HIGGS BUNDLES AND NON-ABELIAN HODGE THEORY

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Contents

1. Introduction

I'll start by telling you the *ending* of the story that we'll discuss in this class. In our story the basic actor is a compact Riemann surface Σ with a *fixed* complex structure. (The next step would be to study a family version of the story over the moduli space of complex structures, but this isn't understood yet.)

One of the fundamental questions is to understand holomorphic vector bundles $F \to \Sigma$. The smooth vector bundles can be understood via characteristic classes, but if we focus on *holomorphic* vector bundles then there are many even within a fixed topological type.

By uniformization we have $\Sigma = \mathbb{H}^2/\Gamma$ where $\Gamma = \pi_1(\Sigma)$. If you take any fixed vector space $\mathbb{C}^n$ and a representation $\rho : \Gamma \to \mathrm{GL}(N, \mathbb{C})$, then one can form a line bundle on Σ by taking the quotient

$$\mathbb{H}^2 \times_\rho \mathbb{C}^N = \{(z, v)\}/\sim,$$

with equivalence relation $(z, v) \sim (z', v')$ if $z' = \gamma(z)$ and $v' = \rho(\gamma)v$ for $\gamma \in \Gamma$. This is holomorphic because the transition functions are essentially constant. In fact these are called "flat" bundles.

Basic Question: Can one "realize" moduli of holomorphic vector bundles via flat structures (i.e. flat connections)?

This would be a huge win if possible, because the flat structures are essentially just topological data, as we saw above.

It is a theorem of Narasimhan-Seshadri that this is true with qualifications. First, it doesn't work for *all* vector bundles; you have to restrict your attention to *stable* holomorphic bundles, which have projectively flat unitary connections. (This means that the curvature tensor is pure trace, so it's not 0 but "all the interesting parts" are 0.) The first proof was probably algebraic; I've never actually looked. In 1985 Donaldson gave a revolutionary gauge-theoretic proof (1985).

This course will focus on a 1987 paper of Hitchin, where he considers a slightly different starting point: $\Sigma \times \mathbb{R}^2$. This is a four-manifold, and you can apply Yang-Mills theory to it. This means considering anti-self-dual connections on it which are $\mathbb{R}^2$-invariant. Let's start out thinking purely in local coordinates x_1, x_2, x_3, x_4. If $A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4$ where the A_i are matrices (properly speaking, valued in the Lie algebra of the structure group), then the ASD equations are

$$\begin{aligned} F_A &= dA + [A, A] \\ xF_A &= -F_A. \end{aligned}$$

You should think of these equations as being $dA = 0$ and $d^*A = 0$ to first order, plus some non-linear junk, which is the "non-abelian" part of "non-abelian Hodge theory".

Let $\Phi = A_3 dx_3 + A_4 dx_4$. Suppose you have a rank 2 bundle $E \to \Sigma$. A connection A on Σ (abusing notation, we mean only the first two components) and we can think of $\Phi \in C^\infty(\Sigma, \mathrm{End}(E) \otimes \bigwedge^i \Sigma)$. If you actually write down the conditions for invariance, the

ASD equations are equivalent to

$$\begin{aligned} F_A + [\Phi \wedge \Phi^a] &= 0 \\ \overline{\partial}_A \Phi &= 0. \end{aligned}$$

The second equation says that we have a connection on E, and Φ is a holomorphic section. The first equation measures "how non-normal". These are called the "Hitchin equations". Now these are make sense independent of our assumptions on the flat coordinates, and we can globalize them accordingly.

A *Higgs pair* is (E, Φ). Higgs showed that *stable* Higgs pairs are essentially the same as solutions to Hitchin's equations modulo gauge transformations. This relates something purely algebro-geometric (stable Higgs pairs) and something analytic.

$$\{\text{Higgs pairs}\} \longleftrightarrow \{\text{Hitchin solutions}\}/\text{gauge} \sim .$$

There is actually another equivalence, with representations of Γ into (say) $\mathrm{SL}_n(\mathbb{R})$ modulo conjugation. This is something a whole different set of people have been interested in.

$$\begin{tikzcd} \{\text{Higgs pairs}\} \arrow[rr, dash] \arrow[dr, dash] & & \{\text{Hitchin solutions}\}/\text{gauge} \sim \arrow[dl, dash] \\ & \mathrm{Rep}(\Gamma, \mathrm{SL}_n(\mathbb{R})) & \end{tikzcd}$$

Strictly speaking we are lying a little bit; the third thing has multiple components, one of which fits the triangle.

The course will cover the circle of ideas sketched above. Where can one go from there? You can consider the moduli space $\mathscr{M}_E$ of objects above. $\mathscr{M}_E$ carries a natural "Weil-Petersson" metric g_{WP}.

The metric g_{WP} has nice properties: it is complete (strictly speaking a lie; it depends on parameters such as the rank, so sometimes has singularities) and is a *hyperKähler* metric. This means that g_{WP} has three distinct complex strutures I, J, K which satisfy $I^2 = -\mathrm{Id}, J^2 = -\mathrm{Id}, K^2 = -\mathrm{Id}$, and $IJ = K, JK = I, KI = J$ (namely the quaternion relations). This is like a "quaternionic manifold" (the precise thing to say is that the holonomy group has reduction to Sp_4). In particular, g_{WP} is Ricci-flat.

The geometry and topology of this manifold are (only) somewhat understood. It carries various interesting data. A rough picture is that $\mathscr{M}_E$ has a natural fibration ($G = \mathrm{SL}_2$) over the holomorphic quadratic differentials of Σ, which is the tangent space to Teichmuller space.

This is called the "Hitchin fibration" (see Figure 1). The pre-images are tori of dimension $6g-6$ where g is the genus of Σ, but you move around the tori degenerate (otherwise the geometry wouldn't be so interesting!).

2. Holomorphic line bundles

We're going to cover some background on holomorphic vector bundles. As references we recommend the books on Riemann surfaces by Gunnings or Donaldson (note that Gunnings also has unpublished notes on his website).

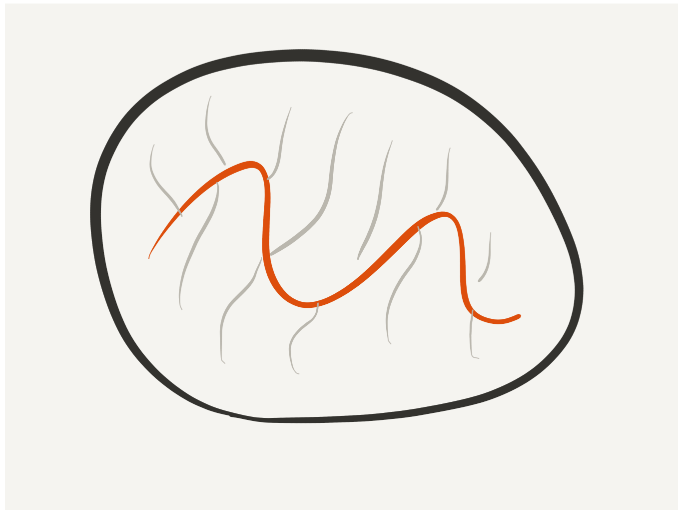


FIGURE 1.0.1. Depiction of the moduli space $\mathscr{M}_E$. The solutions to the Hitchin equation are a complicated subset, drawn in red. The action of the gauge group is depicted in gray.

FIGURE 1.0.2. The Hitchin fibration.

2.1. **Sheaf cohomology.** As before, fix Σ. Let $\{U_\alpha\}$ be an open cover of Σ with each U_α isomorphic to a disk.

