_G)}.$$

As a sanity check, let's verify the degrees. Restricting to X_G *increases* the degree (codimension) by r, while dividing by the top Chern class *decreases* the codimension by r, and pushing forward doesn't change it.

Exercise 13.16. Use this to show that $\mathbb{P}^1$, $\deg(c_1(\mathscr{O}(1))) = 1$.

Example 13.17. Suppose X proper and smooth, and the G-action on X has finitely many fixed points. (For comfort, you can assume that we're over $\mathbb{C}$). Then we have

$$\chi(X) = \#X_G.$$

Why? The Euler characteristic can be expressed as

$$\chi = c_{top}(TX).$$

Taking $\alpha = c_{top}(TX)$ in the Localization Theorem, we have

$$\iota_G^*\alpha = c_{top}(\iota_G^* TX) = c_{top}(N_G).$$

Therefore, the Localization theorem equates α with the puhforward of $\#X_G$ copies of 1.

As a fun application, take $X = G(k,n)$ and let $\mathbb{G}_m^n$ scale every coordinate separately. Then the only fixed k-planes are the coordinates ones, of which there are $\binom{n}{k}$.

13.5. **Example.** Let's go back and (finally!) resolve our confusion about $\mathbb{P}(1,2)$. We previously computed that

$$\chi(\mathbb{P}(1,2),\xi_1^\ell) = \frac{\ell}{2} + \frac{3}{4} + (\text{error terrm}).$$

The error term is the contribution from $\chi(\xi_{-1})$. Using the localization theorem, this should be

$$\int_{X^{-1}/G]} \operatorname{ch}\left(t_{-1}\frac{\iota_{-1}^*\xi^\ell}{\lambda_{-1}(N_{-1}^*)}\right)\cdot \operatorname{Td}([X^{-1}/G]).$$

Let's analyze the ingredients in this formula.

The fixed locus. The locus X^{-1} corresponds to $x = 0$, which is the punctured y-axis. Therefore, $X^{-1}/G = B\mu_2$, which has rational Chow ring $\mathbb{Q}$. Therefore, the degree map simply picks out the constant term. However, we note that since $B\mu_2$ is "half a point" we actually have $\int_{B\mu_2} 1 = \frac{1}{2}$.

Computation of Todd class. What is $\mathrm{Td}(B\mu_2)$? By definitions, this is $\mathrm{Td}(\mathrm{pt})/\mathrm{Td}(\mathfrak{g})$, where $\mathfrak{g}$ is the adjoint representation, but these are both trivial! So we see that $\mathrm{Td}(B\mu_2) = 1$.

Computation of K-theory. Recall (Example 10.7) that $K(B\mu_2) \cong \mathbb{Z}[t]/(t^2-1)$, so we have

$$K(B\mu_2)_{\mathbb{Q}} \cong \mathbb{Q}[t]/(t-1)(t+1).$$

The class of $i_{-1}^*\xi^\ell$ in $K([X^{-1}/G])$ is t^ℓ. Since the y-axis is cut out by the equation $x = 0$ which has weight 1, the conormal bundle has weight 0, so the class of N_h^* is t^{-1}. Therefore,

$$t_{-1}\frac{\iota_{-1}^*\xi^\ell}{\lambda_{-1}(N_{-1}^*)} = \frac{(-1)^\ell t^\ell}{1+t^{-1}}.$$

Only the constant term will matter. Since t_{-1} moves the K-theory class to one supported at 1, it suffices to plug in $t = 1$ to obtain a K-class of $\frac{1}{2}$. Integrating (and using that $B\mu_2$ has "mass 1/2") gives an error term of $\frac{(-1)^\ell}{4}$.

Feeding everything into our formula, we arrive at

$$\boxed{\chi(\mathbb{P}(1,2),\xi^\ell) = \frac{\ell}{2} + \frac{3}{4} + \frac{(-1)^\ell}{4}.}$$

14. RIEMANN-ROCH AND INERTIAL STACKS

14.1. **Group actions and the inertia stack.** Let $G \to S$ be a group scheme. If $S = \operatorname{Spec} k$ and G/S is of finite typoe, then G is an algebraic group. We are interested in linear algebraic groups, i.e. closed subschemes of $\mathrm{GL}(n)/k$. These are all affine obviously, but it turns out conversely that all affine group schemes are linearizable.

