REDUCTIVE GROUPS OVER FIELDS

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These are lecture notes that Tony Feng live-TEXed from a course given by Brian Conrad at Stanford in"winter" 2015, which Feng and Conrad edited afterwards. Two substitute lectures were delivered (by Akshay Venkatesh and Zhiwei Yun) when Conrad was out of town. This is a sequel to a previous course on the general structure of linear algebraic groups; some loose ends from that course (e.g., Chevalley's self-normalizing theorem for parabolic subgroups and Grothendieck's theorem on geometrically maximal tori in the special case of *finite* ground fields) are addressed early on.

The main novelty of the approach in this course to avoid the two-step process of first developing the structure theory of reductive groups over algebraically closed fields and then using that to establish the refined version over general fields. Instead, systematic use of dynamic techniques introduced in the previous course (and reviewed here) make it possible to directly establish the general structure theory over arbitrary fields in one fell swoop (building on the "geometric" theory of general linear algebraic groups from the previous course, where reductivity was not the main focus).

This document sometimes undergoes minor updates to make corrections (typos, etc.) or clarifications. Please send errata/comments to `conrad@math.stanford.edu`.

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1. BASIC STRUCTURE OF REDUCTIVE GROUPS

1.1. **Linear algebraic groups.** Let's review some notions from the previous course.

Definition 1.1.1. For a field k, a *linear algebraic group* over k is a smooth affine k-group scheme (equivalently, a smooth closed k-subgroup of GL_n).

Remark 1.1.2. We allow linear algebraic groups to be disconnected. However, the identity component G^0 is *geometrically* connected over k. This follows from a general fact (an instructive exercise) that if X is finite type over k and $X(k) \neq \emptyset$, then X is connected if and only if X is geometrically connected.

One uses this fact all the time when calculating with geometric points of normalizers, centralizers, etc. to ensure that one does not lose contact with connectedness.

1.2. **Reductive groups.** Recall the brute-force definition of reductivity:

Definition 1.2.1. A *reductive k-group* is a linear algebraic group G over k whose geometric unipotent radical (maximal unipotent normal smooth connected $\overline{k}$-subgroup) $\mathscr{R}_u(G_{\overline{k}})$ is trivial.

Example 1.2.2. Many classical groups are reductive (verified in lecture or homework of the previous course, but some to be revisited from scratch in this course):

- $\mathrm{SO}(q)$ for finite-dimensional quadratic spaces (V, q) with $q \neq 0$ that are non-degenerate. (Non-degeneracy is defined by smoothness of the zero-scheme of the projective quadric $(q = 0)$. This works uniformly in all characteristics.)
- $\mathrm{U}(h)$ and $\mathrm{SU}(h)$ for non-degenerate finite-dimensional hermitian spaces (V', h) with respect to quadratic Galois extensions k'/k.
- $\mathrm{Sp}(\psi)$ for a non-degenerate finite-dimensional symplectic space (V, ψ).
- $\underline{A}^\times$ for A a finite-dimensional central simple algebra over k, representing the functor $R \rightsquigarrow (A \otimes_k R)^\times$ on (commutative) k-algebras.

Remark 1.2.3. We shall use throughout Grothendieck's fundamental theorem, proved in the previous course, that maximal k-tori in a linear algebraic group G over k are geometrically maximal (i.e., remain maximal over $\overline{k}$, or equivalently after any field extension). In particular, all such tori have the same dimension, called the *rank* of G. Our proof of Grothendieck's theorem applied to infinite k; the handout on Lang's theorem (and dynamic methods) takes care of the case of finite fields, so Grothendieck's theorem is thereby established in general.

The following was a major result near the end of the previous course, to be used a lot in this course.

Theorem 1.2.4. *If G is connected reductive and split (i.e. has a split maximal k-torus) and has rank 1 then $G = \mathbf{G}_m, \mathrm{SL}_2$, or PGL_2 as k-groups.*

There are a lot more properties of reductive groups that we would like to investigate (some to be addressed in handouts of this course), such as the following.

- If $G \twoheadrightarrow G'$ is a surjective homomorphism of linear algebraic k-groups then we would like to show $\mathscr{R}_u(G_{\overline{k}}) \twoheadrightarrow \mathscr{R}_u(G'_{\overline{k}})$, so G' is reductive if G is.

- If char $k = 0$ and G is reductive, then all linear representations of G on finite-dimensional k-vector spaces are completely reducible. (The converse is a consequence of the Lie-Kolchin theorem, but in characteristic $p > 0$ only applies to special G such as tori and finite étale groups of order not divisible by p.)
- Structural properties of connected reductive k-groups G, such as:
  - for locally compact k, relate compactness of $G(k)$ to properties of G as an algebraic group. For instance if G contains $\mathbf{G}_m = \mathrm{GL}_1$ then $G(k)$ is not compact since its closed subgroup $k^\times$ is non-compact; we want to show that this is the only way compactness fails.
  - for general k, prove $G(k)$-conjugacy for maximal split k-tori and minimal parabolic k-subgroups. Build a "relative root system".
  - use root systems and root data to analyze the k-subgroup structure of G (e.g., structure of parabolic k-subgroups) and the subgroup structure of $G(k)$ (e.g., simplicity results for $G(k)/Z_G(k)$, at least for k-split G).
  - for $k = \mathbf{R}$, understand $\pi_0(G(\mathbf{R}))$ and prove that if G is semisimple and simply connected in an algebraic sense then $G(\mathbf{R})$ is connected.

1.3. **Chevalley's Theorem.** Recall that a smooth closed k-subgroup $P \subset G$ is *parabolic* if the quasi-projective coset space G/P is k-proper, or equivalently projective. (This can be checked over $\overline{k}$.) By the Borel fixed point theorem, which says that a solvable connected linear algebraic k-group acting on a proper k-scheme has a fixed point over $\overline{k}$, $P_{\overline{k}}$ contains a Borel subgroup (it is a simple group theory exercise to show that $P_{\overline{k}}$ contains a $G(\overline{k})$-conjugate of a Borel $B \subset G_{\overline{k}}$ if B acting on $(G/P)_{\overline{k}}$ has a fixed point).

Here is the key result which enabled Chevalley to get his structure theory over algebraically closed fields off the ground (as Chevalley put it, once the following was proved "the rest follows by analytic continuation"):

Theorem 1.3.1 (Chevalley)**.** *Let G be a connected linear algebraic k-group and $P \subset G$ a parabolic k-subgroup. Then P is connected and $N_{G(K)}P_K = P(K)$ for any extension K/k.*

Remark 1.3.2. One could ask if $P = N_G(P)$, the scheme-theoretic normalizer. [See HW3, Exercise 3 of the previous course for the notion and existence of scheme-theoretic normalizers.] The answer is yes, but the proof uses a dynamic description of P. We'll address this in Corollary 6.3.12.

To prove Chevalley's Theorem (stated without proof in the previous course but used crucially there, such as to prove Theorem 1.2.4!), first we want to pass to an algebraically closed field. For connectedness it is harmless to do this; what about the normalizer property? Note that

$$N_{G(K)}(P_K) = N_{G(\overline{K})}(P_{\overline{K}}) \cap G(K)$$

so if $N_{G(\overline{K})}(P_{\overline{K}}) = P(\overline{K})$ then the right side is $P(K)$. Thus, without loss of generality we may assume that $K = k = \overline{k}$.

Next note that the normalizer property implies the connectedness. Indeed, if P contains a Borel, then P^0 contains a Borel (by definition Borel subgroups are connected), so P^0 is parabolic. Therefore, *if* the normalizer property is proved in general, then we can apply this to P^0. Any group normalizes its own identity component, so we would immediately get that P is connected.

We claim that it suffices to show that $N_{G(k)}(B)=B(k)$ for one Borel subgroup B. Any two such B are $G(k)$-conjugate, so it is the same for that to hold for all Borel subgroups. Grant this equality. For general P, choose $B \subset P$. Consider $n \in N_{G(k)}(P)$; we want to show that $n \in P$. Consider the conjugation action of n on B. The element n doesn't necessarily conjugate B into itself, but for our purposes it is harmless to change n by $P(k)$-translation. Note that nBn^{-1} is a Borel subgroup of P. But any two Borel subgroups of any linear algebraic group are conjugate, so $nBn^{-1} = pBp^{-1}$ for some $p \in P(k)$. This implies that $p^{-1}n \in N_{G(k)}(B) = B(k) \subset P(k)$, so $n \in P(k)$.

Now we focus on the assertion $N_{G(k)}(B) = B(k)$. We proceed by induction on $\dim G$. If G is solvable then the result is easy ($G = B$), so the case $\dim G \leq 2$ is fine. We seek lower-dimensional subgroups that interact well with B, but also contain the element in question; there's a tension between these two things. (In general B does not necessarily intersect an arbitrary smooth connected subgroup H in a Borel subgroup of H; we need H to be in "good position" relative to B and to be rather special.)

Choose $n \in N_{G(k)}(B)$; we want to show that $n \in B(k)$. Fix a maximal torus $T \subset B$; all such choices are $B(k)$-conjugate, so nTn^{-1} is $B(k)$-conjugate to T. Replacing n with a suitable $B(k)$-translate, we may assume without loss of generality that n normalizes T as well, so $n \in N_{G(k)}(T)$ too. Now we would like to find a lower-dimensional subgroup or quotient of G into which n fits, so that we can apply induction. We'll do this by studying

$$H := Z_T(n)_{\mathrm{red}} = \{t \in T \mid ntn^{-1} = t\},$$

the reduced scheme structure on the centralizer of n in T. Then $S := H^0 \subset T$ is a torus in B centralized by n. (The general principle is that if you have a central subgroup, then you pass to the quotient; if it's not central, then you pass to the centralizer.) There are now three cases:

(1) $S \subset G$ is non-trivial and central,
(2) $S \subset G$ is non-trivial and non-central,
(3) S is trivial.

We analyze each in turn.

(1) Then $B/S \subset G/S$ is of lower dimension, so we can apply induction to conclude that $\overline{n} \in B/S$, so $n \in B$.

(2) The centralizer of S contains n and is lower-dimensional, *and is connected* (as we shall review in §1.4). But why does this interact well with Borel subgroups? First, $Z_G(S)$ is a (smooth) connected lower-dimensional subgroup - the smoothness and connectedness are nontrivial properties proved in the previous course (HW8 Exercise 4(i) and Theorem 24.2.5). If $B \cap Z_G(S) = Z_B(S)$ (which is smooth connected and solvable) is a *Borel* subgroup of $Z_G(S)$, then again we would win by dimension induction.

The problem here is that one doesn't necessarily know that $Z_B(S)$ is a Borel subgroup of $Z_G(S)$. This would follow from $Z_G(S)/Z_B(S)$ being proper, but is that true? There is certainly a map

$$Z_G(S)/Z_B(S) \to (G/B)^S$$

which is at least injective on points, and $(G/B)^S$ is complete, being a closed subset of G/B. We want to show that the above map is in fact a closed embedding.

Proposition 1.3.3. *For a smooth subgroup $H \subset G$ containing a k-torus S, the natural map $Z_G(S)/Z_H(S) \to (G/H)^S$ is an isomorphism onto the connected component of the identity point in the target.*

This is proved in a handout on torus centralizers.

Remark 1.3.4. It is essential that S is a torus since a key point is that the representation theory of S is completely reducible. The proof eventually reduces to showing that $Z_G(S) \to (G/H)^S$ is smooth at the identity, and upon passing to tangent spaces at identity points we need that the invariants in a quotient vector space are images of invariants in the original vector space. In general the geometry of $(G/H)^S$ is completely *non*-homogeneous (e.g., components with varying dimensions, some affine and some proper, etc.); examples are given in the handout.

(3) This is the most difficult case. We'll show that in this case G is actually equal to B (i.e. is solvable). Let $N = N_G(B)$.

Lemma 1.3.5. *If S is trivial then $T \subset \mathscr{D}(N)$.*

Proof. Consider the map $T \to T$ given by $t \mapsto ntn^{-1}t^{-1} = (ntn^{-1})t^{-1}$. This is a product of homomorphisms (as T is commutative), hence a homomorphism. Its image is contained in $\mathscr{D}(N)$, and it has finite kernel by the triviality of S, so it is surjective for dimension reasons. □

Now pick a representation $\rho\colon G \to \mathrm{GL}(V)$ such that $N = \mathrm{Stab}_G(L)$ for some line $L \subset V$. The N-action on L is via some character $\chi\colon N \to \mathbf{G}_m$. Necessarily χ kills $\mathscr{D}(N)$, hence T. It also kills $\mathscr{R}_u(B)$, because unipotent groups don't have non-trivial characters. But B is solvable, and over an algebraically closed field a connected solvable group is the semidirect product of any maximal torus and its unipotent radical, so χ even kills B, i.e. B acts trivially on L. Now we're basically done: picking any non-zero $v \in L$, the orbit map on v induces a map $G/B \to V$. But G/B is proper and V is affine, so this orbit map is *constant*, so $N = \mathrm{Stab}_G(L) = G$. So $B \triangleleft G$, which implies that G/B is affine. However it's also proper, hence equal to a point, so $G = B$. □

1.4. **Connectedness of torus centralizers.**

By HW7 Exercise 4 of the previous course, if Y is a smooth (separated) k-scheme with the action of a k-torus T, then the functor

$$R \rightsquigarrow \underline{Y}^T(R) = \{y \in Y(R) \mid {}_{\text{is constant map to } y}^{t \mapsto t\cdot y\colon T_R \to Y_R}\}$$

is represented by a closed subscheme $Y^T \subset Y$ (if $k = k_s$ then $Y^T = \bigcap_{t \in T(k)} \mathrm{Stab}_Y(t)$, and in general one bootstraps from this via Galois descent).

Remark 1.4.1. By infinitesimal methods, in HW8 Exercise 3 of the previous course it was shown that if Y is smooth then Y^T is smooth. The main ingredient in the proof is the complete reducibility for T-representations, so in characteristic 0 (see Proposition 5.3.1) it works for any reductive group in place of T.

Example 1.4.2. If G is a linear algebraic group over k, and $T \subset G$ is a k-torus acting by conjugation, then $Z_G(T) \subset G$ is smooth.

In the proof of Chevalley's Theorem 1.3.1 concerning the self-normalizing property of parabolics, we used the following fact:

Proposition 1.4.3. *For a connected linear algebraic group G and torus $T \subset G$, $Z_G(T)$ is connected.*

We'll give the proof in this section as an illustration and review of the all-important dynamic method (which was used multiple times in the previous course and will pervade this course). Without loss of generality we may assume that $k = \overline{k}$. Then $T = \mathbf{G}_m^r$ and we can assume that $r \geq 1$ (otherwise the result is trivial). We can write $T = \mathbf{G}_m \times T'$.

A homomorphism from $\mathbf{G}_m$ into a k-group scheme is called a *one-parameter subgroup*. We have a one-parameter subgroup

$$\lambda \colon \mathbf{G}_m \to T \subset G$$

given by $s \mapsto (s, 1)$ and

$$Z_G(T) = Z_{Z_G(T')}(\lambda),$$

so by induction on r, it is enough to study $Z_G(\mu)$ for a one-parameter subgroup $\mu \colon \mathbf{G}_m \to T$ and show that it is connected.

Review of the dynamic method. Let G be an affine k-group of finite type and $\lambda \colon \mathbf{G}_m \to G$ a k-homomorphism.

Example 1.4.4. For $G = \mathrm{GL}_4$, an example of a one-parameter subgroup is

$$\lambda(t) = \begin{pmatrix} t^7 & & & \\ & t^7 & & \\ & & t^3 & \\ & & & t^{-2} \end{pmatrix}.$$

Remark 1.4.5. We think of cocharacters additively (especially when valued in a fixed torus), so we define $-\lambda := 1/\lambda$ and write 0 to denote the trivial cocharacter.

There exist closed subschemes $Z_G(\lambda), P_G(\lambda), U_G(\lambda) \subset G$ that represent the functors on k-algebras:

- $\underline{Z}_G(\lambda) \colon R \rightsquigarrow \{g \in G(R) \mid g \text{ commutes with } \lambda_R\}$.
- $\underline{P}_G(\lambda) \colon R \rightsquigarrow \{g \in G(R) \mid \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$. By "limit exists" we mean that the R-scheme map $\mathbf{G}_{m,R} \to G_R$ given by $g \mapsto \lambda(t) g \lambda(t)^{-1}$ factors through $\mathbf{A}^1_R$:

$$\begin{tikzcd} \mathbf{G}_{m,R} \arrow[r, "\lambda(t)g\lambda(t)^{-1}"] \arrow[d] & G_R \\ \mathbf{A}^1_R \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

(Such a factorization is unique if it exists since G_R is affine and $R[t] \to R[t, 1/t]$ is injective.)

- $\underline{U}_G(\lambda)\colon R \to \{g \in G(R) \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} = 1\}$.

Obviously $Z_G(\lambda) \subset U_G(\lambda) \subset P_G(\lambda)$.

Example 1.4.6. In Example 1.4.4,

$$\lambda(t)g\lambda(t)^{-1} = \begin{pmatrix} g_{11} & g_{12} & t^4 g_{13} & t^9 g_{14} \\ g_{21} & g_{22} & t^4 g_{23} & t^9 g_{24} \\ t^{-4} g_{31} & t^{-4} g_{32} & g_{33} & t^5 g_{34} \\ t^{-9} g_{41} & t^{-9} g_{42} & t^{-5} g_{43} & g_{44} \end{pmatrix}$$

So $P_G(\lambda)$ is cut out by the condition that some partial lower-triangular stuff vanishes:

$$P_G(\lambda) = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

Likewise,

$$U_G(\lambda) = \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Observe that a point $g \in U_G(-\lambda) \cap P_G(\lambda)$ valued in a k-algebra R has the property that $\lambda(t)g\lambda(t)^{-1}$ has limits as t goes to both 0 and ∞; i.e. we have a factorization

$$\begin{tikzcd} \mathbf{G}_{m,R} \arrow[r] \arrow[d] & G_R \\ \mathbf{P}^1_R \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

But G_R is affine, so the extended map must be constant. Furthermore, $1 \in \mathbf{G}_{m,R}$ maps to $1 \in G_R$ by the definition of $U_G(-\lambda)$, so $g = 1$ (the identity R-point); i.e., $U_G(-\lambda) \cap P_G(\lambda) = 1$ as k-schemes. This implies that the multiplication map

$$m\colon U_G(-\lambda) \times P_G(\lambda) \to G$$

is monic. On the other hand, it is an isomorphism on tangent spaces at the identity: this was worked out in the previous course. The point is that the Lie algebra decomposes as

$$\mathrm{Lie}(G) = \mathrm{Lie}(G)_{<0} \oplus \mathrm{Lie}(G)_{\geq 0} \text{ for } \mathbf{G}_m\text{-weights.}$$

(A $\mathbf{G}_m$-action on a vector space corresponds to a grading, and in the previous course we saw that the Lie algebra of $U_G(-\lambda)$ is $\mathrm{Lie}(G)_{<0}$ and the Lie algebra of $P_G(\lambda)$ is $\mathrm{Lie}(G)_{\geq 0}$.) *If* we knew that $U_G(-\lambda)$ and $P_G(\lambda)$ were smooth, then it would follow that this map is étale at the identity, hence étale everywhere by the homogeneity coming from the left and right translations by group actions on the source. An étale monomorphism is automatically an open immersion (see [EGA, IV_4, 17.9.1], but it is an instructive exercise to prove this directly for schemes or finite type over a field or for noetherian schemes).

Remark 1.4.7. In general $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ (proved in the previous course without any smoothness hypotheses on G). So we get that

$$m\colon U_G(-\lambda) \times P_G(\lambda) \to G$$

is the same as

$$m\colon U_G(-\lambda) \times Z_G(\lambda) \times U_G(\lambda) \to G$$

However, it turns out that this line of reasoning isn't so easy to push through without knowing smoothness of $U_G(-\lambda)$ and $P_G(\lambda)$; instead we shall show by an entirely different method that $m\colon U_G(-\lambda) \times P_G(\lambda) \to G$ is an open immersion *without* any smoothness hypotheses on G (or on anything), and use that to deduce the smoothness of $U_G(-\lambda)$ and $P_G(\lambda)$ from smoothness of G.

By HW10 of the previous course, for $G = \mathrm{GL}(V)$ we can compute $U_G(\pm\lambda), Z_G(\lambda), P_G(\lambda)$ explicitly over $\overline{k}$ (upon composing with suitable conjugation to make λ valued in the diagonal torus with monotonically decreasing exponents) to see by inspection that they're all smooth. This generalizes the example above with GL_4. Thus, for $G = \mathrm{GL}(V)$ the above argument with tangential and translation considerations shows that the multiplication map

$$U_G(-\lambda) \times P_G(\lambda) \to \mathbf{G}$$

is an open immersion.

Example 1.4.8. Suppose $G = \mathrm{GL}_n$ and

$$\lambda(t) = \begin{pmatrix} t^{e_1} & & \\ & \ddots & \\ & & t^{e_n} \end{pmatrix}$$

For $e_1 > \ldots > e_n$, this gives the usual upper triangular Borel B^+ as $P_G(\lambda)$, and $U_G(-\lambda)$ is the usual lower triangular unipotent subgroup U^-. Thus, $U^- \times B^+ \to \mathrm{GL}_n$ is the "standard open cell".

The general case of the open immersion property for m (without any smoothness hypotheses on G!) is deduced from the settled GL_n in the handout on (Lang's theorem and) dynamic methods as follows. For any subgroup inclusion $j\colon G \hookrightarrow G'$ and the induced one-parameter subgroup $\lambda' = j \circ \lambda$ one does some hard work to prove that inside G' (as subfunctors) we have

$$U_G(-\lambda) \times P_G(\lambda) = G \cap (U_{G'}(-\lambda') \times P_{G'}(\lambda')).$$

Hence, taking $G' = \mathrm{GL}_n$ gives the desired open immersion property in general! This is called the "open cell" in G associated to λ.

Now we *know* that

$$U_G(-\lambda) \times Z_G(\lambda) \times U_G(\lambda) \hookrightarrow G$$

is an open immersion. So if G is smooth then each of its (non-empty) direct factor schemes $Z_G(\lambda), U_G(\lambda), P_G(\lambda)$ are all smooth. Likewise, if G is connected (hence irreducible, since $(G_{\overline{k}})_{\mathrm{red}}$ is smooth and *connected*), so its "open cell" is irreducible and thus connected, then all of these factor schemes are connected too! In particular, we get that $Z_G(\lambda)$ is connected for connected linear algebraic groups G.

Remark 1.4.9. These constructions are functorial. If you have a homomorphism $G \to G'$ then any one-parameter subgroup λ for G induces a one-parameter subgroup λ' for G', and there are natural maps $U_G(\lambda) \to U_{G'}(\lambda')$ and $P_G(\lambda) \to P_{G'}(\lambda')$.

If $G \twoheadrightarrow G'$ is surjective, then between dense open subschemes we have

$$\begin{array}{ccccc} U(-\lambda) & \times & Z(\lambda) & \times & U(\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ U(-\lambda') & \times & Z(\lambda') & \times & U(\lambda') \end{array}$$

is dominant, so each of the factor maps (e.g. $U(-\lambda') \to U(-\lambda)$) is dominant. But we know that a homomorphism of groups has closed image, so $U(-\lambda) \twoheadrightarrow U(-\lambda')$ and likewise for Z, P. Iterating such 1-parameter considerations, much as in our earlier reduction of the connectedness of torus centralizers to the 1-parameter centralizer case, we obtain:

Corollary 1.4.10. *For a surjective k-homomorphism $\pi : G \twoheadrightarrow \overline{G}$ between linear algebraic groups and $S \subset G$ a torus, the natural map $Z_G(S) \to Z_{\overline{G}}(\overline{S})$ is surjective.*

This is amazing for π non-separable or G disconnected. In the separable case it's surjective on Lie algebras, so the result is clear with connectedness. But in the non-separable or disconnected case you can't check on Lie algebras.

2. The unipotent radicals

2.1. **Two important theorems.** Next we'll discuss two big theorems on unipotent radicals. (The omitted details are given in §1 of the handout "Basics of reductive and semisimple groups".)

The definition of reductivity in terms of the unipotent radical is hard to use at first so we want to relate the unipotent radicals to more tangible things.

Suppose G is a connected linear algebraic group over an algebraically closed field k and $T \subset G$ is a maximal torus. Consider

$$I(T) := \Big(\bigcap_{B \supset T} \mathscr{R}_u(B)\Big)^0_{\mathrm{red}}$$

The unipotent radical is contained in a Borel, as it's solvable. But all Borels are conjugate and the unipotent radical is normal, so it is contained in the above.

Theorem 2.1.1. *The containment $I(T) \supset \mathscr{R}_u(G)$ is an equality.*

Remark 2.1.2. Pass to $G/\mathscr{R}_u(G)$, which is reductive (exercise: extension of unipotent by unipotent is unipotent). Under quotient maps, Borels surject onto Borels (exercise). Hence, by the structure of connected solvable groups, under a surjection between linear algebraic groups the image of a maximal torus is a maximal torus; we will use this a lot without comment. In particular, the theorem says exactly that for reductive connected G we have $I(T) = 1$.

One visualization of (and motivation for) the theorem is in terms of opposite Borels, whose unipotent radicals already intersect trivially, e.g. for GL_n

$$B_+ = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \qquad B_- = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix}.$$

The notion of *opposite Borel* in connected reductive groups (admitting a Borel subgroup) will rely on much of what we are developing in the coming lectures!

For G a connected linear algebraic group over $k = \overline{k}$ and $S \subset G$ a k-torus, obviously

$$Z_G(S) \cap \mathscr{R}_u(G) = \mathscr{R}_u(G)^S.$$

Now the smooth group $\mathscr{R}_u(G)^S$ is a connected (by preceding discussion, being a centralizer of a connected group under a torus action, realized as a centralizer against a torus *subgroup* upon passing to a semi-direct product construction), and it is clearly unipotent as well as normal inside $Z_G(S)$. Thus, a priori $\mathscr{R}_u(G)^S \subset \mathscr{R}_u(Z_G(S))$.

Theorem 2.1.3. *The containment $\mathscr{R}_u(G)^S \subset \mathscr{R}_u(Z_G(S))$ is an equality. In particular, a torus centralizer in a reductive group is reductive.*

Theorem 2.1.1 implies Theorem 2.1.3. Why? The first theorem is a formula for the unipotent radical in terms of those of certain Borels. We discussed that Borels play well with centralizers of tori: $Z_G(S) \cap B$ is a Borel of $Z_G(S)$. So you apply the formula to G and $Z_G(S)$, and roughly speaking you have the same collection of Borels in the two cases.

Exercise 2.1.4. Write out a careful proof for the preceding sketch.

Remarks on proof of Theorem 2.1.1 The proof of this Theorem is quite involved, and the argument basically never comes up again, so we won't go through it in detail, referring instead to the handout "Basics of reductive and semisimple groups" for full details. (I once presented this material in extra lectures for a course and it took many hours.) However, we'll point out some of the ingredients now. It uses lots of manipulations with torus centralizers and dynamic constructions; the main serious input is the classification of rank-1 reductive groups over algebraically closed fields.

Lemma 2.1.5. *If H is reductive and $Z \subset H$ is a central torus, then H/Z is reductive.*

(More generally a quotient of reductive is reductive, but that requires much more input and will be proved a bit later; we need this special case now.)

Proof. Without loss of generality $k = \overline{k}$. Let $U = \mathscr{R}_u(H/Z)$ and N be the pre-image of U in H.

$$\begin{tikzcd} 1 \arrow[r] & Z \arrow[r] & H \arrow[r] & H/Z \arrow[r] & 1 \\ 1 \arrow[r] & Z \arrow[r] \arrow[u, equal] & N \arrow[r] \arrow[u, hook] & U \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

Then N is a connected smooth normal in H because U is in H/Z, and being an extension of a unipotent group by a torus it is also solvable. We have that $\mathscr{R}_u(N) \subset N$

is stable under all automorphisms because it is a *characteristic* subgroup, so in particular it is stable under $H(k)$-conjugation, so $\mathscr{R}_u(N) \triangleleft H$ (this argument applies to all characteristic subgroups of normal subgroups of linear algebraic groups). Therefore, $\mathscr{R}_u(N) \subset \mathscr{R}_u(H) = 1$, so N is reductive. But we also saw that N is solvable, and over an algebraically closed field any solvable group is the semidirect product of a torus and its radical, so then N is a torus. (This is the general fact over any field that a connected reductive linear algebraic group is solvable if and only if it is a torus.) As U is unipotent and the quotient of a torus, it must be trivial. $\square$

For the proof of Theorem 2.1.1 we will apply Lemma 2.1.5 to a certain subquotient H of G. Let $\Phi := \Phi(G, T)$ be the set of non-trivial weights, i.e.

$$\Phi = \{a \in \mathrm{X}(T) - \{0\} \mid \mathfrak{g}_a \neq 0\}$$

where $\mathfrak{g}_a$ is the a-weight space for the adjoint action of T on $\mathfrak{g}$. Elements of Φ are called *roots* of G with respect to T. Recall that for this adjoint action there is a decomposition

$$\mathfrak{g} = \mathfrak{g}^T \oplus \left(\bigoplus_{b \in \Phi} \mathfrak{g}_b \right)$$

with $\mathfrak{g}^T = \mathrm{Lie}(Z_G(T))$.

Choose $a \in \Phi$, and view it as a (non-trivial) map $a \colon T \to \mathbf{G}_m$. Let $T_a = (\ker a)^0_{\mathrm{red}} \subset T$ be the codimension-1 subtorus killed by a. Let $G_a = Z_G(T_a)$, a connected linear subgroup of G. Then

$$\mathrm{Lie}(G_a) = \mathfrak{g}^{T_a} \oplus \left(\bigoplus_{b \in \Phi \cap \mathbf{Q}^\times a} \mathfrak{g}_b \right)$$

because the roots that appear in $\mathrm{Lie}(G_a)$ are the roots which are non-zero rational multiples of a (because if χ and χ' are two nontrivial homomorphism $T \to \mathbf{G}_m$ then there is a containment $(\ker \chi)^0 \supset (\ker \chi')^0$ if and only if χ, χ' are $\mathbf{Q}$-linearly dependent; the presence of $\mathbf{Q}$ ultimately comes from the equality $\mathrm{End}(\mathbf{G}_m) = \mathbf{Z}$).

Exercise 2.1.6. Prove this description of $\Phi(G_a, T)$.

Consider $G_a/\mathscr{R}_u(G_a) \cdot T_a =: \overline{G_a}$. This is a reductive group by Lemma 2.1.5 because T_a is by definition central in G_a. Inside here we have a maximal torus T/T_a, which is one-dimensional. But by the classification of (split) rank-1 connected reductive groups there are only three possibilities for $\overline{G_a}$: SL_2 or PGL_2 or $\mathbf{G}_m$, with T/T_a identified with the diagonal torus in the first two cases. For these particular groups and maximal tori it is elementary to find which Borel subgroups contain the indicated maximal torus (as we can write down one and prove the Bruhat decomposition for SL_2 by bare hands so as to see there is only one other such Borel subgroup). The last case is trivial, and in the first two you have only the upper/lower triangular Borels. By inspection $I_{\overline{G_a}}(T/T_a) = 1$ in all of these cases!

The central T_a lives inside all Borel subgroups of G_a (for instance, since such Borel subgroups are all conjugate). Hence, we can upgrade to say $I_{G_a/\mathscr{R}_u(G_a)}(T) = 1$, or equivalently $I_{G_a}(T) = \mathscr{R}_u(G_a)$. This proves the result we want for each G_a in place of G.

This is all we will say here about the proof of the general case (full details in the handout as noted earlier). Rather than attack the full group directly, you study the subgroups

G_a first and then need to bootstrap from such subgroups for varying a to conclude. The classification of (split) rank-1 connected reductive groups is the most serious input; the rest is clever manipulations with torus centralizers, Lie algebras, dynamic constructions, and so on. $\square$

2.2. **Structure of roots.** Theorem 2.1.1 has many good consequences, which we now explain. First we use it to analyze the roots of *connected reductive* G over $k = \overline{k}$, essentially by studying the groups G_a and $\overline{G_a}$ appearing in the proof. Note that while the *proof* of Theorem 2.1.1 might lose contact with $\mathfrak{g}_a$ in quotienting by the unipotent radical $\mathscr{R}_u(G_a)$ (as we don't see a-priori that $\mathfrak{g}_a$ can't be supported inside the Lie algebra of $\mathscr{R}_u(G_a)$), we know *after* the proof that this unipotent radical is in fact trivial. This underlies the proof of:

Corollary 2.2.1. *Let G be connected and reductive over $k = \overline{k}$ and $T \subset G$ be a maximal torus, $a \in \Phi$. Then*

- $\mathbf{Q}a \cap \Phi = \{\pm a\}$ *(so $-\Phi = \Phi$ inside $X(T)$),*
- $\mathfrak{g}_a$ *is one-dimensional.*

Proof. By Theorem 2.1.3 (a consequence of Theorem 2.1.1), we know that G_a is connected reductive. Therefore, $\mathscr{R}_u(G_a) = 1$, so by the argument in the proof of Theorem 2.1.1 we have that $G_a/T_a \simeq \mathrm{SL}_2$ or PGL_2 or $\mathbf{G}_m$.

Actually, we claim that G_a is not solvable, which rules out the possibility $G_a/T_a \simeq \mathbf{G}_m$. Indeed, otherwise the connected reductive G_a would be a torus. But $G_a \supset T$ and the T-action on $\mathrm{Lie}(G_a)$ has a non-trivial weight (namely a), so G_a is not commutative and hence not a torus. So now we know that G_a/T_a is isomorphic to SL_2 or PGL_2.

Choose $\lambda\colon \mathbf{G}_m \to T$ which is an isogeny-complement to $T_a \subset T$; i.e., the map

$$\mathbf{G}_m \times T_a \to T$$

induced by multiplication against λ is an isogeny. (This can be viewed as a splitting of $T \twoheadrightarrow T/T_a \simeq \mathbf{G}_m$ up to isogeny.) Since T_a is already central in G_a, we have $Z_{G_a}(\lambda) = Z_{G_a}(T)$. But inside $G_a/T_a = \mathrm{SL}_2$ or PGL_2, we see by inspection that the diagonal is its own centralizer, so we conclude that the centralizer $Z_{G_a}(\lambda) = Z_{G_a}(T)$ coincides with T.