Definition 2.1.1. A sheaf $\mathscr{S}$ is a topological space equipped with a map $\pi \colon \mathscr{S} \to \Sigma$ such that

(1) π is a local homeomorphism,
(2) $\pi^{-1}(p)$ is an abelian group,
(3) the group operations are continuous, e.g. multiplication is

$$\mathscr{S} \times_\pi \mathscr{S} \to \mathscr{S}$$

is continuous.

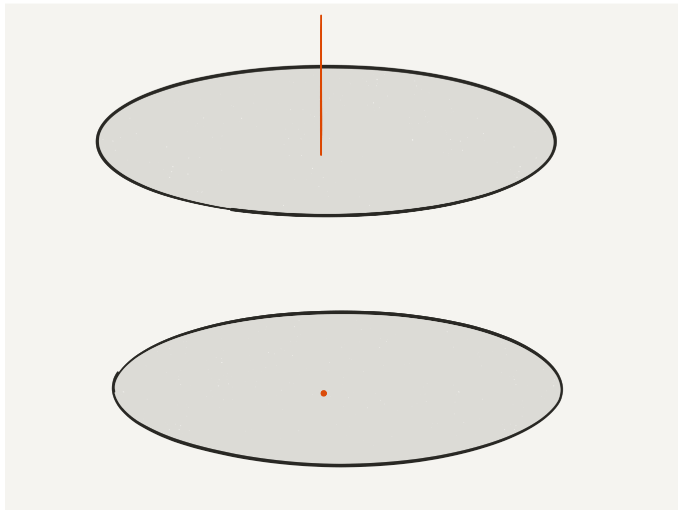


FIGURE 2.1.1. Depiction of a skyscraper sheaf.

We denote the sections of π over an open subset $U \subset X$ by $\mathscr{S}(U)$.

Example 2.1.2. If G is an abelian group with the discrete topology, then $M \times G$ with the natural projection map is a sheaf.

Example 2.1.3. We'll be interested in sheaves which are "germs of functions", where "functions" can mean continuous, smooth, holomorphic, constant. These correspond to $\mathscr{S}(U)$ being

- $C^\infty(U)$,
- $\pi^{-1}(U) = \mathscr{O}(U)$,
- $\mathscr{O}^*(U)$,
- $\mathbb{C}$.

The topology is a bit complicated. In the usual description of sheaves in terms of sections, a basis of open sets for the associated topological space ("éspace étale") is the collection of stalks corresponding to a section over some open subset of M.

Example 2.1.4. For $p \in M$, there is a *skyscraper sheaf* $\mathbb{C}_p$ for any $p \in M$, which is determined by the property

$$\Gamma(U, \mathbb{C}_p) = \begin{cases} 0 & p \notin U \\ \mathbb{C} & p \in U \end{cases}$$

In this case the projection map is an isomorphism away from p, and over p the fiber is $\mathbb{C}$. The open sets are either lifted from an open subset of M not containing p, or an open subset containing p plus a single point in the fiber. (See Figure 2.1.4.)

To define the sheaf cohomology group $H^q(\Sigma, \mathscr{S})$ we use the Cech formalism. For any locally finite open cover $\{U_\alpha\}$, a *cochain* $c \in C^q(\{U_\alpha\}, \mathscr{S})$ associates a $q+1$-tuple $c_{\alpha_0,\ldots,\alpha_q} \in$

$\mathscr{S}(U_{\alpha_0}\cap\ldots\cap U_{\alpha_q})$. The differential $\delta\colon C^q\to C^{q+1}$ is

$$(\delta c)_{q_0,\ldots,\alpha_{q+1}}=\sum(-1)^j c_{\alpha_0\ldots\widehat{\alpha}_j\ldots\alpha_{q+1}}$$

and we define the *cocyles* to be $Z^q=\ker\delta$ and the *coboundaries* to be $B^q=\operatorname{Im}\delta$. This defines a group $H^q(\{U_\alpha\},\mathscr{S})$ with respect to the cover $\{U_\alpha\}$. It is an exercise in diagram chasing to check that refinements of open covers induce maps on cohomology. The Cech cohomology is the direct limit of $H^q(\{U_\alpha\},\mathscr{S})$ over all open covers $\{U_\alpha\}$.

Remark 2.1.5. It seems difficult to compute Cech cohomology, since one has to consider all possible open coverings. In fact, one can show that a covering $\{U_\alpha\}$ for which each intersection $U_{\alpha_0}\cap\ldots\cap U_{\alpha_q}$ has vanishing higher cohomology already computes the Cech cohomology (i.e. no need to take direct limits). For example, if all these intersections are contractible and $\mathscr{S}$ is a sheaf of germs of functions, then such a cover computes the cohomology.

Example 2.1.6. Consider $q=0$. Then $c\in C^0$ is a choice of $c_\alpha\in\mathscr{S}(U_\alpha)$, so $(\delta c)_{\alpha\beta}=c_\alpha-c_\beta=0$. So H^0 (with respect to *any* open cover!) is simply the global sections of $\mathscr{S}$.

Example 2.1.7. A $c\in C^1$ is a choice of $c_{\alpha\beta}$ for all α,β. The differentials are $(\delta c)_{\alpha\beta\gamma}=c_{\alpha\beta}c_{\beta\gamma}c_{\gamma\alpha}$ (writing multiplicatively this time) and $c_{\alpha\beta}c_{\beta\alpha}$. Coboundaries are cocycles of the form $c_{\alpha\beta}=b_\alpha/b_\beta$. Therefore, the cohomology group consists of $(c_{\alpha\beta})$ such that

$$\begin{aligned}c_{\alpha\beta}c_{\beta\gamma}c_{\gamma\alpha}&=1\\ c_{\alpha\beta}c_{\beta\alpha}&=1\end{aligned}$$

modulo coboundaries.

2.2. Holomorphic line bundles.

We claim that $H^1(\Sigma,\mathscr{O}^*)$ is the space of line bundles on Σ modulo isomorphism. To get a line bundle from a cohomology class $(c_{\alpha\beta})$, take the trivial line bundles $U_\alpha\times\mathbb{C}$ and $U_\beta\times\mathbb{C}$ and glue them over $U_\alpha\cap U_\beta$ by $(z,v)\sim(z,c_{\alpha\beta}v)$. The cocycle condition ensures compatibility over triple intersections. It is clear that $c_{\alpha\beta}b_\alpha b_\beta^{-1}$ would define an equivalent line bundle (changing the local trivializations), so that the line bundle is really well-defined. Conversely, from a line bundle one forms a cohomology by reversing this process.

Remark 2.2.1. The space of holomorphic line bundles modulo isomorphism is clearly an abelian group. The multiplication is $\xi,\eta\mapsto\xi\otimes\eta$, the inverse is $\xi\mapsto\xi^*$, and the identity is the trivial bundle.

Let $\mathscr{GL}(N,\mathbb{C})$ be the sheaf of germs of holomorphic maps $M\to\mathrm{GL}(N,\mathbb{C})$. Although we formulated everything only for sheaves of abelian groups, which this certainly isn't, it turns out that with some care one can make same of $H^1(M,\mathscr{GL}(N,\mathbb{C}))$, and show that it is the space of holomorphic vector bundles modulo isomorphism. However, note that it is *just a set*, and not a group.