We have a map $G \times X \to X \times X$ sending $(g,x) \mapsto (gx,x)$. We want to construct a space that keeps track of the objects of the quotient, and *also* the isotropy groups. So we form the fibered product

$$\begin{array}{ccc} I_G X & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ G \times X & \longrightarrow & X \times X \end{array}$$

Here $I_G X$ is the group scheme of stabilizers for G acting on X, which you can think of in terms of the functor of points as

$$I_G X(T) = \{(x,g) \in X(T) \times G(T) \mid g \cdot x = x\}.$$

This is a group scheme over X, whose fiber over a point $x \in X$ is the isotropy group G_x.

Now we define an action of G on $I_G X$, via $h \cdot (g,x) = (hgh^{-1}, hx)$. This is compatible for the G-action on the other three terms of the fibered square, and taking the quotient by G recovers the *inertial stack* (see Example 7.25) together with its natural group scheme structure over $[X/G]$;

$$\begin{array}{ccc} I[X/G] & = & [(I_G X)/G] \\ \downarrow & & \downarrow \\ [X/G] & & [X/G] \end{array}$$

Recall that $I[X/G]$ is the inertia stack $[X/G] \times_{[X/G]\times[X/G]} [X/G]$, which can be thought of as

$$I[X/G](T) = \{(x,\alpha) \mid x \in [X/G](T), \alpha \in \operatorname{Aut}(x)\}.$$

Note that if G acts with finite stabilizer (which we are always assuming to be the case) then $(x,\alpha) \in I[X/G](T)$ only if α has finite order (but this is not necessarily sufficient).

14.2. **Restatement in term of inertial stacks.** Recall that we used the Equivariant Riemann-Roch theorem plus a "twisting" automorphism t_h to obtain an isomoprphism.

$$K([X/G]) = \bigoplus_{h \in G} K([X/G])_{\mathfrak{m}_h} \xrightarrow{\sim} \bigoplus_{h \in G} CH^*([X^h/G]).$$

We claim that

$$\bigoplus_{h \in G} CH^*([X^h/G]) \cong CH^*_G(I_G X).$$

From the identification

$$[(I_G X)/G] \cong I[X/G]$$

it follows more or less by definition that

$$CH^*_G(I_G X) \cong CH^*(I[X/G]).$$

This suggests that perhaps the familiar version of Grothendieck-Riemann-Roch can be salvaged in a different way, via a map $K([X/G]) \to CH^*(I[X/G])$ rather than a map to $CH^*([X/G])$. (Note that in the case where $[X/G]$ is a scheme, which necessarily means that G acts freely on X, we obviously have $I[X/G] = [X/G]$.)

The key technical point is:

Proposition 14.1. *There is a finite G-equivariant decomposition*

$$I_G X = \bigsqcup_g S_g$$

where each $S_g = \{(x,g) \mid g \in \mathrm{Stab}_G(x)\} \subset IX$ is a connected component of $I_G X$.

This G-equivariant decomposition descends to a decomposition into connected components of IX, and the claim is that $CH^*([S_h/G])$ corresponds to the summand $CH^*([X^h/G])$. This is clear.

Again using the decomposition, we can define twisting automorphisms $t\colon K(I[X/G])_{\mathbb{C}} \cong K(I[X/G])_{\mathbb{C}}$, which corresponds to the same one from before.

14.3. **Examples.**

Example 14.2. Consider $\mathbb{P}^1(1,2)$. As we've seen in §11.3, we have

$$\begin{aligned} K(\mathbb{P}(1,2))_{\mathbb{Q}} &\cong \mathbb{Q}[t]/(t-1)^2(t+1) \\ &\cong \mathbb{Q}[t]/(t-1)^2 \oplus \mathbb{Q}[t]/(t+1). \end{aligned}$$

Before, we identified the first summand with $CH^*(\mathbb{P}(1,2))_{\mathbb{Q}}$ using the Baum-Fulton homomorphism. We can uniformly understand both by passing to the inertial stack.

In the language above, we have $X = \mathbb{A}^2 - 0$ and $G = \mathbb{G}_m$. The group scheme $I_G X$ over X has two "layers," namely $S_1 \cong X$ (because the identity stabilizers all of x) and $S_{-1} \cong \mathbb{A}^1 - 0$, the punctured y-axis. This realizes the decomposition

$$IX = S_1 \cup S_{-1}.$$

The Chow group is thus

$$CH^*(I\mathbb{P}(1,2)) \cong CH^*(\mathbb{P}(1,2)) \oplus CH^*(B\mu_2).$$

Note that we could also write $B\mu_2 = \mathbb{P}(2)$, to make the general pattern clearer. And indeed, we have $CH^*(B\mu_2)_{\mathbb{Q}} \cong \mathbb{Q}$, which is isomorphic to the second component.