Next we're going to use the open cell

$$G_a \hookleftarrow U_{G_a}(-\lambda) \times (Z_{G_a}(\lambda) = T) \times U_{G_a}(\lambda).$$

Consider the relation between this and the corresponding open cell of $\overline{G_a}$:

$$\begin{tikzcd}
G_a \arrow[d] & U_{G_a}(-\lambda) \arrow[l, hook'] \arrow[d] & \times & T \arrow[d] & \times & U_{G_a}(\lambda) \arrow[d] \\
\overline{G_a} & U_{\overline{G_a}}(-\lambda) \arrow[l, hook'] & \times & T/T_a & \times & U_{\overline{G_a}}(\lambda)
\end{tikzcd}$$

The kernel T_a of the quotient map $G_a \to \overline{G_a}$ is already eaten up in the middle factor, so the outer maps must be isomorphisms because the middle factor at each level and either of the two flanking factors combined to give a semi-direct product *subgroup* of the ambient group at each level.

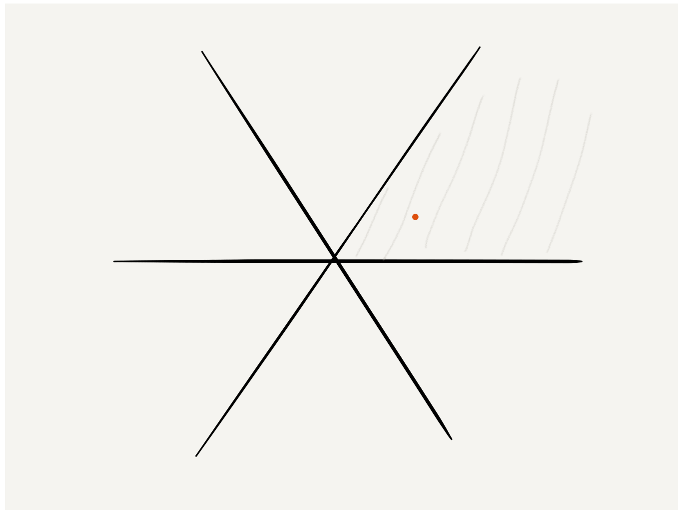


FIGURE 2.2.1. A generic coroot lies in the interior of a chamber cut out by the root hyperplanes.

The Lie algebras $\mathrm{Lie}(U(\pm\lambda))$ account for all nontrivial weights since we have seen that each torus is its own centralizer. Aha! We understand those weight spaces well on the bottom level because $\overline{G_a} = \mathrm{SL}_2$ or PGL_2 with T/T_a the diagonal torus. So we can conclude that $U_{\overline{G_a}}(\pm\lambda)$ are one-dimensional with *opposite* T/T_a weights. We conclude the same upstairs. But we know that upstairs one of the T-roots is a by design, so the other one must be $-a$ (with multiplicity one).

$\square$

The dynamical methods in the of proof of Theorem 2.1.1 yield the following statement, which says essentially that all Borels arise via the dynamic method.

Corollary 2.2.2. *For $G, T/k = \overline{k}$ as in Corollary 2.2.1, the Borel subgroups containing T are* exactly $B(\lambda) = P_G(\lambda)$ *for "generic" $\lambda \in \mathrm{X}_*(T)$.*

What is the meaning of generic? Each root a cuts out a hyperplane $\langle \mu, a\rangle = 0$ in $\mathrm{X}_*(T)_{\mathbf{R}}$ (alternatively, $\mu\colon \mathbf{G}_m \to T$ lands in T_a). A generic λ is one that does not lie in any of these hyperplanes. The proof of this dynamic description of Borel subgroups is not at all trivial, and is a large part of the dynamic work in the proof of Theorem 2.1.1 (upon revisiting that proof once the triviality of $I(T)$ in the connected reductive case has been established).

Remark 2.2.3. We will see later that if we drop the adjective "generic" in Corollary 2.2.2 then we get exactly the parabolic subgroups containing T.

Choose a generic λ. Then the decomposition of the Lie algebra under the adjoint action can be grouped as

$$\mathfrak{g} = \underbrace{\mathfrak{g}^T}_{\mathrm{Lie}(T)} \oplus \bigoplus_{\substack{a\in\Phi \\ \langle a,\lambda\rangle>0}} \mathfrak{g}_a \oplus \bigoplus_{\substack{a\in\Phi \\ \langle a,\lambda\rangle<0}} \mathfrak{g}_a$$

Using that $-\Phi = \Phi$, it is easy to thereby deduce that

$$\dim B = \frac{1}{2}(\dim G + \dim T).$$

This is a cute formula for the dimension of the Borel subgroups of a connected reductive group.

2.3. **Quotient of reductive is reductive.**

Proposition 2.3.1. *If G is connected reductive over a field k and $T \subset G$ is a maximal torus, then $Z_G(T) = T$.*

Proof. We know that $Z_G(T)$ is connected reductive. (This is called a *Cartan subgroup*.) But it's a general fact that Cartan subgroups are always solvable! Indeed, $Z_G(T)/T$ is a connected linear algebraic group with no non-trivial tori (the pre-image would be a non-trivial torus not contained in T, but T is a *central* maximal torus and in general all maximal tori are conjugate). That implies that $Z_G(T)/T$ is unipotent (22.1.4 of the notes for the previous course). In particular $Z_G(T)$ is is solvable. But a connected reductive group is solvable if and only if it's a torus, hence equal to T. □

Corollary 2.3.2. *If $\pi\colon G \twoheadrightarrow G'$ is a surjection of linear algebraic groups, then*

$$\mathscr{R}_u(G_{\overline{k}}) \twoheadrightarrow \mathscr{R}_u(G'_{\overline{k}}).$$

In particular, if G is reductive then so is G'.

Proof. First a couple of reductions. Obviously we can assume without loss of generality that $k = \overline{k}$. Because $\pi(\mathscr{R}_u(G)) \subset \mathscr{R}_u(G')$ is normal in G' (as π is surjective), we can replace G by $G/\mathscr{R}_u(G)$ and G' by $G'/\pi(\mathscr{R}_u(G))$ to assume that G is reductive.

Let $U' = \mathscr{R}_u(G')$. Then $\pi^{-1}(U')^0_{\mathrm{red}}$ is normal in G (easy exercise) and hence reductive (as we have already reviewed in the proof of Lemma 2.1.5 the elementary fact that a normal linear algebraic subgroup in a reductive group is reductive). Thus, replacing G' with U' and replacing G with N allows us to assume without loss of generality that G' is unipotent, and then we want to show that $G' = 1$. Let $T \subset G$ be a maximal torus, so $Z_G(T) \twoheadrightarrow Z_{G'}(\pi(T))$ by Corollary 1.4.10. But $\pi(T) = 1$ since G' is unipotent, so $Z_{G'}(\pi(T)) = G'$. Thus, the unipotent G' is the image of $Z_G(T) = T$ (see Proposition 2.3.1), so $G' = 1$. □

3. Central isogeny decomposition

3.1. **Perfect groups.**

Recall that $\mathrm{SL}_2(k)$ is a *perfect group* (meaning that it is own commutator subgroup) for any field k with $\#k > 3$. In particular, $\mathrm{SL}_2 = \mathscr{D}(\mathrm{SL}_2)$ over *any* field, since this property can be checked on geometric points.

The group SL_n contains many embedded copies of SL_2:

$$\mathrm{SL}_n \supset H_{i<j} := \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & t_i & \dots & x_{ij} & \\ & & \vdots & \ddots & \vdots & \\ & & x_{ji} & \dots & t_i^{-1} & \\ & & & & & \ddots \end{pmatrix}$$

The standard tori $\mathbf{G}_m \subset H_{i<j}$ generate the diagonal torus of SL_n, which implies that the $H_{i<j}$ generate SL_n.

Exercise 3.1.1. Prove this. [Hint: Consider the open cell $U^- \times T \times U^+ \subset \mathrm{SL}_n$, which is evidently contained in the subgroup generated by the $H_{i<j}$.]

Corollary 3.1.2. SL_n *is perfect.*

Example 3.1.3. For $G = \mathrm{GL}_n$, we have an exact sequence of group schemes

$$1 \to \mu_n \xrightarrow{\xi \mapsto (\xi, \xi^{-1})} \mathbf{G}_m \times \mathrm{SL}_n \to G \to 1$$

obtained from the diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SL}_n \arrow[r] \arrow[drr] & \mathrm{GL}_n \arrow[r, "\det"] & \mathbf{G}_m \arrow[r] & 1 \\ 1 \arrow[r] & \mathbf{G}_m \arrow[r, "\text{maximal central torus}"'] \arrow[urr] & \mathrm{GL}_n \arrow[r] & \mathrm{PGL}_n \arrow[r] & 1 \end{tikzcd}$$

The diagonal arrows are isogenies, namely the natural quotient map $\mathrm{SL}_n \to \mathrm{PGL}_n$ and the endomorphism $t \mapsto t^n$ of $\mathbf{G}_m$. In particular, it follows that $\mathscr{D}(\mathrm{GL}_n) = \mathrm{SL}_n$.

We'll see that this isogeny decomposition for GL_n adapts to *any* connected reductive k-group G and that moreover $\mathscr{D}G$ is always perfect for such G (and $\mathscr{D}G$ is certainly reductive, since it is normal in G). The upshot is that the essential case for most work is perfect connected reductive G (we'll see next time that this is the same as connected semisimple).

Proposition 3.1.4. *If G is a connected reductive group over k, then $\mathscr{D}G$ is perfect.*

Example 3.1.5. Here is an example to illustrate that connectedness needs to be respected. Let $H = \mathbf{Z}/2\mathbf{Z} \ltimes \mathbf{G}_m$, with the $\mathbf{Z}/2\mathbf{Z}$ acting by inversion on $\mathbf{G}_m$. This is clearly reductive. Since $(1,t)t(1,t)^{-1}t^{-1} = t^{-2}$, we have $\mathbf{G}_m \subset \mathscr{D}(H)$, and this is an equality since $H/\mathbf{G}_m$ is commutative. Hence, the derived group is not perfect since it is commutative and nontrivial.

Proof. Let $N = \mathscr{D}G \subset G$. This is a characteristic subgroup; i.e. stable under all automorphisms. So its own derived subgroup is still normal (and even characteristic) in G: $\mathscr{D}^2 G \triangleleft G$. Consider the resulting exact sequence

$$1 \to N/\mathscr{D}N \to G/\mathscr{D}^2 G \to G/N \to 1.$$

Now, G/N is commutative connected reductive, hence a torus. Since N is normal in G it is also reductive, and connected because the derived subgroup of a connected group is connected, so $N/\mathscr{D}(N)$ is also a torus. But then $G/\mathscr{D}^2(G)$ is an extension of a torus by a torus, hence is also a torus (by considerations of Jordan decomposition), hence is commutative. Therefore, $\mathscr{D}(G/\mathscr{D}^2G)$ is its own derived group is trivial, but this is $\mathscr{D}G/\mathscr{D}^2G$. $\square$

3.2. **Central isogeny decomposition.** Classically, for a linear algebraic k-subgroup H of a linear algebraic k-group G, the centralizer was built over an algebraic closure to be smooth (i.e., one made $Z_{G_{\overline{k}}}(H_{\overline{k}})$ as a reduced Zariski-closed subgroup of $G_{\overline{k}}$), but it was not clear that this had the expected Lie algebra $\mathfrak{g}_{\overline{k}}^{H_{\overline{k}}}$, and even less clear whether or not it descends to a k-subgroup of G (but classically for H smooth of multiplicative type such descent is proved after some hard work).

We shall need the centralizer $Z_G(H)$ as a scheme over the ground field representing a centralizer functor without smoothness hypotheses on H; see Exercise 3(iii) in HW3 of the previous course for the case of smooth H with no smoothness hypotheses on G. By using k-bases of coordinate rings, one can prove such existence results for $Z_G(H)$ and identify its Lie algebra with $\mathfrak{g}^H$ without any smoothness hypotheses; this will be essential later in this course with smooth G but H a possibly non-smooth k-subgroup scheme of a torus (such as the kernel of a root). See Exercise 4 in HW7 of the previous course for the determination of $\mathrm{Lie}(Z_G(H))$ when G and H are smooth.

In a later handout "Reductive centralizer" we will provide a proof of the existence of $Z_G(H)$ and the identification of its Lie algebra for any affine k-group scheme G and closed k-subgroup scheme H, by using k-bases of coordinate rings in place of the Galois-theoretic method that works in the smooth case. If H has completely reducible representation theory on finite-dimensional k-vector spaces (e.g., a k-group scheme of multiplicative type, such as μ_n, without any smoothness hypotheses!) then $Z_G(H)$ is also *smooth* whenever G is smooth due to the infinitesimal criterion (as presented in Exercise 3 of HW8 of the previous course when H is a torus).

Remark 3.2.1. In HW3 Exercises 3 and 4 of the previous course, for any linear algebraic k-group H we built the *scheme-theoretic* center $Z_H \subset H$, as a closed k-subgroup scheme representing the functor on k-algebras

$$R \rightsquigarrow \{h \in H(R) \mid \text{conjugation by } h \text{ is trivial on } H_R\}.$$

From this definition it is clear that $(Z_H)_K = Z_{(H_K)}$ for any K/k, but $\mathrm{Lie}(Z_H) \subset \mathfrak{h}$ is a mystery. For instance, $\mathrm{Lie}(Z_{\mathrm{SL}_p}) = \mathrm{Lie}(\mu_p) \neq 0$ inside $\mathfrak{sl}_p$ if $p = \mathrm{char}\, k > 0$.

Theorem 3.2.2. *Let G be a connected reductive group over k and $Z \subset G$ the maximal central k-torus.*

(1) $Z = (Z_G)^0_{\mathrm{red}}$ and for all K/k, $Z_K \subset G_K$ is the maximal central K-torus.
(2) The multiplication homomorphism $m : Z \times \mathscr{D}(G) \to G$ is a central isogeny (i.e. an isogeny with central kernel).

Remark 3.2.3. (i) In general, for an affine k-group scheme H of finite type, the underlying reduced scheme $H_{\mathrm{red}} \subset H$ can *fail* to be a k-subgroup scheme when k

is imperfect. A *connected* counterexample is given in [SGA3, $\mathrm{VI_A}$, 1.3.2]. A much easier disconnected counterexample is the p-torsion of a Tate curve over a local function field of characteristic p when the Tate q-parameter is not a pth power.

(ii) This Theorem easily implies that $Z \to G/\mathscr{D}(G)$ and $\mathscr{D}G \to G/Z$ are central isogenies. For $G = \mathrm{GL}_n$, the first map is $t \mapsto t^n$ and the second is $\mathrm{SL}_n \to \mathrm{PGL}_n$. The centrality of kernels of isogenies is automatic in characteristic 0 because a finite group scheme in characteristic 0 is étale and if H is a *connected* linear algebraic group over k with any characteristic then every finite normal *étale* k-subgroup N in H is central (much as a discrete normal subgroup of a connected Lie group is central): conjugation of H on N is classified by a map $H \to \mathrm{Aut}_{N/k}$, and the latter group is finite étale, so this classifying map must be trivial.

By contrast, isogenies in characteristic p can have non-central kernel: the relative Frobenius isogeny $F_{H/k} : H \to H^{(p)}$ for a linear algebraic group H, whose infinitesimal kernel $N \subset H$ has full Lie algebra (very non-commutative when $H = \mathrm{SL}_n$ with $n > 1$, for example).

Proof. For (1), let $T \subset G$ be a maximal k-torus. Then $T = Z_G(T)$ by Lemma 2.3.1, hence contains Z_G. It is a general fact that for *tori*, the underlying reduced scheme is always a subgroup whose formation commutes with extension of the ground field. More specifically for our purposes, for any closed k-subgroup $M \subset T$, M_{red} is a k-subgroup whose formation commutes with extension of k [for a proof, see Lemma 1.3 of the "Basics of reductive groups" handout]. (The idea of the proof is that you can pass to the separable closure, and so assume that $k = k_s$, so T splits as $\mathbf{G}_m^r$. Then the subgroups of T correspond to quotients of the character group, i.e. $M = \underline{\mathrm{Hom}}(\Lambda, \mathbf{G}_m)$ for some quotient Λ of $\mathbf{Z}^r$. But by the structure of $\mathbf{G}_m$, M is necessarily of the form $\mathbf{G}_m^s \times \prod \mu_{\ell_i^{e_i}}$ for some primes ℓ_i, so it reduces to the fact that $(\mu_{p^e})_{\mathrm{red}} = 1$ if $p = \mathrm{char}\, k > 0$.)

This shows that $(Z_G)^0_{\mathrm{red}}$ is a torus, but $Z \subset (Z_G)^0_{\mathrm{red}}$ by definition, so we must have $Z = (Z_G)^0_{\mathrm{red}}$. Since $(Z_G)^0_{\mathrm{red}}$ commutes with formation of extension on k (by the preceding discussion plus the fact that the formation of Z_G and of identity components of group schemes of finite type each commute with extension of the ground field), Z does too.

(2) Let's first show that $\ker m$ is central and finite. The centrality is easy: the kernel is $Z \cap \mathscr{D}G \hookrightarrow Z \times \mathscr{D}G$ embedded via (t, t^{-1}), and anything intersected with Z is obviously central.

For finiteness, we use:

Lemma 3.2.4. *Let H be a connected linear algebraic group over k and $C \subset H$ a central k-torus. Then $C \cap \mathscr{D}(H)$ is finite.*

Proof. The idea is to reduce to the case $H = \mathrm{GL}_n$. Without loss of generality, $k = \overline{k}$. Choose a closed immersion $H \hookrightarrow \mathrm{GL}(V)$ as k-groups. Under the C-action on V we get a weight space decomposition

$$V = \bigoplus_{\chi \in \mathrm{X}(C)} V_\chi.$$

Since $C \subset H$ is central, the H-action on V preserves the C-eigenspaces, so we get an embedding $H \hookrightarrow \prod \mathrm{GL}(V_\chi)$ under which $C \hookrightarrow \prod_\chi \mathbf{G}_m$. But $\mathscr{D}(H) \subset \prod_\chi \mathrm{SL}(V_\chi)$, so it is enough to observe that $\mathbf{G}_m \cap \mathrm{SL}_n = \mu_n$ is finite for any $n \geq 1$. □

Finally, we show the surjectivity of $m\colon Z \times \mathscr{D}G \to G$. It is the same to show that G/Z is perfect. But G/Z is connected reductive with trivial maximal central torus, since a *normal* torus S inside a connected linear algebraic group H is always central (by Exercise 3 of HW6 of the previous course: conjugation on S is classified by a homomorphism $H \to \mathrm{Aut}_{S/k}$ whose target is étale; e.g. if S is split, then this automorphism scheme is the constant group $\mathrm{GL}_n(\mathbf{Z})$). The upshot is that by replacing G with G/Z, we can reduce to the case where Z is trivial, and then we want to show that G is perfect.

Without loss of generality we may assume that $k = \overline{k}$, so all k-tori are split. Let T be a maximal torus in G. For $a \in \Phi(G,T)$, recall that we defined a smooth connected subgroup $G_a := Z_G(T_a) \subset G$ where $T_a = (\ker a)^0_{\mathrm{red}}$ and $G_a/T_a \simeq \mathrm{SL}_2$ or PGL_2. In particular, G_a/T_a is perfect, so the multiplicative homomorphism

$$T_a \times \mathscr{D}(G_a) \to G_a$$

with finite central kernel is *surjective* too; i.e., for G_a (with maximal central torus T_a of codimension 1 in the non-central maximal torus $T \subset G_a$) the desired central-isogeny result is known to hold. Moreover, we saw via dynamic considerations with $U(\pm\lambda)$ for suitable $\lambda \in \mathrm{X}_*(T)$ that the T-equivariant map $\mathscr{D}(G_a) \to G_a/T_a$ induces an isomorphism on the T-weight spaces for nontrivial weights, so the Lie algebra of $\mathscr{D}(G_a)$ has a as a T-weight.

Since $T = Z_G(T)$ has Lie algebra $\mathfrak{g}^T$, it follows that $\mathrm{Lie}(G)$ is spanned by the Lie algebras of T and the $\mathscr{D}(G_a)$'s (using all roots a). Thus, the connected linear algebraic subgroup $\langle T, \mathscr{D}(G_a)\rangle_{a\in\Phi}\rangle$ has full Lie algebra, so it is equal to G. So far we have not used the triviality hypotheses on Z.

Now it suffices to show that $T \subset \mathscr{D}G$ when $Z = 1$. This is immediate from the following claim.

□

Proposition 3.2.5. *If* $Z = 1$ *then* $T = [T, N_G(T)]$.

Proof. Of course the containment $\supset$ is trivial because $tnt^{-1}n^{-1} = t(ntn^{-1}) \in T$. The substance is in the other direction.

Recall the Weyl group $W = N_G(T)/Z_G(T) = N_G(T)/T$, which is *finite* (see Exercise 4 in HW6 of the previous course). We consider the map

$$T \rtimes N_G(T) \to G.$$

For $w \in W$ represented by $n \in N_G(T)$, we have

$$ntn^{-1}t^{-1} = (ntn^{-1})t^{-1} = (w\cdot t)/t.$$

We want to show that T is generated by commutators, and this shows that $(w\cdot t)/t$ is a commutator. So it suffices to show that T is generated by elements of the form $(w\cdot t)/t$.

To this end, consider the subtorus

$$S_w = \mathrm{Im}(T \xrightarrow{t\mapsto (w\cdot t)/t} T).$$

It's enough to show that the S_w generate T. We can check this by showing that any character $\chi \in \mathrm{X}(T)$ which is trivial on each S_w must be trivial on all of T, i.e. if $\chi|_{S_w} = 1$ for all w then $\chi = 1$. The condition that $\chi|_{S_w} = 1$ is simply that $\chi(t) = \chi(w \cdot t)$ for all $t \in T$, i.e. $\chi \in \chi(T)^w$. So it is equivalent to show that $\mathrm{X}(T)^W_{\mathbf{Q}} = 0$.

The reason for passing to rational coefficients is that we have a good theory of representations of finite groups (such as W) over fields of characteristic 0 (such as $\mathbf{Q}$). For instance we know that $\mathbf{Q}[W]$-modules are semisimple, so $\mathrm{X}(T)^W_{\mathbf{Q}} = 0$ if and only if the *dual* module has trivial W-invariants. The dual is the familiar object $\mathrm{X}_*(T)_{\mathbf{Q}}$, so we have reduced to showing that $\mathrm{X}_*(T)^W = 0$.

Now this is a more concrete statement, since $\mathrm{X}_*(T) = \mathrm{Hom}(\mathbf{G}_m, T^W)$. This is equivalent to T^W having no non-trivial subtori, but to say that a closed subgroup scheme M of a torus contains no non-trivial subtori is to say that M is finite. So we have reduced to showing that T^W is finite when $Z = 1$.

Example 3.2.6. Consider $G = \mathrm{GL}_n$ versus $G = \mathrm{SL}_n$. Take T to be the diagonal torus in either case, so $W = S_n$ via coordinate permutation. What is T^W? It is the group of scalar matrices (since W acts by permutation on the coordinates). So T^W is μ_n for SL_n and $\mathbf{G}_m$ for GL_n. Note that $Z = 1$ for SL_n, but $Z \neq 1$ for GL_n.

Lemma 3.2.7. *If $Z = 1$, then $\bigcap_{a \in \Phi} \ker(n_a a)$ is finite for any collection of $n_a \in \mathbf{Z} - \{0\}$.*

(The converse is obviously true).

Proof. Since a closed subgroup scheme M of a torus contains a torus if and only if M is not finite (as M^0_{red} is a torus), it is the same to show that if a subtorus $S \subset T$ is killed by all characters $n_a a$, then $S = 1$. Consider the restriction map

$$\mathrm{X}(T) \to \mathrm{X}(S)$$

and note that $\mathrm{X}(S)$ is torsion-free (since $S \xrightarrow{t^n} S$ is surjective for all $n \neq 0$). Therefore

$$\begin{aligned} n_a \cdot a|_S = 1 &\iff a|_S = 1 \\ &\iff \mathfrak{g}^S \supset \mathfrak{g}_a \\ &\iff \mathrm{Lie}(Z_G(S)) \supset \mathfrak{g}_a. \end{aligned}$$

Certainly $Z_G(S) \supset Z_G(T)$, so $\mathrm{Lie}(Z_G(S)) \supset \mathrm{Lie}(Z_G(T)) \supset \mathfrak{g}^T$. But then $\mathrm{Lie}(Z_G(S))$ contains $\mathfrak{g}^T$ *and* all the root lines, so $\mathrm{Lie}(Z_G(S)) \supset \mathfrak{g}^T \oplus \left(\bigoplus_{a \in \Phi} \mathfrak{g}_a\right) = \mathfrak{g}$, so $Z_G(S) \subset G$ as a connected smooth subgroup is full, i.e. $S \subset Z_G$. The assumption $Z = 1$ obviously finishes off the proof, but notice that none of the preceding argument required that hypothesis. $\square$

Now the proof is (finally!) completed by the following general Lemma (which requires no hypotheses on Z). $\square$

Lemma 3.2.8. *We have $T^W \subset \ker(2a)$ for all $a \in \Phi = \Phi(G, T)$.*

Proof. Choose a and consider $G_a = Z_G(T_a)$ for $T_a = (\ker a)^0_{\mathrm{red}} \subset T$. We're going to use a special case of the theorem (which we're trying to prove) that we already know: the map

$$\pi_a : T_a \times \mathscr{D}(G_a) \to G_a$$

is a central isogeny, because G_a/T_a is perfect (since it's SL_2 or PGL_2).

Remark 3.2.9. Since $\mathscr{D}(G_a) \to G_a/T_a = \mathrm{SL}_2$ or PGL_2 is a central isogeny, it follows that $\mathscr{D}(G_a)$ has rank one and is not solvable (being perfect), so is either SL_2 or PGL_2.

Remark 3.2.10. It is shown in Example 3.4 of the handout "Basics of reductive groups" that for PGL_n, $n \geq 3$ we always have $\mathscr{D}(G_a) = \mathrm{SL}_2$ whereas for SO_5 we sometimes gets SL_2 and sometimes get PGL_2 (depending on a).

We want to reduce to the rank-1 case. Note that for any *central* quotient map $f : H \twoheadrightarrow H'$ (i.e. a quotient by central subgroup scheme) between connected reductive groups over an *arbitrary* field k, it is always the case that there is a bijection

$$\{\text{maximal } k\text{-tori of } H\} \Longleftrightarrow \{\text{maximal } k\text{-tori of } H'\}$$

which is given explicitly by $f \mapsto f(T)$ and $f^{-1}T' \hookleftarrow T'$ (using scheme-theoretic preimage, as always). Why? To prove the recipes give maximal k-tori (e.g., that $f^{-1}(T')$ is k-smooth) and are inverse to each other, it suffices to check after an extension on k; hence, we may and do assume $k = \overline{k}$. Thus, all maximal tori are conjugate, so we just have to show that the (scheme-theoretic) preimage of one maximal torus is a maximal torus. Start with a maximal torus T in H, and we want to know if it is the full preimage of its image. In other words, we need to know that T contains the kernel, and this follows from $\ker f$ being central since $Z_H \subset Z_H(T) = T$.

Apply this discussion to $f = \pi_a$. Then $\pi^{-1}(T)$ is a maximal torus in $T_a \times \mathscr{D}(G_a)$. What can we say about this maximal torus? It contains T_a, since the maximal torus in a direct product of linear algebraic groups is a direct product of maximal tori in the factors (as we may check after passage to an algebraically closed field and using conjugacy considerations), so $\pi_a^{-1}(T) = T_a \times \mathscr{T}_a$ for $\mathscr{T}_a = T \cap \mathscr{D}(G_a)$. Thus, $\mathscr{T}_a$ is a 1-dimensional torus (in particular, smooth and connected!).

Note that

$$T_a \times \mathscr{T}_a \to T$$

is an isogeny, with T_a central in G_a, so it induces an isomorphism of the rational character groups:

$$\mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(\mathscr{T}_a)_{\mathbf{Q}} \xleftarrow{\sim} \mathrm{X}(T)_{\mathbf{Q}}.$$

The whole game is to produce elements of the Weyl group that visibly cut down the invariant space. We know that

$$(\mathscr{D}(G_a), \mathscr{T}_a) \simeq (\mathrm{SL}_2 \text{ or } \mathrm{PGL}_2, \text{diagonal torus})$$

so we have an element $n_a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_{G(a)}(\mathscr{T}_a)$. Note that $n_a \in N_G(T_a \cdot \mathscr{T}_a) = N_G(T)$ since it centralizes T_a and also normalizes $\mathscr{T}_a$, which together generate T, and so represents an element $w_a \in W$. We'll show that $T^{w_a} \subset \ker(2a)$, which will do the job.

Look at the decomposition

$$\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T_a) \oplus \mathrm{X}(\mathscr{T}_a)_{\mathbf{Q}}.$$

The first summand is a hyperplane and the second summand is a line. What does w_a do? It restricts to the identity on the hyperplane, and negation on the line by inspection. This is a reflection (by definition, an automorphism which is the identity on a hyperplane and negation on the quotient line). We claim that the following diagram commutes:

$$\begin{tikzcd} T \arrow[r, "a"] \arrow[d, "n_a\text{-conj}"'] & \mathbf{G}_m \arrow[d, "-1"] \\ T \arrow[r, "a"'] & \mathbf{G}_m \end{tikzcd}$$

It suffices to check this on $T_a, \mathscr{T}_a$ separately. On T_a, the compositions are both trivial. On $\mathscr{T}_a$, they are both $-a$.

Now, suppose that you have a fixed point $t \in T^W$. Then commutativity of the diagram gives that $a(t) = 1/a(t)$, so one has

$$a(t)^2 [= t^{2a}] = 1$$

as desired (by definition of the notation $2a \in \mathrm{X}(T)$). $\square$

Corollary 3.2.11. *For G connected reductive and perfect and a split maximal torus $T \subset G$, $\mathbf{Z}\Phi \subset \mathrm{X}(T)$ has finite index.*

Proof. We always have the central isogeny

$$Z \times \mathscr{D}G \to G$$

so $G = \mathscr{D}G \iff Z = 1$. On the other hand, $\mathbf{Z}\Phi$ has finite index in $\mathrm{X}(T)$ if and only if $\bigcap_{a\in\Phi} \ker a$ contains no $\mathbf{G}_m$ (this is an easy exercise concerning the relationship between subtori and quotient lattices), which as seen above is equivalent to $Z = 1$. $\square$

Corollary 3.2.12. *Let G be a connected linear algebraic group over a field k. Then the following are equivalent:*

(1) the geometric solvable radical $\mathscr{R}(G_{\overline{k}})$ is trivial (the definition of semisimplicity!);
(2) G is reductive and $Z = 1$;
(3) G is reductive and perfect.

Proof. The equivalence of (2) and (3) is clear from central isogeny decomposition.

For the equivalence of (1) and (2), the point is that we know that the formation of Z is compatible with base change, so we can assume that $k = \overline{k}$. Being central and solvable, Z is always in the solvable radical. Conversely, a normal connected solvable subgroup in a connected reductive group is a normal torus (as it is a solvable connected reductive group, hence a torus) and this is central torus (as normal tori in connected linear algebraic groups are always central, due to étaleness of the automorphism scheme of a torus). $\square$

All the real work in representation theory is in the semisimple case, but for inductive purposes it's better to work in the reductive setting because one wants to make constructions, e.g. torus centralizers, which pass out of the semisimple realm.

3.3. **Applications.** For any *central* isogeny $G \twoheadrightarrow G'$ (or more generally central quotient map) between connected reductive groups over *any* field k, we have seen that there is a natural bijection

$$\{\text{maximal tori of } H\} \Longleftrightarrow \{\text{maximal tori of } H'\}$$

given by

$$\begin{aligned} T &\mapsto f(T) \\ f^{-1}(T') &\leftarrow\!\shortmid T'. \end{aligned}$$

The same applies for parabolic k-subgroups, as well as for Borel k-subgroups (if any exist!), by the same argument.

Apply this to $Z \times \mathscr{D}G \to G$ for connected reductive G over a field k to obtain that the set of maximal k-tori of $\mathscr{D}G$ is in bijection with the set of maximal k-tori of G via the constructions

$$\begin{aligned} \mathscr{T} &\mapsto Z \cdot \mathscr{T} \\ T \cap \mathscr{D}G &\leftarrow\!\shortmid T \end{aligned}$$

and likewise for parabolics k-subgroups and Borel k-subgroups. We emphasize that these bijections work with k-subgroups; e.g., G contains a Borel k-subgroup if and only if $\mathscr{D}G$ does.

Example 3.3.1. This fails without the centrality assumption. Example 4.2 of the "Basics of reductive groups" handout shows that such a counterexample over any local function field k of characteristic $p > 0$ is given by the non-central isogeny $f : \underline{D}^\times \to \mathrm{GL}_p$ arising from the relative Frobenius isogeny for $\underline{D}^\times$ where D is a central division algebra of dimension p^2 over k. (If $T \subset \mathrm{GL}_p$ is a split maximal k-torus then $f^{-1}(T)$ is a *non-smooth* k-subgroup scheme of $\underline{D}^\times$, and $f^{-1}(T)_{\mathrm{red}}$ is *not* a smooth k-subgroup scheme of $\underline{D}^\times$; the same goes for a Borel k-subgroup of GL_p.)

The key point is that $\underline{D}^\times$ has no Borel k-subgroup nor a split maximal k-torus due to D being a central division algebra, and D is split by any degree-p extension (such as the p-power endomorphism $k \to k$) due to local class field theory. (The invariant of a central simple algebra multiplies by d under a degree-d extension, so a central division algebra with invariant $j/p \in \mathbf{Q}/\mathbf{Z}$ splits after a degree-p extension.) It is this latter splitting that allows us to identify the Frobenius base change of $\underline{D}^\times$ with GL_p.

3.4. **Central isogenies preserve roots.** The central isogeny decomposition shows that up to central isogeny, for problems with connected reductive k-groups G it is often sufficient to examine $\mathscr{D}G$, which has the advantage of being semisimple. (This applies, for instance, in studying problems involving maximal k-tori, or Borel k-subgroups, and so on.) Here is another instance of the same principle:

Proposition 3.4.1. *For a central isogeny $f : G \to G'$ of connected reductive k-groups, and $T \subset G$ a split maximal k-torus with $T' = f(T') \subset G'$ (necessarily also split maximal), under the inclusion $\mathrm{X}(f) : \mathrm{X}(T') \hookrightarrow \mathrm{X}(T)$ the set of roots $\Phi' := \Phi(G', T')$ is mapped bijectively to $\Phi(G, T) =: \Phi$.*

(We need a split maximal torus to have a weight space decomposition. The formation of weight space decomposition commutes with field extension, so the facts we know about root spaces over algebraically closed fields, such as 1-dimensionality and $\Phi \cap \mathbf{Q} \cdot a = \{\pm a\}$, apply over k.)

