ALGEBRA QUAL PREP: ALGEBRAIC GEOMETRY

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These are hints; they are not a model for what to write on the quals.

1. Spring 2010 A5

See problem 7.

2. Fall 2010 M3

If $P = (f)$ when f is irreducible. If P is not principle, we take two irreducibles $f, g \in P$ which are coprime. In $\mathbf{C}(y)[x]$, which is a Euclidean domain, we can write

$$af + bg = 1, \quad a, b \in \mathbf{C}(y).$$

Clearing denominators, we find $a', b' \in \mathbf{C}[x, y]$ and $h \in \mathbf{C}[y]$ such that

$$a'f + b'g = h.$$

Hence the vanishing set of f and g is contained in a union of horizontal lines. Then $V(P)$ is contained in a single horizontal line, at which we reduce to the statement for polynomials in one variable.

3. Spring 2011 M4

(a) The ideal generated by the f_i and d in $K[x_1, \ldots, x_n]$ have no zero, so (since K is algebraically closed) it must be the unit ideal. Hence we can find $a_1, \ldots, a_n, b \in K[x_1, \ldots, x_n]$ such that

$$\sum a_i f_i + bd = 1.$$

Then $bg = r$ on S, since $r = r \cdot 1 = \sum r a_i f_i + rbd$.

(b) This is equivalent to: there does not exist $h, f \in \mathbf{Q}[x, y]$ such that

$$h \cdot (y - x) - 1 = f(x, y)(x^2 + y^2 - 1).$$

Setting $x = y$ in this relation, it would give $f' \in \mathbf{Q}[x]$ such that $1 = f'(x)(2x^2 - 1)$. But$2x^2 - 1$ is not a unit in $\mathbf{Q}[x]$, since for example it vanishes on $\sqrt{2}$.

4. Spring 2013 A4

(i) Vanishing on $x \in K^n$ is a linear equation on the coefficients of $p \in \mathbf{C}[x_1, \ldots, x_n]$. Therefore vanishing on $Z \cap K^n$ is a system of such equations. The solution space to this system over $\mathbf{C}$ is tensored up from that of K. So we deduce $p = \sum c_i p_i$ with $c_i \in \mathbf{C}$, $p_i \in K[x_1, \ldots, x_n]$ vanishing on $Z \cap K^n$. Then by the Nullstellensatz, $p_i \in \sqrt{J}$, hence vanishes on Z.

(ii) The statement is equivalent to saying that any $f \in \mathbf{C}[x_1,\dots,x_n]]$ vanishing on $Z \cap K^n$ vanishes on Z. But that is what we just proved in (i).

(iii) Given $V(I) \subset \mathbf{C}^n$, we want $V(I) \cap K^n = V(?)$ to be closed. The Zariski closure $V(I) \cap K^{n^{Zar}}$ is the vanishing set of an ideal which is extended from $J \subset K[x_1,\dots,x_n]$ by the same argument as in part (i), namely that the ideal is defined by a system of linear equations with coefficients in K.

Therefore we reduce to showing that for such a J, $V_{\mathbf{C}}(J) \cap K^n$ is Zariski closed. We claim that it is in fact the vanishing set $V_K(J)$ taken in $K[x_1,\dots,x_n]$. We have $\mathfrak{m} \in V_{\mathbf{C}}(J) \cap K^n$ if and only if all $f \in J \otimes_K \mathbf{C}$ vanish on $\mathfrak{m}$, while $\mathfrak{m} \in V_K(J)$ if and only if all $g \in J$ vanish on $\mathfrak{m}$. Clearly the second condition is stronger, so $V_{\mathbf{C}}(J) \cap K^n \subset V_K(J)$. But in part (i) we proved the converse.

5. Fall 2013 A5

Omitted.

6. Spring 2014 A1

(i) Omitted.

(ii) Let $b_1,\dots,b_m$ be generators for B as an A-algebra. By Noether normalization applied to $K \otimes_A B$ as a K-algebra, the images of $b_1,\dots,b_m$ in $K \otimes_A B$ are finite over K, hence they satisfy monic polynomials $p_1,\dots,p_m$ of degrees $d_1,\dots,d_m$ over K. Since there are only finitely many coefficients, these monic polynomials all lie in $B \otimes_A A_a$ for some $a \in A - 0$.

We show that the (finite!) set of monomials $\{\prod b_i^{e_i} : 0 \le e_i < d_i\}$ generates B_a as an A_a-module. It is clear that $b_1,\dots,b_m$ are algebra generators for B_a over A_a, so any $x \in b$ can be written as a polynomial in the b_i with coefficients in A_a. Using the polynomials p_i as relations, we can arrange that this polynomial involves only monomials of the form in our generating set.

(iii) Since $A_a \hookrightarrow B_a$ is finite, the map Spec $B_a \to$ Spec A_a is surjective. Therefore Spec(f) contains the open subset Spec A_a.

7. Fall 2014 A5

(a) Write $f = \sum a_i \otimes b_i$ with the a_i independent over k. Since $B/\mathfrak{m}_x \cong k$ (the Nullstellensatz implies it is a finite extension of k, which is k since it is algebraically closed).

(b) By writing a finite expression for a zero-divisor, we deduce that any counterexample lives in $A \otimes_k R'$ with R' finitely generated over k, hence we may rename R' to B and assume that B is finitely generated over k.

Suppose $x, y \in A \otimes_k B$ are such that $xy = 0$. For maximal $\mathfrak{m} \subset B$, we write $\phi_{\mathfrak{m}} \colon A \otimes_k B \to A$. Clearly $\phi_{\mathfrak{m}}(xy) = 0$ still. Moreover, if $x, y \neq 0$ then by (a) there is a Zariski-open subset of $\mathfrak{m}$ such that $\phi_{\mathfrak{m}}(x) \neq 0$ and $\phi_{\mathfrak{m}}(y) \neq 0$, contradicting that A is a domain.

(c) Done before.

8. Spring 2015 M1

This problem was a typo.

9. Fall 2015 A5

(a) They are all of the following form: $(x_1 - a_1, \dots, x_n - a_n)$ with $a_i \in F$.
(b) Evaluation at a defines a homomorphism $K[x_1, \dots, x_n] \to K(a)$, where $K(a)$ is the finite extension of K generated by the coordinates of a. This finiteness implies that the homomorphism is surjective.
(c) By the Nullstellensatz there is a natural bijection between zero sets and radical ideals. Furthermore, radical ideals are determined by the maximal ideals containing them (namely, as the intersection). So Z determines a collection of maximal ideals of $\overline{K}[x_1, \dots, x_n]$. Those collections that come from ideals of $K[x_1, \dots, x_n]$ are characterized by the ones that are Galois-stable, by (b). So we find that Z is determined by an ideal I in $K[x_1, \dots, x_n]$; it remains to explain why it is principal.

We know that $I \otimes_K \overline{K}$ is principal. So, by looking at an element of I of minimal degree, we find that there is $g \in I$ such that $g = \alpha f$ for $\alpha \in \overline{K}^*$. The any $h \in I$ can be written as $h = \beta f$ for some $\beta \in \overline{K}[x_1, \dots, x_n]$, and so $\beta\alpha \in K[x_1, \dots, x_n]$ as it is fixed by $\mathrm{Gal}(\overline{K}/K)$.