A key short exact sequence of sheaves that we will use is the so-called *exponential exact sequence*

$$0\to\mathbb{Z}\to\mathscr{O}\xrightarrow{\exp}\mathscr{O}^\times\to 0. \tag{1}$$

From a short exact sequence of sheaves we always get a long exact sequence of cohomology, which in this case contains

$$\dots H^1(M,\mathscr{O}) \to H^1(M,\mathscr{O}^\times) \to H^2(M,\mathbb{Z}) \to H^2(M,\mathscr{O}) \to \dots$$

Theorem 2.2.2. *For a Riemann surface M, we have $H^q(M,\mathscr{O}) = 0$ for $q \geq 2$ and $H^1(M,\mathscr{O}) = \Gamma(M,\mathscr{E}^{0,1})/\overline{\partial}\Gamma(M,\mathscr{E}^{0,0})$.*

Let's explain where this comes from. Denote by $\mathscr{E}^{p,q}$ the sheaf of germs of C^∞ sections of $\bigwedge^{p,q} T^*M$. In local coordinates, a frame for T^*M consists of $dz^1,\dots,dz^n$ and $d\overline{z}^n,\dots,d\overline{z}^n$ where $dz_j = dx_j + i dy_j$ and $d\overline{z}_j = dx_j - i dy_j$. We have $dz \wedge dz = 0$ and $d\overline{z} \wedge d\overline{z} = 0$. The Dolbeault operator is $\overline{\partial}$. In local coordinates, if

$$\omega = \sum \omega_{IJ} dz^I \wedge d\overline{z}^J$$

then

$$\overline{\partial}\omega = \sum \frac{\partial \omega_{IJ}}{\partial \overline{z}_j} d\overline{z}_j \wedge (dz^I \wedge d\overline{z}^J).$$

On a Kähler (or even a complex) manifold, the differential lands

$$d \colon \wedge^{p,q} \to \wedge^{p+1,q} + \wedge^{p,q+1}.$$

However, in some settings such as an almost complex manifold, we only have a priori that

$$d \colon \wedge^{p,q} \to \oplus_k \wedge^{p+k+1,q-k}.$$

Anyway, the Dolbeault operator globalizes to

$$\overline{\partial} \colon \Gamma(U,\mathscr{E}^{0,0}) \to \Gamma(U,\mathscr{E}^{0,1}).$$

We claim that there is an exact sequence

$$0 \to \mathscr{O} \to \mathscr{E}^{0,0} \xrightarrow{\overline{\partial}} \mathscr{E}^{0,1} \to 0.$$

The injection and exactness at the middle are clear. The only statement that requires some nontrivial analysis is showing that any $\mu \in \mathscr{E}^{0,1}(U)$ for small enough U is of the form $\frac{\partial f}{\partial \overline{z}}$. This is what is called "Korn's Lemma", and one writes down an explicit f in terms of integrating μ. From the long exact sequence is

$$\dots \to H^{q-1}(\mathscr{E}^{0,0}) \to H^{q-1}(\mathscr{E}^{0,1}) \to H^q(\mathscr{O}) \to H^q(\mathscr{E}^{0,0}) \to \dots$$

To show the claimed isomorphism, we need to show that $H^i(\mathscr{E}^{0,0}) = 0$ for $i > 0$.

Definition 2.2.3. A sheaf $\mathscr{S}$ is called *fine* if "it has partitions of unity". More precisely, let $\{U_\alpha\}$ be a locally finite open cover. Then there exists a sheaf homomorphism

$$r_\alpha \colon \mathscr{S} \to \mathscr{S}$$

such that $r_\alpha(s) = 0$ if $\pi(s) \notin U_\alpha$ and $\sum r_\alpha = \mathrm{Id}$.

Theorem 2.2.4. *If $\mathscr{S}$ is fine, then $H^q(M,\mathscr{S}) = 0$ if $q > i$.*

Proof. This is essentially a combinatorial version of Poincaré's Lemma.

If $s \in Z^q$ is such that $s \equiv 0$ outside U_β, then we define a chain homotopy by

$$s^{\beta}_{\alpha_0,\dots,\alpha_{q-1}}(a) := s_{\beta\alpha_0\dots\alpha_{q-1}}(a)$$

if $a \in U_\beta$, and extend by 0 (this is where the hypothesis is used). Let's unravel the condition $\delta s = 0$:

$$(\delta s)_{\beta\alpha_0\dots\alpha_q} = s_{\alpha_0\dots\alpha_q} - \sum (-1)^j s_{\beta\alpha_0\dots\alpha_j\alpha_{j+1}\dots\alpha_q}$$

so

$$s = \sum (-1)^j \delta s^{\beta}_{\alpha_0\dots\widehat{\alpha_j}\dots\alpha_q}.$$

This expresses s as a coboundary.

In general, we use a partition of unity (η_β) to write

$$s = \sum \eta_\beta s.$$

Since by defintion $s \mapsto \eta_\beta s$ are sheaf homomorphisms, $\eta_\beta s$ is also coclosed and we can apply the above argument to each summand. $\square$

So from the exponential exact sequence (1) we get

$$\dots H^1(M,\mathscr{O}) \to H^1(M,\mathscr{O}^\times) \to H^2(M,\mathbb{Z}) \to H^2(M,\mathscr{O}) \to \dots$$

By our discussion, we get a short exact sequence

$$0 \to H^1(M,\mathscr{O})/H^1(M,\mathbb{Z}) \to H^1(M,\mathscr{O}^\times) \xrightarrow{c} H^2(M,\mathbb{Z}) \to 0.$$

We know that $H^1(M,\mathscr{O}^\times)$ is the space of line bundles modulo isomorphism and $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$ since M is a closed connected manifold. The map c is takes a bundle to its first Chern class, and this completely classifies the topological information in the bundle. The group $H^1(M,\mathscr{O})/H^1(M,\mathbb{Z})$ is actually a complex torus, denoted $\mathrm{Pic}^0(M)$. So we see that even after fixing a topological type, the space of holomorphic line bundles has interesting moduli.

Lemma 2.2.5. *Let $\xi \in H^1(M,\mathscr{O}^*)$. Then $\xi = 1$ if and only if there exists $s \in \Gamma(M,\mathscr{O}(\xi))$ with $s \neq 0$ on all of M.*

Proof. If ξ is trivial, then there is obviously a global section. If s exists, then choose a locally finite open cover (U_α) for ξ, with transition functions $\varphi_{\alpha\beta}$. Then

$$s_\alpha = \varphi_{\alpha\beta} s_\beta \text{ on } U_\alpha \cap U_\beta.$$

By the non-vanishing assumption, $s_\alpha \in \mathscr{O}^*(U_\alpha)$, so $\varphi_{\alpha\beta} = s_\alpha/s_\beta$ is an explicit expression of $\varphi_{\alpha\beta}$ as a coboundary. $\square$

Example 2.2.6. The same proof implies that the existence of a meromorphic section implies that $\varphi_{\alpha\beta} = 1$ in $H^1(M,\mathscr{M}^*)$ where $\mathscr{M}^*$ is the sheaf of germs of nontrivial meromorphic functions. This group itself is trivial, so every line bundle admits a meromorphic section.

Example 2.2.7. Let $T^2 = \mathbb{C}/\Lambda$. We can assume that $\Lambda = \langle 1, \tau \rangle$. The topological line bundles with trivial Chern class are classified by a representation $\rho : \Lambda \to \mathbb{C}^*$. Indeed, from such a representation one can form a line bundle by quotienting $\mathbb{C}\times\mathbb{C}$ by the equivalence relation $(z, v) \sim (z+\lambda, \rho(\lambda)v)$ for $\lambda \in \Lambda$.

This information is slightly redundant. What does it mean for the bundles obtained by ρ, ρ' to be isomorphic? It means that there is an isomorphic of the trivial bundles over $\mathbb{C}$ which is equivariant with respect to the two actions of Λ. In other words, that there is a non-vanishing function f on $\mathbb{C}$ such that the diagram commutes:

$$f(z+\lambda)\rho(\lambda) = f(z)\rho'(\lambda).$$

By choosing f of the form $\exp(2\pi i \zeta)$, we see that we can choose ρ so that $\rho(1) = 1$ (where Λ is generated by 1 and τ). This constrains f to be $\exp(2\pi i n z)$ for some n, which means that the only remaining ambiguity is if $\rho(z) = \exp(2\pi i n \tau)$. So we see that an equivalence for line bundles is represented by choices of $\rho(\tau) \in C^*/q^{\mathbb{Z}} \cong \mathbb{C}/\langle 1, \tau\rangle$ where $q = \exp(2\pi i n \tau)$.