Before proving this result, we make some general observations.

Remark 3.4.2. This is only interesting in characteristic p since it is trivial when f is étale (as then $\mathrm{Lie}(f)$ is an isomorphism). If f is not étale (e.g. $\mathrm{SL}_p \to \mathrm{PGL}_p$) then the finite group scheme $\ker f$ must have nonzero Lie algebra, so $\mathfrak{g} \to \mathfrak{g}'$ is not injective and hence is not surjective (as G and G' have the same dimension). This failure of surjectivity is what gives the result some substance, as it is not clear a priori that a root line in the target is in the image of $\mathrm{Lie}(f)$. The proof will show that it is in the image (of a unique root line in the source).

Remark 3.4.3. Apply this result to the central isogeny

$$Z \times \mathscr{D}G \to G.$$

If $T \subset G$ is a split maximal torus, then it pulls back to a split maximal torus of the form $Z \times \mathscr{T} \subset Z \times \mathscr{D}G$. This induces a finite-index inclusion

$$\mathrm{X}(T) \hookrightarrow \mathrm{X}(Z) \oplus \mathrm{X}(\mathscr{T}).$$

Clearly $\Phi(G,T)$ restricts to 0 on $\mathrm{X}(Z)$, so must map bijectively to $\Phi(\mathscr{D}G, \mathscr{T})$. In other words, every root of the derived group arises from a unique root of the ambient group. Informally, this says that the root system of G only knows $\mathscr{D}G$. (The roots also only know about G up to central isogeny, which is why one needs to keep track of a *root datum* in the classification up to isomorphism.)

Exercise 3.4.4. Centrality is crucial! For the relative Frobenius isogeny $F_{G/k} \colon G \to G^{(p)}$ in characteristic p (e.g. $\mathrm{SL}_n \to \mathrm{SL}_n$ by $(x_{ij}) \mapsto (x_{ij}^p)$), the effect on the roots is to identify $\mathrm{X}(T^{(p)})$ with $p \cdot \mathrm{X}(T)$ and $\Phi(G^{(p)}, T^{(p)})$ with $p\Phi(G,T)$. (Hint: show that the composition of the natural bijection $\mathrm{X}(T) \simeq \mathrm{X}(T^{(p)})$ induced by base change with the pullback map $\mathrm{X}(T^{(p)}) \to \mathrm{X}(T)$ is multiplication by p).

Example 3.4.5. There are even more bizarre examples, coming from exceptional isogenies in low characteristic such as $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$ in characteristic 2. This has commutative non-central kernel α_2^{2n}. When $n = 1$, this is $\mathrm{PGL}_2 \to \mathrm{SL}_2$, which is an instance of the more general disorienting map $\mathrm{PGL}_p = \mathrm{SL}_p\,\mu_p \to \mathrm{SL}_p$ (induced by $F_{\mathrm{SL}_p/k}$) from the adjoint type to the simply connected type.

Proof of Proposition 3.4.1. Choose $\lambda \in \mathrm{X}_*(T)$ and let $\lambda' = f \circ \lambda \in \mathrm{X}_*(T')$. Since f induces an isomorphism $\mathrm{X}_*(T)_{\mathbf{Q}} \simeq \mathrm{X}_*(T')_{\mathbf{Q}}$, we can choose the cocharacter λ so that *both* λ and λ' are generic (i.e. lie outside the root hyperplanes). We have the map of open cells

$$\begin{array}{ccccccc}
G & \hookleftarrow & U(-\lambda) & \times & T & \times & U(\lambda) \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
G' & \hookleftarrow & U(-\lambda') & \times & T' & \times & U(\lambda')
\end{array}$$

Since $\ker f \subset Z_G \subset Z_G(T) = T$, we have $\ker f = \ker(T \to T')$. So $U(-\lambda) \simeq U(-\lambda')$ and $U(\lambda) \simeq U(\lambda')$. Since the roots are contained in the Lie algebras of these unipotent factors, the result now follows from passing to the map on Lie algebras. □

4. Borel's covering theorem

4.1. **Statement and proof.** We need one more big theorem before we get into the fine structure theory of reductive groups, which asserts that a linear algebraic group is covered by Borel subgroups, a vast generalization of the statement that any $n \times n$ matrix over an algebraically closed field can be conjugated to be upper triangular.

Theorem 4.1.1 (Borel's Covering Theorem)**.** *Let G be a connected linear algebraic group over $k = \overline{k}$. Let $B \subset G$ be a Borel subgroup. Then*

$$G(k) = \bigcup_{g \in G(k)} gB(k)g^{-1}.$$

Remark 4.1.2. We'll see later that this has *many* useful consequences. For $G = \mathrm{GL}_n$ it is the familiar fact that every matrix can be conjugated to be upper triangular.

Outline of proof. For a linear algebraic subgroup $H \subset G$, we define

$$\Sigma_H = \bigcup_{g \in G(k)} gH(k)g^{-1}.$$

Step 1. We'll show that Σ_B is closed in $G(k)$ (for the Zariski topology).

Step 2. Choose $T \subset B$ a maximal torus (necessarily also maximal in G). Then $Z_G(T) \subset B$ (e.g. pass to $G/\mathscr{R}_u(G)$ and use the fact that maximal tori are their own centralizers in connected reductive groups) so

$$\Sigma_{Z_G(T)} \subset \Sigma_B.$$

We'll show that $\Sigma_{Z_G(T)}$ is *dense* in $G(k)$. (This is like the statement that a random matrix is semisimple with distinct eigenvalues.)

Now we begin the proof of Theorem 4.1.1. Consider the map

$$f_H \colon G \times G \xrightarrow{\mu} G \times G \xrightarrow{p_1 \times 1} (G/H) \times G = (G \times G)/(H \times 1).$$

where μ sends $(g, g') \mapsto (g, gg'g^{-1})$, which is an isomorphism. We track the subgroup $G \times H$:

$$\begin{tikzcd}
f_H \colon G \times G \arrow[r, "\mu", "\sim"'] & G \times G \arrow[r, "p_1 \times 1"] & (G/H) \times G \\
G \times H \arrow[u, hook] \arrow[r] & \mu(G \times H) \arrow[u, hook] \arrow[r] & f_H(G \times H) \arrow[u, hook]
\end{tikzcd}$$

Now $\mu(G \times H) = \{(g, ghg^{-1})\}$ which is stable by H-translation on the right on the first factor. So the map $\mu(G \times H) \to f_H(G \times H)$ is a *quotient* map by $H \times 1$. In particular, by descent theory for closed subschemes, $f_H(G \times H)$ is closed in $(G/H) \times G$.

Note that $p_2(f_H(G\times H))(k)=\Sigma_H$. So if G/H were proper, then $(G/H)\times G\xrightarrow{p_2}G$ would be closed, hence Σ_H would be closed in G. But that properness holds for $H=B$. This completes Step 1.

Next, we want to show that Σ_H is dense when $H\subset G$ is a Cartan subgroup. Working with general H, consider the map π defined by composing f_H with p_2:

$$\begin{tikzcd} f_H(G\times H) \arrow[r] \arrow[dr, "\pi"'] & (G/H)\times G \arrow[d, "p_2"] \\ & G \end{tikzcd}$$

which has image Σ_H on k-points. We want to show that this is dominant, so we should get a handle on the dimension of the source. The map f_H is the composition of μ, which is an isomorphism, with a quotient by $H\times 1$, so

$$\begin{aligned} \dim f_H(G\times H) &= \dim \mu(G\times H)-\dim(H\times 1)\\ &= \dim G+\dim H-\dim H\\ &= \dim G \end{aligned}$$

If $\Sigma_H\subset G(k)$ is dense then π has finite fibers over a dense open in $G(k)$ (as for any dominant map between varieties of the same dimension). Even better, the converse holds with just *one* finite non-empty fiber over $G(k)$, by semicontinuity for fiber dimension.

So what is the fiber of π over a point $g_0\in G(k)$? Chasing through the definition,

$$\pi^{-1}(g_0)=\{(gH,g_0)\mid g^{-1}g_0g\in H\}.$$

Note that $g^{-1}g_0g\in H \iff g_0\in gHg^{-1}$. This is finite if and only if

$$\{\overline{g}\in G(k)/H(k)\,|\,g_0\in gHg^{-1}\}$$

is finite. We'd like to massage this a little bit, and replace G/H with $G(k)/N_G(H)(k)$. Of course, the condition on g only depends on its right $N_G(H)$-coset, not just its right H-coset, but we have to make sure that the finiteness aspect of the counting isn't screwed up either. So *if* $H\subset N_G(H)$ has finite index on k-points, then we can index by $\overline{g}\in G(k)/N_G(H)(k)$. We claim that this finite-index property holds for Cartan subgroups H:

Example 4.1.3. The group $H:=Z_G(T)$ for maximal $T\subset G$ has finite index in $N_G(H)$. Indeed, $T\subset Z_G(T)$ is the *unique* maximal torus in $Z_G(T)$, so any (smooth) subgroup of G normalizing $Z_G(T)$ must normalize T. Hence,

$$Z_G(T)=H\subset N_G(H)\subset N_G(T)$$

but the outer inclusion is finite index on k-points!

The upshot is that it is enough to find g_0 which lies in only finitely many (and at least one!) Cartan subgroups. We will show that most points in a maximal torus T of G lie in a unique Cartan subgroup, namely $Z_G(T)$; that will finish the proof of Theorem 4.1.1.

Definition 4.1.4. For $T\subset G$, we say that $t\in T(k)$ is *regular* if $a(t)\neq 1$ for all $a\in\Phi(G,T)$.

This means that

$$t \in T - \left(\bigcup_{a \in \Phi} \ker a \right)$$

so there are many such t.

Remark 4.1.5. A priori this definition depends on a choice of T, but we will later see that it is in fact an intrinsic property of a semisimple element $t \in G(k)$.

Example 4.1.6. For $G = \mathrm{GL}_n$ and T the diagonal torus, then

$$\Phi := \Phi(G,T) = \{a_{ij} \colon t \mapsto t_i/t_j\}_{i \neq j}$$

so $t \in T$ is regular with respect to T if and only if t has distinct eigenvalues.

Example 4.1.7. For $G = \mathrm{SL}_2$ and T the diagonal torus

$$T = \left\{ \begin{pmatrix} c & \\ & c^{-1} \end{pmatrix} \right\}$$

the roots are $\left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right) \mapsto c^{\pm 2}$, so regularity is the condition $c^2 \neq 1$.

Example 4.1.8. For $G = \mathrm{PGL}_2$ and T the diagonal torus

$$T = \left\{ \begin{pmatrix} c & \\ & 1 \end{pmatrix} \right\}$$

the roots are $\left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right) \mapsto c^{\pm 1}$, so regularity is the condition $c \neq 1$.

We wish to analyze the scheme-theoretic centralizer $Z_g(t)$ and its Lie algebra. Since t is an element rather than a k-group, we shall relate this to constructions involving the Zariski closure $M = \overline{\langle t \rangle}$ of the cyclic subgroup generated by t. Note that M is a smooth (possibly disconnected) closed subgroup of T, so in the short exact sequence

$$1 \to M^0 \to M \to M/M^0 \to 1,$$

M^0 is a torus and M/M^0 is a finite group whose order is not divisible by char k (since M/M^0 is a finite constant subgroup of the *torus* T/M^0, and over a field of characteristic $p > 0$ a torus has no non-trivial geometric p-torsion, as follows from the case of $\mathbf{G}_m$).

We claim that that the scheme-theoretic centralizer $Z_G(M)$ is smooth. We know that $Z_G(M^0)$ is smooth: the proof by infinitesimal criteria in the previous course came down to the fact that the finite-dimensional representation theory of a k-torus is semisimple. Since M/M^0 is a finite constant group of order not divisible by the characteristic (so its representation theory is also semisimple), we can push through the infinitesimal criterion for M as well. See Lemma 3.2 of the handout on "Applications of Borel's covering theorem" for further details.

Since $t^{\mathbf{Z}} \subset M(k)$ is Zariski-dense in M, it is schematically dense (i.e. the map $k[M] \hookrightarrow \prod_{n \in \mathbf{Z}} k$ via $f \mapsto (f(t^n))_n$ is injective). Although tensoring does not commute with infinite direct products in general, *over a field* the natural map map

$$V \otimes \prod W_i \hookrightarrow \prod (V \otimes W_i)$$

is automatically injective. Therefore, for any k-algebra A it follows that the coordinate ring of M_A injects into the direct product of copies of A indexed by $t^{\mathbf{Z}}$ (via evaluation at t^n's), so $Z_G(t) = Z_G(M)$ as *schemes* (i.e., as functors on k-algebras). Thus, $Z_G(t) = Z_G(M)$ is smooth, with Lie algebra

$$\mathfrak{g}^M = \mathfrak{g}^{t=1} = \mathfrak{g}^T \oplus \bigoplus_{\substack{a\in\Phi \\ a(t)=1}} \mathfrak{g}_a.$$

It follows that $\dim Z_G(t)^0 \geq \dim Z_G(T) = \dim \mathfrak{g}^T$ with equality if and only if $a(t) \neq 1$ for all $a \in \Phi$; i.e. t is regular. In particular, $Z_G(t)^0 = Z_G(T)$ in the regular case.

Proposition 4.1.9. *Let G be a connected linear algebraic group over $k = \overline{k}$ and $T \subset G$ a maximal torus. For $t \in T(k)$ regular (with respect to T),*

(1) T is the unique maximal torus containing t,
(2) $Z_G(T)$ is the unique Cartan subgroup containing t (and it is $Z_G(t)^0$).

Remark 4.1.10. Part (2) settles the proof of Borel's Covering Theorem 4.1.1.

Proof. We saw that regularity implies $Z_G(t)^0 = Z_G(T)$, which has T as a central (hence unique) maximal torus. Since tori are commutative, tori containing t lie in $Z_G(t)^0 = Z_G(T)$. This implies (1).

For (2), suppose $C = Z_G(S)$ is a Cartan subgroup containing t (for $S \subset G$ a maximal torus). Since C is connected and solvable (as for Cartan subgroups in general), by the structure theorem for connected solvable groups over algebraically closed fields and the fact that S is central in C we obtain

$$C = S \ltimes U = S \times U$$

for $U := \mathscr{R}_u(C)$. But $t \in C$ is a semisimple element, so t is necessarily killed by the quotient map $C \twoheadrightarrow C/S = U$ because U is unipotent, so $t \in S$. This forces $S = T$ by (1). $\square$

Example 4.1.11. Here is an example to show that $Z_G(t)$ can be disconnected in general. For $G = \mathrm{PGL}_2$, T the diagonal torus, and $t = \mathrm{diag}(-1, 1)$ over a field k with char $k \neq 2$ one computes:

$$Z_G(t) = T \rtimes \langle w \rangle$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

4.2. **Some applications of Borel's covering theorem.** See the handout "Applications of Borel's covering theorem" for a comprehensive discussion of many applications of the covering theorem, with proofs that are sometimes quite different from the ones in standard textbooks. Here we will only discuss the statements of the main applications. None of what follows is used in the rest of this course, but these results are important in practice.

Let G be *any* connected linear algebraic group over *any* field k.

(i) If $g \in G(k)$ is semisimple, then $g \in T(k)$ for some maximal k-torus $T \subset G$. (Warning: you can't try the identity component of the closure of the subgroup generated by g because g could have finite order, in which case $g \notin \overline{\langle g \rangle}^0$). Furthermore, it is always the case that $g \in Z_G(g)^0$.

Remark 4.2.1. This implies, in view of our arguments with $Z_G(t)$ above, that for *any* semisimple $g \in G(k)$, $Z_G(g)^0$ is smooth with Lie algebra $\mathfrak{g}^{g=1}$ (which you can check over $\overline{k}$) with dimension $\geq \dim(\text{Cartan})$, and that equality of dimensions holds if and only if $Z_G(g)^0$ is itself a Cartan subgroup, in which g lies in a unique maximal k-torus. In such cases we say that g is *regular*. (This shows that regularity is an intrinsic property of semisimple elements of $G(k)$.)

(ii) If $g \in G(k)$ is unipotent (so necessarily of finite order in characteristic $p > 0$) then $g \in U(k)$ for some unipotent smooth connected k-subgroup $U \subset G$ *provided* that k is perfect. (Example 3.5 of the handout gives counterexamples over every imperfect field.)

In characteristic 0 this is easy because $\overline{\langle g \rangle}$ is unipotent and unipotent groups in characteristic 0 are automatically connected. (The reason for connectedness in characteristic 0 is that semisimplicity and unipotence are well-behaved with respect to subgroups and homomorphisms, so if a unipotent group were disconnected then its finite component group would be too. But in characteristic 0 nontrivial unipotent elements always have infinite order, as we see by computing in GL_n - the exact opposite of what happens in characteristic p.)

For the proofs of (i) and (ii), one first settles the case over $\overline{k}$ by using Borel's Theorem (e.g., a unipotent element u is in some Borel subgroup B by the covering theorem, and dies in the quotient of B by $\mathscr{R}_u(B)$ since that is a torus, and hence $u \in U := \mathscr{R}_u(B)$). More work is needed to bootstrap from the result over $\overline{k}$ to get the result over k.

Remark 4.2.2. Steinberg defines *regularity* for $g \in G(k)$ with $k = \overline{k}$ by the condition

$$\dim Z_G(g) = \dim(\text{Cartan}).$$

(The inequality $\geq$ always holds, but is not obvious.) Warning: [Bor] defines an element to be regular if its semisimple part is regular, which unfortunately would never hold (for instance) for a unipotent element in a nontrivial connected semisimple group, whereas Steinberg's definition is satisfied by many unipotent elements in general.

(iii) Using (i) and the fact that any two maximal tori are conjugate over a separably closed field, we see that over $k = k_s$ there is a bijection

$$\{g \text{ semisimple} \in G(k)\}/G(k)\text{-conj} \longleftrightarrow T(k)/W$$

for $W = N_G(T)/Z_G(T)$, a finite constant group (since $k = k_s$).

It makes sense to seek Lie algebra versions of (i) and (ii), since we have notions of semisimplicity and nilpotence for elements of $\mathfrak{g}$ (definitions which rest on the specification of G too, especially in positive characteristic). Namely, we can ask if a semisimple element of the Lie algebra is tangent to a k-torus, or if a nilpotent element of the Lie algebra is tangent to a (connected) unipotent smooth closed k-subgroup. These work out well in the geometric case, as follows.

(iv) If $k = \overline{k}$ and $X \in \mathfrak{g}$ then

- X semisimple $\Longrightarrow X \in \mathrm{Lie}(T)$ for some maximal torus $T \subset G$,

- X nilpotent $\implies$ $X \in \mathrm{Lie}(U)$ for some unipotent connected linear algebraic subgroup $U \subset G$.

For the semisimple case, the proof uses the smoothness of the scheme-theoretic centralizer $Z_G(X)$ for X under Ad_G. For the nilpotent case, the proof uses the full force of the structure theory for connected reductive groups (Bruhat decomposition, root systems, and so on); note that for the treatment of the nilpotent case one can reduce the general case to the reductive case by quotienting by the unipotent radical.

Remark 4.2.3. For connected linear algebraic groups H over a general field k, there is a finer notion than solvable called *nilpotence*, which is defined as the descending central series reaching $\{1\}$. The main example is that of (connected) unipotent groups. The handout on nilpotence shows that nilpotence is equivalent to solvability with a central maximal k-torus. For example, a Cartan k-subroup is always nilpotent.

5. Exponentiating root spaces

5.1. **The reductive case: examples.** Let G be a connected reductive group over a field k. We know that if G has a split maximal k-torus T (which it may not), then

- $\Phi := \Phi(G,T) = -\Phi$,
- $\mathbf{Q}a \cap \Phi = \{\pm a\}$ for $a \in \Phi$,
- $\mathfrak{g}_a$ is 1-dimensional for all $a \in \Phi$.

We want to "exponentiate" $\mathfrak{g}_a$ to a canonical k-subgroup $U_a \subset G$ (the uniqueness is obvious in characteristic 0, but can fail characteristic $p > 0$ without further requirements on U_a, as we shall see below!). More generally, for suitable spans $\mathfrak{h} \subset \mathfrak{g}$ of root lines, we seek a canonical connected unipotent linear algebraic subgroup $U \subset G$ with $\mathrm{Lie}(U) = \mathfrak{h}$ *and* such that U is normalized by T. (The second property is automatic in characteristic 0, but not so in characteristic $p > 0$.)

Example 5.1.1. Let $G = \mathrm{SL}_4$ and T the diagonal subgroup

$$\left\{\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_4 \end{pmatrix} : \prod t_i = 1\right\}$$

Then $\mathrm{X}(T) = \mathbf{Z}^4/\Delta$ and $\Phi = \{a_{ij} : t \mapsto t_i/t_j\}_{i \neq j}$. Letting

$$U_{a_{ij}} = \left\{\begin{pmatrix} 1 & & & \\ & 1 & \dots & x \\ & & \ddots & \\ & & & 1 \end{pmatrix} : x = ij \text{ entry}\right\}$$

we have

$$U_{a_{12}} \times U_{a_{34}} = \left\{ u(x,y) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & y \\ & & & 1 \end{pmatrix}\right\} \simeq \mathbf{G}_a \times \mathbf{G}_a$$

and $t\,u(x,y)t^{-1} = u(t_1 t_2^{-1} x, t_3 t_4^{-1} y)$ for $t \in T$. If char $k = p > 0$, consider the subgroup

$$U = \mathbf{G}_a \hookrightarrow \mathbf{G}_a \times \mathbf{G}_a = U_{a_{12}} \times U_{a_{34}}$$

embedded by $z \mapsto (z, z^p)$. Then $\mathrm{Lie}(U) = \mathfrak{g}_{a_{12}}$ is a T-root line, but U is not T-stable because conjugation by t scales the first coordinate scales by $a_{12}(t)$ and the second by $a_{34}(t)$, and $a_{34}(t) \neq a_{12}(t)^p$.

Example 5.1.2. For $G = \mathrm{SL}_3$, let $a = a_{12}$, $b = a_{23}$, and $c = a_{13} = a + b$. In terms of

$$\left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right\}$$

note that x is the coordinate on U_a, z is the coordinate on U_b, and y is the coordinate on U_c. Then

$$U_b \cdot U_{a+b} = \begin{pmatrix} 1 & 0 & y \\ & 1 & z \\ & & 1 \end{pmatrix}$$

is a subgroup of G with Lie algebra having weights b and $a+b$. On the other hand,

$$U_a \cdot U_b = \begin{pmatrix} 1 & x & 0 \\ & 1 & z \\ & & 1 \end{pmatrix}$$

is not a subgroup of G. You can see this by explicit computation, or by observing that the commutator of $\mathfrak{g}_a$ and $\mathfrak{g}_b$ contains $\mathfrak{g}_{a+b}$. Note that there is a subsemigroup $A \subset \mathrm{X}(T)$ meeting the set of roots in exactly b and $a+b$, but no such A exists for the pair a, b (since such an A would obviously have to contain $a+b$); see Figure 5.1.2. The significance of the existence (or not) of such an A will become apparent later, when we study which spans of root lines can be "exponentiated" to a (unique) T-smooth unipotent connected linear algebraic subgroups.

Next time we'll discuss in the general setting of split tori S acting on connected linear algebraic groups H how to use *dynamic methods* to construct S-stable connected linear algebraic subgroups of H whose Lie algebra realizes sets of S-weights determined by a subsemigroup of $\mathrm{X}(S)$. (The use of subsemigroups will later be related to the notion of *closed sets of roots* that cannot consider at this stage since we have not yet shown that sets of nontrivial weights occurring on the Lie algebra constitute a root system.) This will be the essential key to how we can set up the structure theory of reductive groups over fields without needing to first develop the entire theory in the split case (or over algebraically closed fields) first.

5.2. **Subgroups corresponding to root semigroups.** Let G be a connected linear algebraic group over a field k equipped with an action by a k-torus S. (Reductivity will play *no* role in this discussion.)

For $H = S \ltimes G$, we have $Z_H(S) = S \times G^S$, where G^S is the functorial centralizer of S, defined as the fixed points of the S-action. Passing to H reduces many problems to the more familiar case of the conjugation action by a subtorus.

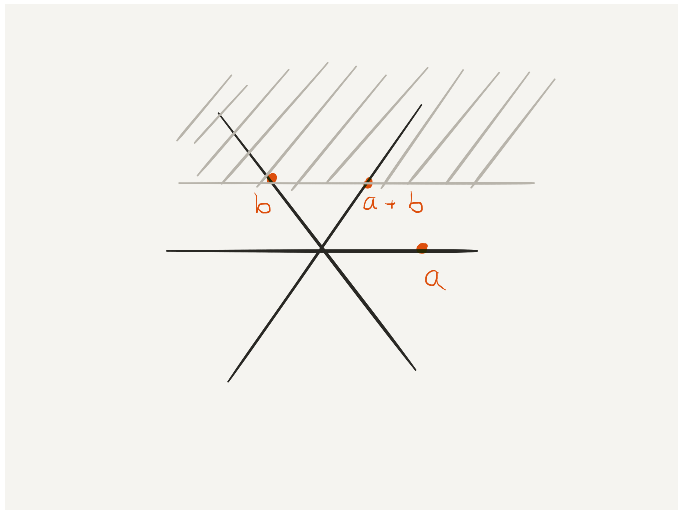


FIGURE 5.1.1. The A_2 root system corresponding to SL_3. We can find a subsemigroup, such as an affine half-space, meeting the set of roots in $\{b, a+b\}$ but not in $\{a, b\}$.

Example 5.2.1. For $S = \mathbf{G}_m$, we have $S \times G^S = Z_H(\lambda)$ for

$$\begin{aligned}\lambda\colon \mathbf{G}_m = S &\hookrightarrow S \ltimes G\\ s &\mapsto (s,1)\end{aligned}$$

Setup. Let $\mathrm{wt}_S(G) = \{a \in \mathrm{X}(S) \mid \mathfrak{g}_a \neq 0\}$. We want to exponentiate (spans of) suitable S-weight spaces to S-stable connected linear algebraic subgroups (with S-stability ensuring a uniqueness that cannot be guaranteed by the Lie algebra alone in positive characteristic).

Lemma 5.2.2. *The set of S-weights on the augmentation ideal $I_G \subset k[G]$ lies in the subsemigroup $\langle wt_S(G)\rangle$ generated by $wt_S(G)$.*

Proof. Since G is connected, its coordinate ring $k[G]$ is a domain and hence injects into the local noetherian ring $k[G]_{I_G}$. This local ring is filtered by powers of its maximal ideal and so each nonzero element of its maximal ideal has nonzero image in I_G^n/I_G^{n+1} for some $n > 0$ by the Krull Intersection Theorem. Any successive quotient I_G^n/I_G^{n+1} is a quotient of $(I_G/I_G^2)^{\otimes n}$, and $I_G/I_G^2 = \mathfrak{g}^*$. This is all S-equivariant (giving $\mathfrak{g}^*$ the linear dual action by S). $\square$

Remark 5.2.3. Depending on how you define the action of G on its coordinate ring, the weights which show up are those claimed or their negations (of course, this difference blurs in the case of the action of the commutative S, especially when the set of weights which occur on the Lie algebra is stable under negation as for the conjugation on a connected reductive group by a split maximal torus).

Proposition 5.2.4. [CGP, Prop. 3.3.6] *Let $A \subset \mathrm{X}(S)$ be a subsemigroup (i.e., a subset set, possibly empty, stable under addition). There is a unique S-stable connected linear algebraic k-subgroup $H_A(G) \subset G$ with the property that*

$$\mathrm{Lie}(H_A(G)) = \bigoplus_{a \in A \cap wt_S(G)} \mathfrak{g}_a.$$

Moreover, $H_A(G)$ enjoys the following properties.

(1) (maximality) If $H \subset G$ is an S-stable connected linear algebraic k-subgroup such that $wt_S(\mathfrak{h}) \subset A$ then $H \subset H_A(G)$.
(2) If $0 \notin A$ then $H_A(G)$ is unipotent (denoted $U_A(G)$ for emphasis).

Remark 5.2.5. The weight spaces $\mathfrak{g}_a$ for nonzero a can be pretty large because of the generality: in particular, they are not necessarily lines!

Idea of proof. We're going to explain the idea of the proof if $0 \notin A$, as happens in most cases of interest. (When $0 \in A$, one has to include G^S in the construction below.)

To build $H_A(G)$, we want to construct things with $A \cap \mathrm{wt}_S(G)$ showing up in the Lie algebra. For each $a \in A \cap \mathrm{wt}_S(G)$, consider the k-subgroup scheme $Z_G(\ker a) \subset G$, which is smooth even if $\ker a$ is non-reduced since $\ker a \subset S$ (see near the start of §3.2); this subgroup might be disconnected.

How does S act on $Z_G(\ker a)$? Well, $\ker a$ acts trivially, so the S-action factors through the quotient $S/\ker a$, which can be identified *via* a with $\mathbf{G}_m$. Let

$$\lambda_a : \mathbf{G}_m \times Z_G(\ker a) \to Z_G(\ker a)$$

be the induced action. Note that λ_a is *not* a one-parameter subgroup, but by the semi-direct product trick as discussed above we can apply all of the dynamic formalism to $\mathbf{G}_m$-actions on affine group schemes of finite type. Hence, we can make sense of $U_{Z_G(\ker a)}(\lambda_a)$ as a unipotent smooth k-group that is moreover connected (as groups $U_H(\mu)$ arising in the dynamic formalism are always connected, even when H is not, since the limiting definition provides an affine line from each geometric point to the identity point.).

This construction extracts the part of the Lie algebra with weights which are *positive* with respect to λ_a. That is, the Lie algebra of $U_{Z_G(\ker a)}(\lambda_a)$ consists of the *positive* integral multiples of a that are S-weights on $\mathfrak{g}$:

$$\mathrm{Lie}\, U_{Z_G(\ker a)}(\lambda_a) = \bigoplus_{b \in \mathrm{wt}_S(G) \cap \mathbf{Z}_{\geq 1} a} \mathfrak{g}_b.$$

Define the S-stable connected linear algebraic subgroup

$$H_A(G) = \langle U_{Z_G(\ker a)}(\lambda_a) \rangle_{a \in A \cap \mathrm{wt}_S(G)} \subset G.$$

The real content is to show that its Lie algebra doesn't have unexpected weights. The key is that we control the weights showing up in the coordinate ring of each $U_{Z_G(\ker a)}(\lambda_a)$, and the group generated by these has a dominant multiplication map from a direct product of finitely many of these groups (perhaps with high multiplicity); that realizes the coordinate ring of $H_A(G)$ as an S-equivariant k-subalgebra of a tensor product of finitely many copies of the $U_{Z_G(\ker a)}(\lambda_a)$'s. So the fact that this has the expected Lie algebra is ultimately due to Lemma 5.2.2.

How do we show unipotence of $H_A(G)$ when $0 \notin A$? Let $H := S \ltimes H_A(G)$. Then $Z_H(S)$ has Lie algebra $\mathfrak{h}^S = \mathrm{Lie}(S)$ since $0 \notin A$, so the inclusion $S \subset Z_H(S)$ is equality. Hence S is a *maximal* k-torus in H, so in particular the dimension of the maximal tori of H (i.e., its *rank*) is $\dim S$. This implies that $H_A(G)$ has trivial maximal torus - for instance, by working over $\overline{k}$ we see that

$$\dim S = \mathrm{rank}\, H = \dim S + \mathrm{rank}\, H_A(G),$$

so $\mathrm{rank}\, H_A(G) = 0$, implying that $H_A(G)$ is unipotent.

□

Example 5.2.6. Let G be a connected reductive group with *split* maximal k-torus T. By Proposition 5.2.4, for $a \in \Phi(G,T)$ there is a unique connected linear algebraic group $U_a \subset G$ that is T-stable with $\mathrm{Lie}(U_a)$ having as its set of T-weight $\mathrm{wt}_T(G) \cap \langle a \rangle = \{a\}$, where $\langle a \rangle$ denotes the subsemigroup $\{na\}_{n \geq 1}$. So we know that U_a is unipotent with $\mathrm{Lie}(U_a) = \mathfrak{g}_a$ a *line*. Therefore, $\dim U_a = 1$. We call this the *a-root group*.

We would like to show that $U_a \simeq \mathbf{G}_a$; this is a special case of Proposition 5.2.12 below. Once this is proved, since $\mathrm{Aut}_{k_s}(\mathbf{G}_a) = k_s^\times$ (true over *fields* but not arbitrary bases!) it would follows that transporting the T-action on U_a over to $\mathbf{G}_a$ becomes

$$t \cdot x = a(t)x$$

(as the action of each $t \in T(k_s)$ is scaling by some element of $k_s^\times$, and the actual scalar can be read off from the action on the Lie algebra of $\mathbf{G}_a$).

The unique characterization of U_a implies that if $f : G \to G'$ is a central isogeny and for the split maximal k-torus $T' := f(T) \subset G'$ we denote by $a' \in \Phi(G', T')$ the root corresponding to a as in Proposition 3.4.1 then f carries U_a *isomorphically* on $U_{a'}$ (so the formation of root groups is insensitive to passing to central isogenous quotients!). Indeed, if we choose generic $\lambda \in \mathrm{X}_*(T)$ (i.e., $\langle b, \lambda \rangle \neq 0$ for all $b \in \Phi(G,T)$) then $\lambda' := f \circ \lambda$ is generic for (G', T') and in the proof of Proposition 3.4.1 we saw that f carries $U_G(\lambda)$ isomorphically onto $U_{G'}(\lambda')$. Since necessarily $U_a \subset U_G(\lambda)$ and $U_{a'} \subset U_{G'}(\lambda')$, the unique characterization forces the isomorphism $f : U_G(\lambda) \simeq U_{G'}(\lambda')$ to carry U_a isomorphically onto $U_{a'}$. This reasoning applies *verbatim* when $G' = G/Z$ for any (not necessarily finite) central closed k-subgroup scheme $Z \subset G$; i.e., the formation of root groups is compatible with central quotients.

Example 5.2.7 (Rosenlicht). If k is not perfect, then we can always make unipotent, connected, 1-dimensional k-groups which are *not* $\mathbf{G}_a$! If $c \in k - k^p$ for $p = \mathrm{char}(k)$ then take

$$U = \{y^p = x - c x^p\} \subset \mathbf{A}^2 = \mathbf{G}_a \times \mathbf{G}_a.$$

Its closure in $\mathbf{P}^2$ has one point, which is regular, at the line at ∞. However, this point is not rational (it is Spec $k(\sqrt[p]{c})$). Therefore, U is not isomorphic to $\mathbf{A}^1_k$, since the unique regular compactification of $\mathbf{A}^1_k$ has a unique point at ∞ that is moreover k-rational.