Exercise 14.3. Write down the explicit isomorphism afforded by the equivariant Riemann-Roch theorem.

Example 14.4. Consider $\mathbb{P}^1(1,3)$. Then

$$K(\mathbb{P}(1,3)) \cong \mathbb{Q}[x]/(x-1)^2(x^2+x+1).$$

Tensoring with $\mathbb{C}$ splits this as

$$K(\mathbb{P}(1,3)) \cong \mathbb{C}[x]/(x-1)^2 \oplus \mathbb{C}[x]/(x-\omega) \oplus \mathbb{C}[x]/(x-\omega^2).$$

Again, the "first version" of the equivariant Riemann-Roch theorem only recovers the first summand as $CH^*(\mathbb{P}(1,3))$. To get the rest, we study the inertia stack.

We first have $I_G X = X^1 \coprod X^\omega \coprod X^{\omega^2}$. Each of X^ω and X^{ω^2} is a line, so $S_1 \cong X$, $S_\omega \cong S_{\omega^2} \cong \mathbb{A}^1 - 0$. Passing to quotients, we obtain the decomposition

$$I[X/G] = \mathbb{P}(1,3) \sqcup \mathbb{P}(3) \sqcup \mathbb{P}(3)$$

Therefore,

$$CH^*(I[X/G]) \cong CH^*(\mathbb{P}(1,3)) \oplus \mathbb{Q} \oplus \mathbb{Q},$$

which is compatible with the K-theory.

Exercise 14.5. Write down the explicit isomorphism afforded by the equivariant Riemann-Roch theorem.

Example 14.6. Consider $\mathbb{P}^2(1,2,4)$. Here $X = \mathbb{A}^3 - 0$, and we have

$$\begin{aligned} K(\mathbb{P}(1,2,4))_{\mathbb{C}} &\cong \mathbb{C}[\chi]/(\chi-1)(\chi^2-1)(\chi^4-1) \\ &\cong \mathbb{C}[\chi]/(\chi-1)^3(\chi+1)^2(\chi-i+)(\chi+i). \end{aligned}$$

Let's start decomposing $I_G X$ into connected components. Again, we have $S_1 \cong X$. We also have non-trivial fixed loci for -1 and $\pm i$. Note that -1 fixes an entire punctured plane, so $S_{-1} \cong \mathbb{A}^2 - 0$. Each of $\pm i$ fixes a punctured line $\mathbb{A}^1 - 0$.

Taking the quotient by G, we obtain

$$\mathbb{P}(1,2,4) = \mathbb{P}(1,2,4) \sqcup \mathbb{P}(2,4) \sqcup \mathbb{P}(4) \sqcup \mathbb{P}(4)$$

and taking Chow groups recovers something at least abstractly isomorphism to K-theory.

Exercise 14.7. Write down the explicit isomorphism afforded by the equivariant Riemann-Roch theorem.

The formula for general weighted projective spaces should now be obvious.

References

[Edi00] Dan Edidin, *Notes on the construction of the moduli space of curves*, Recent progress in intersection theory (Bologna, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 85–113. MR 1849292 (2002f:14039)

[Edi13] ______, *Riemann-Roch for Deligne-Mumford stacks*, A celebration of algebraic geometry, Clay Math. Proc., vol. 18, Amer. Math. Soc., Providence, RI, 2013, pp. 241–266. MR 3114943

[EG00] Dan Edidin and William Graham, *Riemann-roch for equivariant chow groups*, Duke Math. Journal **102** (2000), no. 3, 567–594.

[EHKV01] Dan Edidin, Brendan Hassett, Andrew Kresch, and Angelo Vistoli, *Brauer groups and quotient stacks*, Amer. J. Math. **123** (2001), no. 4, 761–777. MR 1844577 (2002f:14002)

[Hei10] Jochen Heinloth, *Lectures on the moduli stack of vector bundles on a curve*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 123–153. MR 3013029

[Tot99] Burt Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281. MR 1743244 (2001f:14011)