Why is the map $H^1(M, \mathscr{O}^*) \xrightarrow{c} H^2(M, \mathbb{Z})$ the usual Chern class? Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathscr{E}^{0,0} \to (\mathscr{E}^{0,0})^* \to 0.$$

The long exact sequence on cohomology contains

$$H^1(M, \mathscr{E}^{0,0}) \to H^1(M, (\mathscr{E}^{0,0})^*) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathscr{E}^{0,0})$$

but both flanking terms vanish since they are higher cohomology of fine sheaves, so the middle map is an isomorphism. This is the usual Chern class. From the map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathscr{O} \arrow[r] \arrow[d] & \mathscr{O}^* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathscr{E}^{0,0} \arrow[r] & (\mathscr{E}^{0,0})^* \arrow[r] & 0 \end{tikzcd}$$

we get maps of associated long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & H^1(M, \mathscr{O}^*) \arrow[r, "c"] \arrow[d] & H^2(M, Z) \arrow[r] \arrow[d] & \dots \\ 0 \arrow[r] & H^1(M, (\mathscr{E}^{0,0})^*) \arrow[r, "\cong"'] & H^2(M, \mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

The bottom map is an isomorphism.

2.3. **Divisors.**

Definition 2.3.1. A *divisor* λ on a Riemann surface M is a finite formal sum $\lambda = \sum n_i \cdot p_i$ for $p_i \in M, n_i \in \mathbb{Z}$.

Example 2.3.2. If f is a meromorphic function then we can produce an associated divisor

$$\lambda(f) = \sum (\text{order zero}) p_i - \sum (\text{order pole}) q_i.$$

We can generalize this construction to $f \in H^0(M, \mathscr{M}(\xi))$, using local coordinates $f_\alpha = \phi_{\alpha\beta} f_\beta$.

Let $\mathscr{D}$ be the sheaf of germs of divisors, i.e. $\Gamma(U, \mathscr{D})$ is the divisors over $\mathscr{U}$

Exercise 2.3.3. Show that $\mathscr{D}$ is a fine sheaf.

The association of a divisor to a meromorphic function leads to a short exact sequence

$$0 \to \mathscr{O}^* \to \mathscr{M}^* \to \mathscr{D} \to 0.$$

The associated long exact sequence is

$$0 \to \Gamma(M, \mathscr{O}^*) \to \Gamma(M, \mathscr{M}^*) \to \Gamma(M, \mathscr{D}) \to H^1(M, \mathscr{O}^*) \to H^1(M, \mathscr{M}^*) \to 0.$$

From this we get a short exact sequence

$$0 \to \Gamma(M, \mathscr{D})/\Gamma(M, \mathscr{M}^*) \to H^1(M, \mathscr{O}^*) \to H^1(M, \mathscr{M}^*) \to 0.$$

This gives another description of the space of line bundles.

Example 2.3.4. Let $p \in M$. We claim that there exists a bundle ξ_p which has a global section s_p vanishing precisely at p (to order 1). Choose $U_0 = M \setminus \{p\}$ and $U_1 = D(p)$, a disc around p. Then $U_0 \cap U_1 = D^* = \{z : 0 < |z| < 1\}$, a punctured disc about p. The transition function is simply $\varphi_{01}(z) = z$.

A global section is described by two compatible local sections s_0, s_1. Since this is so canonical, we had better take $s_0 \equiv 1$ and $s_1 = z$, and indeed this fits the bill.

Given any divisor $\lambda = \sum n_i p_i$, we can construct a line bundle

$$\xi_\lambda := \xi_{p_1}^{n_1} \cdot \ldots \cdot \xi_{p_k}^{n_k}.$$

Remark 2.3.5. It is a fact that every line bundle arises from a divisor if and only if $H^1(M, \mathscr{M}^*) = 0$. We have asserted above that this is true; we will prove it eventually.

2.4. **The Chern class.**

Theorem 2.4.1. *For $f \in H^0(M, \mathscr{M}(\xi))$ and $\lambda_f = (f)$ a divisor associated to f, $c(\xi) = \deg \lambda$.*

Proof. We'll check this for $\xi = \xi_\lambda$. Since the Chern class is a homomorphism, it suffices to check that $c(\xi_p) = 1$ for all $p \in M$.

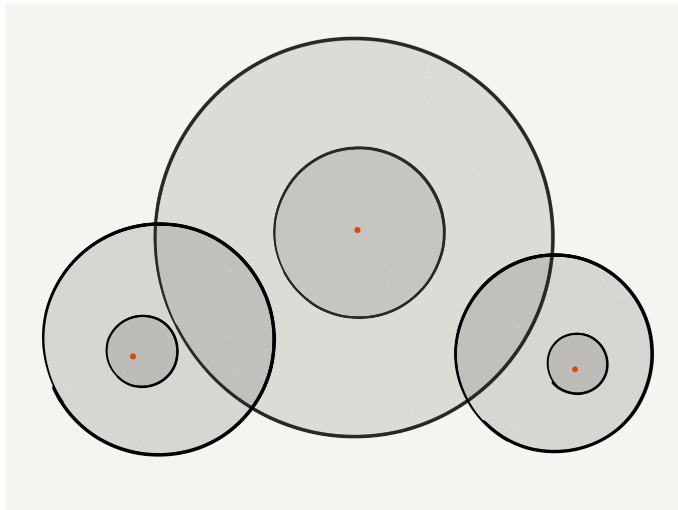
Introduce a Hermitian metric on ξ. For a good open cover U_α, this is simply the choice of a positive real number $h_\alpha > 0$ on each U_α, satisfying the overlap condition

$$h_\alpha = |\varphi_{\alpha\beta}|^2 h_\beta.$$

(This is a glimpse of one of the big themes of the course, namely how to do choose the best Hermitian metric. For line bundles there is not much subtlety, but for higher dimensional vector bundles we will see that there are tensions between different choices.)

Consider the local form $\overline{\partial}(\log h_\alpha)$. A priori this depend on our choice of coordinates, but using the overlap condition we check that:

$$\begin{aligned}
\overline{\partial}\partial(\log h_\alpha) &= \overline{\partial}\partial(\log|\varphi_{\alpha\beta}|^2 + \log h_\beta) \\
&= \overline{\partial}\partial(\log \varphi_{\alpha\beta} + \log \overline{\varphi_{\alpha\beta}} + \log h_\beta) \\
&= \overline{\partial}\partial \log h_\alpha.
\end{aligned}$$



The upshot is that $\gamma := \overline{\partial}\partial \log h_\alpha$ is well-defined globally on M.

Theorem 2.4.2. *We have*

$$\frac{1}{2\pi i}\int_M \gamma = c(\xi) \text{ for any } h.$$

Proof. First we check independence of h. Given any other $k_\alpha = |\varphi_{\alpha\beta}|^2 k_\beta$, we have $k_\alpha = \mu h_\alpha$ for all α for some $\mu \in C^\infty(M)$ with $\mu > 0$. Then $k_\alpha/h_\alpha = k_\beta/h_\beta$ for all α, β, so

$$\overline{\partial}\partial \log k_\alpha = \overline{\partial}\partial \log \mu + \overline{\partial}\partial \log h_\alpha.$$

Noting that $\overline{\partial}\partial f = d\partial f$ for any f, the integral of $\overline{\partial}\partial \log \mu$ vanishes by Stokes' Theorem.

Now choose $f \in \Gamma(M, \mathscr{M}(\xi))$ and set $h_\alpha = |f_\alpha|^2$ away from the zeros and poles. We can choose the open cover U_α so that every zero and pole is in exactly one of our open sets. We then extend our choice of $h_\alpha > 0$ on the open sets V_α containing a zero or pole. Then the integral of $\overline{\partial}\partial \log h_\alpha$ vanishes away from $V_{\alpha_1} \cup \ldots \cup V_{\alpha_k}$.