In general, $U_A(G)$ is filtered by $\mathbf{G}_a$'s in the following sense:

Definition 5.2.8. Let U be a unipotent connected linear algebraic k-group. We say that U is *k-split* if there exists a composition series $\{U_j\}$ of connected linear algebraic k-subgroups such that

$$U_j/U_{j+1} \simeq \mathbf{G}_a.$$

We say that U is *k-wound* if it has no copy of $\mathbf{G}_a$ as a k-subgroup; i.e. if there is no k-subgroup inclusion $\mathbf{G}_a \hookrightarrow U$.

Example 5.2.9. Inside GL_n, $\mathscr{R}_u(B)$ is split for a Borel subgroup B. On the other hand, Rosenlicht's construction in Example 5.2.7 is k-wound.

Tits' structure theory of unipotent groups is developed in a self-contained modern manner in [CGP, App. B] (addressed in the handout "Structure of solvable groups over fields"), including split and wound connected unipotent linear algebraic groups and properties of these groups.

Warnings. The notions of *split* and *wound* for connected unipotent linear algebraic k-groups are not as robust as in the theory of k-tori (i.e. where there is a notion maximal split k-torus the quotient by which is k-anisotropic, and maximal k-anisotropic k-subtorus the quotient by which is k-split).

(1) Rosenlicht's wound group is a k-subgroup of a split k-group $\mathbf{G}_a \times \mathbf{G}_a$.
(2) [CGP, Ex. B.2.3] gives a $\mathbf{G}_a$-quotient of a wound k-group.

Proposition 5.2.10. *If k is perfect, then every unipotent connected linear algebraic k-group U is k-split.*

Proof. See [Bor, 15.5(i)] or [CGP, B.2.5]. □

Actually, we'll need a slightly more general result, as follows. For tori we noted above that there is a robust theory of split subtori with anisotropic quotient and vice-versa. For unipotent groups, we only have one of the two: a maximal split smooth connected k-subgroup U_s such that U/U_s is k-wound. More precisely, by [CGP, B.3.4], for any unipotent connected linear algebraic k-group U, there exists a unique k-split connected smooth k-subgroup $U_s \triangleleft U$ such that

$$\overline{U} := U/U_s$$

is k-wound (and the formation of U_s is compatible with separable extension of k).

Remark 5.2.11. Whereas the formation of U_s is compatible with separable extensions on k, the analogue for tori is only compatible with purely inseparanble extensions.

Proposition 5.2.12. *If $0 \notin A$, then $U_A(G)$ is k-split.*

Proof. For $U = U_A(G)$, we want to show that $\overline{U} = 1$. By design, if $\overline{U} \neq 1$ then $\mathrm{wt}_S(\overline{U}) \subset \mathrm{wt}_S(U) \subset A$ is non-empty. But it is a general fact [CGP, B.4.4] that a wound group (such as $\overline{U}$) can only admit a trivial action by a torus (such as S). □

Example 5.2.13. If k'/k is a nontrivial finite separable extension, and G' is a split connected reductive k'-group with $T' \simeq \mathbf{G}_m^r \subset G'$ the split maximal torus, consider the Weil restriction

$$G := \mathrm{R}_{k'/k}(G').$$

We claim that this is connected reductive, with maximal k-torus $T := \mathrm{R}_{k'/k}(T')$ since this can be checked over k_s (and we know that $G_{k_s} \simeq \prod_{\sigma: k' \hookrightarrow k_s} G' \otimes_{k,\sigma} k_s$, and likewise for T). So a maximal torus for G is $T = \mathrm{R}_{k'/k}(\mathbf{G}_m^r) \supset \mathbf{G}_m^r =: S$. (Later we will show that in a connected reductive group over a field, all maximal split tori are rationally conjugate to each other; S is such a k-torus for G.) As an exercise, check that via the identification $\mathrm{X}(S) = \mathrm{X}(T')$, we have

$$\Phi(G, S) = \Phi(G', T')$$

and $\mathfrak{g}_b = \mathfrak{g}'_b$ as k-vector spaces (so of dimension $[k' : k] > 1$), and at the level of roots groups $U_b = \mathrm{R}_{k'/k}(\mathbf{G}_a) \simeq \mathbf{G}_a^{[k':k]}$.

5.3. **Direct spanning of root subgroups.** Let (G, T) be a *split reductive pair* over k (i.e. G is a connected reductive k-group and T a split maximal k-torus). Let $\Phi = \Phi(G, T) \subset \mathrm{X}(T) - \{0\}$ be the set of roots with respect to this pair.

Some reminders. Recall that we showed that the central isogeny

$$Z \times \mathscr{D}(G) \to G,$$

where Z is the maximal central k-torus of G, induces isomorphisms

$$\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(\mathscr{T})_{\mathbf{Q}} \oplus \mathrm{X}(Z)_{\mathbf{Q}}.$$

where $\mathscr{T} := T \cap \mathscr{D}(G)$ is a (split) maximal k-torus of $\mathscr{D}(G)$, and

$$\Phi = \Phi(\mathscr{D}(G), \mathscr{T}) \times \{0\}.$$

Since $\mathscr{D}(G)$ has trivial maximal central torus, we know that the inclusion $\mathbf{Z}\Phi \subset \mathrm{X}(\mathscr{T})$ has finite index, so $\mathbf{Q}\Phi = \mathrm{X}(\mathscr{T})_{\mathbf{Q}}$ inside $\mathrm{X}(T)_{\mathbf{Q}}$.

For $c \in \Phi$, we have a root group $U_c \simeq \mathbf{G}_a$ inside $Z_G(T_c)$, and

$$\mathscr{D}(Z_G(T_c)) = \mathrm{SL}_2 \text{ or } \mathrm{PGL}_2$$

since it is connected semisimple and split of rank 1: the codimension-1 torus $T_c \subset T$ must be maximal central in $Z_G(T_c)$ (as T isn't central) and we have the central isogeny

$$\mathscr{D}(Z_G(T_c)) \times T_c \to Z_G(T_c)$$

satisfying

$$\mathscr{T}_c \times T_c \twoheadrightarrow T$$

where $\mathscr{T}_c := T \cap \mathscr{D}(Z_G(T_c))$ is a 1-dimensional *split* torus. The identification of $\mathscr{D}(Z_G(T_c))$ with SL_2 or PGL_2 can be chosen to carry $\mathscr{T}_c$ over to the diagonal. This implies that the only non-trivial dependencies over $\mathbf{Q}$ among a pair roots are for $\pm c$, and that $\langle U_c, U_{-c} \rangle = \mathscr{D}(Z_G(T_c))$.

As an application of the ubiquity of SL_2 inside split connected semisimple groups, we obtain an important fact in characteristic 0 (never used in this course):

Proposition 5.3.1. *If G is connected reductive over a field k of characteristic 0 then every linear representation of G on a finite-dimensional k-vector space V is completely reducible.*

In contrast, for any field K whatsoever and any linear representation V of smooth connected *unipotent* K-group U, the subspace V^U is nonzero (as we may check over $\overline{K}$ via the Lie–Kolchin theorem).

Proof. Let's check that we can assume $k = k_s$ (so tori become split). This will be mainly an application of the structure theory of finite-dimensional algebras over fields [L, XVII]. Suppose complete reducibility holds for $V' := V_{k_s}$. Let $A \subset \mathrm{End}_k(V)$ be the Galois descent of $k_s[G(k_s)] \subset \mathrm{End}_{k_s}(V')$. Assuming V' is a semisimple G_{k_s}-representation, so it is a semisimple A_{k_s}-module (as $G(k_s)$ is Zariski-dense in G_{k_s} since G is k-smooth), A_{k_s} is a semisimple ring by [L, XVII, Prop. 4.7]. Thus, A_{k_s} has no nonzero nilpotent 2-sided ideal, so the same holds for A. Such vanishing characterizes semisimplicity for finite-dimensional algebras over fields (see [L, XVII, Thm. 6.1]), so A is semisimple. The A-module V therefore decomposes as a direct sum of simple A-modules [L, XVII, Prop. 4.1]. But a k-subspace of V is G-stable if and only if it is an A-submodule (as each can be checked over k_s), so V is completely reducible for G as desired.

Now we may assume $k = k_s$. Let $Z \subset G$ be the maximal central torus and $G' = \mathscr{D}(G)$. Since Z is split, V decomposes into weight spaces for Z. The groups G' and Z commute inside G, so the G'-action on V preserves each Z-weight space. It suffices to treat these weight spaces as G-representations separately, so we are reduced to the case where Z acts through a $\mathbf{G}_m$-valued character. Now a subspace is stable under G if and only if it is stable under G', so we may focus on G'; i.e., we may assume G is semisimple. So far we haven't used that k has characteristic 0!

The crucial fact is that $\mathfrak{g} := \mathrm{Lie}(G)$ is a semisimple Lie algebra. In the theory of finite-dimensional Lie algebras in characteristic 0 as developed in [Bou1, §6], semisimplicity is defined as the vanishing of commutative Lie ideals and it is proved that the linear representation theory of such Lie algebras is completely reducible. Hence, it suffices to prove two things: $\mathfrak{g}$ has no nonzero commutative Lie ideal (so it is semisimple) and that a subspace of V is G-stable if and only if it is $\mathfrak{g}$-stable (under the Lie algebra representation $\mathfrak{g} \to \mathrm{End}(V)$). The proofs of each will use characteristic zero in a crucial way (and are false in every positive characteristic).

Let $W \subset V$ be a subspace. We claim that W is G-stable if and only if it is $\mathfrak{g}$-stable. The "only if" direction is obvious. To prove the converse, inside $\mathrm{GL}(V)$ consider the subgroup H of linear automorphisms preserving W. It is easy to check by computations with a basis of V extending one of W that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(V)$ consists of the endomorphisms of V preserving W. The hypothesis gives that for our representation $\rho : G \to \mathrm{GL}(V)$, $\mathrm{Lie}(\rho)$ lands inside $\mathfrak{h}$, and we want to prove that the scheme-theoretic preimage $\rho^{-1}(H)$ coincides with G. By Cartier's theorem (characteristic 0!) $\rho^{-1}(H)$ is smooth, so it suffices to show that it has full Lie algebra inside $\mathfrak{g}$. The formation of tangent spaces is compatible with fiber products, so $\rho^{-1}(H)$ has Lie algebra $\mathrm{Lie}(\rho)^{-1}(\mathfrak{h}) = \mathfrak{g}$.

Finally, we check that any commutative Lie ideal $\mathfrak{c}$ in $\mathfrak{g}$ vanishes. Since $\mathrm{Lie}(\mathrm{Ad}_G) = \mathrm{ad}_{\mathfrak{g}}$, any Lie ideal is stable under Ad_G by the equivalence of G-stability and $\mathfrak{g}$-stability for subspaces of G-representations. Thus, $\mathfrak{c}$ is a G-subrepresentation of $\mathfrak{g}$. Let $T \subset G$ be a maximal torus, so $\mathfrak{c}$ is a T-subspace of $\mathfrak{g}$. If $\mathfrak{c}$ supports a nonzero T-weight then it contains some root line $\mathfrak{g}_a$. But the subgroup $\langle U_a, U_{-a}\rangle \subset G$ that is SL_2 or PGL_2 contains the standard Weyl element that swaps a and $-a$, so by G-stability $\mathfrak{c}$ would also contain $\mathfrak{g}_{-a}$.

The common Lie algebra $\mathfrak{sl}_2$ of SL_2 and PGL_2 away from characteristic 2 is generated as a Lie algebra by its opposite root lines, so $\mathfrak{c}$ would have to contain $\mathfrak{sl}_2$, contradicting that $\mathfrak{c}$ is commutative.

Hence, the T-action on $\mathfrak{c}$ is trivial, which is to say $\mathfrak{c} \subset \mathfrak{g}^T = \mathrm{Lie}(Z_G(T)) = \mathrm{Lie}(T)$. But T was *arbitrary*! By noetherian induction, the intersection $\bigcap_T T$ of all maximal tori coincides with the intersection $T_1 \cap \cdots \cap T_n$ of finitely many. This intersection coincides with the center Z_G by Corollary 2.4 in the handout "Basics of reductive groups", so

$$\mathfrak{c} \subset \cap_j \mathrm{Lie}(T_j) = \mathrm{Lie}(\cap_j T_j) = \mathrm{Lie}(Z_G).$$

But Z_G is finite since G is semisimple, and it is *smooth* (Cartier's theorem, once again), so $\mathrm{Lie}(Z_G) = 0$. $\square$

Now we consider linearly independent roots $c, c' \in \Phi$ (i.e. $c' \neq \pm c$). Note that

$$\begin{aligned} A :=& \langle c \rangle + \langle c' \rangle = \mathbf{N}\, c + \mathbf{N}\, c \\ =& \{ic + jc' \in \Phi \mid i, j \geq 1\} =: (c, c') \end{aligned}$$

does *not* contain 0. Thus, by Proposition 5.2.4 we have a unipotent connected linear algebraic group

$$U_A(G) = \langle U_a \rangle_{a \in A \cap \Phi}.$$

(The inclusion $\supset$ follows from maximality of $U_A(G)$, and $\subset$ because we know the Lie algebras coincide by our description of roots for a split reductive group, and the fact that an inclusion of smooth connected groups with the same Lie algebras is an isomorphism.) In fact we can do much better: we will show that such groups U_a "directly span" $U_A(G)$ under multiplication in any desired (but fixed) enumeration of $A \cap \Phi$. Before we discuss such a property, we record a general result on commutators.

Notation. For smooth closed k-subgroups H and H' of a linear algebraic group G over k, we denote by (H, H') the subgroup generated by commutators of H by H'. In the past this has usually been denoted $[H, H']$.

Proposition 5.3.2. [CGP, Prop. 3.3.5] *Let G be a connected linear algebraic k-group with action by a split k-torus S.*

(1) If $H, H' \subset G$ are S-stable connected linear algebraic k-subgroups then

$$wt_S((H, H')) \subset wt_S(H) + wt_S(H').$$

(2) If $A, A' \subset \mathrm{X}(S)$ are subsemigroups, then

$$(H_A(G), H_{A'}(G)) \subset H_{A+A'}(G).$$

Remarks on proof. Clearly (1) implies (2), by the maximality of $H_{A+A'}$ with respect to containing weights within $A + A'$. For (1), we get intuition from characteristic 0, since in such cases

$$\mathrm{Lie}((H, H')) = [\mathfrak{h}, \mathfrak{h}']$$

and we have functoriality of the Lie bracket, so

$$t.[X, X'] = [t.X, t.X'] \text{ for } t \in S$$

via the induced S-action on $\mathfrak{g}$. A characteristic-free proof is given in [CGP] by studying the S-action on coordinate rings. $\square$

Example 5.3.3. Let (G,T) be a split reductive pair. For $c\in\Phi$, *choose* a k-isomorphism

$$u_c:\mathbf{G}_a\simeq U_c.$$

By Proposition 5.3.2(2) we have

$$(U_c,U_{c'})\subset U_{\langle c\rangle+\langle c'\rangle}(G)=\langle U_a\rangle_{a\in(c,c')}.$$

The right side is "directly spanned" by $\{U_a\}_{a\in(c,c')}$ (where $(c,c'):=(\langle c\rangle+\langle c'\rangle)\cap\Phi$) in the sense of the following definition.

Definition 5.3.4. A linear algebraic k-group H is *directly spanned* by smooth closed k-subgroups $H_1,\dots,H_n\subset H$ if for *any* permutation σ of $\{1,\dots,n\}$ the multiplication map

$$H_{\sigma(1)}\times\ldots\times H_{\sigma(n)}\to H$$

is an isomorphism of k-schemes.

The direct-spanning result to be recorded shortly has nothing to do with reductive groups (not even in the proofs), being instead about the action of a split k-torus S on any connected linear algebraic k-group G. Let $A\subset \mathrm{X}(S)$ be a subsemigroup not containing 0. Consider a decomposition

$$A\cap \mathrm{wt}_S(G)=\coprod \Psi_i$$

where Ψ_i is disjoint from the subsemigroup $A_j:=\langle\Psi_j\rangle$ for all $j\neq i$; i.e. the subset $A\cap \mathrm{wt}_S(G)$ breaks up into pieces which do not intersect even when some mild addition is permitted.

Example 5.3.5. Here is a prototypical example for the setup: let (G,S) be a split reductive pair and

$$A=\{\chi\in\mathrm{X}(S):\langle\chi,\lambda\rangle>0\}\text{ for a regular }\lambda\in\mathrm{X}_*(S).$$

We may take $\Psi_i=\{a_i\}$ for $\{a_1,\dots,a_n\}=\Phi_{\lambda>0}=\Phi_{\lambda\geq 0}$.

Theorem 5.3.6. [CGP, Thm. 3.3.11] *The k-group $U_A(G)$ is directly spanned by $\{U_{A_j}(G)\}$.*

Idea of proof. Note that each $U_{A_i}(G)=U_{A_i}(U_A(G))$ because $U_{A_i}\subset U_A(G)$ a priori from the maximality property. Hence, we can replace G with $U_A(G)$ so that G is *unipotent*. Now this is one of the rare instances where want to use the descending central series (remember that unipotent is nilpotent, so the descending central series terminates). This is a good thing to do because the difficulty with these direct spanning results is the rampant non-commutativity.

If G is non-commutative then we have via the *canonical* descending central series an S-equivariant exact sequence

$$1\to Z\to G\to G/Z\to 1$$

with $Z\subset G$ a nontrivial smooth connected central k-subgroup. We perform dimension induction via such centrality to reduce to the case of *commutative* G.

In characteristic 0, a commutative unipotent group is a vector group. In characteristic $p>0$ we use the composition series $\{p^iG\}$ to reduce to the case when G is also p-torsion. Now the key is to use the structure theory in [CGP, App. B] for unipotent connected linear algebraic groups equipped with a sufficiently nontrivial action by a split torus (via the hypothesis $0 \notin A$) to ensure that such a commutative G also killed by p in characteristic p is necessarily a *vector group* (i.e., $\mathbf{G}_a^n$) on which the S-action is *linear* relative to *some* linear structure (beware that $\mathbf{G}_a^2$ admits many non-linear automorphisms in characteristic $p > 0$, such as $(x, y) \mapsto (x + y^p, y)$). The complete reducibility of linear representations of a split torus then yields the desired result. $\square$

Example 5.3.7. If (G, T) is a split reductive pair then

$$\{\text{Borels} \supset T\} = \{P_G(\lambda) \supset T \text{ for regular } \lambda \in \mathrm{X}_*(T)\}.$$

Thus, for any such B we have $\mathscr{R}_{u,k}(B) = U_G(\lambda)$ using such a regular λ, so $\mathscr{R}_{u,k}(B) = U_{c_1} \times \ldots \times U_{c_n}$ via multiplication for any enumeration $\{c_1, \ldots, c_n\}$ of $\Phi_{\lambda>0} = \Phi(B, T)$.

Note in particular (as can also be proved directly from the dynamic description) that a Borel k-subgroup containing T is determined by the set of roots $\Phi(B, T)$ (as $B = T \ltimes \mathscr{R}_{u,k}(B)$). A consequence of this is that there is a unique *opposite Borel* $B' \subset G$ containing T and satisfying $B' \cap B = T$; this characterization of B' in terms of B and T makes sense without requiting T to be split, and in that generality its existence and uniqueness can be deduced immediately from the split case over k_s via Galois descent.

Proposition 5.3.8. *Consider a split reductive pair (G, T) and $c, c' \in \Phi := \Phi(G, T)$ with $c' \neq \pm c$. Fix a parametrization $u_{c''} \colon \mathbf{G}_a \simeq U_{c''}$ for all $c'' \in \Phi$. Then*

$$(u_c(x), u_{c'}(x)) = \prod_{a = ic + jc' \in (c, c')} u_a(m_{i,j,c,c'} x^i y^j)$$

for some $m_{i,j,c,c'} \in k$ using a fixed enumeration of (c, c').

(*The structure constants $m_{i,j,c,c'}$ depend on the enumeration and parametrizations* $\{u_a\}$).

Before we prove this result, we make some remarks. Chevalley showed that for suitable parameterizations one can always arrange the structure constants to arise from specific *integers* determined up to sign solely by the enumeration and (root system) Φ. It turns out that such integers lie in $\{\pm 1, \pm 2, \pm 3\}$; see [C1, Prop. 6.3.5, Rem. 6.3.5] and references therein.

Example 5.3.9. Let $G = \mathrm{Sp}_4$ (root system Φ of type $B_2 = C_2$) with the symplectic form

$$\psi(v, v') = v^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} v'$$

and maximal torus

$$T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1^{-1} & \\ & & & t_2^{-1} \end{pmatrix}.$$

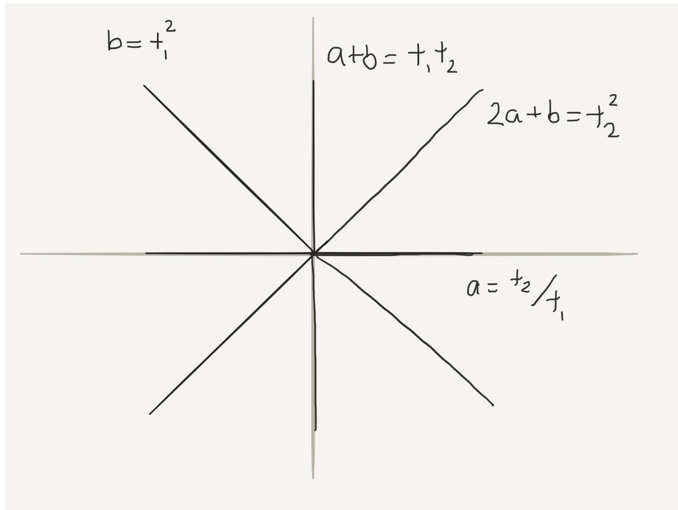


FIGURE 5.3.1. Positive roots for the $B_2 = C_2$ root system corresponding to Sp_4.

We have parameterizations of corresponding root groups:

$$u_a(x) = \begin{pmatrix} 1 & & & \\ -x & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix}$$

$$u_b(x) = \begin{pmatrix} 1 & & x & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$u_{a+b}(x) = \begin{pmatrix} 1 & & & x \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$u_{2a+b}(x) = \begin{pmatrix} 1 & & & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

The nontrivial commutation relations are

$$(u_a(x), u_b(y)) = u_{2a+b}(x^2 y) u_{a+b}(-xy),$$
$$(u_a(x), u_{a+b}(y)) = u_{2a+b}(-2xy);$$

U_b and U_{a+b} commute, as do U_a and U_{2a+b} (either by inspection or more conceptually because the sets of roots $(b, a+b)$ and $(a, 2a+b)$ are empty!). Note in particular that in characteristic 2 the second commutation relation degenerates, or more specifically the root groups for a and $a+b$ *commute* in characteristic 2.

Proof of Proposition 5.3.8. We are considering a split reductive pair (G,T) over k. Let $c,c' \in \Phi(G,T)$ be two roots with $c' \neq \pm c$. In Proposition 5.3.2 we noted that there is a containment $[U_c, U_{c'}] \subset U_A(G)$ for $A = \langle c\rangle + \langle c'\rangle$ for formal reasons. (Note that $(c,c') = A \cap \Phi$.) Fix k-isomorphisms $u_b : \mathbf{G}_a \simeq U_b$ for all roots b. Upon *fixing an enumeration* of the set (c,c') of roots $a = ic + jc'$ with $i,j \geq 1$, we have

$$[u_c(x), u_{c'}(y)] = \prod_{a \in (c,c')} u_a(F_a(x,y))$$

for some $F_a : \mathbf{A}^2_k = U_c \times U_{c'} \to U_a = \mathbf{A}^1_k$; i.e., $F_a \in k[x,y]$ (depending on the choice of enumeration and on the parameterizations u_b for roots b). Our task is to show that F_a is a monomial, and more specifically of bi-degree (i,j) where $a = ic + jc'$.

The proof is a simple consequence of behavior under conjugation against T. There is no harm in assuming that $k = \overline{k}$. For any $t \in T(k)$, we have

$$\begin{aligned} t[u_c(x), u_{c'}(y)]t^{-1} &= [t u_c(x) t^{-1}, t u_{c'}(y) t^{-1}] \\ &= [u_c(c(t)x), u_{c'}(c'(t)y)] \\ &= \prod_a u_a(F_a(c(t)x, c'(t)y)). \end{aligned}$$

On the other hand,

$$\begin{aligned} t \cdot \prod_{a \in (c,c')} u_a(F_a(x,y)) \cdot t^{-1} &= \prod_{a = ic+jc'} (t u_a(F_a(x,y)) t^{-1}) \\ &= \prod_a u_a(a(t) F_a(x,y)). \end{aligned}$$

Therefore we must have a termwise equality:

$$F_a(c(t)x, c'(t)y) = a(t)F_a(x,y) = c(t)^i c'(t)^j F_a(x,y)$$

where $a = ic + jc'$. Now we are saved by the condition that the characters are linearly independent. We have a surjective map

$$T \xrightarrow{(c,c')} \mathbf{G}_m \times \mathbf{G}_m.$$

Therefore

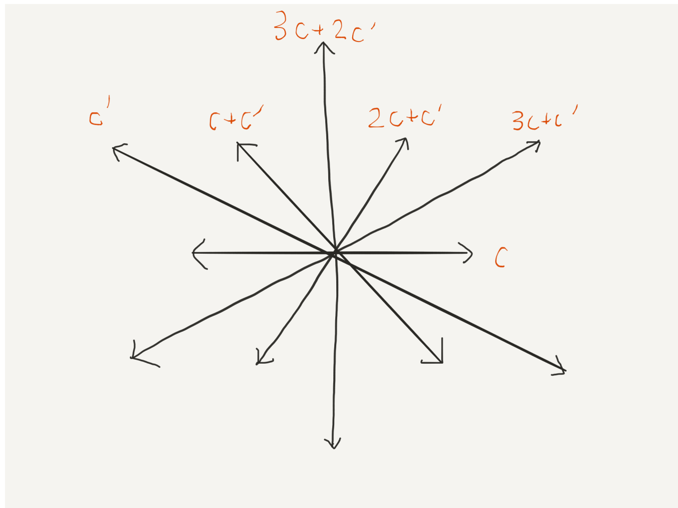
$$F_a(ux, vy) = u^i v^j F_a(x,y)$$

for all $x, y \in k$ and $u, v \in k^\times$. By considering the contribution to both sides from each monomial term appearing in F_a, this implies that $F_a(x,y) = f_{ij} x^i y^j$ for $f_{ij} \in k$. $\square$

An important fact due to Chevalley, which we mentioned last time, is that there is a systematic choice of enumeration of (c,c') and parameterizations $\{u_b\}$ which leads to $f_{ij} \in \{\pm 1, \pm 2, \pm 3\}$ determined solely by Φ.

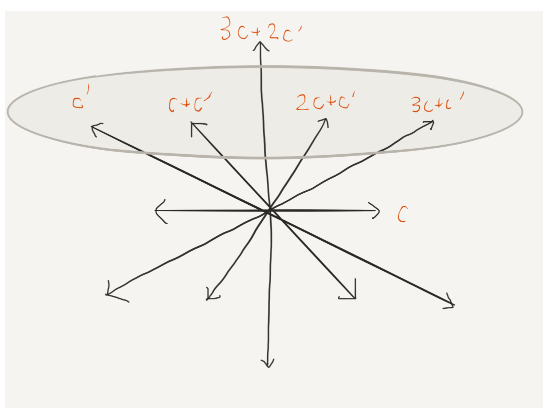
Example 5.3.10. Consider the split connected semisimple k-group $G = G_2$; this is the automorphism scheme of the unique (up to isomorphism) split octonion algebra over k (see [C2, App. B]). In this case, a split maximal k-torus $T \subset G$ has rank 2 and $\Phi := \Phi(G,T)$

is a root system of type G_2. Let $\{c, c'\}$ be a basis for this root system with c short and c' long as shown in the accompanying Figure.



The set (c, c') consists of $c+c', 3c+2c', 2c+c', 3c+c'$. It turns out that $f_{ij} = \pm 3$ does occur for this root system, using an appropriate enumeration of (c, c') (see [C1, Lemma 6.2.8, Ex. 6.2.9] and references therein or [H, §33.5] for the general form of Chevalley's commutation relations for G_2). We now explain the consequence this has for the structure of G, especially in characteristic 3; this will not be used in anything that follows.

From the picture and Chevalley's commutation relations, the span of the root lines for $c', c+c', 2c+c', 3c+c'$ is a 4-dimensional representation for the k-subgroup $\mathrm{SL}_2 \simeq G_c \to G = G_2$ generated by the root groups for $\{\pm c\}$.



The corresponding four root groups together with the root group for $3c+2c'$ directly span a unipotent group U normalized by G_c; explicitly, $U = U_A(G)$ for an evident subsemigroup A not containing 0. The form of the commutation relations implies that the root group $U_{3c+2c'}$ corresponding to the highest root is the center of U and that

$\overline{U} := U/U_{3c+2c'}$ is abelian, so there is a short exact sequence

$$1 \to Z_U = U_{3c+2c'} \to U \to \overline{U} \to 1$$

for which $\overline{U}$ is a vector group of dimension 4 identified with the direct product of the other 4 root groups. That direct product structure on $\overline{U}$ yields a linear structure on this vector group that is the *unique* one equivariant with respect to the T-action; in this way we identify $\overline{U}$ with its Lie algebra $\mathfrak{h} = \mathrm{Lie}(\overline{U})$.

The commutator of the root lines for $c + c'$ and $2c + c'$ is nonzero and valued in the root line for $3c + 2c'$, and similarly for the longer roots c' and $3c + c'$ (by a computation in the root system for $\mathfrak{sl}_3$). Hence, U is a Heisenberg group: there is a symplectic pairing on the quotient $\mathfrak{h}$ of U valued in the k-subgroup $U_{3c+2c'} \subset U$.

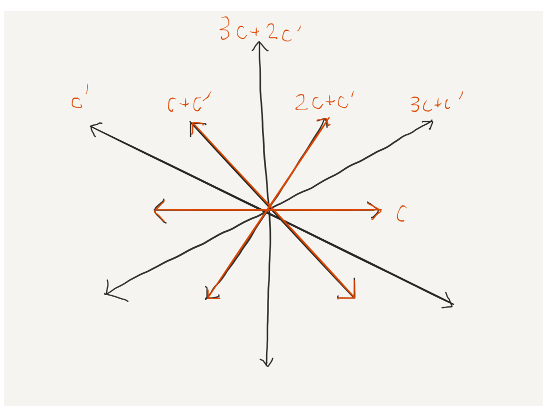
We have built $\mathrm{SL}_2 = G_c$ acting on the 4-dimensional space $\mathfrak{h}$, and this action is linear (as we can check using the actions of U_c and U_{-c}). When $\mathrm{char}(k) \neq 2, 3$, we can conclude that $\mathfrak{h}$ is the unique 4-dimensional irreducible representation of SL_2, so $\mathfrak{h} \simeq \mathrm{Sym}^3(V)$ where V is the standard 2-dimensional representation of SL_2. Letting $\{e, e'\}$ denote the standard ordered basis for V, a basis for $\mathfrak{h}$ is $\{e^3, e^2e', ee'^2, e'^3\}$ relative to which the "raising operator" $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \mathrm{SL}_2$ acts by $e \mapsto e$ and $e' \mapsto e + e'$. When we apply this operator to e'^3, for instance, the coefficient 3 shows up against ee'^2.

Assume $\mathrm{char}(k) = 3$. The irreducible 4-dimensional SL_2-representations are $(V^{\otimes 3})^{S_3}$ and $(V^{\otimes 3})_{S_3}$. Inside $(V^{\otimes 3})^{S_3}$ is a copy of V spanned by the symmetrizers of $e^{\otimes 2} \otimes e'$ and $e \otimes e'^{\otimes 2}$, the quotient by which is the Frobenius twist $V^{(3)}$ (using $e^{\otimes 3} \mapsto e^{(3)}$ and $e'^{\otimes 3} \mapsto e'^{(3)}$). Likewise, $(V^{\otimes 3})_{S_3}$ contains a copy of $V^{(3)}$ spanned by the classes of $e^{\otimes 3}$ and $e'^{\otimes 3}$, the quotient by which is V (spanned by images of the classes of $e^{\otimes 2} \otimes e'$ and $e \otimes e'^{\otimes 2}$). This gives non-trivial extensions of SL_2-representations:

$$0 \to V \to (V^{\otimes 3})^{S_3} \to V^{(3)} \to 0$$

$$0 \to V^{(3)} \to (V^{\otimes 3})_{S_3} \to V \to 0$$

These are distinguished by the unique 2-dimensional subrepresentations with weights for the diagonal torus given by ± 1 for $(V^{\otimes 3})^{S_3}$ and ± 3 for $(V^{\otimes 3})_{S_3}$ respectively.



Is $\mathfrak{h}$ isomorphic to $(V^{\otimes 3})^{S_3}$ or $(V^{\otimes 3})_{S_3}$? To answer this we shall calculate use some notions to be discussed later, namely coroots (see §6.2) and their encoding in terms of the Dynkin diagram or geometry of the root system. This gives $\langle c, c'^{\vee}\rangle = -1$ and $\langle c', c^{\vee}\rangle = -3$, so the weights of the (split) maximal torus $c^{\vee}(\mathbf{G}_m) \subset G_c = \mathrm{SL}_2$ on the short root lines $\mathfrak{g}_{c+c'}$ and $\mathfrak{g}_{2c+c'}$ inside $\mathfrak{h}$ are ∓ 1 and the weights of $c^{\vee}(\mathbf{G}_m)$ on the long root lines $\mathfrak{g}_{c'}$ and $\mathfrak{g}_{3c+c'}$ inside $\mathfrak{h}$ are ∓ 3. Thus, $\mathfrak{h} \simeq (V^{\otimes 3})^{S_3}$ precisely when the short root lines in $\mathfrak{h}$ span a G_c-stable subspace (and then the span of the long root lines in $\mathfrak{h}$ is not G_c-stable). We shall prove that the span of the short root lines is indeed G_c-stable.

