ALGEBRA QUAL PREP: COMMUTATIVE ALGEBRA

TONY FENG

1. SPRING 2012 A1

(a) Do it.
(b) Let $x_1, \dots, x_n$ be a (finite) set of generators for J, which exist because A is Noetherian. We make a map $A[x_1', \dots, x_n'] \to G_J(A)$ by sending $(x_1')^{e_1} \dots (x_n')^{e_n}$ to the element $x_1^{e_1} \dots x_n^{e_n}$ viewed in $J^{e_1+\dots+e_n} \subset G_J(A)$. Check that this is a ring homomorphism, and is surjective. Since a quotient of a Noetherian ring is Noetherian, and $A[x_1', \dots, x_n']$ is Noetherian by Hilbert Basis theorem, we deduce that $G_J(A)$ is Noetherian.

2. FALL 2012 M4

(i) We claim that the matrix of f has determinant a unit. This will show that f is invertible, hence is injective. To see the claim, note that the surjectivity of f implies the surjectivity of $\wedge^n f : R \to R$, which is multiplication by $\det f$.
(ii) Let $I_n = \ker(f^{\circ n})$. Then we have an increasing chain

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

Since R is Noetherian, eventually $I_k = I_{k+1} = I_{k+2} \dots$. This means $\ker(f^{\circ k}) = \ker(f^{\circ k+1})$. But since f is surjective, so is $f^{\circ k}$. Hence we can find x such that $f^{\circ k}(x) \in I_1 - 0$, so that $x \in I_{k+1} - I_k$.
(iii) Take $f : k[x_1, x_2, \dots] \to k[x_2, \dots]$ sending $x_1 \mapsto 0$.

3. FALL 2011 A4

(i) We show that A is integral over A^G. Indeed, any $a \in A$ satisfies the monic polynomial

$$\prod_{g \in G} (X - g(a))$$

which has coefficients in A^G.

Since A is finitely generated (as an algebra) over k, it is finitely generated (as an algebra) over A^G. Since it is also integral over A^G as we just showed, it is finite as a module over A^G.
(ii) It suffices to prove the following general lemma (sometimes called the "Artin-Tate Lemma"): if A is finitely generated over R, and $B \subset A$ is a subalgebra such that A is finite over B, then B is finitely generated over R.

Take a set of generators for A as an algebra over R, say $x_1, \dots, x_n$. These are all integral over B, so they satisfy monic polynomials with coefficients in B; let $y_1, \dots, y_N$ be all such coefficients. These generate a subring C of B. Furthermore it is clear that A is finite over C. Since R is noetherian, so is C, hence B is also finite over C. Therefore B is finitely generated as an algebra over A.

Note: one can find a counterexample when R is not noetherian. Take the square-zero extension structure on $R\oplus R$, which contains $R\oplus I$ as a subring for any ideal $I\subset R$.

4. Spring 2012 A3

(a) You should know the definition of integral. If B is finitely generated as an A-module, then we can pick a finite set of generators $x_1,\dots,x_n$. Given $b\in B$, multiplication by b can be given by a matrix (b_{ij}). This satisfies its own characteristic polynomial, so multiplication by b satisfies that characteristic polynomial. So we find a monic polynomial in b such that multiplication by it annihilates $1\in B$, hence is 0.

(b) The map $A\to B$ factors as $A\twoheadrightarrow A/I\hookrightarrow B$. Since $\operatorname{Spec}(A/I)\subset\operatorname{Spec}A$ is closed, we can reduce to the case where f is injective. We then claim that $\operatorname{Spec}B\twoheadrightarrow\operatorname{Spec}A$. Pick a prime $\mathfrak{p}$ of A, which we want to show is in the image of Spec B. Since localization is exact, we have

$$A_{\mathfrak{p}}\hookrightarrow B\otimes A_{\mathfrak{p}}$$

is still an injective integral extension.

Let $\mathfrak{q}$ be a maximal ideal of $B\otimes_A A_{\mathfrak{p}}$. Then we have

$$A_{\mathfrak{p}}/(\mathfrak{q}\cap A_{\mathfrak{p}})\hookrightarrow B\otimes_A A_{\mathfrak{p}}/\mathfrak{q}$$

is still an injective integral extension. But since $\mathfrak{q}$ is maximal, $B\otimes_A A_{\mathfrak{p}}/\mathfrak{q}$ is a field. If a field is integral over a domain, that domain must be a field. So $\mathfrak{q}\cap A_{\mathfrak{p}}$ is the unique maximal ideal of $A_{\mathfrak{p}}$, which is $\mathfrak{p}$.