Now it only remains to examine what happens on some V_{α_j}. There h_α is $|f_\alpha|^2$ near ∂V_{α_j} and is positive inside. (See Figure 2.4.)

Then

$$\int_{V_{\alpha_j}} \overline{\partial}\partial \log h_\alpha = \int_{\partial V_{\alpha_j}} \partial \log |f_\alpha|^2 = \int_{\partial V_{\alpha_j}} \partial \log f_\alpha = 2\pi i n_j.$$

It's not obvious from the formula that this really is the usual definition of the first Chern class. This requires a diagram chase, but we'll skip it. The cleanest method is a hypercohomology spectral sequence relating the Cech and de Rham theorems.

□

This shows that for $\xi = \xi_p$ and s_p, the section with $\lambda_{s_p} = 1 \cdot p$, we have $c(\xi_p) = 1$. □

Corollary 2.4.3. *If ξ, η are two line bundles then $c(\xi \otimes \eta) = c(\xi) + c(\eta)$.*

Proof. This is clear from the fact that a logarithm of a product is the sum of the logarithms. (Of course, it also follows formally from the fact that the Chern class is a group homomorphism.) ☐

This is the beginning of the *Chern-Weil theory*, which represents characteristic classes by integrals of differential forms.

2.5. **Serre duality.**

Definition 2.5.1. The *canonical line bundle* K is the holomorphic line bundle on M associated to the local sections dz.

Theorem 2.5.2 (Serre duality)**.** *For* $E \to M$ any *holomorphic vector bundle, we have a canonical isomorphism*

$$H^1(M, \mathscr{O}(E)) \cong H^0(M, \mathscr{O}(E^* \otimes K))^*.$$

These cohomology groups are all de Rham type cohomology groups, hence can be identified with kernels/cokernels of operators. This (combined with analytic properties of elliptic operators) is what underlies the analytic proofs.

Proof. We have a bilinear pairing

$$H^0(M, \mathscr{O}(K \otimes E^*)) \times H^0(M, \mathscr{E}^{0,1}(E)) \to \mathbb{C}$$

defined as follows. For $\sigma \in H^0(M, \mathscr{O}(E^* \otimes K))$ and $\tau \in C^\infty(M, \overline{K} \otimes E)$, we have that $(\sigma, \tau) \in H^0(M, \mathscr{E}^{1,1})$ under the pairing of E and E^*, so we can define

$$\langle \sigma, \tau \rangle = \int_M (\sigma, \tau).$$

A general observation is that if $F \colon H^0(M, \mathscr{E}^{0,1}(E)) \to \mathbb{C}$ and $F|_{\overline{\partial} H^0(M, \mathscr{E}^{0,0}(E))} = 0$ then F is given by integrating against some distribution ϕ which is in the kernel of $\overline{\partial}^*$, and hence by elliptic regularity is holomorphic. The short exact sequence

$$0 \to \mathscr{O} \to \mathscr{E}^{0,0} \xrightarrow{\overline{\partial}} \mathscr{E}^{0,1} \to 0$$

induces as part of the long exact sequence an isomorphism

$$H^0(M, \mathscr{E}^{0,1}(E))/\overline{\partial} H^0(M, \mathscr{E}^{0,0}(E)) \cong H^1(M, \mathscr{O}) \to 0.$$

Therefore, F descends to $\widetilde{F} \colon H^1(M, \mathscr{O}(E)) \to \mathbb{C}$.

This defines a pairing

$$H^0(M, \mathscr{O}(E^* \otimes K)) \times H^1(M, \mathscr{O}(E)) \to \mathbb{C}$$

which is non-degenerate by formal properties of Fredholm operators and linear algebra. ☐

2.6. **Riemann-Roch.**

Theorem 2.6.1 (Riemann-Roch)**.** *We have*

$$\dim H^0(M, \mathscr{O}(\xi)) - \dim H^1(M, \mathscr{O}(\xi)) = c(\xi) + 1 - g$$

where g is the genus of M.

Remark 2.6.2. The quantity $\dim H^0(M, \mathscr{O}(\xi)) - \dim H^1(M, \mathscr{O}(\xi))$ is the index of a certain operator, while the right hand side is something topological. This is a feature of index formulas, fitting into the story of the Atiyah-Singer Index Theorem. The pattern of proof for these types of formulas is that they are so stable, you can reduce them to very simple cases.

Proof. First we check this for ξ trivial. The left hand side is $1 - \dim H^0(M, \mathscr{O}(K))$ by Serre duality. Therefore, we simply want to show that

$$g' := \dim H^0(M, \mathscr{O}(K)) = g$$

where g is the genus of the underlying surface. Well, consider the short exact sequence

$$0 \to \mathbb{C} \to \mathscr{O} \xrightarrow{d} \mathscr{O}(K) \to 0.$$

(The substance of exactness is surjectivity, which follows from a variant of the Poincaré lemma.) Note that we have a decomposition $df = \partial f + \overline{\partial} f$. In the long exact sequence, we have

$$\begin{aligned} 0 &\to H^0(M, \mathbb{C}) \to H^0(M, \mathscr{O}) \to H^0(M, \mathscr{O}(K)) \\ &\to H^1(M, \mathbb{C}) \to H^1(M, \mathscr{O}) \to H^1(M, \mathscr{O}(K)) \\ &\to H^2(M, \mathbb{C}) \to 0. \end{aligned}$$

By Serre duality, $H^1(M, \mathscr{O}) \cong H^0(M, \mathscr{O}(K))^*$ and $H^1(M, \mathscr{O}(K)) \cong H^0(M, \mathscr{O})^*$. In any long exact sequence the Euler characteristic is 0, which gives

$$1 - 1 + g' - 2g + g' - 1 + 1 = 0.$$

This shows that $g = g'$, as desired.

Next, we check that the statement for ξ is equivalent to the statement for $\xi \otimes \xi_p$. We have a short exact sequence

$$0 \to \mathscr{O}(\xi) \xrightarrow{\times s_p} \mathscr{O}(\xi \otimes \xi_p) \to \mathscr{S}_p \to 0.$$

The $\mathscr{S}_p$ is a skyscraper sheaf of degree 1 supported at p. The long exact sequence on cohomology is

$$\begin{aligned} 0 \to &H^0(M, \mathscr{O}(\xi)) \to H^0(M, \mathscr{O}(\xi \otimes \xi_p)) \to H^0(M, \mathscr{S}_p) \\ \to &H^1(M, \mathscr{O}(\xi)) \to H^1(M, \mathscr{O}(\xi \otimes \xi_p)) \to 0 \end{aligned}$$

since the higher cohomology of skyscraper sheaves vanishes. The vanishing of the Euler characteristic implies that

$$\chi(\xi) - \chi(\xi \xi_p) + 1 = 0.$$

We also have that $c(\xi)-c(\xi\xi_p)=1$, so indeed the formula changes compatibly.