The group $G_c = \mathrm{SL}_2$ is generated by the torus $c^{\vee}(\mathbf{G}_m)$, the root group U_c, and any representative for the nontrivial element of $W(G_c, c^{\vee}(\mathbf{G}_m))$. The adjoint action of $c^{\vee}(\mathbf{G}_m)$ preserves all root lines, and the effect of a Weyl element swaps root lines according to the reflection r_c, so this swaps the two short root lines inside $\mathfrak{h}$ as well as swaps the two long root lines inside $\mathfrak{h}$. Hence, everything comes down to determining whether the adjoint action of U_c preserves the span of the short root lines inside $\mathfrak{h}$ or preserves the span of the long root lines inside $\mathfrak{h}$. We shall prove that the former holds.

For suitable parameterizations of the root groups the commutation relations for U_c against $U_{c+c'}$ and $U_{2c+c'}$ degenerate in characteristic 3 to

$$\begin{aligned} u_{c+c'}(y)u_c(x) &= u_c(x)u_{c+c'}(y)u_{2c+c}(-2xy) \\ u_{2c+c'}(y)u_c(x) &= u_c(x)u_{2c+c'}(y). \end{aligned}$$

Thus, the adjoint action of U_c on $\mathfrak{g}_{2c+c'}$ is trivial and on $\mathfrak{g}_{c+c'}$ is valued in $\mathfrak{g}_{c+c'} + \mathfrak{g}_{2c+c'}$, so the span of the short root lines inside $\mathfrak{h}$ is G_c-stable as claimed. That is, $\mathfrak{h} \simeq (V^{\otimes 3})^{S_3}$ as a representation for $G_c = \mathrm{SL}_2$.

Note that the long roots constitute a root system of type A_2, and likewise for the short roots. Hence, it is natural to ask if the long root groups are the root groups for a copy of SL_3 or PGL_3 (the two split connected semisimple groups with root system A_2) inside G_2 containing T, and likewise for the short root groups. In all characteristics (and even over $\mathbf{Z}$ with appropriate definitions) the long root groups generate an SL_3 containing T with those long roots as its T-root groups, whereas precisely in characteristic 3 the short root groups are the root groups for a connected semisimple group of type A_2 containing T, and this special subgroup in characteristic 3 is PGL_3. In [CGP, §7.1] this phenomenon is explained from a broader point of view.

6. Dynamic description of parabolic subgroups

6.1. **Main result.** We want to prove the following characterization of parabolic subgroups. It is difficult to overestimate the importance of this result.

Theorem 6.1.1. *Let G be a connected, reductive group (not necessarily split) over any field k.*

1. *For every k-homomorphism $\lambda : \mathbf{G}_m \to G$ the k-subgroup $P_G(\lambda)$ is parabolic, and every parabolic k-subgroup of G arises in this manner.*
2. *If P is a parabolic k-subgroup of G and $T \subset P$ is any maximal k-torus, then there exists a k-homomorphism $\lambda : \mathbf{G}_m \to T$ such that $P_G(\lambda) = P$.*

Remark 6.1.2. The generality is quite non-trivial. For instance, if $P \neq G$ then λ is obviously nontrivial, so this implies that any maximal torus $T \subset P$ must have a non-trivial one-parameter k-subgroup. Thus, a consequence of this Theorem is that any maximal k-torus T in a proper parabolic k-subgroup of G cannot be k-anisotropic.

Since proper parabolic k-subgroups of G correspond bijectively to those of $\mathscr{D}(G)$, and the center of $\mathscr{D}(G)$ is finite with the multiplication map $Z \times \mathscr{D}(G) \to G$ (where Z is the maximal central k-torus of G) being an isogeny, it follows from this Theorem that G has a *proper* parabolic k-subgroup if and only if $\mathscr{D}(G)$ contains a nontrivial split k-torus (i.e., $\mathscr{D}(G)$ is k-isotropic, or equivalently G contains a *non-central* split k-torus).

Indeed, by finiteness of the center of $\mathscr{D}(G)$ it follows that if there is a nontrivial split k-torus S in $\mathscr{D}(G)$ and $\lambda \in \mathrm{X}_*(S) - \{0\}$ then the *parabolic* k-subgroup $P_G(\lambda)$ must be proper (as otherwise $U_G(\lambda) = \mathscr{R}_{u,k}(G) = 1$, forcing $G = P_G(\lambda) = Z_G(\lambda)$, contradicting that the nontrivial $\lambda : \mathbf{G}_m \to \mathscr{D}(G)$ cannot be central in $Z \cdot \mathscr{D}(G) = G$).

We have already seen in Corollary 1.11 of the handout "Basics of reductivity" that for a maximal k-torus $T \subset G$ the Borel subgroups of $G_{\overline{k}}$ containing $T_{\overline{k}}$ are $P_G(\lambda)$ for regular $\lambda \colon \mathbf{G}_m \to T_{\overline{k}}$. This will be used in the proof of Theorem 6.1.1.

To begin the proof of this Theorem, we first prove $P_G(\lambda)$ is a parabolic k-subgroup of G for any k-homomorphism $\lambda : \mathbf{G}_m \to G$. For this, we may assume that $k = \overline{k}$. We need to exhibit a Borel subgroup in $P_G(\lambda)$. Let $T \subset P_G(\lambda)$ be a maximal torus (so T is also maximal in G). We'll produce a regular $\mu \colon \mathbf{G}_m \to T$ such that $P_G(\mu) \subset P_G(\lambda)$. Since $P_G(\mu)$ is a Borel subgroup of G, the parabolicity of $P_G(\lambda)$ would follows.

Visualizing the finite subset $\Phi \subset \mathrm{X}(T)_{\mathbf{Q}} - \{0\}$, it is clear what to do: choose μ to be a small perturbation of λ. More precisely, we have

$$\Phi(P_G(\lambda), T) = \{a \in \Phi \mid \langle a, \lambda \rangle \geq 0\}.$$

We can decompose this into two subsets $\Phi_{\lambda>0}$ and $\Phi_{\lambda=0}$. Choose $\mu_0 \in \mathrm{X}_*(T)_{\mathbf{Q}}$ such that for each of the finitely many $a \in \Phi_{\lambda>0}$, we have $\langle a, \mu_0 \rangle > 0$. (Any μ_0 close enough to λ works.) Then take $\mu = N\mu_0$ for a positive integer N sufficiently divisible so that $\mu \in \mathrm{X}_*(T)$, and μ will still have the same property. Let $B = P_G(\mu)$, a Borel subgroup.

The crucial step is to show that B is contained in $P_G(\lambda)$. The set

$$\Phi(B, T) = \{a \in \Phi \mid \langle a, \mu \rangle \geq 0\}$$

is a subset of

$$\Phi(P_G(\lambda), T) = \{a \in \Phi \mid \langle a, \lambda \rangle \geq 0\}$$

because the choice of μ forces

$$\langle a, \lambda \rangle < 0 \implies \langle -a, \lambda \rangle > 0 \implies \langle -a, \mu \rangle > 0 \implies \langle a, \mu \rangle < 0.$$

Hence, for the two smooth connected k-subgroups $B, P_G(\lambda) \subset G$ containing T we have

$$\mathrm{Lie}(B) \subset \mathrm{Lie}(P_G(\lambda))$$

since each Lie algebra is the direct sum of $\mathrm{Lie}(T)$ and the root lines for the respective roots for T that occur in these Lie algebras. This does the job in characteristic 0, but we need to do more work to give an argument applicable in all characteristics:

Lemma 6.1.3. *For $\lambda, \mu \in \mathrm{X}_*(T)$, if $\Phi(P_G(\mu), T) \subset \Phi(P_G(\lambda), T)$ then $P_G(\mu) \subset P_G(\lambda)$.*

Proof. We use the functoriality of the dynamical construction. Let $P = P_G(\lambda)$ and $Q = P_G(\mu)$; these are smooth and connected. Thus, $Q \cap P = Q \cap P_G(\lambda) = P_Q(\lambda)$ is also smooth and connected (and contains T). But the containment of Lie algebras $\mathrm{Lie}(P_Q(\lambda)) \subset \mathrm{Lie}(Q)$ is an equality by comparing T-roots that occur in each (precisely by the hypothesis $\Phi(P,T) \subset \Phi(Q,T)$). A containment of smooth connected groups with the same Lie algebras is always an equality. □

We have finished the proof over general k (not just algebraically closed) that $P_G(\lambda)$ is always parabolic. To complete the proof of (1) we need to show that every parabolic k-subgroup arises from the dynamic construction over k, and this is a consequence of (2), so it suffices to prove (2).

Let us reduce the proof of (2) to the case $k = \overline{k}$. Given P over k and a maximal k-torus $T \subset P$, suppose $\lambda' \in \mathrm{X}_*(T_{\overline{k}})$ such that $P_{\overline{k}} = P_{G_{\overline{k}}}(\lambda')$. We want to find a k-homomorphism $\lambda : \mathbf{G}_m \to T$ such that $P = P_G(\lambda)$.

We know that $\lambda' \in \mathrm{X}_*(T_{k_s})$, and the equality $P_{k_s} = P_{G_{k_s}}(\lambda')$ holds since it can be checked over $\overline{k}$. There exists a finite Galois extension k'/k splitting T, so $\lambda' \in \mathrm{X}_*(T_{k'})$ and

$$P_{k'} = P_{G_{k'}}(\lambda').$$

This λ' may not be defined over k, so the idea to overcome this problem is to average over a finite Galois group.

The k'-homomorphism $\mathbf{G}_m \to T_{k'}$ defined by

$$\lambda := \sum_{\sigma \in \mathrm{Gal}(k'/k)} \sigma\lambda'$$

is visibly $\mathrm{Gal}(k'/k)$-invariant, so it descends (uniquely) to a k-homomorphism $\mathbf{G}_m \to T$ that we also denote as λ. We shall prove that $P_G(\lambda) = P$.

It suffices to check equality after scalar extension to k': we claim that

$$P_G(\lambda)_{k'} = P_{k'}.$$

By functoriality of the dynamic construction with respect to base change, this is equivalent to showing

$$P_{G_{k'}}(\lambda) = P_{G_{k'}}(\lambda').$$

By Lemma 6.1.3 (applied over k'), this reduces to proving the equality

$$\{a \in \Phi \mid \langle a, \lambda\rangle \geq 0\} = \{a \in \Phi \mid \langle a, \lambda'\rangle \geq 0\}$$

for $\Phi := \Phi(G_{k'}, T_{k'})$. Hence, we want to show

$$\langle a, \lambda\rangle < 0 \iff \langle a, \lambda'\rangle < 0 \tag{6.1.1}$$

for each $a \in \Phi$.

Now we use the fact that the k'-subgroup $P_{k'} \subset G_{k'}$ arises from the k-subgroup $P \subset G$, so the set of roots occurring in $\mathrm{Lie}(P_{k'}) = \mathrm{Lie}(P)_{k'} \subset \mathrm{Lie}(G)_{k'} = \mathrm{Lie}(G_{k'})$ is stable under the natural Galois action on $\mathrm{X}(T_{k'})$. Therefore, the condition imposed by the right side of (6.1.1) is actually *Galois-invariant*. This is the key observation! So

$$\begin{aligned}\langle a, \lambda'\rangle < 0 &\iff \langle \sigma(a), \lambda'\rangle < 0 \\ &\iff \langle a, \sigma^{-1}\lambda'\rangle < 0\end{aligned}$$

by the Galois-invariance of the pairing. Now we claim that

$$\langle a, \lambda' \rangle < 0 \iff \langle a, \sum_\sigma \sigma\lambda' \rangle < 0.$$

The direction $\implies$ is clear. For the other direction, examine the contrapositive: if a pairs non-negatively with some $\sigma\lambda'$, then it pairs non-negatively with each term in the sum (by our Galois-invariance observation) and hence with the sum of the Galois orbit of λ'.

Now we show that for any maximal torus $T \subset P$, we can find a $\lambda \colon \mathbf{G}_m \to T$ such that $P = P_G(\lambda)$. Let $\Phi := \Phi(G, T) \supset \Phi(P, T) =: \Psi$.

Lemma 6.1.4. *If $\lambda \in \mathrm{X}_*(T)$ satisfies $\Phi_{\lambda \geq 0} := \{a \in \Phi \mid \langle a, \lambda \rangle \geq 0\} = \Psi$ then $P = P_G(\lambda)$.*

Proof. The proof again rests on the functoriality of the dynamical construction. Note that $P \cap P_G(\lambda) = P_P(\lambda)$ is a smooth connected subgroup of P containing T and having the same Lie algebra (as it suffices to compare T-root lines contained in each), so $P_P(\lambda) = P$. This implies that $P \subset P_G(\lambda)$. Again, this is an inclusion of smooth connected groups containing T and having the same Lie algebra, so $P = P_G(\lambda)$. □

We now make a second reduction. Consider the central isogeny decomposition

$$Z \times \mathscr{D}G \to G.$$

Take the corresponding tori:

$$Z \times \mathscr{T} \to T$$

where $\mathscr{T} = (T \cap \mathscr{D}G)^0_{\mathrm{red}}$. Since this is an isogeny, it induces a decomposition at the level of rational cocharacter groups:

$$\mathrm{X}_*(T)_{\mathbf{Q}} \simeq \mathrm{X}_*(Z)_{\mathbf{Q}} \oplus \mathrm{X}_*(\mathscr{T})_{\mathbf{Q}}$$

such that $\Phi \perp \mathrm{X}_*(Z)_{\mathbf{Q}}$. Also recall that $\Phi(G, T) = \Phi(\mathscr{D}(G), \mathscr{T})$. It suffices to find $\lambda \in \mathrm{X}_*(\mathscr{T})_{\mathbf{Q}}$ such that $\Phi_{\lambda \geq 0} = \Psi$, so we may assume that $G = \mathscr{D}(G)$ is semisimple. In particular, the $\mathbf{Z}$-span of Φ has finite index in $\mathrm{X}(T)$, so $\Phi(G, T)$ spans $\mathrm{X}(T)_{\mathbf{Q}}$.

To complete the proof of Theorem 6.1.1, we need some basic theory of root systems. We'll digress to discuss this and then come back to the proof.

6.2. **Root systems and coroots.**

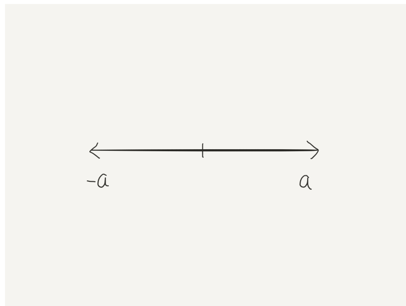
Definition 6.2.1. Let V be a $\mathbf{Q}$-vector space of finite dimension. Let $\Phi \subset V - \{0\}$ be a finite set that spans V (e.g. the roots $\Phi(G, T)$ for a connected semisimple group G admitting a split maximal torus T, with $V = \mathrm{X}(T)_{\mathbf{Q}}$). Then (V, Φ) is a *root system* if for all $a \in \Phi$, there exists $a^\vee \in V^*$ satisfying the following two properties:

1. $a^\vee(a) = 2$, and $a^\vee(\Phi) \subset \mathbf{Z}$.
2. Define $r_{a,a^\vee} \colon V \to V$ by $x \mapsto x - a^\vee(x)a$. This is a reflection across the hyperplane $\ker a^\vee$, satisfying $a \mapsto -a$. (Since there is no Euclidean structure imposed on V, we need to specify the action on the quotient $V/(\ker a^\vee) = \mathbf{Q} \cdot a$ is negation to speak of a "reflection" on V.) Then $r_{a,a^\vee}$ is required to preserve the roots; i.e.,

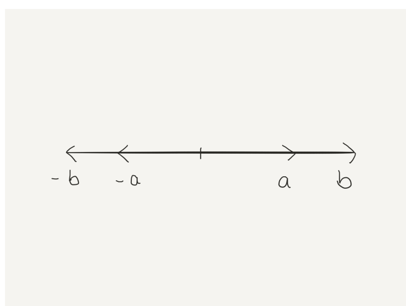
$$r_{a,a^\vee}(\Phi) = \Phi.$$

Note in particular that since $r_{a,a^\vee}(a) = -a$, we have $-\Phi = \Phi$ for any root system (V, Φ). We say Φ is *reduced* if $\Phi \cap \mathbf{Q}a = \{\pm a\}$ for each $a \in \Phi$.

Example 6.2.2. What are the 1-dimensional root systems? There is one that looks like



(with $a^\vee$ uniquely determined by the condition $a^\vee(a) = 2$); this is denoted A_1. Are there any others (up to isomorphism)? Suppose we tried to add another root $b \neq \pm a$, so Φ must contain $\{-b, -a, a, b\}$. We may and do assume (by relabeling if necessary) that $b = qa$ for some $q \in \mathbf{Q}$ with $q > 1$. This is usually not contained in any root sytem:



the second condition in the definition is not a problem since reflections in dimension 1 are simply negation, but the first condition implies

$$\langle a^\vee, a \rangle = 2$$
$$\langle a^\vee, b \rangle \in \mathbf{Z}$$
$$\langle b^\vee, b \rangle = 2$$
$$\langle b^\vee, a \rangle \in \mathbf{Z}$$

Writing $b = qa$ with $q > 1$, we have $2q \in \mathbf{Z}$, $b^\vee = (1/q)a^\vee$, and $2/q \in \mathbf{Z}$. Hence, $q = n/2$ for some integer $n > 2$ such that $2/q = 4/n$ is an integer, so $n = 4$; i.e., $q = 2$. In other words, there is exactly one other 1-dimensional root system, obtained by taking $b = 2a$ (and it cannot be made any larger); this is called BC_1 and it is non-reduced.

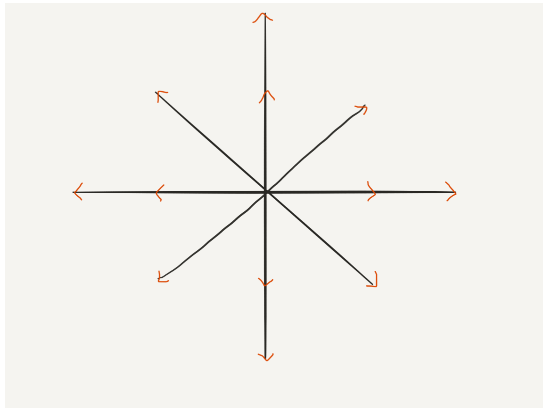
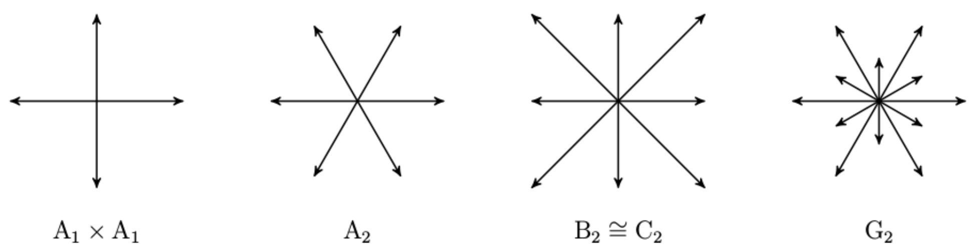


FIGURE 6.2.1. Nonreduced root system of rank 2.

Example 6.2.3. The reduced 2-dimensional root systems are $\mathrm{A}_1 \times \mathrm{A}_1, \mathrm{B}_2 = \mathrm{C}_2, \mathrm{A}_2, \mathrm{G}_2$.



These are all *reduced*. There is a unique non-reduced root system including $\mathrm{B}_2 = \mathrm{C}_2$, called BC_2 (it is the union of a copy of B_2 and a copy of C_2 for which the short roots of a copy of C_2 are the long roots of a copy of B_2), and the only other non-reduced 2-dimensional root systems turn out to be $\mathrm{A}_1 \times \mathrm{BC}_1$ and $\mathrm{BC}_1 \times \mathrm{BC}_1$.

Remark 6.2.4. It is not immediate from the definition, but $a \mapsto a^\vee$ is well-defined; i.e. $a^\vee$ is *uniquely* determined by a. See [SGA3, XXI, 1.1.4] for a self-contained proof of this fact. Therefore, we can write $r_a := r_{a,a^\vee}$. Since r_a and a uniquely determined the linear form $a^\vee$, it follows that $\Phi \to V^*$ defined by $a \mapsto a^\vee$ is a bijection onto a finite subset of $V^* - \{0\}$ denoted $\Phi^\vee$. It turns out that $(V^*, \Phi^\vee)$ is a root system, called the *dual* of (V, Φ). Elements of $\Phi^\vee$ are called *coroots*.

In the handout "Root datum of a split reductive group" it is shown that the roots of a split connected semisimple group form a root system. (A more refined result is proved there involving the notion of *root datum*, which we will discuss this later.) Let's explain where the coroots come from in group-theoretic terms.

Let G be a connected semisimple group over a field k and suppose it contains a split maximal k-torus T, so

$$V := \mathrm{X}(T)_{\mathbf{Q}} \supset \Phi(G, T) =: \Phi.$$

For $a \in \Phi$, we seek $a^\vee \in V^* = \mathrm{X}_*(T)_{\mathbf{Q}}$ whose pairing with all roots is in $\mathbf{Z}$. We will find $a^\vee$ inside $\mathrm{X}_*(T)$ (even when $\mathbf{Z}\Phi$ is a proper sublattice of $\mathrm{X}(T)$). To get started, a corresponds to a k-homomorphism

$$a\colon T \to \mathbf{G}_m.$$

For the torus $T_a := \ker(a)^0_{\mathrm{red}}$ of codimension-1 in T, the connected reductive k-group $Z_G(T_a)$ contain T as a split maximal k-torus and T_a is central. In fact, T_a is the *maximal* central k-torus in $Z_G(T_a)$ since the maximal k-torus T of dimension $1+\dim T_a$ is non-central in G_a, as its action on $\mathrm{Lie}(Z_G(T_a)) = \mathfrak{g}^{T_a}$ supports root lines for $\pm a$.

Hence, for $G_a := \mathscr{D}(Z_G(T_a))$, multiplication $T_a \times G_a \to Z_G(T_a)$ is a central isogeny with G_a connected semisimple of rank 1 having

$$\mathscr{T}_a := T \cap G_a$$

as a *split* maximal k-torus. (Recall the general link between maximal tori of a connected reductive group and of its derived group; apply that to $Z_G(T_a)$.) Hence, by the classification of rank one semisimple groups (from the previous course), $G_a \simeq \mathrm{SL}_2$ or PGL_2 carrying $\mathscr{T}_a$ over to the diagonal k-torus.

We will find a k-homomorphism $a^\vee\colon \mathbf{G}_m \to \mathscr{T}_a \subset T$ which does the job. Since $a : T \to \mathbf{G}_m$ is nontrivial and kills T_a, the restriction $a|_{\mathscr{T}_a}$ to the canonical isogeny-complement torus $\mathscr{T}_a$ to T_a inside T is nontrivial. But $\mathscr{T}_a$ is 1-dimensional (and split), so the desired $\mathscr{T}_a$-valued cocharacter $a^\vee$ is uniquely determined (if it exists!) by the requirement $\langle a, a^\vee\rangle = 2$. This uniqueness enables us to make some choices in the construction of $a^\vee$ without any concern that the end result will depend on choices beyond (G, T, a).

To build such an $a^\vee$, we treat separately the cases that G_a is SL_2 or PGL_2.

- In the SL_2-case, fix an isomorphism $G_a \simeq \mathrm{SL}_2$ carrying $\mathscr{T}_a$ over to the diagonal torus $\{\left(\begin{smallmatrix} t & 0 \\ 0 & 1/t\end{smallmatrix}\right)\}$. This isomorphism must carry the $\mathscr{T}_a$-root group $U_a \subset G_a$ over to one of the two root groups $U^\pm$ in SL_2 for the diagonal torus. We may conjugate the chosen isomorphism $G_a \simeq \mathrm{SL}_2$ against the standard Weyl element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0\end{smallmatrix}\right)$ if necessary so that U_a is carried over to U^+. Hence, the root $a|_{\mathscr{T}_a}$ computing the $\mathscr{T}_a$-action on $\mathrm{Lie}(G_a) = \mathfrak{sl}_2$ is the root $\left(\begin{smallmatrix} t & 0 \\ 0 & 1/t\end{smallmatrix}\right) \mapsto t^2$ for U^+. Then we can write down the 1-parameter subgroup $a^\vee$ explicitly as
$$t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix},$$
so $a^\vee : \mathbf{G}_m \to \mathscr{T}_a$ is an *isomorphism* in the SL_2-case.
- In the PGL_2-case we proceed similarly, the only difference being that the diagonal torus is described in terms of matrices $\left(\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right)$ for which the corresponding conjugation on U^+ induces multiplication by t on the k-line $\mathrm{Lie}(U^+)$. Consequently, $a^\vee : \mathbf{G}_m \to \mathscr{T}_a$ is a degree-2 isogeny corresponding to $t \mapsto \left(\begin{smallmatrix} t^2 & 0 \\ 0 & 1\end{smallmatrix}\right)$ since we require $\langle a, a^\vee\rangle = 2$.

Since $T = T_a \cdot \mathscr{T}_a$ with T_a central in $Z_G(T_a)$ and $\mathrm{Aut}(\mathscr{T}_a) = \{\pm 1\}$, we have naturally

$$W(G_a, \mathscr{T}_a) \subset W(G, T) \subset \mathrm{GL}(\mathrm{X}(T)) \subset \mathrm{GL}(\mathrm{X}(T_a)_{\mathbf{Q}} \oplus \mathrm{X}(\mathscr{T}_a)_{\mathbf{Q}})$$

as a subgroup of $\mathrm{GL}(\mathrm{X}(T_a)) \times \mathrm{GL}(\mathrm{X}(\mathscr{T}_a)) = \mathrm{GL}(\mathrm{X}(T_a)) \times \{\pm 1\}$ with trivial effect on $\mathrm{X}(T_a)$. That is, $W(G_a, \mathscr{T}_a)$ has order at most 2, as its elements induces the identity on the hyperplane $\mathrm{X}(T_a)_{\mathbf{Q}} = (\ker a^\vee)_{\mathbf{Q}}$ and induce ± 1 on the complementary line $\mathrm{X}(\mathscr{T}_a)_{\mathbf{Q}} = \mathbf{Q} \cdot a$ (!), so $W(G_a, \mathscr{T}_a)$ really has order exactly 2 since conjugation against the standard Weyl element w in SL_2 or PGL_2 identified with G_a as above gives a nontrivial element.

This nontrivial element is a reflection that *must* be the canonical refection r_a on $\mathrm{X}(T)_{\mathbf{Q}}$ since both automorphisms of $\mathrm{X}(T)_{\mathbf{Q}}$ satisfy the *same* properties. In other words, r_a is also characterized uniquely as being the only nontrivial element in $W(G_a, \mathscr{T}_a)$.

Remark 6.2.5. As an element of $N_{G_a}(\mathscr{T}_a)(k)$ rather than merely its quotient $W(G_a, \mathscr{T}_a)$ we do *not* get a canonical element. Indeed, it is ambiguous up to translation by $\mathscr{T}_a(k)$, precisely the ambiguity in the choice of isomorphism from G_a onto SL_2 or PGL_2 carrying $\mathscr{T}_a$ onto the diagonal torus and U_a onto U^+. Note also that if $\mathrm{char}(k) \neq 2$ then the standard Weyl element in $\mathrm{SL}_2(k)$ has order 4 rather than order 2.

6.3. **Parabolic sets of roots.** Given $\Phi \supset \Psi = \Phi(P, T)$ we want to find $\lambda \in \mathrm{X}_*(T)_{\mathbf{Q}}$ such that $\Phi_{\lambda \geq 0} = \Psi$. We need to find an *intrinsic* characterization of subsets of Φ of the form $\Phi_{\lambda \geq 0}$, mirroring the one for parabolics as subgroups containing a Borel.

Definition 6.3.1. For a root system Φ, a subset $\Psi \subset \Phi$ is called *closed* if for all $a, b \in \Psi$ such that $a + b \in \Phi$, we also have $a + b \in \Psi$.

Example 6.3.2. Subsets of the form $\Phi_{\lambda \geq 0}$ and $\Phi_{\lambda > 0}$ (for $\lambda \in V^*$), or more generally $\Phi \cap A$ for a subsemigroup $A \subset V$, are closed.

This condition isn't enough to characterize subsets of the form $\Phi_{\lambda \geq 0}$, since (for instance) the latter contain at least half of the roots. (Recall $\Phi \subset V - \{0\}$ is a finite subset stable under negation.)

Definition 6.3.3. A subset Ψ of a root systemΦ is *parabolic* if

(1) Ψ is closed,
(2) $\Psi \cup (-\Psi) = \Phi$.

Example 6.3.4. For $\lambda \in \mathrm{X}_*(T)$, $\Phi_{\lambda \geq 0}$ is a parabolic subset.

Proposition 6.3.5. [CGP, Prop. 2.2.8] *Any parabolic subset of a root system Φ is of the form $\Phi_{\lambda \geq 0}$ for some $\lambda \in V^*$.*

The proof (given in [CGP]) is an application of the early developments about root systems in [Bou, VI, §1.7]. Here we just give an intuitive explanation for how to find λ, given a parabolic subset. First consider the case $\Psi \cap (-\Psi) = \emptyset$, so the parabolic set Ψ is a closed subset consisting of *exactly* half of the roots. In this case we seek λ that is regular (since $\Phi_{\lambda \geq 0}$ will need to be disjoint from its negative).

We need to appeal to the theory of "bases" of root systems. A *basis* of Φ is a subset Δ with two properties:

- Δ is a basis for V,
- every element of Φ has its Δ-coefficients either all in $\mathbf{Z}_{\geq 0}$ or all in $\mathbf{Z}_{\leq 0}$.

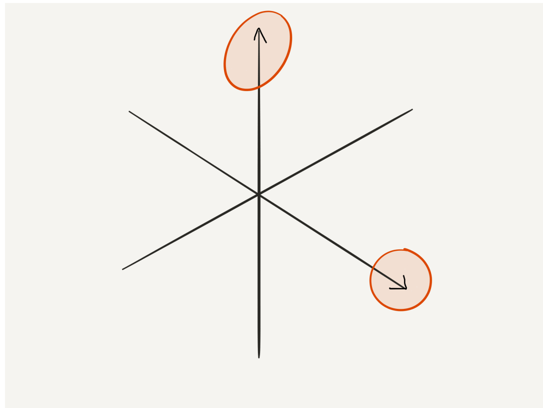


FIGURE 6.3.1. A basis for the root system A_2.

The general fact is that for any parabolic subset there exists a basis Δ of Φ such that the parabolic subset contains $(\mathbf{Z}_{\geq 0}\Delta) \cap \Psi$. (This is the counterpart of the characterization of parabolic subgroups of connected reductive groups over algebraically closed fields as smooth closed subgroups contains a Borel subgroup.) For any basis Δ of Φ we get the dual $\mathbf{Q}$-basis $\Delta^* = \{b^*\}_{b \in \Delta}$ of V^*.

Warning 6.3.6. Do not confuse b^* with $b^\vee$; they are not equal in general! Indeed, we can have $\langle a, b^\vee \rangle \neq 0$ for linearly independent $a, b \in \Delta$, as already happens for B_2 and G_2, whereas by design $\langle a, b^* \rangle = 0$ for distinct $a, b \in \Delta$.

Returning to a parabolic Ψ of exactly half the size of Φ, define $\lambda = \sum_{b \in \Delta} b^*$ for Δ whose non-negative $\mathbf{Z}$-linear combinations in Φ are contained in Ψ, and hence coincide with Ψ by counting reasons. This works precisely from the definition of Δ being a basis contained in Ψ.

If instead $\Psi \cap (-\Psi)$ is non-empty, consider $\Phi' = \Psi \cap (-\Psi)$. It turns out that this is also a root system in its own $\mathbf{Q}$-span inside V. (The image of Φ in the quotient $V/(\mathbf{Q} \cdot \Phi')$ may not be a root system.) The idea is to find a basis $\Delta' \subset \Phi'$ and extend it to a basis Δ of Φ contained in Ψ. Then one takes $\lambda = \sum_{b \in \Delta - \Delta'} b^*$.

We finish the proof of Theorem 6.1.1 by proving:

Proposition 6.3.7. *Let G be a connected semisimple group over $k = \overline{k}$. If $P \subset G$ is a parabolic subgroup and $T \subset P$ is a maximal torus then the subset $\Psi := \Phi(P, T) \subset \Phi(G, T)$ is parabolic.*

Since P contains a Borel subgroup B and the T-roots occurring in $\mathrm{Lie}(B)$ cover Φ up to signs, clearly $\Psi \cup (-\Psi) = \Phi$. Hence, by Proposition 6.3.5, it remains to check that Ψ is closed. This will rest on a general criterion for closedness of a subset of a *reduced* root system in terms of reflections, the proof of which involves calculations with (reduced) rank-2 root systems.

Let G be a connected *semisimple* group over $k = \overline{k}$, $P \subset G$ a parabolic group, and T a maximal torus inside P. Note that $\mathrm{X}(T)_{\mathbf{Q}} = \mathbf{Q} \cdot \Phi$ since $\mathbf{Z}\Phi$ has finite index inside $\mathrm{X}(T)$ in the semisimple case. We need to show that the subset $\Phi(P,T) \subset \Phi := \Phi(G,T)$ is closed. We know that $P \supset B \supset T$ for a Borel subgroup $B = P_G(\mu)$ for some regular cocharacter $\mu \in X_*(T)$. Therefore,

$$\Phi(P,T) \supset \Phi^+ := \Phi(B,T) = \Phi_{\mu \geq 0} = \Phi_{\mu > 0}.$$

We shall use the following sufficient criterion for a subset of roots in a *reduced* root system (such as $\Phi(G,T)$) to be closed:

Proposition 6.3.8. *Let (V,Φ) be a reduced root system. A subset $\Psi \subset \Phi$ containing $\Phi_{\mu>0}$ for a regular cocharacter μ is closed if for all $\{c,-c\} \subset \Psi$ the action of the reflection r_c on Φ preserves Ψ.*

We'll prove this soon using case-checking for all rank-2 reduced root systems ($A_1 \times A_1$, A_2, $B_2 = C_2$, G_2). But first let's see how to use this criterion. For root systems coming from an actual split reductive pair (G,T), the reflections come from the standard Weyl element in SL_2 or PGL_2 identified with $G_c = \langle U_c, U_{-c}\rangle = \mathscr{D}(Z_G(T_c)) \subset G$ (equipped with the split maximal torus $\mathscr{T}_c := T \cap G_c$). So in that setting for $\Psi = \Phi(P,T)$ it suffices to show that such Weyl elements arise from $P(k)$ when $\{c,-c\} \subset \Psi$. The key claim is:

Lemma 6.3.9. *If $c \in \Phi(P,T)$ then $U_c \subset P$.*

Granting this lemma, it follows that if $\{c,-c\} \subset \Phi(P,T)$ then

$$P \supset \langle U_c, U_{-c}\rangle = G_c := \mathscr{D}(Z_G(T_c)) \simeq SL_2 \text{ or } PGL_2$$

where the final isomorphism carries the split maximal k-torus $\mathscr{T}_c := T \cap G_c$ over to the diagonal. Then P contains the k-point n_c corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in SL_2 or PGL_2, and n_c-conjugation preserves $T_c \cdot \mathscr{T}_c = T$ (as the identity on T_c and -1 on $\mathscr{T}_c$), inducing r_c on $X(T)$. Since conjugation on G by $n_c \in P(k)$ trivially preserves P (and T), it follows that the induced effect r_c of n_c-conjugation on $X(T)$ preserves $\Phi(P,T)$ as desired. Here is the proof of Lemma 6.3.9.