5. Fall 2013 M4

(a) Done before.

(b) Done before.

(c) First we check that B is closed under addition and multiplication. Let $x,y\in B$. Then $A[x]\subset B$ is a finite A-module, hence $A[x,y]\subset B$ is a finite $A[x]$-module. As this contains $x+y$ and xy, we find that they are also integral over A by (a). Now, we need to show that any $x\in L$ can be represented as p/q where $p,q\in B$. Certainly x satisfies a monic equation over K:

$$x^n+\frac{p_{n-1}}{q_{n-1}}x^{n-1}+\ldots+\frac{p_1}{q_1}x+\frac{p_0}{q_0}=0.$$

Set $q:=\prod q_i$ and $r_j=\prod_{j\neq i}q_j$. Multiply by q^n to get

$$(qx)^n+p_{n-1}r_{n-1}(qx)^{n-1}+p_{n-2}r_{n-2}(qx)^{n-2}+\ldots+p_1r_1(qx)+p_0r_0=0$$

with each coefficient in B. This shows that qx is integral over A, hence lies in B.

6. Spring 2013 M1

(i) We use the going-up theorem: let $A \hookrightarrow B$ be a integral. Then for any chain $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ in A and $\mathfrak{q}_1$ of B mapping to $\mathfrak{p}_1$, we can extend to $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2$ restricting to the chain $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. (Replacing $A \hookrightarrow B$ by $A/\mathfrak{p}_1 \hookrightarrow B/\mathfrak{q}_1$, this reduces to the surjectivity statement proved earlier.)

This shows that $\dim B \geq \dim A$. On the other hand, if $\mathfrak{q}_1 \subset \mathfrak{q}_2$ is a proper inclusion of ideals of B with both $\mathfrak{q}_i$ restricting to $\mathfrak{p} \in \operatorname{Spec} A$, then $A_\mathfrak{p}/\mathfrak{p} \hookrightarrow B_\mathfrak{p}/\mathfrak{q}_1$ is the inclusion of a field into a domain integral over it, which shows that $B_\mathfrak{p}/\mathfrak{q}_1$ is a field and hence that $\mathfrak{q}_2 = \mathfrak{q}_1$.

(ii) Noether normalization: any finitely generated algebra A over k is finite over $k[x_1, \dots, x_n]$. Proof sketch: embed Spec A in affine space, and choose generic presentations to hyperplanes until the map has finite fibers.

(iii) By Noether normalization we have a finite map $k[x_1, \dots, x_n] \to B$, hence $\dim B = \dim k[x_1, \dots, x_n] = n$. As finiteness is preserved by tensoring, the map $K[x_1, \dots, x_n] \to K \otimes_k B$ is also finite. Hence by (a) we also have $\dim K \otimes_k B = \dim K[x_1, \dots, x_n] = n$.

7. Spring 2013 A1

(i) Done before.

(ii) It suffices to show that if $A \hookrightarrow B$ is integral, then we can extend an embedding $A \hookrightarrow K$ to $B \hookrightarrow K$. By induction/Zorn's Lemma we can do this in the case where B is generated as an A-algebra by a single element x. Then $B \cong A[x]/f(x)$ for some monic polynomial f. We can find an extension just by sending x to a root of f in K.

(iii) Consider a non-trivial separable field extension E/F. Let $G = \operatorname{Gal}(E/F)$ and $A = E$. Then we know that there are $|G|$ F-embeddings $E \hookrightarrow \overline{F}$.

8. Spring 2014 M4

(i) We have $B_Q/Q \hookrightarrow B' \otimes_B B_Q/P'$. This is an integral extension, so $B' \otimes_B B_Q/P'$ is a field. But then P' coincides with Q' in $B' \otimes_B B_Q$, hence $P' = Q'$.

(ii) Let the $\mathfrak{p}_i$ be the primes of B' over $\mathfrak{p} \in \operatorname{Spec} B$. By Prime Avoidance, we can find $x \in \mathfrak{p}_1 - \bigcup_{i>1} \mathfrak{p}_i$. Consider $\prod_{g \in G}(gx)$. This lies in $\bigcap(g\mathfrak{p}_1) \cap B = \mathfrak{p} \subset \mathfrak{p}_i$ for any $\mathfrak{p}_i$. A product of elements not in $\mathfrak{p}_i$ cannot lie in $\mathfrak{p}_i$, so some $gx \in \mathfrak{p}_i$. This forces $g\mathfrak{p}_1 = \mathfrak{p}_i$.