Finally, we have to check that every line bundle is of the form ξ_λ.

$$0 \to \mathscr{O}(\xi) \xrightarrow{s_p^n} \mathscr{O}(\xi\xi_p^n) \to \mathscr{S}_{p,n} \to 0.$$

From the long exact sequence we find that

$$\dim H^0(M,\mathscr{O}(\xi\xi_p^n)) = n + \dim H^0(M,\mathscr{O}(\xi)) - \dim H^1(M,\mathscr{O}(\xi)) + \dim H^1(M,\mathscr{O}(\xi\xi_p^n)).$$

We don't know what the last term is, but it's at least non-negative. So for large enough n, the left hand side is positive. That means that there is an $s \in H^0(M,\mathscr{O}(\xi\xi_p^n))$. So

$$(s) = \sum n_j p_j.$$

Then $(\xi \otimes \xi_p^n)\xi_{p_1}^{-n_1} \ldots \xi_{p_k}^{-n_k}$ has the section $s s_{p_1}^{-n_1} \ldots s_{p_k}^{-n_k}$ whose divisor is empty. But then the bundle is necessarily trivial, i.e.

$$\xi \otimes \xi_p^n \cong \xi_{p_1}^{-n_1} \ldots \xi_{p_k}^{-n_k}.$$

□

Remark 2.6.3. Let L be an elliptic operator. Then $L\colon C^\infty(M,E) \to C^\infty(M,F)$ is *Fredholm*. Actually, C^∞ spaces are terrible to work with, because they are only Frechet. It is better to work with $L^2(M,E)$. However, L is not defined on $L^2C^\infty(M,E)$, since an L^2-function need not be derivative (or square-integrable even if it is), so one has to regard L as an *unbounded operator*, meaning that it's only defined on a Sobolev subspace. By Fredholm theory, the kernel is finite-dimensional, so it is interesting to consider

$$\operatorname{ind} L = \dim\ker L - \dim\ker L^*.$$

The basic idea is that this is so stable, you can wiggle the space around topologically and it should stay the same. Therefore, it should be expressible topologically. The original proof worked by showing that the formula was bordism invariant, and so one could replace M by simpler manifolds.

2.7. **What have we achieved?**

(1) We saw that

$$\begin{aligned} H^1(M,\mathscr{M}^*)=0 &\iff \text{ every } \xi = \xi_{p_1}^{n_1}\ldots\xi_{p_k}^{n_k} \\ &\iff H^1(M,\mathscr{O}^*) =: \overline{\mathscr{A}}(M) = H^0(M,\mathscr{D})/H^0(M,\mathscr{M}^*) \end{aligned}$$

(2) A special case of Riemann-Roch is that $\chi(K) = c(K)+1-g$. But $\chi(K) = g-1$, so $c(K) = 2g-2$. This has the following nice consequence. An extremely important class of objects on any Riemann surface is the quadratic differentials, which comprise the line bundle $\mathscr{O}(K^2)$. Then we know that $\deg K^2 = 4g-4$ (and we can usually predict its dimension).

(3) We have a short exact sequence

$$0 \to H^1(M,\mathscr{O})/H^1(M,\mathbb{Z}) \to H^1(M,\mathscr{O}^*) \to H^2(M,\mathbb{Z}) \to 0$$

which gives a picture of holomorphic line bundles as being parametrized by a discrete invariant (corresponding to the underlying topological structure) and a space with the structure of a complex torus.

2.8. **Flat connections.** Can we find a "best" Hermitian metric on ξ?

The claim is that we can, and it is the one for which the associated Chern connection is flat. Let's recall this story.

For any hermitian metric H, we have a *Chern connection*

$$\nabla^H : C^\infty(M,\xi) \to C^\infty(M, \xi \otimes T^*M).$$

Then

$$(\nabla^H)^2 : C^\infty(M,\xi) \to C^\infty(M, \xi \otimes (T^*M)^2)$$

turns out quite remarkably to be an *endomorphism* (i.e. C^∞-linear). This is the *curvature* $F^H \in C^\infty(M, \mathrm{End}(\xi, \xi \otimes \wedge^2 T^*M))$.

Theorem 2.8.1. *Given ξ, there is a unique H such that*

$$F^H = i\lambda\omega$$

where ω is the Kähler form on M, and

$$\frac{1}{2\pi i}\int \mathrm{Tr}\, F^H = c(\xi).$$

This is called a "Hermitian-Einstein Yang-Mills equation".

Remark 2.8.2. Since $F^H \in \mathrm{End}(\xi, \xi \otimes \wedge^2 T^*M)$ it has a trace in $\wedge^2 TM$. For line bundles, this turns out to simply be $\partial\overline{\partial} h$.

First, let's review the definition of the Chern connection ∇^H. Fix the hermitian metric H.

Definition 2.8.3. We say that a ∇ is *compatible with H* if

$$dH(\sigma,\tau) = H(\nabla\sigma,\tau) + H(\sigma,\nabla\tau).$$

The splitting $T^*M = T^{1,0}M \oplus T^{0,1}M$ induces $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ via

$$\nabla : C^\infty(M,\xi) \to C^\infty(M,\xi\otimes T^*M) \cong C^\infty(M,\xi\otimes T^{1,0}M) \oplus C^\infty(M,\xi\otimes T^{0,1}M)$$

Since ξ is holomorphic, if s is any section then $\overline{\partial} s \in C^\infty(M, \xi \otimes T^{0,1})$. The point is that if $s = f_\alpha s_\alpha = f_\beta s_\beta$, then $\overline{\partial} s = (\overline{\partial} f_\alpha) s_\beta = (\overline{\partial} f_\beta) s_\beta = (\overline{\partial} f_\alpha)\varphi_{\alpha\beta} s_\beta$.

Definition 2.8.4. We say that ∇^H is *compatible with the holomorphic strcture* if $\nabla^{0,1} = \overline{\partial}$.

Proposition 2.8.5. *Fixing H, there exists a unique connection ∇ which is compatible with H and the holomorphic structure.*

Proof. Since the uniqueness lets us patch, it suffices to work in a local holomorphic chart. For a holomorphic function f, we have in local coordinates

$$\nabla = d + A, \quad A \in C^\infty(M, T^*M).$$

Then $\nabla s = ds + As$. This means that if $s = f s_\alpha$, then $\nabla s = (df + Af)s_\alpha$. Now what are the constraints? We have

$$\begin{aligned} dH(f_1 s_\alpha, f_2 s_\alpha) &= d(h_\alpha f_1 \overline{f_2}) \\ &= dh_\alpha + f_1 \overline{f_2} + h_\alpha(\overline{f_2} df_1 + f_1 d\overline{f}_2) \end{aligned}$$

On the other hand, the compatibility condition with H forces

$$dH(f_1 s_\alpha, f_2 s_\alpha) = H((df_1 + Af_1)s_\alpha, f_2 s_\alpha) + H(f_1 s_\alpha, (df_2 + f_2 A)s_\alpha).$$

Equating these boils down to the equation

$$dh_\alpha = Ah_\alpha + \overline{A}h_\alpha.$$

We also have compatibility with the holomorphic structure:

$$\nabla = d + A = (\partial + A^{1,0}) + (\overline{\partial} + A^{0,1}) \implies A^{0,1} = 0.$$

Thus $\partial h_\alpha = Ah_\alpha$, so $A = (\partial h_\alpha)h_\alpha^{-1} = \partial \log h_\alpha$. (It was important to work with holomorphic frame instead of unitary; unitary would kill the h_α but the ∂s_α would be nontrivial). Thus, $\nabla^H = d + (\partial \log h_\alpha)$. □

Example 2.8.6. What's the curvature of the Chern form?

$$\begin{aligned} \nabla^H \circ \nabla^H &= (\partial + \partial \log h_\alpha + \overline{\partial})(\partial + \partial \log h_\alpha + \overline{\partial}) \\ &= \overline{\partial}(\partial \log h_\alpha) + \partial \log h_\alpha \wedge \overline{\partial} \\ &= \overline{\partial}\partial \log h_\alpha. \end{aligned}$$

Theorem 2.8.7. *Given H and ξ, there exists $K_\mu = e^\mu H$ such that $F^{K_\mu} = i\lambda\omega$.*

Proof. Just write down the equations we need. We have $\log k_\alpha = \log h_\alpha + \mu$. Then

$$\partial\overline{\partial} \log k_\mu = \partial\overline{\partial} \log h_\mu + \partial\overline{\partial}\mu.$$

Now fix the metric, and contract with ω. Write $\Lambda\gamma = \langle \gamma, \omega \rangle$. The contraction of the metric against $\partial\overline{\partial}\mu$ is the *Laplacian*, so we find that

$$\Lambda F^{K_\mu} = \Lambda F^H + \Delta\mu.$$

Integrating, we find that

$$\int \Delta\mu + \Lambda F^H = \int \Lambda F^{K_\mu} = c(\xi) 2\pi i.$$

We want to choose μ so that $\Lambda F^{K_\mu} = \Lambda(i\lambda\omega) = i\lambda$ is constant, which amounts to solving

$$\Delta\mu = -f + \lambda.$$

The obstruction to a solution to this Laplace equation is that the right hand side has integral 0. So we need to set $\lambda = \int f / \int 1$. □

There is an alternate point of view on the preceding discussion. We adopted a fixed metric and holomorphic structure as our starting point, and thought of the connection as the variable. We could instead fix the the hermitian metric H and the *connection A*, and considering varying the holomorphic structure.