Proof. We have $P \supset B \supset T$, so

$$Z_P(T_c) = P \cap Z_G(T_c) \supset B \cap Z_G(T_c).$$

Now, $Z_P(T_c)$ is smooth and connected and $B \cap Z_G(T_c)$ is a Borel subgroup of $Z_G(T_c)$ (recall that intersecting a Borel subgroup with the centralizer of a subtorus of the Borel always yields a Borel subgroup of that torus centralizer), so $P \cap Z_G(T_c)$ is parabolic in $Z_G(T_c)$.

We know that for any connected reductive group H, there is a natural bijection

$$\{\text{parabolics of } H\} \Longleftrightarrow \{\text{parabolics of } \mathscr{D}H\}$$

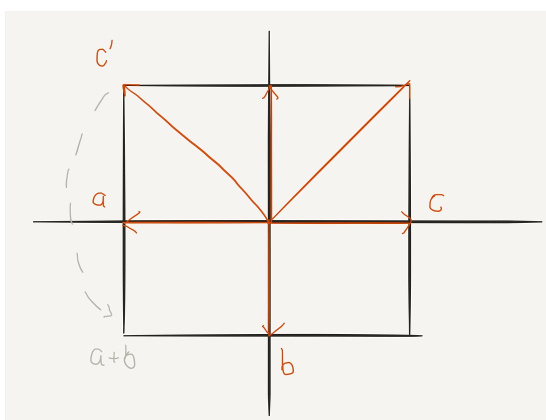
due to the central isogeny $Z \times \mathscr{D}(H) \to H$ for the maximal central torus $Z \subset H$ that is contained in every maximal torus and hence in every parabolic subgroup. Thus, the fact that $P \cap Z_G(T_c)$ is *parabolic* in $Z_G(T_c)$ implies that $P \cap G_c$ is parabolic in G_c. The (split) maximal torus $\mathscr{T}_c \subset G_c$ acts on $\mathrm{Lie}(G_c)$ with the only roots being $c|_{\mathscr{T}_c}$ and $-c|_{\mathscr{T}_c}$. Hence, we can pass to G_c (and $\mathscr{T}_c$) to reduce to the case of SL_2 or PGL_2 with diagonal T. In this case the possibilities for parabolic subgroups P containing T are G, B^+, and B^-, so the containment $U_c \subset P$ follows by inspection. $\square$

Proof of Proposition 6.3.8. For reduced (V,Φ) and regular $\mu \in V^*$ and $\Phi_{\mu>0} \subset \Psi \subset \Phi$, we want to show that Ψ is closed if Ψ is r_c-stable for all $\{c,-c\} \subset \Psi$.

Choose $a, b \in \Psi$ such that $a+b \in \Phi$. Note that a and b can't be dependent because $\Phi \cap \mathbf{Q} \cdot a = \{\pm a\}$ and $0, 2a \notin \Phi$ by the reducedness of Φ. Thus, $V' := \mathbf{Q}a + \mathbf{Q}b \subset V$ is a plane, containing the finite spanning $\Phi' := (\mathbf{Z}a + \mathbf{Z}b) \cap \Phi \subset V' - \{0\}$. This is itself a reduced root system (using the coroots $\{c^\vee|_{V'}\}_{c\in\Phi'}$) since the reflection of a root c' through a reflection $r_{c''}$ is an integral combination of c' and c''. Using $\Psi' = \Psi \cap \Phi'$, the problem reduces to that of $(V', \Phi', \Psi', \mu|_{V'})$, so now our problem considered reduced root systems of rank 2.

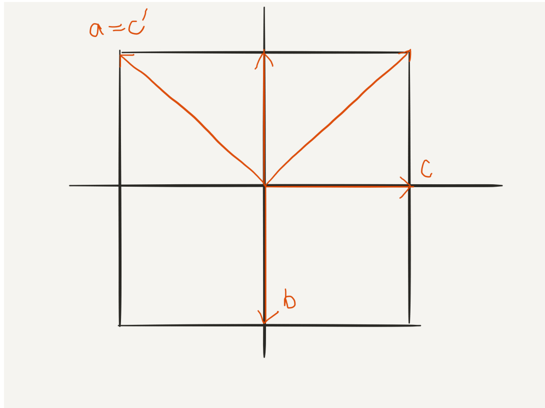
But there is a classification of reduced root systems (V,Φ) equipped with a choice of $\Phi^+ = \Phi_{\mu>0}$ for regular $\mu \in V^*$, rather explicit in the rank-2 case. We will discuss this more fully later when we need to make more serious use of Dynkin diagrams and related concepts, but for now we simply state the classification and then check each case. Of course, we can also assume at least one of a or b is not contained in Φ^+ or else there is nothing to do (as $\Phi^+ \subset \Psi$ by hypothesis and obviously Φ^+ is closed).

The possibilities, as mentioned already, are $\mathrm{A}_1 \times \mathrm{A}_1$, A_2, $\mathrm{B}_2 = \mathrm{C}_2$, G_2. The first case is irrelevant since no two roots have sum equal to a root in that case. For the other cases, to be systematic one separately considers the cases $a \in \Phi^+$, $b \notin \Phi^+$ and then $a, b \notin \Phi^+$ (exhausting all possibilities since the task is symmetric in a and b). Taking into account symmetries of a regular hexagon, the case of A_2 involves just two possibilities for $\{a, b\}$ to check, each of which is easy, and to illustrate the more interesting case-checking we now present the case of B_2.



The above Figure is for one of the few cases to consider with $a, b \notin \Phi^+$ (where Φ^+ consists of the $\mathbf{Z}_{\geq 0}$-linear combinations of c and c' as shown). We want to show that $a+b \in \Psi$. Since b and $-b$ are in Ψ, we have by assumption that r_b preserves Ψ. But then $a + b = r_b(c') \in \Psi$.

Among the few cases with $a \in \Phi^+$ and $b \notin \Phi^+$, here is one:



In this case we have $b, -b \in \Psi$ so $-a = r_b(c' + 2c) \in \Psi$. Then $\pm a \in \Psi$, so Ψ contains $r_a(b) = a + b$.

For the root system G_2, the subset of roots with a common length in G_2 is a copy of A_2. We have already settled A_2, so for the case-checking with G_2 we may assume that if a and b have the same length then each is short and $a + b$ is long. $\square$

The following consequence of Theorem 6.1.1 in the split case will later be generalized to incorporate the non-split case.

Corollary 6.3.10. *For a split reductive pair* (G, T), *there is a natural inclusion-preserving bijection*

$$\{\textit{parabolics} \supset T\} \Longleftrightarrow \{\textit{parabolic subsets of}\, \Phi(G, T)\}$$

Proof. The maps are

$$P \longmapsto \Phi(P, T)$$
$$P_G(\lambda) = \langle T, \{U_c\}_{c \in \Psi} \rangle \longleftarrow\!\shortmid\; \Psi = \Phi_{\lambda \geq 0}$$

The inclusion-preserving nature follows from the explicit descriptions. $\square$

Remark 6.3.11. The preceding proof shows that parabolic k-subgroups $P, P' \supset T$ we have $P \supset P'$ if and only if $\mathrm{Lie}(P) \supset \mathrm{Lie}(P')$. The conditions on both sides of this equivalence do not refer to T, so one might wonder if it is true even without the assumption $P, P' \supset T$ or more generally without assuming G contains a split maximal k-torus at all.

Later we'll show that for parabolic k-subgroups P, P' in a connected reductive group G over any field k, $P \cap P'$ is smooth and contains a maximal k-torus of G. (Note that to prove this assertion it suffices to work over $\overline{k}$.) Therefore, $P \subset P'$ if and only if $\mathrm{Lie}(P) \subset \mathrm{Lie}(P')$ because it is enough to check over $\overline{k}$, where a choice of common maximal torus in P and P' becomes split.

As another application, we can settle a question raised in Remark 1.3.2:

Corollary 6.3.12. *If P is a parabolic k-subgroup of a connected reductive k-group G then the scheme-theoretic normalizer $N_G(P)$ coincides with P.*

Proof. We can assume $k = \overline{k}$. Since equality holds on geometric points (Theorem 1.3.1) and P is smooth and connected, it suffices to check equality of tangent spaces. From the functorial meaning of $N_G(P)$, if $X \in \mathrm{Lie}(N_G(P))$ then $\mathrm{Ad}_G(g)(X) - X \in \mathrm{Lie}(P)$ for all $g \in G(k)$. If $T \subset P$ is a maximal torus and $\mathrm{Lie}(N_G(P))$ is strictly larger than $\mathrm{Lie}(P)$ then we can pick such $X \neq 0$ that is an eigenvector for T, say with weight a. Hence, for all $t \in T(k)$ we have $(a(t)-1)X \in \mathrm{Lie}(P)$, forcing $a = 1$. Thus, it suffices to show that all T-weights on $\mathrm{Lie}(G)/\mathrm{Lie}(P)$ are nontrivial. But we can pick $\lambda \in \mathrm{X}_*(T)$ such that $P = P_G(\lambda)$, so $U_G(-\lambda) \to G/P$ is an open neighborhood of the identity. In particular, all T-weights a on $\mathrm{Lie}(G/P) = \mathfrak{g}/\mathfrak{p}$ satisfy $\langle a, \lambda\rangle < 0$, so $a \neq 1$. □

7. Maximal split tori and minimal parabolic subgroups

7.1. **Conjugacy results and first steps of proof.** The following theorem (to be proved!) underlies the structure theory of connected reductive groups over general fields.

Theorem 7.1.1. *Let G be a connected reductive group over a field k.*

(1) The maximal split k-tori are $G(k)$-conjugate.
(2) The minimal parabolic k-subgroups are $G(k)$-conjugate.

Note that (1) is only interesting if there is a non-central split k-torus, and (2) is only interesting when there is a proper parabolic k-subgroup. But we have seen in Remark 6.1.2 via the dynamic description of parabolic k-subgroups there is a non-central split k-torus if and only if there is a proper parabolic k-subgroup, so the interesting cases of (1) and (2) either both occur or both do not occur.

Remark 7.1.2. Warnings:

- Do not confuse maximal split k-tori with split maximal k-tori.
- In general there may not be Borel k-subgroups, so the minimal parabolic k-subgroups may not be solvable.

Exercise 7.1.3. Read the handout "Root datum for reductive groups", especially Examples 2.1 and 2.2 for comparing root data of G and $\mathscr{D}G$, as well as comparing root data for G versus G/Z_G (including the special case $G = \mathrm{GL}_n$). See the October 25, 2011 Eilenberg lecture by B. Gross (at Columbia) on YouTube.

Before we prove Theorem 7.1.1 we record some consequences.

Corollary 7.1.4. *If there exists a Borel k-subgroup $B \subset G$, then every minimal parabolic k-subgroup is a Borel, and all such are $G(k)$-conjugate.*

Proof. By dimension reasons every parabolic k-subgroup contains a minimal one, and certainly B is minimal (as it is even minimal over $\overline{k}$). Thus, by part (2) of Theorem 7.1.1, every parabolic k-subgroup of G contains a $G(k)$-conjugate of B, so the minimal parabolic k-subgroups of G are precisely the $G(k)$-conjugates of B. In particular, such k-subgroups are Borel k-subgroups. □

Corollary 7.1.5. *If G is split, then every Borel k-subgroup contains a split maximal k-torus.*

Proof. By hypothesis G contains a split maximal torus T. We know that we can construct Borel k-subgroups containing T via the dynamic method: these are $P_G(\lambda)$ for a regular cocharacter $\lambda \in \mathrm{X}_*(T)$. Since Theorem 7.1.1 tells us that all Borel k-subgroups are conjugate, it follows that every Borel k-subgroup contains a split maximal torus (in fact a $G(k)$-conjugate of T). □

Example 7.1.6. Later we'll see that for a non-degenerate finite-dimensional quadratic space (V, q) and a non-degenerate finite-dimensional hermitian space (W, h) (relative to separable quadratic extensions k'/k), usually $\mathrm{SO}(q)$ and $\mathrm{SU}(h)$ are *not* quasi-split (the minimal parabolic k-subgroups and maximal split k-tori will be related to maximal isotropic subspaces for the bilinear form B_q and sesquilinear form B_h respectively).

The handout "Compactness and anisotropicity" shows that for connected semisimple G over a local field k (including $k = \mathbf{R}$), G has no proper parabolic k-subgroup (which is equivalent to G being k-anisotropic) if and only if $G(k)$ is compact. Of course if $G(k)$ is compact then G is k-anisotropic (equivalently, G contains no proper parabolic k-subgroup), so the converse is the interesting part.

Example 7.1.7. If k is any field and D is a finite-dimensional central division algebra over k then by Exercise 1 of Homework 8 of the previous course the connected semisimple k-group $\mathrm{SL}_1(D)$ of units of reduced norm 1 in D (i.e., the k-group scheme whose points valued in a k-algebra R are the units with reduced norm 1 in the Azumaya R-algebra $R \otimes_k D$) is k-anisotropic.

It is a deep result of Bruhat and Tits (beyond the scope of this course) that over a non-archimedean local field k, up to central isogeny these are the *only* nontrivial connected semisimple k-groups that are k-anisotropic and absolutely simple (i.e., have no nontrivial proper normal subgroups over $\overline{k}$).

7.2. **Proof of part (1) of Theorem 7.1.1.** We will prove (1) by induction on $\dim G$. We may assume that there are non-central split k-tori (as otherwise the only maximal split k-torus is the maximal split k-subtorus of the unique maximal central k-torus $Z \subset G$), so any *maximal* split k-torus $S \subset G$ is non-central. In particular, there must exist a proper parabolic k-subgroup P. We will apply dimension induction to a Levi k-subgroup $L \subset P$ (e.g., $L = Z_G(\lambda)$ for $\lambda : \mathbf{G}_m \to G$ such that $P = P_G(\lambda)$). To carry out such dimension induction with L, we require:

Proposition 7.2.1. *For a connected reductive k-group G and a parabolic k-subgroup P, the natural map $G(k) \to (G/P)(k)$ is surjective.*

A natural first attempt at proving such surjectivity is to note that for $\xi \in (G/P)(k)$, the fiber $\pi^{-1}(\xi) \subset G$ is a (right) torsor for P, so we can try to show that *all* P-torsors are trivial (i.e. have a k-point) when $P \neq G$ (the case $P = G$ being trivial in Proposition 7.2.1). This works if $P = B$ is a Borel k-subgroups containing a split maximal k-torus T, since then $B = T \ltimes \mathscr{R}_{u,k}(B)$ is filtered by $\mathbf{G}_a$'s and $\mathbf{G}_m$'s, so $\mathrm{H}^1(k, B) = 1$.

However, the triviality of P-torsors more generally is *false*, as we'll see in a moment. The upshot is that we *cannot* expect to prove Proposition 7.2.1 by a purely cohomological argument. More specifically, the fiber of $\pi : G \to G/P$ over any $\xi \in (G/P)(k)$ is a P-torsor, but even if G is split it can happen (for many k) that there exist non-trivial

P-torsors over k. Hence, the absence of such torsors in the fibers of π over $(G/P)(k)$ is remarkable. Here is an example:

Example 7.2.2. Let $G = \mathrm{PGO}(q) = \mathrm{PGO}_N$ for the standard split non-degenerate quadratic form

$$q = x_1 x_N + x_2 x_{N-1} + \dots$$

in N variables with $N \geq 4$; this is SO_N for odd N and SO_N/μ_2 for even N. The 1-parameter subgroup $\lambda(t) = \mathrm{diag}(t, 1, 1, \dots, 1, t^{-1}) \bmod \mathbf{G}_m$ gives rise to a proper parabolic k-subgroup $P = P_G(\lambda) = L \ltimes U$ with k-split $U = U_G(\lambda)$ and $L = Z_G(\lambda)$. We have $L = \mathrm{GO}(q')$ for $q' := x_2 x_{N-1} + \dots$ a split non-degenerate quadratic form in $N-2$ variables (check separately for odd N and even N, using the good behavior of torus centralizers under quotient maps in the latter case).

For any L-torsor E whose pushout along $L \to P$ is trivial, E is trivial since there is a retraction homomorphism $P \twoheadrightarrow P/U = L$. (Using the insensitivity of the k-split property under k_s/k-twisting of unipotent connected linear algebraic k-groups, one can even show that the sets of isomorphism classes of L-torsors and P-torsors over k are in natural bijection under such pushout.) Thus, to exhibit a nontrivial P-torsor over k it suffices to construct a non-trivial torsor over k for $L = \mathrm{GO}_{N-2}$.

Since GO_{N-2} is the group scheme of conformal automorphisms of the standard split quadratic space of rank $N-2$, the isomorphism class of an L-torsor over k is "the same" as the conformal isometry class (i.e., homothety class) of a non-degenerate quadratic space of rank $N-2 \geq 2$ over k. Hence, the class of such a k-anisotropic quadratic space (of which there are many examples over any global field k) does the job.

Proof of Proposition 7.2.1. For finite k, all torsors for a smooth connected k-group are trivial by Lang's Theorem. Thus, the naive idea works in this case, so we may and do assume k is infinite. Now the crucial fact is that $G(k) \subset G$ is Zariski dense! This follows from:

Proposition 7.2.3. *For any field K, a connected linear algebraic K-group H is unirational (over K) if K is perfect or H is reductive.*

Recall that by definition a geometrically integral scheme X of finite type over a field K is *unirational* (over K) when there is dominant K-morphism $\Omega \to X$ with Ω dense open in an affine space over K.

Proposition 7.2.3 is proved in a handout, where it is shown that the hypotheses are optimal by giving a non-unirational unipotent H over any imperfect field.

Let's give some brief comments on the proof of unirationality in the reductive case over any field (to which the general case over perfect fields is reduced, using the split property of unipotent connected linear algebraic groups over perfect fields). The idea for infinite K is to show that H is generated by its maximal K-tori (reducing unirationality to the case of tori). More precisely, for infinite K one uses Zariski-density considerations in Lie algebras (reminiscent of the proof of Grothendieck's theorem on maximal tori) to build K-tori $T_1, \dots, T_n \subset H$ such that the multiplication map of K-schemes

$$T_1 \times \dots \times T_n \to H$$

is dominant. A natural method is to try to find tori whose Lie algebras span the Lie algebra of H, though that does not actually work in general (e.g., in characteristic 2 all maximal tori in SL_2 have the *same* 1-dimensional Lie algebra, namely the Lie algebra of the central μ_2), forcing one to instead use dimension induction via centralizers of semisimple elements in the Lie algebra.

The unirationality of tori can be handled directly by arguing with Galois lattices: any Galois lattice occurs inside a direct product of finitely many induced Galois lattices, so any torus over a field K is a quotient of a direct product of finitely many Weil-restricted tori of the form $\mathrm{R}_{K'/K}(\mathbf{G}_m)$ for separable finite extensions K'/K. But as a K-scheme such a Weil restriction is dense open inside $\mathrm{R}_{K'/K}(\mathbf{A}^1_K) = \mathbf{A}^{[K':K]}_K$.

Returning to the proof of Proposition 7.2.1, we'll show that $G \to G/P$ has Zariski-local sections. (This obviously implies that the induced map of rational points is surjective.) So we want to show that there is an open cover $\{\Omega_\alpha\}$ of G/P such that $\pi^{-1}(\Omega_\alpha) \to \Omega_\alpha$ has sections (over k) for all α. How can we do this?

One such dense open subset of G/P is provided by the dynamic method: we know that $P = P_G(\lambda)$ for some k-homomorphism $\lambda : \mathbf{G}_m \to G$, so

$$U_G(-\lambda) \times P \to G$$

is an open immersion. Then by descent theory $U_G(-\lambda) \to G/P$ is an open immersion, so the image $\Omega \subset G/P$ has a section

$$\begin{tikzcd} U_G(-\lambda) \arrow[r, hook] \arrow[d, "\simeq"'] & G \arrow[d] \\ \Omega \arrow[r, hook] & P/G \end{tikzcd}$$

Clearly we have sections over $g \cdot \Omega$ for all $g \in G(k)$ (by applying g-translation to the section over Ω). Hence, it is enough to show that $\{g \cdot \Omega\}_{g \in G(k)}$ covers G/P. But $G(k) \subset G$ being Zariski dense implies that $G(k) \subset G_{\overline{k}}$ is Zariski-dense (by expressing the injectivity in terms of injectivity of the total evaluation map

$$k[G] \to \prod_{x \in G(k)} k$$

and then tensoring with $\overline{k}$, using that $V \otimes_k \prod_i W_i \to \prod_i (V \otimes_k W_i)$ is injective for any vector space V and collection of vector spaces $\{W_i\}$). It then follows that $\{g\Omega_{\overline{k}}\}$ is an open cover of $(G/P)_{\overline{k}}$ because (by the Nullstellensatz) such an "open covering" property for $\overline{k}$-schemes of finite type may be checked on $\overline{k}$-points: if $x \in (G/P)(\overline{k})$ and $g_0 \in G(\overline{k})$ represents x then the non-empty open set $\Omega_{\overline{k}} g_0^{-1}$ has non-empty open preimage in $G_{\overline{k}}$ that must contain some point g in the dense subset $G(k)$, so $g.x \in \Omega_{\overline{k}}$ and hence $x \in g^{-1}\Omega_{\overline{k}}$. □

The following important consequence of Proposition 7.2.1 is the analogue over general fields of the fact that a Borel subgroup over an algebraically closed field contains a conjugate of any connected solvable closed subgroup:

Corollary 7.2.4. *Let G be a connected reductive group over a field k, and P a parabolic k-subgroup. Let $H \subset G$ be a connected linear algebraic k-subgroup that is split-solvable (i.e. has a composition over k with successive quotients $\mathbf{G}_a$ or $\mathbf{G}_m$). There exists $g \in G(k)$ such that $gHg^{-1} \subset P$.*

This result will be used with H a torus in our proof of Theorem 7.1.1(1), and with H unipotent in our proof of Theorem 7.1.1(2).

Proof. The connected split-solvable H acts by left translation on the proper k-scheme $X = G/P$ for which $X(k) \neq \emptyset$. Then the Borel fixed point theorem, which applies for *connected split-solvable* groups acting on proper schemes with a rational point over a field (as shown in the previous course), says that there is a rational fixed point $x \in X(k)$ fixed by H. But $X(k) = G(k)/P(k)$ by Proposition 7.2.1, and if $g \in G(k)$ represents x then $g^{-1}Hg \subset P$. □

We have carried out the preliminaries needed to finally give the proof of part (1) of Theorem 7.1.1! Let $S \subset G$ be a maximal split k-torus. We all such S to be $G(k)$-conjugate to each other. If one such S is central then we are done (as for any split k-torus S' the k-group generated by S and S' is a quotient of $S \times S'$ by centrality of S, so it is a split k-torus containing S and thus equal to S by maximality, forcing $S' \subset S$).

Now we can assume all such S are non-central. In particular, there exists a *proper* parabolic k-subgroup $P = P_G(\lambda) = L \ltimes U$ for $L = Z_G(\lambda)$ a lower-dimensional connected reductive group, and $U = U_G(\lambda)$ a connected unipotent k-subgroup that is *split* (by Proposition 5.2.12 applied with the action of the torus $\mathbf{G}_m$ through λ and the subsemigroup $A \subset \mathrm{X}(\mathbf{G}_m) = \mathbf{Z}$ consisting of positive integers). By Corollary 7.2.4, every S admits a $G(k)$-conjugate contained in P, so we may limit ourselves to only study maximal k-split tori of G contained in P.

The composition

$$S \hookrightarrow P \twoheadrightarrow P/U \simeq L$$

realizes S as a split k-torus of P/U, with $\dim(P/U) < \dim G$. By dimension induction, the image of S in P/U admits a $(P/U)(k)$-conjugate contained inside a *fixed choice* of maximal split k-torus $S_0 \subset P/U$. But $P(k) \to (P/U)(k)$ is surjective because the k-subgroup $L \subset P$ maps isomorphically onto P/U. Identifying S_0 with a k-subtorus of L in this manner, it is enough to study maximal split k-tori of G lying in $H_0 = S_0 \ltimes U$ for the split torus S_0 and the split connected unipotent U normalized by S_0.

Our general conjugacy claim for maximal split k-tori in arbitrary connected reductive k-groups has been reduced to the same for k-groups of the form $H_0 = S_0 \ltimes U_0$ where S_0 is a split k-torus acting on a split connected unipotent linear algebraic k-group U_0. For this we shall use induction on $\dim U_0$ to eventually reduce to the case of a linear representation of S_0 on a vector space, which we will be able to treat by bare hands.

Remark 7.2.5. Every k-torus $T \subset H_0$ is split since the composition $T \hookrightarrow H_0/U_0 = S_0$ has trivial kernel. Hence, we aim to prove $H_0(k)$-conjugacy of all maximal k-tori in H_0.

Warning 7.2.6. If $\mathrm{char}(k) = p > 0$ and the split connected unipotent U_0 is commutative and p-torsion then $U_0 \simeq \mathbf{G}_a^n$ (see Exercise U.1 in the handout on "Structure of solvable groups") but $\mathrm{Aut}(\mathbf{G}_a^n)$ is much bigger than GL_n for $n > 1$. The standard example of a non-linear automorphism is

$$\mathbf{G}_a \times \mathbf{G}_a \simeq \mathbf{G}_a \times \mathbf{G}_a$$
$$(x, y) \mapsto (x + y^p, y)$$

Thus, the $\mathbf{G}_a$-module structure on such U_0 is highly non-unique in general, so finding one that is S_0-equivariant is not obvious.

The situation is better in characteristic 0: the endomorphism functor of $\mathbf{G}_a$ on the category of $\mathbf{Q}$-algebras is represented by $\mathbf{G}_a$ (as the only additive polynomials over a $\mathbf{Q}$-algebra are the degree-1 monomials), so the automorphism functor of $\mathbf{G}_a^n$ on the category of $\mathbf{Q}$-algebras is represented by GL_n. Hence, any action on $\mathbf{G}_a^n$ by a group scheme in characteristic 0 is necessarily linear.

We'll argue by induction on $\dim U_0$ via S_0-equivariant composition series of U_0 over k in *split* connected linear algebraic k-subgroups of U_0. The point is to use such a composition series to simplify the situation, eventually reaching the case that U_0 is a linear representation of S_0. As a first step, we record some good properties of the "split" condition for connected unipotent linear algebraic groups over a general field. (Recall that over imperfect fields a split connected unipotent linear algebraic group can have non-split smooth connected subgroups, such as Rosenlicht's example of a non-split form of $\mathbf{G}_a$ as a subgroup of $\mathbf{G}_a^2$, in contrast with the analogue for tori.)

We need three facts about splitness in the connected unipotent case.

(1) It is inherited by quotients (*not* by k-subgroups in general).
(2) It always holds for k perfect (e.g. characteristic 0 or finite fields).
(3) It is inherited by the derived group.

For (1) and (2) see §20 of the notes from the first course. Assertion (3) is more subtle, so we sketch the idea. Recall that the derived group is generated by a bounded number of commutators; i.e., for the commutator map of k-schemes $c: U \times U \to \mathscr{D}(U)$, there exists some integer $n > 1$ such that the multiplication of n commutators

$$(U \times U)^n \to \mathscr{D}(U)$$

is dominant. Thus, (3) follows from a geometric characterization of splitness for connected unipotent linear algebraic k-groups U: it is equivalent to the existence of a dominant k-morphism

$$\mathbf{A}_k^n - Z \to U$$

for a generically k-smooth closed proper subscheme $Z \subset \mathbf{A}_k^n$. This characterization is proved in Corollary 3.9 in the handout on solvable groups. (There is also a counterexample in Remark 3.6 of that handout, giving a 1-dimensional U in characteristic 2 demonstrating the necessity of the generic smoothness hypothesis on Z.) The easy direction is that if U is split, then it is even an affine space over k. This is proved by successive application of the fact that any $\mathbf{G}_a$-torsor (for the étale topology) over an affine space over a field is itself an affine space, due to the vanishing of the étale cohomology group $\mathrm{H}^1(\mathbf{A}^n, \mathbf{G}_a)$ (via the comparison with Zariski cohomology for quasi-coherent coefficients, for example).

Using (3), we have an S_0-equivariant composition series $\{\mathscr{D}^j(U_0)\}$ with each $\mathscr{D}^j(U_0)$ actually k-split. Rather generally, whenever we have an S_0-equivariant exact sequence

$$1 \to U_0' \to U_0 \to U_0'' \to 1$$

where U_0', U_0'' are *split* connected unipotent linear algebraic k-groups, then this induces an exact sequence on k-points (because $\mathrm{H}^1(k, U_0') = 1$), so the associated exact sequence of k-groups

$$1 \to H_0' \to H_0 \to H_0'' \to 1$$

(built via semi-direct products against S_0) is also exact on k-points. In this way, we reduce the problem of showing that all maximal (split) tori are k-rationally conjugate to each other for H_0 to the same for H_0' and H_0''. By applying this with the successive quotients of the derived series $\{\mathscr{D}^j(U_0)\}$ we thereby reduce to the case where U_0 is commutative, as we now may and do assume.

In characteristic $p > 0$, we can further reduce to the case when the commutative U_0 is p-torsion by using the successive quotients of (split!) *images* $p^j(U_0)$ of multiplication by powers of p. (One might try to instead use the successive kernels $U_0[p^j]$, but this runs into numerous problems: such kernels might not be smooth or connected, if non-smooth their underlying reduced schemes might not be subgroup schemes when k is not perfect, and even if smooth and connected perhaps they may not be split!) In this way, we reduce to the case when the split connected unipotent commutative U_0 is also p-torsion when $\mathrm{char}(k) = p > 0$. Hence, in all characteristics U_0 is a vector group.

We now must confront the problem noted earlier that if $\mathrm{char}(k) > 0$ it isn't at all clear that the vector group U_0 admits a $\mathbf{G}_a$-module structure that is S_0-equivariant (i.e., can we obtain U_0 from a linear representation of S_0 on a vector space?). Incredibly, by studying the S_0-action on the huge k-vector subspace $\mathrm{Hom}_{k\text{-gp}}(U_0, \mathbf{G}_m) \subset k[U_0]$, Tits was able to realize U_0 inside a linear representation of S_0 and then refine the description to establish:

Theorem 7.2.7 (Miracle Theorem 4.3 of handout on structure of solvable groups)**.** *There exists an S_0-equivariant decomposition $U_0 = U_0' \times V'$ where*

- *S_0 acts trivially on U_0',*
- *V' is a linear representation of S_0 with no occurrence of the trivial weight.*

Since $H_0 := S_0 \ltimes U_0 = U_0' \times (S_0 \ltimes V')$, and any torus contained in H_0 obviously projects trivially to the unipotent direct factor U_0', every such torus is contained inside $S_0 \ltimes V'$. Thus, by replacing U_0 with V' we may assume that U_0 is a linear representation of S_0 with no trivial weight. Using a filtration of such a representation space built from spans of eigenlines for S_0, we further reduce to the case when there is a non-trivial character $\chi : S_0 \to \mathbf{G}_m$ such that $U_0 = \mathbf{G}_a(\chi)$ is $\mathbf{G}_a$ on which S_0 acts through scaling against χ.

Let $S_0' = (\ker)^0_{\mathrm{red}} \subset S_0$, the codimension-1 subtorus killed by χ. Every subtorus of a split k-torus always arises as a direct factor because every $\mathbf{Z}^r$-quotient of $\mathbf{Z}^n$ always splits off as a direct summand. Hence, $S_0 = S_0' \times \mathbf{G}_m$ with χ arising from a nontrivial character on the second factor. This gives

$$H_0 = S_0' \times (\mathbf{G}_m \ltimes \mathbf{G}_a(\chi))$$

where $\chi(t) = t^n$ for some $n \neq 0$. Clearly every maximal torus inside H_0 contains the *central* k-torus S_0', so we can reduce to the case $H_0 = \mathbf{G}_m \ltimes \mathbf{G}_a$ with action $t.x = t^n x$ for some $n \neq 0$. Now we're facing a very tractable-looking task!

For a maximal (split) k-torus $T \subset H_0$ (necessarily 1-dimensional), the composition

$$\sigma : T \twoheadrightarrow H_0/\mathbf{G}_a = \mathbf{G}_m$$

is an isomorphism since the kernel is the (scheme-theoretic) intersection of T with a unipotent group. Using σ^{-1} to identify T with $\mathbf{G}_m$, the inclusion of T into H_0 becomes an inclusion having the special form

$$j : \mathbf{G}_m \xrightarrow{(t, f(t))} \mathbf{G}_m \ltimes \mathbf{G}_a$$

for some Laurent polynomial $f(t) \in k[t, 1/t]$. What is the condition on f for j to be a homomorphism? It turns out to be precisely that

$$f(tt') = f(t') + \frac{f(t)}{(t')^n}.$$

Exercise 7.2.8. Inspect monomial terms in f to show that it is necessary and sufficient that $f(t) = c(t^{-n} - 1)$ for some $c \in k$. Then it is easy to check that $j(\mathbf{G}_m)$ is the $(1, c)$-conjugate of $S_0 = \mathbf{G}_m \times \{0\}$.

7.3. **Proof of part (2) of Theorem 7.1.1.** We'll shortly begin the proof of the second part of Theorem 7.1.1: for a connected reductive k-group G, all minimal parabolic k-subgroups $P \subset G$ are $G(k)$-conjugate to each other. The proof is again a game of conjugating the object of interest into smaller groups and then applying dimension induction, and we will use the settled $G(k)$-conjugacy of maximal split k-tori as part of the argument.