To dive into the details, our condition for A to be compatible with the holomorphic structure was ξ was that if

$$\nabla^A = (\nabla^A)^{1,0} + (\nabla^A)^{0,1}$$

then $(\nabla^A)^{0,1} = \overline{\partial}$. Alternatively, we could view A as a connection on a *smooth* vector bundle ξ, and *define* $(\nabla^A)^{0,1}$ to be be $\overline{\partial}$. One obstructionto this is that

$$\overline{\partial} \circ \overline{\partial} = ((\nabla^A)^{0,1})^2 = 0.$$

Note that this is always satisfied on a Riemann surface, but is a genuine obstruction in higher dimensions. In order that $(\nabla^A)^{0,1}$ define a holomorphic structure, we need that there exist $s \neq 0$ solving $(\nabla^A)^{0,1} s = 0$. This is a differential equation. Locally we can write $\nabla^A = d + A$, where A is a complex-valued one-form. In this case

$$(\nabla^A)^{0,1} = \overline{\partial} + A^{0,1}$$

and we can reformulate our equation as

$$\overline{\partial} \log s = -A^{0,1}.$$

2.9. **The gauge group.** Now what if we let the gauge group act? Fix H and A.

Definition 2.9.1. The *gauge group* (for unitary line bundles) is

$$\mathscr{G} := C^\infty(M, \mathrm{U}(1)).$$

The *complexified gauge group* (for line bundles) is

$$\mathscr{G}^c := C^\infty(M, \mathbb{C}^*).$$

In general, a gauge group is the group of automorphisms of a principal G-bundles $P \to M$ (lying over the identity in M). In particular, a $\mathrm{GL}(n)$ bundle (resp. $\mathrm{U}(n)$ bundle) is equivalent to a vector bundles (resp. unitary vector bundle), by using the same transition functions. The automorphism bundle $\mathrm{Aut}(P)$ is also a G-bundle over M with the same transitions acting by conjugation, rather than translation.

For line bundles elements of the gauge group are just function, but for more complicated bundles they are sections of nontrivial bundles. Explicitly, for $s \in C^\infty(M, \xi)$ and $g \in \mathscr{G}$, $g \cdot s$ is the section of $C^\infty(M)$ which takes the value $g(p)s(p)$ at p.

In particular, the gauge group acts on connections. For $g \in \mathscr{G}$, we define $g \cdot \nabla^A$ by the diagram

$$\begin{tikzcd}
C^\infty(M,\xi) \arrow[r, "\nabla^A"] \arrow[d, "g"'] & C^\infty(M,\xi\otimes T^*) \arrow[d, "g"'] \\
C^\infty(M,\xi) \arrow[r, "\nabla^{g\cdot A}"'] & C^\infty(M,\xi\otimes T^*)
\end{tikzcd}$$

In other words,

$$(g \cdot \nabla^A)s = (g \circ \nabla^A \circ g^{-1})s = \nabla^A s - dg \otimes g^{-1} s.$$

Let's see what this comes out in local coordinates. In a local frame, we can write $\nabla^A = d + A_\alpha$, so that

$$\nabla^A(f s_\alpha) = (df + A_\alpha f) s_\alpha.$$

Then we have in general

$$\nabla^{g\cdot A} = d + gA_\alpha g^{-1} - dg \otimes g^{-1}$$

but in the abelian case (e.g. for line bundles) we have $gA_\alpha g^{-1} = A_\alpha$, so we can write

$$g \cdot A = A - d(\log g).$$

The action of the *complexified* gauge group is different. To express it pick a local holomorphic frame s_α. By our assumption of compatibility with the holomorphic structure,

$$A = A^{1,0} \implies \nabla^A = A^{1,0} + \overline{\partial}.$$

Then the action of $g \in \mathscr{G}^c$ is

$$g \cdot \overline{\partial} = g \circ \overline{\partial} \circ g^{-1}$$

or alternatively

$$g \cdot \overline{\partial} = \overline{\partial} - (\overline{\partial} g \cdot g^{-1}).$$

This defines an action of $\mathscr{G}^c$ on the space of holomorphic line structures, because the square of $g \circ \overline{\partial} \circ g^{-1}$ is still zero!

Remark 2.9.2. Why the difference between these two cases? For $g \in \mathscr{G}$, the action on connection is $A \mapsto gAg^{-1} - dg \cdot g^{-1} =: A_g$. You can check that if A was unitary then A_g is still unitary, essentially because the logarithm maps unitary matrices to their Lie algebra.

On the other hand, if $g \in \mathscr{G}^c$ is not unitary then A_g is not unitary. So the reason for modifying the action is to ensure that its acts on unitary matrices.

What is the associated action on *unitary* connections? By definition

$$A^{0,1} \mapsto A^{0,1} - \overline{\partial} g \cdot g^{-1}$$

but what about $A^{1,0}$? You can compute the action by considering the compatibility condition

$$\begin{aligned} dH(fs_\alpha, ks_\alpha) &= d(fkH_\alpha) \\ &= H((df + A_\alpha f)s_\alpha, (dk + A_\alpha k)s_\alpha) \\ &= H_\alpha \langle df + A_\alpha f, dk + A_\alpha k\rangle \end{aligned}$$

Exercise 2.9.3. Complete the computation.

After some work you find an answer of

$$A^{1,0} \mapsto g^* \circ \partial_A \circ (g^*)^{-1} = \overline{g} \circ \partial_A \circ \overline{g}^{-1}$$

(here g^* is the conjugate transpose). So

$$A_g^{1,0} = \overline{g} A^{1,0} \overline{g}^{-1} - (\partial \overline{g}) \cdot \overline{g}^{-1} = A^{1,0} - (\partial \overline{g}) \cdot \overline{g}^{-1}.$$

The conclusion is that $\mathscr{G}^c$ acts on $\mathscr{A}$, the space of unitary connections.

In general, the curvature of A will be non-zero:

$$F^A = \nabla^A \circ \nabla^A \neq 0.$$

(Suppose for simplicity that $\deg \xi = 0$, so it is reasonable to look for flat connections.) Can we modify it by some $g \in \mathscr{G}^c$ such that $F^{A_g} = 0$? (Why do we have to pass to the complexified Gauge group? The action of the real Gauge group on the curvature is by conjugation, $F^{A_g} = gF^a g^{-1} = F^A$, and in this case of line bundles is therefore trivial.) You can compute that for $g \in \mathscr{G}^c$,

$$F^{A_g} = F_a - \overline{\partial}_A \overline{\partial}_A \log(|g|^2)$$

(in higher rank $|g|^2$ is replaced by gg^*). Now this is essentially the Laplace equation, so we can solve it to find some g that works.