First we record an interesting consequence of this second conjugacy result:

Corollary 7.3.1. *For any extension field K/k and parabolic k-subgroups $P, Q \subset G$, if P_K and Q_K are $G(K)$-conjugate then P and Q are $G(k)$-conjugate.*

Proof. We know that the minimal parabolics are rationally conjugate. Therefore, by using Theorem 7.1.1 to replace Q by some $G(k)$-conjugate we can assume that P and Q both contain a *common* (minimal) parabolic k-subgroup. Of course the property of containing a common parabolic (with reference to minimality!) is preserved after ground field extension (minimality may be destroyed in this way).

Under such a property, we claim that $G(K)$-conjugacy implies $P_K = Q_K$ (and thus $P = Q$). For the purpose of proving this refined claim, we may assume $k = K = \overline{K}$. Correspondingly, P and Q contain a common Borel subgroup B (by choosing a Borel subgroup of a common parabolic subgroup).

We have by assumption $Q = gPg^{-1} \supset B$ for some $g \in G(k)$, and also $Q \supset gBg^{-1}$ since P contains B. By conjugacy of Borel subgroups of Q over the algebraically closed field k, there exists some $q \in Q(k)$ such that $q(gBg^{-1})q^{-1} = B$. Therefore, $qg \in N_G(B) = B$ (!). Hence, $qg \in B \subset Q$, so $g \in Q$ and thus $P = g^{-1}Qg = Q$ as claimed. □

We now move on to the proof of $G(k)$-conjugacy of any two minimal parabolic k-subgroups $P, Q \subset G$. By the dynamic method, there is a k-homomorphism $\lambda : \mathbf{G}_m \to G$

such that

$$P = P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda),$$

and we know that $U_G(\lambda) = R_{u,k}(P)$ is k-split. Choose a maximal split k-torus $S \subset G$. Since P is parabolic, we can find a $G(k)$-conjugate of P containing S. Therefore, without loss of generality we can assume that $P \supset S$, so

$$P \supset S \ltimes R_{u,k}(P).$$

Since $S \ltimes R_{u,k}(P)$ is connected and split-solvable, by Lemma 7.2.4 we can replace Q with a $G(k)$-conjugate so that $Q \supset S \ltimes R_{u,k}(P)$.

Proposition 7.3.2. *Let G be a connected reductive group over a field k, and let S be a maximal split k-torus of G contained in a parabolic k-subgroup P. Then*

(1) $Z_G(S) \subset P$ (so $Z_P(S) = Z_G(S)$),
(2) P is minimal if and only if $P = Z_G(S) \cdot R_{u,k}(P)$.

Before giving the proof of the proposition, let's see why this finishes the proof of part (2) of Theorem 7.1.1. Applying Proposition 7.3.2 to P and Q that each contain $S \ltimes R_{u,k}(P)$, with P minimal, we get

$$P = Z_G(S) \cdot R_{u,k}(P) \subset Q.$$

But since Q is minimal, we must have $P = Q$.

Proof of Proposition 7.3.2. Since $S \subset P$, we can find a maximal k-torus $T \subset P$ containing S. By Theorem 6.1.1, we can find $\lambda\colon \mathbf{G}_m \to T$ such that $P = P_G(\lambda)$. Since S is a maximal split torus of G, it is a fortiori the maximal split torus of T. Therefore, $\lambda(\mathbf{G}_m) \subset S$ inside T. Then anything centralizing S certainly centralizes λ, so

$$Z_G(S) \subset Z_G(\lambda) \subset P_G(\lambda) = P.$$

This proves (1).

Now onto (2). Write $U = R_{u,k}(P)$ and consider the quotient map

$$P \twoheadrightarrow P/U =: \overline{P}$$

onto a connected reductive k-group. Clearly $P = Z_G(S) \cdot U$ if and only if $Z_G(S) \twoheadrightarrow \overline{P}$. But recall that torus centralizers always behave well under quotient maps: the image of the torus centralizer is the centralizer of the image torus. Applying that to the surjection $P \twoheadrightarrow \overline{P}$, the image of $Z_G(S) = Z_P(S)$ in $\overline{P}$ is $Z_{\overline{P}}(\overline{S})$ for the image torus $\overline{S}$ under $S \twoheadrightarrow \overline{S} \subset \overline{P}$. But $S \twoheadrightarrow \overline{S}$ is an isomorphism since $S \cap U = 1$ (as S is a torus and U is unipotent), so for dimension reasons $\overline{S}$ is maximal as a split torus of $\overline{P}$ (as $\overline{P} \simeq Z_G(\lambda) \subset G$).

The upshot is that $P = Z_G(S) \cdot U$ if and only if the maximal split k-torus $\overline{S}$ in $\overline{P}$ is central. This is equivalent to saying that *all* split tori of the connected reductive $\overline{P}$ are central, which is equivalent to the connected reductive $\overline{P}$ having no proper parabolic k-subgroups. Therefore, it suffices to show that P has no proper parabolic k-subgroups of G if and only if $\overline{P}$ has no proper parabolic k-subgroups of itself.

We claim that a linear algebraic k-subgroup $Q \subset P$ is parabolic in G if and only if Q is parabolic in P. This is because we can always check the property of being parabolic over $\overline{k}$, where it is equivalent to containing a Borel, and by dimension considerations the Borel subgroups of $P_{\overline{k}}$ are the same as the Borel subgroups of $G_{\overline{k}}$ contained in $P_{\overline{k}}$.

Also, *every* parabolic k-subgroup of P contains $U := R_{u,k}(P)$ because $U_{\overline{k}} = R_{u,\overline{k}}(P_{\overline{k}})$ (the dynamic construction commutes with field extension) and the unipotent radical of a linear algebraic group over $\overline{k}$ is contained in every Borel (being connected and solvable). Hence, we have established a bijection

$$\{\text{parabolic } k\text{-subgroups of } P\} \longleftrightarrow \{\text{parabolic } k\text{-subgroups of } P/U\}$$

that completes the proof. □

7.4. **Consequences.**

Proposition 7.4.1. *For P a parabolic k-subgroup of a connected reductive k-group G, all maximal split k-tori of P are $P(k)$-conjugate.*

Proof. Write $P = L \ltimes U$ as usual, and consider the quotient map

$$P \twoheadrightarrow P/U \xleftarrow{\sim} L.$$

We saw above that any maximal split k-torus $S \subset P$ maps isomorphically onto a maximal split k-torus $\overline{S} \subset P/U$. By the settled reductive case, all such $\overline{S}$'s are $(P/U)(k)$-conjugate. Also, $P(k) \twoheadrightarrow (P/U)(k)$ because $P = L \ltimes U$. Therefore, we can reduce to considering those S for which $\overline{S}$ is equal to a fixed $\overline{S}_0 \subset P/U$ for a fixed maximal split k-torus $S_0 \subset P$. Hence, $S \subset S_0 \ltimes U$. But rational conjugacy is already known for semidirect products of a split torus against a split connected unipotent linear algebraic k-group, as we saw in the *proof* of Theorem 7.1.1. □

We want to refine some of the conclusions above. For instance, we showed that $P = Z_G(S) \cdot R_{u,k}(P)$ for a minimal parabolic k-subgroup P of a connected reductive k-group G, but we want to upgrade this to a semidirect product (e.g., that is useful for calculations with rational points).

Example 7.4.2. We have $\mathrm{GL}_n = \mathrm{SL}_n \cdot \mathbf{G}_m$ with $\mathbf{G}_m$ embedded diagonally, but this is rarely an equality at the level of rational points: the subgroup $\mathrm{SL}_n(k)\mathbf{G}_m(k) \subset \mathrm{GL}_n(k)$ consists of the elements whose determinant is an nth power inside $k^\times$.

Proposition 7.4.3. *Let G be a connected reductive k-group, $S \subset G$ a maximal split k-torus, and $P \subset G$ a minimal parabolic k-subgroup such that $S \subset P$. Let $U = R_{u,k}(P)$ (which is k-split).*

Then $P = Z_G(S) \ltimes U$; i.e., $Z_G(S) \cap U = 1$ as k-group schemes.

Proof. Since U is normal in P and $S \subset P$, certainly S normalizes U. Thus, $Z_G(S) \cap U$ coincides with the S-centralizer U^S in U under the conjugation action of S on U. For general reasons U^S is smooth and connected because we can express it in terms of a torus centralizer:

$$Z_{S \ltimes U}(S) = S \times U^S$$

and for any torus acting on a connected linear algebraic group we know that the centralizer is always smooth and connected.

But U^S is normal in $Z_G(S)$ because U is normal in P and moreover $Z_G(S) \subset P$. Thus, U^S is a connected normal unipotent linear algebraic k-subgroup of $Z_G(S)$. Since $Z_G(S)$ is connected reductive, this forces $U^S = 1$. □

Example 7.4.4. We claim that there exist connected semisimple groups G with a non-trivial unipotent normal (infinitesimal) subgroup scheme. To make sense of this assertion, recall that by [SGA3, XVII, Def. 1.1], an affine k-group scheme of finite type is called *unipotent* if over $\overline{k}$ it admits a composition series whose successive quotients are subgroup schemes of $\mathbf{G}_a$. By [SGA3, XVII, Theorem 3.5(i)⟺(v)], this is equivalent to being a k-subgroup scheme of the upper-triangular unipotent subgroup of GL_n for some $n > 0$.

Assume $\mathrm{char}(k) = 2$ and consider the non-degenerate quadratic space (V, q) over k with $V = k^{2n+1}$ and $q = x_0^2 + x_1x_2 + \ldots + x_{2n-1}x_{2n}$. The associated symmetric bilinear form B_q on V has defect space $V^\perp$ equal to the line ke_0 since $\mathrm{char}(k) = 2$, with induced non-degenerate bilinear form $\overline{B}_q$ on $V/V^\perp \simeq k^{2n}$ that is symplectic (as $\mathrm{char}(k) = 2$).

Consider the natural map

$$\mathrm{SO}(q) = \mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n} = \mathrm{Sp}(V/V^\perp, \overline{B}_q)$$

induced by passage to the quotient by $V^\perp$ (preserved by the smooth $\mathrm{SO}(q)$ via consideration of k_s-points). The kernel group scheme is

$$\begin{pmatrix} 1 & & & \\ \alpha_2 & 1 & & \\ \vdots & 0 & \ddots & \\ \alpha_2 & 0 & \ldots & 1 \end{pmatrix} \simeq \alpha_2^{2n}$$

This is commutative, unipotent, and non-trivial (and non-central; e.g., it does not lie inside any maximal torus).

Corollary 7.4.5. *In the setting of Proposition* 7.4.3*, we have* $N_G(S) \cap P = Z_G(S)$*. Moreover, for any* $\lambda \in \mathrm{X}_*(S)$ *such that* $P = P_G(\lambda)$ *we have* $Z_G(S) = Z_G(\lambda)$*.*

Remark 7.4.6. The existence of λ as above is guaranteed by Theorem 6.1.1 (applied with any maximal k-torus $T \subset P$ containing S, since any k-rational cocharacter of T must be valued in S by maximality of S as a k-split subtorus of T). The equality $N_G(S) \cap P = Z_G(S)$ is Step 0 towards the general *Bruhat decomposition* asserting the bijectivity of the natural map

$$N_G(S)(k)/Z_G(S)(k) \simeq P(k)\backslash G(k)/P(k)$$

(namely, the Corollary asserts that only the identity coset on the left side goes over to the identity double-coset on the right side). The finite group $N_G(S)(k)/Z_G(S)(k)$ is called the *relative Weyl group*, and later we will relate it to the Weyl group of a "relative root system" associated to the S-action on $\mathrm{Lie}(G)$.

Proof. We know that $P = Z_G(S) \ltimes U$, so $N_G(S) \cap P = Z_G(S) \ltimes (N_G(S) \cap U)$. Hence, we want to prove

$$N_G(S) \cap U = 1.$$

What can we say about this intersection?

Note that $N_G(S) \cap U \hookrightarrow N_G(S)/Z_G(S)$ since we know that $Z_G(S) \cap U = U^S = 1$, and the target is finite étale because it embeds as a finite type k-subgroup of the étale automorphism scheme $\mathrm{Aut}_{S/k}$ (as discussed in the first course). That forces the intersection $N_G(S) \cap U$ to be finite étale. But it is also stable under S (it is straightforward to check that conjugation by S stabilizes $N_G(S)$), and S is *connected*.

A smooth connected group can only act trivially on a finite étale group (by contrast, $\mathbf{G}_m$-scaling on $\mathbf{G}_a$ restricts to a nontrivial action on the finite group scheme α_p when $p = \mathrm{char}(k) > 0$), so $N_G(S) \cap U$ is centralized by S. Thus, $N_G(S) \cap U = Z_G(S) \cap U = 1$.

Finally, we pick $\lambda \in \mathrm{X}_*(S)$ such that $P = P_G(\lambda)$, so $R_{u,k}(P) = U_G(\lambda)$. Clearly $Z_G(S) \subset Z_G(\lambda)$, yet $P = Z_G(S) \ltimes R_{u,k}(P)$ by minimality of P (see Proposition 7.4.3), so the dynamic equality $P_G(\lambda) = Z_G(\lambda) \ltimes U_G(\lambda)$ forces $Z_G(S) = Z_G(\lambda)$. □

8. STRUCTURE THEORY OF REDUCTIVE GROUPS I

8.1. **Main goals.** We explain the goals for the structure theory of connected *semisimple* groups G over a field k.

I. (RELATIVE ROOT SYSTEMS) Let S be a maximal split k-torus (we now know that this is unique up to $G(k)$-conjugacy.) We define a set

$$ {}_k\Phi := \Phi(G,S) = \{\text{non-trivial } S\text{-weights on } \mathfrak{g}\} \subset \mathrm{X}(S) - \{0\}. $$

This is a root system spanning $\mathrm{X}(S)_{\mathbf{Q}}$ (that breaks down for reductive G if there are non-trivial central split tori, but the semisimplicity of G rules out that possibility). However,

- the root system ${}_k\Phi$ can be non-reduced (when S is not a maximal k-torus)
- for $a \in {}_k\Phi$, the weight space $\mathfrak{g}_a \subset \mathfrak{g}$ can be huge,
- the k-anisotropic group $M := Z_G(S)/S$ is a "black hole" in the sense that one can't say much about its structure. (We call $\mathscr{D}(M/Z_M)$ the *anisotropic kernel*.)

Note that ${}_k\Phi = \emptyset \iff G = Z_G(S) \iff S = 1$ (because Z_G is finite). The handout "Compactness and anisotropicity" proves:

Theorem 8.1.1. *For k a local field (allowing $\mathbf{R}$), a connected reductive k-group G is k-anisotropic if and only if $G(k)$ is compact.*

Remark 8.1.2. A deeper fact is that for non-archimedean k and "absolutely simple" G (i.e., G is connected semisimple and nontrivial with $G_{\overline{k}}$ having no non-trivial connected proper normal subgroup), G is k-anisotropic exactly for central quotients of algebraic k-groups of units with reduced norm 1 in central division algebras over k. The case $k = \mathbf{R}$ is exactly the classical and highly-developed story of connected compact Lie groups.

An important global counterpart to this beyond the scope of the present course is:

Theorem 8.1.3. *For k a global field, G is k-anisotropic if and only if $G(\mathbf{A}_K)/G(k)$ is compact.*

Remark 8.1.4. Connected reductive groups over finite fields k always have a Borel k-subgroup, so they have non-central split k-tori.

Example 8.1.5. (Example of large root spaces, repeated from Example 5.2.13) For k'/k a finite separable extension and G' a split connected semisimple k'-group (e.g. SL_n), and $T' \subset G'$ a split maximal k'-torus, consider $G := R_{k'/k}(G')$. The k-group G is connected semisimple since

$$G_{k_s} \simeq \prod_{\sigma : k' \hookrightarrow k_s/k} G' \otimes_{k',\sigma} k_s$$

is connected semisimple of dimension $[k':k]\dim(G')$, and likewise $R_{k'/k}(T')$ of dimension $[k':k]\dim(T')$ is a k-torus that is maximal in G. The maximal split subtorus S has dimension $\dim(T')$ (since the canonical $\mathbf{G}_m \subset \mathrm{R}_{k'/k}(\mathbf{G}_m)$ is an isogeny complement to the norm-1 subtorus of $\mathrm{R}_{k'/k}(\mathbf{G}_m)$ that is k-anisotropic); for example, if $T' \simeq \mathbf{G}_m^{n-1}$ is the diagonal of SL_n over k' then $S(k)$ is the "k-diagonal" in $\mathrm{SL}_n(k')$.

Later on we will analyze this construction in more detail and show that the k-torus S is maximal split in G and that the restriction map $\mathrm{X}(T') \to \mathrm{X}(S)$ (which has image $[k':k]\mathrm{X}(S)$, as we can see by computing with the k'-split T' replaced by $\mathbf{G}_m$) induces an isomorphism $\mathrm{X}(T')_{\mathbf{Q}} = \mathrm{X}(S)_{\mathbf{Q}}$. Moreover, we will see that $\mathfrak{g}$ is naturally identified with the Lie algebra over k (not just vector space over k) underlying $\mathfrak{g}'$, under which the k-subspaces corresponding to the T'-root lines over k' are the S-root spaces, so ${}_k\Phi$ is identified with $\Phi(G',T')$ via restriction $\mathrm{X}(T')_{\mathbf{Q}} \simeq \mathrm{X}(S)_{\mathbf{Q}}$ and all S-root spaces have dimension $[k':k]$.

Since the preceding example is not absolutely simple (i.e., over $\overline{k}$ it has nontrivial connected proper normal subgroups) when $k' \neq k$, the higher dimension of its root spaces may not be so interesting. In the next example, which is absolutely simple, there are nonetheless 2-dimensional root spaces.

Example 8.1.6. (Example of non-reduced root systems) Fix k'/k a quadratic separable extension and $\{1,\sigma\} = \mathrm{Gal}(k'/k)$. For $n \geq 2$ and $0 \leq q \leq n/2$, consider the non-degenerate hermitian form

$$h : (k')^n \times (k')^n \to k'$$

(i.e., $h(\vec{x},\vec{y})$ is linear in $\vec{x}$ and σ-semilinear in $\vec{y}$, and $h(\vec{y},\vec{x}) = \sigma h(\vec{x},\vec{y})$) given by

$$h(\vec{x},\vec{y}) = \vec{x}^T \begin{pmatrix} & 1_q & \\ \epsilon 1_q & & \\ & & \mathrm{diag}(c_i) \end{pmatrix} \sigma(\vec{y})$$

where $\sigma(c_i) = c_i$ (ensuring the hermitian property) and $\sum_{i=1}^{n-2q} c_i x_i \sigma(x_i)$ has no nontrivial zeros ("anisotropic").

Consider the special unitary group $G = \mathrm{SU}(h)$; this is a k-subgroup of $R_{k'/k}(\mathrm{SL}_n)$ and is a k'/k-form of SL_n (see Exercise 3 in HW 7 of the first course). We will later show that this has q-dimensional maximal split k-torus

$$S = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix}$$

with diagonal $t \in \mathbf{G}_m^q$ and that

$$_k\Phi = \begin{cases} \mathrm{C}_q, & n = 2q \\ \mathrm{BC}_q, & n > 2q \end{cases}$$

where BC_q is the unique non-reduced irreducible root system of rank q.

II. (PARAMETRIZATION OF PARABOLIC k-SUBGROUPS) The $G(k)$-conjugacy classes of parabolic k-subgroups are labelled by subsets $I \subset \Delta$, where Δ is a basis of $_k\Phi$. One can even write down an explicit cocharacter λ_I such that $P_I = P_G(\lambda_I)$, and $P_I \subset P_J \iff I \subset J$. In particular, $P_\emptyset$ is a minimal parabolic k-subgroup of G (and it contains S).

Moreover, we have a bijection

$$\{\text{min. parabolic } k\text{-subgroups} \supset S\} \longleftrightarrow \{\text{pos. systems of roots } \Phi^+ \subset_k \Phi\}$$

given by $P \mapsto \Phi(P, S)$.

III. (RELATIVE WEYL GROUP) Let $N := N_G(S)$ and $Z := Z_G(S)$, so N/Z is a finite constant k-group since it is a finite type subgroup of the constant automorphism scheme $\mathrm{Aut}_{S/k} = \mathrm{GL}(\mathrm{X}(S))_k$ (see Exercises 3 and 4 of Homework 6 of the first course).

We will see that the finite *relative Weyl group* $_kW := N(k)/Z(k) \subset \mathrm{Aut}_k(S)$ coincides with $(N/Z)(k)$ (which is remarkable because it doesn't follow from cohomological reasons; typically $\mathrm{H}^1(k, Z) \neq 1$) and that

$$_kW = W(_k\Phi) := \langle r_a \rangle_{a \in \Delta}$$

inside $\mathrm{GL}(\mathrm{X}(S))$. By design each reflection r_a on $\mathrm{X}(S)_{\mathbf{Q}}$ will arise from $N(k)$, but it is not evident that every element of $_kW$ arises from $W(_k\Phi)$.

IV. (BRUHAT DECOMPOSITION) The natural map

$$\frac{N_G(S)(k)}{Z_G(S)(k)} \longrightarrow P_\emptyset(k) \backslash G(k) / P_\emptyset(k)$$

is bijective; i.e.,

$$G(k) = \coprod_{w \in W(_k\Phi)} P_\emptyset(k) n_w P_\emptyset(k)$$

where $n_w \in N(k)$ is a representative for w. Moreover, the locally closed subsets $P_\emptyset n_w P_\emptyset$ (orbits for $P_\emptyset \times P_\emptyset$ acting on G via $(p, p').g = pgp'^{-1}$) are pairwise disjoint, with

$$(P_\emptyset n_w P_\emptyset)(k) = P_\emptyset(k) n_w P_\emptyset(k).$$

For split G these locally closed sets cover G and in fact constitute a stratification (with closure relations among the strata given by the Bruhat order on the Weyl group). But in general we only get a covering at the level of k-points, so the Bruhat decomposition has only group-theoretic rather than geometric meaning beyond the split case.

V. (TITS CLASSIFICATION) The possibilities for G up to k-isomorphism will be determined by the following data (and some relations among them which we omit here):

- the anisotropic kernel $\mathscr{D}(M/Z_M)$ for $M := Z_G(S)/S$ (anisotropic with trivial center, by design);
- G_{k_s} (equivalent to a *root datum*);

- a continuous $\mathrm{Gal}(k_s/k)$-action on the Dynkin diagram $\mathrm{Dyn}(G_{k_s})$.

In the split case the anisotropic kernel is trivial (as holds even in the quasi-split case, since a Levi factor $Z_G(S)$ of a minimal parabolic k-subgroup is obviously a torus when the minimal parabolic is a Borel), and the Galois action on the diagram is trivial in the split case.

The merit of this approach is that it enables one to sometimes prove that certain constructions based on linear algebra structures (quadratic forms, hermitian forms, central simple algebras, etc.) exhaust all possibilities for a given root datum over k_s. The most important case is when G is absolutely simple, and a precise understanding of $\mathrm{Aut}_{k_s}(G_{k_s})$ is essential to making exhaustive lists of possibilities.

To illustrate, for $n \geq 2$ and G_{k_s} of type B_n with trivial center, the absence of nontrivial diagram automorphisms will underlie the proof that the possibilities are exactly $\mathrm{SO}(q)$ for non-degenerate (V, q) of dimension n, with $\mathrm{SO}(q) \simeq \mathrm{SO}(q')$ if and only if (V, q) is conformally equivalent to (V', q'). On the other hand, for type D_n (informally, twisted forms of SO_{2n}) with $n \geq 3$, the possibilities are more complicated because the Dynkin diagram admits a nontrivial diagram automorphism.

The starting point for the Tits classification is the "Existence/Isomorphism Theorem". A crude formulation of this remarkable theorem is that there is a bijection

$$\left\{\text{split connected reductive } k\text{-groups}\right\}/\simeq \;\longleftrightarrow\; \{\text{root data}\}/\simeq$$

along with a description of $\mathrm{Aut}_k(G)$ in terms of $(G/Z_G)(k)$ and diagram automorphisms (upon fixing a split maximal k-torus). In particular, for any connected reductive group G over k there exists a (unique) split k-descent of G_{k_s}; this is really amazing.

The upshot is that if we can understand the split groups *and* their automorphisms, then we can hope to describe their Galois twists and thereby find all possibilities. This becomes a nontrivial problem in Galois cohomology, guided by the Galois action on the diagram and knowledge of the relationship between relative and absolute roots (i.e., the root systems for G and G_{k_s}).

Remark 8.1.7. The book [Spr] proves the Existence and Isomorphism Theorems by a method with heavy computations that works only over algebraically closed fields. An alternative proof of the Isomorphism Theorem (determining a split group by its root datum, up to isomorphism) due to Steinberg [St] works over any field, or one can use descent theory to deduce the Isomorphism Theorem over general fields from the case of algebraically closed fields (see the proof of [CGP, Thm. A.4.6]).

But all proofs of the Existence Theorem are very hard (especially to handle the exceptional groups in all characteristics, which is the crux of the problem). In [C1, App. D], the Existence Theorem is proved over $\mathbf{C}$ by an analytic method (building on results for Lie algebras in characteristic 0 and on the theory of compact Lie groups), and in [C1, §6.3] this is used to deduce a version of the Existence Theorem over $\mathbf{Z}$ following the method of Demazure in [SGA3, Exp. XXV] (based on birational group laws over Dedekind domains), from which the result follows over general fields via scalar extension. This Existence Theorem over $\mathbf{Z}$ is one the main results in [SGA3]. The method is very abstract but also rather clean (when based on the dynamic method over rings, as in [C1]).

There are a variety of alternative approaches over fields due to Steinberg, Chevalley, Lusztig, and others, often working directly with **Z**-structures on Lie algebras. But ultimately the Bruhat–Tits structure theory for reductive groups over non-archimedean local fields rests on a robust theory of reductive group schemes over discrete valuation rings (possibly of mixed characteristic), and to work with such integral models the framework of SGA3 is essential.

8.2. **Central isogeny decomposition for semisimple groups.** Root systems will turn out to classify split connected semisimple groups "up to central k-isogeny". The notion of "central quotient" for general connected linear algebraic groups is not stable under composition (consider any nilpotent connected nontrivial linear algebraic group that is non-commutative, such as any smooth connected unipotent group). We will soon see that in characteristic $p > 0$ even the notion of central isogeny is not preserved under composition, but let's first prove that everything works well in the reductive case for all characteristics:

Proposition 8.2.1. *For central quotient maps*

$$G'' \xrightarrow{f'} G' \xrightarrow{f} G$$

between connected reductive k-groups, the composite map $f \circ f' \colon G'' \twoheadrightarrow G$ has central kernel. Also, if $M \subset Z_G$ is a closed k-subgroup then

$$Z_G/M \hookrightarrow Z_{G/M}$$

Proof. The first assertion is easily deduced from the second (as the second essentially says that under a central quotient the center does not become "larger than expected"). So we focus on the second assertion, and we can assume $k = \overline{k}$ since the formation of the scheme-theoretic center of a linear algebraic group commutes with any extension of the ground field. In Corollary 2.4 of the "Basics of reductive groups" handout there is a description of the center in the connected reductive case over an algebraically closed field:

$$Z_G = \bigcap_{T \subset G} T$$

where T varies through the set of maximal tori of G.

(This description of Z_G uses crucially that G is reductive! The main content is that all points of the intersection, valued in any k-algebra A, centralize the A-scheme G_A. That intersection scheme centralizes each T, so it suffices to show that maximal tori generate G in the sense of linear algebraic groups. Consider the closed linear algebraic subgroup N generated by all maximal tori. This is clearly normal, so G/N is connected reductive but contains no nontrivial tori by design, so G/N is unipotent. Reductivity then forces $G/N = 1$, which is to say $N = G$ as desired.)

But for $M \subset Z_G$ we have a bijection between the sets of maximal tori

$$\{T \subset G\} \longleftrightarrow \{\overline{T} \subset G/M\}$$

via

$$\begin{aligned} T &\mapsto T/M \\ \pi^{-1}(\overline{T}) &\hookleftarrow \overline{T} \end{aligned}$$

for the quotient map $\pi\colon G \to G/M$. Thus, for formula for $Z_{G/M}$ in terms of intersecting such $\overline{T}$'s yields that the inclusion $Z_G/M \subset Z_{G/M}$ is an equality. $\square$

Remark 8.2.2. The preceding proof shows that what really matters about reductivity is that it ensures generation by tori (over algebraically closed fields). Hence, if we seek an example beyond the reductive case for which the formation of the center "grows" under a central quotient map we are led to search among connected unipotent linear algebraic groups.

Consider the connected unipotent Heisenberg group

$$G = U_3 = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_3$$

over any field k of characteristic $p > 0$. Note that Z_G is the copy of $\mathbf{G}_a$ in the upper right corner, the quotient by which is the direct product of the two $\mathbf{G}_a$'s coming from the other matrix entries, so we have a short exact sequence

$$1 \to \mathbf{G}_a \to G \to \mathbf{G}_a \times \mathbf{G}_a \to 1$$

This recovers our earlier observation (valid in any characteristic) with nilpotent connected linear algebraic groups that the second statement in Proposition 8.2.1 can fail for non-reductive groups.

For a counterexample to the second assertion in Proposition 8.2.1 using central *isogenies* among *connected* linear algebraic groups in characteristic $p > 0$, consider the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbf{G}_a \arrow[r] & G \arrow[r] & \mathbf{G}_a \times \mathbf{G}_a \arrow[r] & 1 \\ 1 \arrow[r] & \alpha_p \arrow[r] \arrow[u, hook] & \ker F_{G/k} \arrow[r] \arrow[u, hook] & \alpha_p \times \alpha_p \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

in which the bottom row is the induced diagram among Frobenius kernels of the corresponding groups in the top row. The bottom row is left exact a priori, and by counting orders of finite k-group schemes it is short exact because by inspection $\ker F_{G/k}$ has order p^3 for $G = U_3$ (or more generally the infinitesimal Frobenius kernel of a smooth k-group of dimension d has order p^d since the relative Frobenius map for a smooth k-scheme of pure dimension d is finite flat with constant degree p^d, as we verify via an étale-local calculation to reduce to the case of an affine space).

The key feature to prove is the centrality of the k-subgroup scheme

$$\alpha_p \times \alpha_p = (\ker F_{G/k})/\alpha_p \subset G/\alpha_p.$$

Once this is known, the isogeny $F_{G/k}\colon G \to G^{(p)}$ with visibly *non-central* kernel will be expressed as a composition of central isogenies $G \to G/\alpha_p \to G/\ker F_{G/k} = G^{(p)}$.

To justify the centrality claim made above, observe that for any k-algebra A and A-valued point

$$h = \begin{pmatrix} 1 & u & * \\ & 1 & v \\ & & 1 \end{pmatrix} \in (\ker F_{G/k})(A)$$

with $u, v \in A$ satisfying $u^p = v^p = 0$, and for any

$$g = \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \in G(A)$$

we have

$$hgh^{-1} = g \cdot \begin{pmatrix} 1 & 0 & zu - xv \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in \alpha_p(A) \subset Z_G(A),$$

so consideration with fppf group sheaves shows that the k-subgroup scheme $\alpha_p \times \alpha_p \subset G/\alpha_p$ is indeed central.

Discussion of semisimplicity. Recall that for **C**-linear representations of a finite group, we have a decomposition

$$V = \bigoplus V_i^{e_i}$$

for pairwise non-isomorphic *irreducible* V_i (and $e_i > 0$), but only the *isotypic subrepresentations* $W_i = V_1^{e_i} \subset V$ and their associated multiplicities e_i are intrinsic; there is no intrinsic choice of V_i as a subrepresentation of V when $e_i > 1$.

Similarly, for abelian varieties A over an arbitrary field k we know from the graduate course on abelian varieties that there is a k-isogeny ("Poincaré complete reducibility" over k)

$$A \sim \prod A_i^{e_i}$$

for pairwise non-isogenous k-simple A_i (with $e_i > 0$), but only the isotypic abelian subvarieties $B_i = \mathrm{im}(A_i^{e_i} \to A)$ and their associated multiplicities $e_i > 0$ are intrinsic; there is no intrinsic choice of a k-isogenous quotient of A_i inside A when $e_i > 1$.

In these cases, for $e > 1$ we have $\mathrm{GL}_{e_i}(\mathbf{C})$ acts on $V_i^{e_i}$ creating many interactions among the factors V_i when $e_i > 1$, and likewise $\mathrm{GL}_{e_i}(\mathbf{Z})$ acts on $A_i^{e_i}$ creating many interactions among the factors A_i when $e_i > 1$. In particular, there are many inclusions $V_i \hookrightarrow V_i \times V_i$ of representation spaces and $A_i \hookrightarrow A_i \times A_i$ of abelian varieties that are not the images of factor inclusions (e.g., the diagonal homomorphism in each case). But for *non-commutative* G there are no "obvious" automorphisms of $G \times G$ beyond swapping factors, and likewise no evident *normal* subgroups other than $G \times 1$ and $1 \times G$.

Example 8.2.3. The diagonal $\Delta_G \subset G \times G$ is *not* normal in $G \times G$ whenever G is not commutative! (Hint: Consider conjugation against $1 \times g$ non-central g.)

In contrast with the preceding examples, over any field k connected semisimple k-groups admit a canonical central isogeny decomposition. Let's first illustrate the result:

Example 8.2.4. For $G = \mathrm{SL}_n \times^{\mu_n} \mathrm{SL}_n$ or $\mathrm{SO}_{10} \times^{\mu_2} \mathrm{Sp}_6$ (gluing factors along the evident anti-diagonal central inclusions of μ_n and μ_2 respectively), the *only* non-trivial proper normal connected linear algebraic k-subgroups of G are the evident "factors".

The "central isogeny decomposition" for connected semisimple k-groups will be proved later via Galois descent from the split case over k_s, and it goes as follows:

Theorem 8.2.5. *Let G be a connected semisimple k-group G, and $\{G_i\}$ its set of minimal non-trivial smooth connected normal k-subgroups (all semisimple by normality, since for any smooth connected normal k-subgroup $N \subset G$ the solvable radical of $N_{\overline{k}}$ is stable under all $G(\overline{k})$-conjugations and hence is contained in $\mathscr{R}_u(G_{\overline{k}}) = 1$). The following hold:*

(1) the set $\{G_i\}$ is finite and these pairwise commute, with the multiplication homomorphism $\prod G_i \to G$ a central k-isogeny;
(2) each G_i is k-simple;
(3) the smooth connected normal k-subgroups of G are exactly

$$N_J = \langle G_i \rangle_{i \in J}$$

for subsets $J \subset I$. Moreover, $N_J \subset N_{J'}$ if and only if $J \subset J'$ (so $G_i \subset N_J$ if and only if $i \in J$).