2.10. **Summary.** We showed that there are correspondences

$$\begin{tikzcd} & \mathrm{Pic}^0(M) = H^1(M,\mathscr{O}^*) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{Hom}(\pi_1(M),\mathrm{U}(1)) \arrow[rr, dash] & & \{\text{flat unitary connections}\}. \end{tikzcd}$$

(Normally we would have to consider homomorphisms $\pi_1(M) \to \mathrm{U}(1)$ up to conjugacy in the bottom left, but no need in the abelian case.) Now, $\mathrm{Pic}^0(M)$ is the space of holomorphic line bundles of degree 0. The space of the flat unitary connections can also be thought of as

$$\mathrm{Hom}(\Gamma, U(1)) = H^1(M,S^1) = H^1(M,\mathbb{R})/H^1(M,\mathbb{Z}).$$

Remark 2.10.1. To get a correspondence beyond degree 0, we should replace by "Hermitian Einstein Yang-Mills".

The map from a line bundle of degree 0 to a flat unitary connection is by solving the differential equation discussed above. Conversely, given a flat unitary connection on a smooth line bundle one can find a holomorphic structure by take the $0,1$ part and showing that it has enough holomorphic sections.

To go from a flat unitary connection to a representation of the fundamental group, we want to solve a differential equation

$$\nabla^H s = 0.$$

We get local solutions, but they may not be monodromy invariant. However, they do patch to a well-defined solution over the universal cover of M. That is, given ∇^H we find a nonzero *parallel* section s which is locally constant. This extends globally on $\widetilde{M} = \mathbb{H}^2$ to a nonvanishing section, so $\pi^*\xi$ is holomorphically trivial. So we have an action of Γ on $\mathbb{H}^2 \times \mathbb{C}$,

$$(z,v) \mapsto (\gamma z, \rho(\gamma)v).$$

This furnishes a representation of the fundamental group on $\mathbb{C}$, which is unitary because ∇^H is.

To finish off, we do one last computation with short exact sequence of sheaves. Recall that $H^1(M,\mathscr{O}^*) = H^1(M,\mathscr{O})/H^1(M,\mathbb{Z})$, coming from

$$0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0.$$

This presents Pic^0 as $\mathbb{C}^g/\Lambda_{2g}$.

We also have a short exact sequence

$$0 \to \mathbb{C} \to \mathscr{O} \to \mathscr{O}(K) \to 0.$$

From the associated long exact sequence, we have

$$0 \to \Gamma(M, \mathscr{O}(K)) \to H^1(M, \mathbb{C}) \to H^1(M, \mathscr{O}) \to 0.$$

This is compatible with what we got above:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(M, \mathscr{O}(K)) \arrow[r] & H^1(M,\mathbb{C}) \arrow[r] & H^1(M,\mathscr{O}) \arrow[r] & 0 \\
 & & H^1(M,\mathbb{Z}) \arrow[u, hook] \arrow[r] & H^1(M,\mathscr{O}) \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

Therefore,

$$H^1(M, \mathscr{O}) = H^1(M, \mathbb{C})/\delta\Gamma(M, \mathscr{O}(K)).$$

This gives

$$H^1(M, \mathscr{O}^*) = H^1(M, \mathscr{O})/H^1(M, \mathbb{Z}) = H^1(M, \mathbb{C})/(\delta H^1(M, \mathscr{O}(K)) + H^1(M, \mathbb{Z})).$$

What is the map δ? Take $\gamma \in \Gamma(M, \mathscr{O}(K))$. Locally $\gamma_\alpha = f_\alpha dz = dk_\alpha$. We know that $d(k_\alpha - k_\beta) = 0$, so $k_\alpha - k_\beta \in \mathbb{C}$. That define a one-cycle valued in $\mathbb{C}$, which represents $\delta\gamma$.

Finally, we have a short exact sequence

$$0 \to \mathbb{C}^* \to \mathscr{O}^* \to \mathscr{O}(K) \to 0.$$

The right map is $f \mapsto \frac{1}{2\pi i} d \log f$. This induces

$$0 \to \Gamma(M, \mathscr{O}(M)) \to H^1(M, \mathbb{C}^*) \to H^1(M, \mathscr{O}^*) \to \mathbb{Z} \to 0.$$

Then $\mathrm{Pic}^0 = H^1(M, \mathbb{C}^*)/\Gamma(M, \mathscr{O}(K))$, which can be thought of as the space of flat bundles modulo the natural equivalence relation of flat bundles.

3. Holomorphic vector bundles

3.1. **Definitions.** Let $E \to M$ be a holomorphic vector bundle. Let (U_α, s_α) be a trivializing cover. On $U_\alpha \cap U_\beta$, we have $s_\alpha = \varphi_{\alpha\beta} s_\beta$, for some holomorphic maps

$$\varphi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{C}).$$

Remark 3.1.1. If we fix a trivialization of $\det E := \wedge^n E$ then the structure group might instead be $\mathrm{SL}(n, \mathbb{C})$.

We can generalize the usual multilinear algebra operations: $E, F \rightsquigarrow E \oplus F, E \otimes F$, etc. Given

$$0 \to E' \to F \to E'' \to 0$$

the sheaves of sections $\mathscr{E} := \mathscr{O}(E)$, etc. fit into

$$0 \to \mathscr{E}' \to \mathscr{F} \to \mathscr{E}'' \to 0.$$

What is different from the C^∞ situation? In the smooth case, any short exact sequence has a splitting. Why? You can put a hermitian metric on F, and take an orthogonal complement. This defines a vector subbundle of F that maps isomorphically to E''. This is a very nonholomorphic construction, so it is not surprising that short exact sequences of holomorphic vector bundles don't split. To what extent can we measure this nonsplitting?

Assume $n=2$, and we have

$$0 \to E' \to F \to E'' \to 0$$

where E'' and E' are line bundles, inducing

$$0 \to \mathscr{E}' \to \mathscr{F} \to \mathscr{E}'' \to 0.$$

This is called an *extension of E'' by E'*.

Theorem 3.1.2. *The space of extensions of E'' by E' modulo equivalence is isomorphic to $H^1(M, \mathscr{O}(E') \otimes (E'')^{-1})$.*

Note: $H^1(M, \mathscr{O}(\xi_1 \xi_2^{-1})) = H^0(M, \mathscr{O}(K\xi_1^{-1}\xi_2))^*$. This vanishes if $\deg K\xi_1^{-1}\xi_2 < 0$, or

$$2g-2-c_1(\xi_1)+c_1(\xi_2) < 0.$$

Proof. Take transition functions $\varphi_{j\alpha\beta}$ for ξ_j, $j=1,2$. We have

$$0 \to \xi_1 \to E \to \xi_2 \to 0.$$

The transition functions of E are of the form

$$\Phi_{\alpha\beta} = \begin{pmatrix} \varphi_{1\alpha\beta} & \lambda_{\alpha\beta} \\ & \varphi_{2\alpha\beta} \end{pmatrix}.$$

The $\lambda_{\alpha\beta}$ are measuring the nontriviality of the extension.

What is equivalence?

$$\begin{tikzcd}
0 \arrow[r] & \xi_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\Theta"'] & \xi_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \xi_1 \arrow[r] & E' \arrow[r] & \xi_2' \arrow[r] & 0
\end{tikzcd}$$

The equivalence is exactly that

$$\Phi'_{\alpha\beta}\Theta_\beta = \Theta_\alpha \Phi_{\alpha\beta}.$$

If $\Theta_\alpha = 1$ on ξ_1 and 1 on ξ_2, then

$$\Theta_\alpha = \begin{pmatrix} 1 & h_\alpha \\ & 1 \end{pmatrix}.$$

What we need is then

$$\varphi_{1\alpha\beta} h_\beta + \lambda'_{\alpha\beta} = \lambda_{\alpha\beta} + h_\alpha \varphi_{2\alpha\beta}.$$

$\square$