We call the k-subgroups G_i the "k-simple factors" of G. The precise formulation of the full result in the *split* case makes it possible to prove the general result because we can use Galois descent to pull everything down from the split case applied to G_{k_s}. (Note that since normality is not transitive in non-commutative groups, it is not a tautology that each G_i is k-simple!)

To prove the theorem we will begin by settling the split case via input from the theory of root systems, such as an "irreducible decomposition" theorem for root systems. So now we digress to review some more facts concerning root systems.

9. Root systems

9.1. **Decompositions of root systems.**

Definition 9.1.1. For root systems (V,Φ) and (V',Φ'), their *direct sum* is $(V \oplus V', \Phi \coprod \Phi')$. We say that a root system (W,Ψ) is *reducible* if it is isomorphic to such a direct sum with $V, V' \neq 0$, and (W,Ψ) is *irreducible* if it's not $(0,\emptyset)$ and is not reducible.

Example 9.1.2. For a central isogeny

$$f : G' \times G'' \twoheadrightarrow G$$

of split connected semisimple k-groups, with corresponding k-isogeny of split maximal k-tori

$$T' \times T'' \twoheadrightarrow T$$

(so $f^{-1}(T) = T' \times T''$ since $\ker f$ is central), we have $\Phi(G,T) = \Phi(G' \times G', T' \times T'')$ inside $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T')_{\mathbf{Q}} \oplus \mathrm{X}(T'')_{\mathbf{Q}}$ by the known invariance of the formation of root systems with respect to central quotients.

Clearly $\Phi(G' \times G'', T' \times T'') = \Phi(G', T') \coprod \Phi(G'', T'')$, so we conclude that the root system $\Phi(G,T)$ is the direct product of the root systems $\Phi(G',T')$ and $\Phi(G'',T'')$ via the equality $\mathrm{X}(T)_{\mathbf{Q}} = \mathrm{X}(T')_{\mathbf{Q}} \oplus \mathrm{X}(T'')_{\mathbf{Q}}$. (We use the standard abuse of notation that $\Phi(G,T)$ may denote the ordered pair $(\mathrm{X}(T)_{\mathbf{Q}}, \Phi(G,T))$ without warning.)

Example 9.1.3. We will prove later that for a *split* semisimple pair (G,T), the following conditions are equivalent: G is k-simple, $G_{\overline{k}}$ is $\overline{k}$-simple, and $\Phi(G,T)$ is irreducible. Thus, the irreducibility of a root system is a combinatorial replacement for simplicity (in the sense of smooth connected normal subgroups) for split connected semisimple groups.

Now it is time to record some facts concerning the decomposition of root systems. These are all proved in [Bou, VI, §1.2] (largely relying on [Bou, IV, V], where the main work of the proofs is often found).

Proposition 9.1.4. *Every root system (V,Φ) is uniquely a direct sum of irreducible root systems (V_i,Φ_i). Here, "uniqueness" means that the subspaces $V_i \subset V$ (and so subsets $\Phi_i = V_i \cap \Phi$)are uniquely determined.*

This is much stronger than the uniqueness aspect of irreducible decomposition for characteristic-0 representations of a finite group, or of the isogeny decomposition for an abelian variety, insofar as even in the presence of multiplicities the actual subspaces $V_i \subset V$ that arise are uniquely determined.

Proposition 9.1.5. *For irreducible (V,Φ), the action of the Weyl group $W(\Phi)$ on V is absolutely irreducible.*

9.2. **Euclidean structure on root systems.** The absolute irreducibility of the $W(\Phi)$-action for irreducible (V,Φ) explains the role of Euclidean geometry in the study of root systems over $\mathbf{R}$ (even though there is no axiom concerning a preferred inner product in the definition of a root system or of irreducibility of root systems). To explain this, first note that since $W(\Phi)$ is a finite group, by averaging we obtain a $W(\Phi)$-invariant positive-definite symmetric bilinear form $V \times V \to \mathbf{Q}$ (or equivalently a $W(\Phi)$-invariant positive-definite quadratic form $q : V \to \mathbf{Q}$). Such a bilinear form is non-degenerate, and so it corresponds to a $W(\Phi)$-equivariant isomorphism $V \simeq V^*$. Hence, by Schur's Lemma (due to the absolute irreducibility!), such an equivariant isomorphism is *unique up to* $\mathbf{Q}^\times$*-scaling.*

Thus, for irreducible root systems (V,Φ) there is a unique homothety class of non-degenerate $W(\Phi)$-invariant bilinear forms on V, so it is well-posed to speak of *ratios* of (squared) root-lengths $q(a)/q(b)$ for $a, b \in \Phi$ and any $W(\Phi)$-invariant nonzero (necessarily definite) quadratic form $q : V \to \mathbf{Q}$. Irreducible root systems that are *reduced* have at most two root lengths (see [Bou, VI, §1.4, Prop. 12]). Hence, for reduced and irreducible root systems with at least two (and hence exactly two) root lengths it is always well-posed to speak of a root being "long" or "short".

Working over $\mathbf{R}$ (so positive elements have square roots), we can uniquely normalize a $W(\Phi)$-invariant nonzero $Q : V_{\mathbf{R}} \to \mathbf{R}$ by the requirement that the shortest root-length is 1. (This involves replacing an initial choice of q by $q(a_0)^{-1/2} \cdot q$ for $a_0 \in \Phi$ with shortest q-length.) Consequently, any root system admits a *canonical* Euclidean structure on its realification. This explains why it is meaningful to draw pictures of irreducible rank-2 root systems with specified Euclidean structure, and more generally to express the study of irreducible root systems in terms of the Euclidean language of reflection groups, even though there is *no Euclidean structure* specified in the initial definition of a root system as in [Bou]. This makes contact with other references that use Euclidean structure right from the beginning of the study of root systems. Scaling the inner product has no effect on the angle between two roots, so that notion is the same regardless of the choice of $W(\Phi)$-invariant q above.

Remark 9.2.1. For an *arbitrary* (not necessarily irreducible) root system (V,Φ) there is a "canonical" $W(\Phi)$-invariant positive-definite bilinear form $B_\Phi : V \times V \to \mathbf{Q}$, given by

$$B_\Phi(v,v') = \sum_{a\in\Phi} \langle v, a^\vee\rangle \cdot \langle v', a^\vee\rangle$$

where $\langle v,\ell\rangle := \ell(v)$ for $\ell \in V^*$. Indeed, B_Φ is visibly bilinear, and it is $W(\Phi)$-invariant since the effect of a reflection r_b on a coroot $a^\vee$ is to bring it to the coroot $r_b(a)^\vee$; i.e., such reflections applied to V have the effect on B_Φ of permutting the terms in the defining sum.

The positive-semidefiniteness of B_Φ is clear, and positive-definiteness holds because the coroots span V^*.

To summarize, we have just seen that if a root system (V,Φ) is irreducible then the $W(\Phi)$-action on V is absolutely irreducible over $\mathbf{Q}$, so the $W(\Phi)$-invariant symmetric isomorphism $V \simeq V^*$ is unique up to $\mathbf{Q}^\times$-scaling, and the corresponding symmetric bilinear form form $(-,-)\colon V \times V \to \mathbf{Q}$ is unique up to $\mathbf{Q}_{>0}$-scaling as a *positive*-definite form.

What about a root system which is not irreducible? In general, if (V_i,Φ_i) are the irreducible components of (V,Φ) then via the decomposition

$$V = \bigoplus V_i$$

we have a decomposition of Weyl groups

$$W(\Phi) = \prod W(\Phi_i).$$

(To see this recall that from the very construction of a direct sum of roots systems, a reflection corresponding to a root in Φ_i has no effect on the Φ_j for $j \neq i$. Thus we see explicitly that the Weyl group decomposes.) Therefore, if $B\colon V \times V \to \mathbf{Q}$ is a non-degenerate symmetric bilinear form invariant under $W(\Phi)$ then the induced isomorphism $V \simeq V^*$ of $W(\Phi) = \prod W(\Phi_i)$-representations must carry V_i to V_i^* (by looking at the $W(\Phi_i)$-isotypic decompositions). So necessarily V_i is B-orthogonal to V_j for $j \neq i$ and B is the orthogonal sum of the $W(\Phi_i)$-invariant symmetric non-degenerate forms $B_i\colon V_i \times V_i \to \mathbf{Q}$.

Lemma 9.2.2. *If we fix such $B\colon V \times V \to \mathbf{Q}$ (positive-definite) then under the induced isomorphism $V \simeq V^*$, for each root $a \in \Phi$ the coroot $a^\vee \in \Phi^\vee \subset V^*$ corresponds to*

$$a' := \frac{2a}{B(a,a)} = \frac{2a}{\|a\|_B^2}.$$

Remark 9.2.3. This formula "explains" why $a \mapsto a^\vee$ is usually not additive.

Proof. Without loss of generality, we work in $V_{\mathbf{R}}$. By $W(\Phi)$-invariance, the action of $W(\Phi)$ is *orthogonal*. (We originally defined a reflection to be an involution that is the identity on a hyperplane and negation on the quotient. Since we now have a Euclidean structure on which $W(\Phi)$ acts orthogonally, the reflections in $W(\Phi)$ are "orthogonal reflections" in the usual sense.)

For $a \in \Phi$, we have by definition

$$r_a(x) = x - \langle x, a^\vee \rangle a \tag{9.2.1}$$

for $a^\vee \in V_{\mathbf{R}}^*$. On the other hand, we have the classical formula for reflection: if $u \in V_{\mathbf{R}}$ is a unit vector orthogonal to the hyperplane $\ker(a_{\mathbf{R}}^\vee)$ of r_a-fixed points then

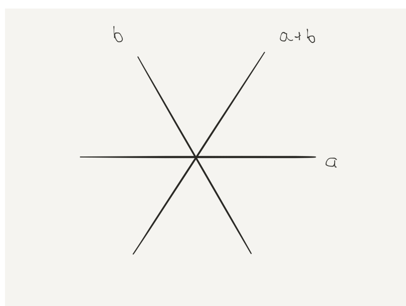
$$r_a(x) = x - 2(x \cdot u)u. \tag{9.2.2}$$

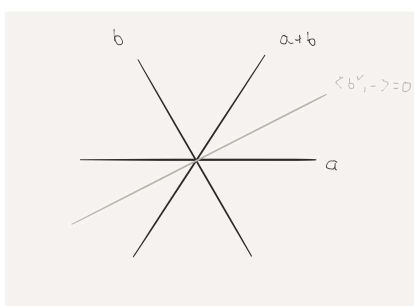
From this we see that $u \in \mathbf{R}^\times \cdot a$ (comparing (9.2.1) and (9.2.2) for x chosen generically so that $\langle x, a^\vee \rangle, x \cdot u \neq 0$). Hence, $u = \pm a/\|a\|$, and comparing the formulas (9.2.1) and (9.2.2) again gives

$$\langle x, a^\vee \rangle a = \frac{x \cdot 2a}{\|a\|^2} a.$$

Thus, the linear functional $a^\vee \in V_{\mathbf{R}}^*$ is given by dot product against $2a/\|a\|^2$. $\square$

Example 9.2.4. For the root system A_2 corresponding to SL_3 we have $c^\vee = 2c$ when the Euclidean structure is normalized to make all roots unit vectors.





Example 9.2.5. For the root system $\mathrm{B}_2 = \mathrm{C}_2$ arising from SO_5 and Sp_4 we have $c^\vee = 2c$ for c short and $c^\vee = c$ for c long (with length $\sqrt{2}$) when shortest roots are given length 1; see Figure 9.2.5.

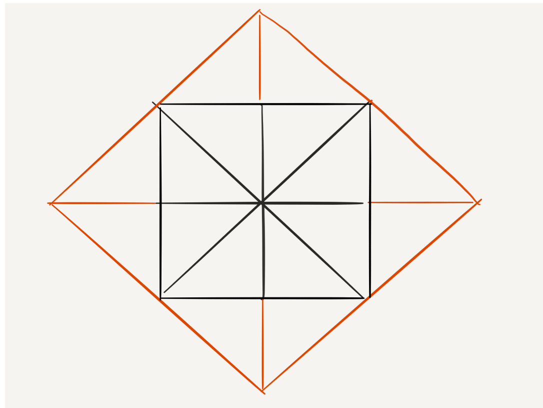


FIGURE 9.2.1. The root system $\mathrm{B}_2 = \mathrm{C}_2$ depicted in black (with short roots of length 1) and its dual in red.

Lemma 9.2.2 implies that $\Phi^\vee$ spans V^* over $\mathbf{Q}$, so $(V^*, \Phi^\vee)$ forms a root system with coroots $(a^\vee)^\vee := a$, called the *dual* root system. Since $a/\|a\|^2$ has length $1/\|a\|$, passage to the dual root system "interchanges" the long and short roots when there are two root-lengths.

We have noted and used that the Weyl group acts absolutely irreducibly on the underlying vector space for an irreducible root system, but how transitive is the Weyl action on the set of roots? The answer is: as transitive as possible. This is the first of the following two important properties of irreducible root systems:

Proposition 9.2.6. [Bou, VI, §1.3, Prop. 11] *The group $W(\Phi)$ acts transitively on the set of roots of a common length.*

Proposition 9.2.7. [Bou, VI, §1.4, Prop. 12] *If an irreducible root system Φ is reduced, then there are at most two root lengths, and in such cases the ratio of square root-lengths (long divided by short) is 2 or 3.*

9.3. **Simply connected groups.** Let $\Phi = \Phi(G, T)$ for a split semisimple pair (G, T). We have the finite index inclusions

$$\mathbf{Z}\Phi \subset X := \mathrm{X}(T) \subset (\mathbf{Z}\Phi^\vee)^*$$

with the latter $\mathbf{Z}$-dual to the finite-index inclusion

$$\mathbf{Z}\Phi^\vee \subset \mathrm{X}_*(T) =: X^\vee.$$

(Recall that coroots rationally span the dual $\mathbf{Q}$-vector space.) Hence, there are only finitely many possibilities for $\mathrm{X}(T)$, namely the intermediate groups between $\mathbf{Z}\Phi$ and $(\mathbf{Z}\Phi^\vee)^*$. The lattice $\mathbf{Z}\Phi$ is denoted Q and called the *root lattice*, and the dual lattice $(\mathbf{Z}\Phi^\vee)^*$ is denoted P and called the *weight lattice* ("poid" is French for "weight").

We have seen that the formation of $\Phi(G, T)$ is invariant under *central* isogeny (but not under general isogenies; recall the exceptional isogeny $\mathrm{SO}_{2n+1} \to \mathrm{Sp}_{2n}$ in characteristic 2; we will later see that SO_{2n+1} has root system B_n whereas Sp_{2n} has root system C_n, and these are not isomorphic for $n > 2$).

For any (G,T) and intermediate lattice $Q\subset X'\subset \mathrm{X}(T)$, the corresponding finite subgroup $\mu\subset T$ corresponding to the finite quotient $\mathrm{X}(T)/X'$ is *central* in G since it is killed by all roots (as the restriction $\Phi|_\mu\subset \mathrm{X}(\mu)=\mathrm{X}(T)/X'$ is trivial since $Q\subset X'$). Hence, the central quotient G/μ with split maximal torus T/μ has root system Φ realizing X' as the character lattice for T/μ. A calculation similar to that in Example 2.1 of the handout on root data shows that $\mathrm{X}(T/Z_G)=Q$. Furthermore, Z_G is Cartier dual to X/Q.

In the proof of Proposition 2.5 of the handout on root data it is shown that $r_{a^\vee,a}(b^\vee)=r_a(b)^\vee\in\Phi^\vee$ for all $a,b\in\Phi$, so for X equal to either of the two extremes P and Q the 4-tuple $(X,\Phi,X^\vee,\Phi^\vee)$ really is a root datum. Hence, the Existence Theorem ensures that both extremes arise from actual pairs (G,T), so taking $X=P$ then implies by the preceding considerations that *every* group X between P and Q does arise from some split pair (G,T).

To summarize, for a central isogeny $G\twoheadrightarrow \overline{G}$ satisfying $T\twoheadrightarrow\overline{T}$ we have $\mathrm{X}(\overline{T})\hookrightarrow \mathrm{X}(T)$ with finite index, and under central isogenies X moves around between Q and P exhausting all possibilities, with P being the "biggest" possibility and Q the smallest.

Definition 9.3.1. A split connected semisimple k-group G is *simply connected* if $X=P$ (equivalently, $\Phi^\vee$ spans $\mathrm{X}_*(T)$), and the *fundamental group* of G is the Cartier dual of P/X. More generally, a connected semisimple k-group is *simply connected* when G_{k_s} is so.

There are a few reasons that these are appropriate definitions. Firstly, if $k=\mathbf{C}$ then $\pi_1(G(\mathbf{C}))=\mathrm{Hom}(P/X,\mathbf{C}^\times)$ (the essential case is $X=P$). This can be proved in at least two ways: using maximal compact subgroups and Lie theory (see [C1, Prop. D.4.1]) or using the Riemann Existence Theorem that finite-degree covering spaces of the analytification of a finite type $\mathbf{C}$-scheme have a unique compatible finite étale algebraizations [SGA1, Exp. XII, Thm. 5.1] Beware however that this algebraic definition is not compatible with the usual one for real Lie groups. For instance, $\mathrm{SL}_2(\mathbf{R})=\mathrm{SO}_2(\mathbf{R})\times\mathbf{R}^2$ as manifolds, so $\pi_1(\mathrm{SL}_2(\mathbf{R}))=\mathbf{Z}$, but SL_2 is "simply connected" as a connected semisimple $\mathbf{R}$-group.

Next, and perhaps more compellingly, if G is simply connected then it has a mapping property with respect to central covers that is reminiscent of the classical definition of being simply connected and doesn't refer to a maximal torus:

Proposition 9.3.2. *Let G be a connected semsimple k-group that is simply connected. For any k-homomorphism $f:G\to H$ to a smooth affine k-group and $H'\twoheadrightarrow H$ a central extension of H by a k-group scheme μ of multiplicative type, there exists a unique k-homomorphism $f':G\to H'$ lifting f:*

$$\begin{tikzcd} & H' \arrow[d, two heads] \\ G \arrow[r] \arrow[ur, dotted, "\exists!"] & H \end{tikzcd}$$

Before we prove this result, we note that the case $H=G$ subsumes the definition of "simply connected" in the split case *granting* the Existence Theorem, since if G is connected semisimple and split then we can apply the mapping property over k with

$H=G$ and $H' \to H$ the central isogenous cover of degree $[P:X]$ corresponding to the root datum using character lattice P to get a homomorphic section, forcing $[P:X]=1$ (i.e., G is simply connected). The mapping property over k similarly forces G to be simply connected *without* a split hypothesis, using Corollary 9.3.3 below.

Proof. The uniqueness of such a lift follows from G being perfect and smooth. Indeed, the difference between any two lifts $G \to H'$ lands in $\ker(H' \to H)$, which is central in H'. The centrality implies that this difference is actually a group homomorphism. But since $G = \mathscr{D}G$, an abelian quotient of G must be trivial.

With uniqueness established, Galois descent implies that for the purpose of showing existence, we can assume without loss of generality that $k = k_s$. The advantage of this setting is that $[G(k), G(k)] \subset G$ is Zariski-dense and hence (by reducedness) schematically dense.

Schematic density is preserved by base change against *any* k-algebra (such as the artin local ring $k' \otimes_k k'$ for a finite extension k' of k), so we can use fppf descent to reduce to the existence over $\overline{k}$. (The point of passing to $\overline{k}$ is to guarantee that the underlying reduced scheme of a group scheme of finite type is a subgroup). The schematic density ensures he uniqueness of such a lift over bases which are *not* necessarily fields, such as the artin rings that come up for fppf descent from a finite extension of k. This uniqueness comes from the same argument as before: the discrepancy of two such lifts would be a homomorphism to a commutative group scheme, and that is trivial due to the extension of a schematically dense collection of commutator points over any k-algebra.

Now with $k = \overline{k}$, consider the central extension $G' := f^{-1}(H')$ of G by a k-group scheme $\mu = \ker(H' \twoheadrightarrow H)$ of multiplicative type:

$$\begin{tikzcd} G' \arrow[r] \arrow[d] & H' \arrow[d, two heads] \\ G \arrow[r] & H \end{tikzcd}$$

This reduces us to the case $H = G$ and $H' = G'$, and we want to find a homomorphic section. We can replace G' with $\mathscr{D}(G'_{\mathrm{red}})^0$ (as the latter certainly maps onto G). Now we have reduced to the case where G' is perfect, smooth, and connected. If G' were reductive (hence semisimple) then we would be done since the central kernel μ would have to be finite yet the hypothesis on G ensures that it admits no central isogeny of degree > 1 from a connected semisimple group. Since μ is of multiplicative type and the quotient $G'/\mu = G$ is reductive, the unipotent radical of G' is forced to be trivial. $\square$

Corollary 9.3.3. *Let G be a connected semisimple k-group. Up to unique isomorphism, there exists a central isogeny $\pi : \widetilde{G} \to G$ with connected semisimple $\widetilde{G}$ that is simply connected. This is initial among all central extensions of G by a k-group scheme of multiplicative type. In particular, $\widetilde{G}$ is functorial with respect to isomorphisms in such G.*

We call $\ker \pi$ is the *fundamental group* of G, recovering Definition 9.3.1 for split G.

Proof. The uniqueness assertions permit us to use Galois descent to reduce to the case $k = k_s$, so G is split. The Existence Theorem then provides a central isogeny onto G

from a simply connected $\widetilde{G}$. It suffices to show that for any surjection $q : G' \twoheadrightarrow G$ of affine k-group schemes of finite type with central kernel μ of multiplicative type, there is a unique k-homomorphism $\widetilde{G} \to G'$ over G. It is equivalent to say that the central extension $\widetilde{G} \times_G G'$ of $\widetilde{G}$ by μ admits a unique section, which in turn is the special case $H = G$ in Proposition 9.3.2. $\square$

Example 9.3.4. For SL_n with $n \geq 2$,

$$\begin{array}{ccc} \mathrm{SL}_n \supset & T = \{\mathrm{diag}\} \\ \downarrow & \downarrow \\ \mathrm{PGL}_n \supset & \overline{T} \end{array}$$

we have the finite-index inclusion

$$\mathrm{X}(\overline{T}) = (\mathbf{Z}^n)^{\Sigma=0} \hookrightarrow (\mathbf{Z}^n)/\Delta = \mathrm{X}(T).$$

obtained by the recipe in Example 2.1 of the handout on root data applied to the derived group SL_n and central quotient PGL_n of GL_n.

Explicitly, the description of the character group of the diagonal of GL_n implies that for the derived group GL_n a character of T is of the form

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod t_i^{a_i}$$

for $\vec{a} \in \mathbf{Z}^n$ that only matters modulo the diagonal since $\prod t_i = 1$. (We are just passing to character lattices on the exact sequence $1 \to T \to \mathbf{G}_m^n \to \mathbf{G}_m \to 1$ whose right map is multiplication.) Similarly, character of $\overline{T}$ is exactly a map

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto \prod t_i^{a_i}$$

that is well-defined (i.e., unaffected by scaling all t_i's against a common unit) precisely when $\sum a_i = 0$.

Using these explicit descriptions, the set of roots

$$\Phi = \{e_i - e_j \mod \Delta\}$$

spans $\mathrm{X}(\overline{T})$ over $\mathbf{Z}$ (consistent with the fact that PGL_n has trivial center) and the set of coroots

$$\Phi^\vee = \{e_i^* - e_j^*\}$$

spans the dual $\mathrm{X}_*(T) = ((\mathbf{Z}^n)^*)^{\Sigma=0}$ of $\mathrm{X}(T)$, affirming that SL_n is simply connected.

Remark 9.3.5. The preceding example is part of the type-A_{n-1} case in the handout on root systems for split classical groups. In particular, it is verified there by computing $\Delta^\vee$ that the groups Sp_{2n} for $n \geq 2$ (type C_n) are also simply connected and that special orthogonal groups SO_{2n+1} for $n \geq 2$ (type B_n) and SO_{2n} for $n \geq 3$ (type D_n) have a degree-2 simply connected central cover. The simply connected double cover of SO_N

is denoted Spin_N, and it naturally extends to a central double cover $\mathrm{Pin}_N \to \mathrm{O}_N$; all of these double covers can be constructed explicitly by using Clifford algebras.

A general discussion of Spin groups, Pin groups, and related notions (e.g., the spinor norm) associated to non-degenerate quadratic spaces is given over any ring in [C1, App. C], building on a development there of orthogonal and special orthogonal groups over any ring (which is also given in Appendix A of the notes for the first course).

9.4. **Weyl chambers.** For split reductive (G,T) we have

$$W(\Phi(G,T)) \subset \frac{N_G(T)(k)}{Z_G(T)(k)} =: W(G,T)$$

(all sitting inside $\mathrm{GL}(\mathrm{X}(T))$) as the subgroup generated by the reflections r_a, where

$$W(\mathscr{D}(Z_G(T_a)), \mathscr{T}_a) = \{1, r_a\} \subset W(G,T).$$

We want to prove that this is an equality (and likewise for relative roots ${}_k\Phi = \Phi(G,S)$ more generally). This will use Euclidean geometry to relate different-looking notions of "positive systems of roots".

We know that $W(G,T)$ acts simply transitively on the set of Borel k-subgroups containing T. We'll show that $W(\Phi(G,T))$ acts simply transitively on the set of Weyl chambers. So we need to relate Weyl chambers and Borel subgroups, and that will go through two notions of "positive systems of roots". The punchline will be that via the inclusion between the two types of Weyl groups, they compatibly act simply transitively on "the same thing" and hence the inclusion between these groups is an equality:

$$\begin{array}{ccc}
\left\{\text{Weyl chambers in } \mathrm{X}(T)_{\mathbf{R}}\right\} & & \{\text{Borel } k\text{-subgroups} \supset T\} \\
\| & & \| \\
\left\{\begin{matrix}\text{combinatorial notion of} \\ \text{positive roots } \Phi^+ \subset \Phi \\ \text{using a root basis}\end{matrix}\right\} & \xlongequal[\text{(in Bourbaki)}]{\text{Theorem}} & \left\{\begin{matrix}\text{dynamic notion of} \\ \text{positive roots } \Phi^+ \subset \Phi\end{matrix}\right\}.
\end{array}$$

Remark 9.4.1. The overall method will apply verbatim to pairs (G,S) in general (S a maximal split k-torus, and using minimal parabolic k-subgroups containing S in place of the Borel k-subgroups containing T) once we show that the subset ${}_k\Phi := \Phi(G,S) \subset \mathrm{X}(S)$ is a root system in its $\mathbf{Q}$-span and that the minimal parabolic k-subgroups $P \supset S$ have the "expected" dynamic characterization (using $\lambda \in \mathrm{X}_*(S) \cap \mathbf{Q} \cdot {}_k\Phi$ non-vanishing on all relative roots) to get

$$W({}_k\Phi) = \frac{N_G(S)(k)}{Z_G(S)(k)}$$

inside $\mathrm{GL}(\mathrm{X}(S))$,

To carry out our plan, we shall introduce Weyl chambers and related notions for an *arbitrary* root system (V,Φ) (possibly reducible or non-reduced) so that we can apply it later to ${}_k\Phi$ (that we have noted can be non-reduced for special unitary groups even over $k = \mathbf{R}$). For $a \in \Phi$, let

$$H_a := \ker(a^\vee_{\mathbf{R}}) \subset V_{\mathbf{R}}$$

be the hyperplane of r_a-fixed points (so $H_a = (\mathbf{R}a)^\perp = H_{qa}$ for any $q \in \mathbf{Q}^\times$ such that $qa \in \Phi$). The roots $\pm a$ lie in distinct connected components of $V_{\mathbf{R}} - H_a$; i.e. $\pm a$ lie on "opposite sides" of H_a.

Definition 9.4.2. A *Weyl chamber* for (V, Φ) is a connected component of $V_{\mathbf{R}} - \left(\bigcup_{a\in\Phi} H_a\right)$.

It is not at all evident a-priori how to describe such connected components in terms of choices of signs for each root (i.e., which collections of choices correspond to a connected component?), so this is one reason that passing to $V_{\mathbf{R}}$ in place of V is very helpful: we can use topological conditions to define concepts that at the outset would be difficult to describe in explicit purely algebraic terms.

Theorem 9.4.3. [Bou, VI, §1.5, Thm. 2] *We have:*

(1) $W(\Phi)$ *acts simply transitively on the set of Weyl chambers* C*, and each* C *has the form*

$$C = \{v \in V_{\mathbf{R}} \mid \langle v, a_i^\vee\rangle > 0 \ [\iff (v, a_i) > 0]\}$$

where $\{a_i\}$ *are exactly the non-divisible roots* a *such that* $\partial C \cap H_a$ *has non-empty interior in* H_a *and* a *is on the same side of* H_a *as* C *is. Also,*

$$\partial C = \bigcup_i (\partial C \cap H_{a_i}),\ \ \overline{C} = \{v \in V_{\mathbf{R}} \mid \langle v, a_i^\vee\rangle \geq 0\}.$$

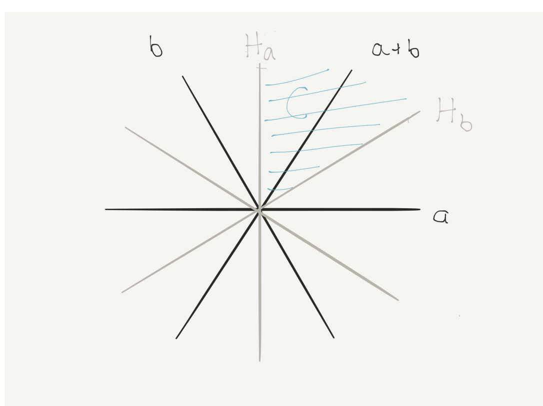
(*These* H_{a_i} *are called the "walls" of* C.)

(2) *The set* $\{a_i\}$ *is a* $\mathbf{Q}$*-basis for* (V, Φ). (*This is called a "base" or "root basis" or "set of simple positive/dominant" roots.*) *Furthermore,* $W(\Phi)$ *has a presentation*

$$W(\Phi) = \langle r_a \mid r_a^2 = 1, (r_a r_b)^{m_{ab}} = 1\rangle_{a,b\in\Delta}$$

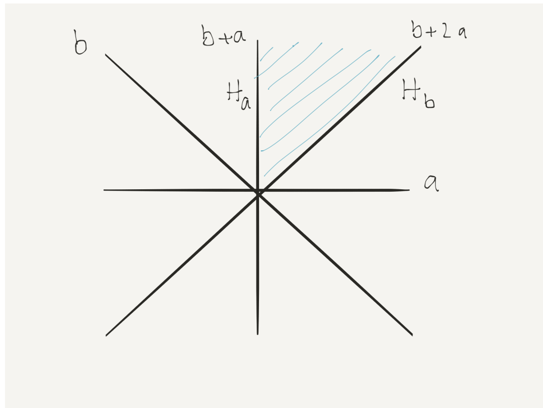
where $\measuredangle(a, b) = \pi - \frac{\pi}{m_{ab}}$*, with* $m_{ab} \in \{2, 3, 4, 6\}$. (*A group with such a presentation is called a "Coxeter group".*)

Example 9.4.4. There are six distinct Weyl chambers for the root system A_2.



In this case $W = \langle r_a, r_b\rangle = S_3$ with $(r_a r_b)^3 = 1$.

Example 9.4.5. For the root system $B_2 = C_2$, there are 8 Weyl chambers.



In this case the Weyl group admits a description $W = S_2 \ltimes (\mathbf{Z}/2\mathbf{Z})^2$ with S_2 generated by the reflection r_a in the short simple root and $(\mathbf{Z}/2\mathbf{Z})^2$ generated by the reflections r_b, r_{2a+b} in the long positive roots.

9.5. **Root bases.**

Theorem 9.5.1. [Bou, VI, §1.6, Thm. 3] *If* $\Delta \subset \Phi$ *is a base, then*

$$\Phi \subset \mathbf{Z}_{\geq 0}\Delta \sqcup (-\mathbf{Z}_{\geq 0})\Delta.$$

In other words, when a root is written in terms of a base then all coefficients are integers, with the nonzero coefficients having the same sign. In particular, Δ is a $\mathbf{Z}$-basis of the root lattice $Q = \mathbf{Z}\Phi$, so if (G, T) is a split semisimple pair with root system Φ then $Z_G = 1$ if and only if Δ spans $\mathrm{X}(T)$.

Definition 9.5.2. The subset $\mathbf{Z}_{\geq 0}\Delta$ is denoted $R_+(C)$ or $R_+(\Delta)$, and called the *positive system of roots* associated to C (or to Δ; we can reconstruct C from Δ).

Remark 9.5.3. The *Coxeter matrix* of Φ is $(m_{ab})_{a,b\in\Delta}$. This is a symmetric matrix.

Corollary 9.5.4. [Bou, VI, §1.7, Cor. 1] *A subset* $\Psi \subset \Phi$ *is a positive system of roots in the above sense if and only if* Ψ *is closed and* $\{\Psi, -\Psi\}$ *is a partition of* Φ. *In such cases, the* C *such that* $\Psi = R_+(C)$ *is unique.*

Example 9.5.5. If $\lambda \in V^*$ is regular (i.e. non-zero on all roots) then $\Phi_{\lambda>0}$ is evidently closed and together with its negative partitions Φ. Hence $\Phi_{\lambda>0} = R_+(C)$ for some C. In fact, since Δ is a $\mathbf{Q}$-basis of V *every* $R_+(\Delta)$ has the form $\Phi_{\lambda>0}$ for some regular $\lambda \in V^*$. (For instance, one could take λ to be the sum over the dual basis to Δ.)

Thus, the dynamic notion of "positive system of roots" that we have met in the study of Borel subgroups of split connected reductive groups *coincides* with Bourbaki's notion defined in terms of Weyl chambers (which in turn admits a combinatorial characterization in terms of closedness and giving a partition up to signs).

Now we return to studying $\Phi = \Phi(G,T)$ for a split reductive pair (G,T). We already know that there is a bijection

$$\{\text{Borels} \supset T\} \longleftrightarrow \{\Phi^+ \subset \Phi\}$$
$$B \mapsto \Phi(B,T)$$

with compatible actions of $W(G,T)$ on the set of Borels containing T and of $W(\Phi)$ on the set of positive systems of roots. Each action is simply transitive: for the left side this uses the general k-rational conjugacy results that we have already proved and that

$$N_G(T)(k) \cap B(k) = Z_G(T)(k),$$

(a special case of equality $N_G(S)(k) \cap P(k) = Z_G(S)(k)$ for S a maximal split torus and P a minimal parabolic k-subgroup containing S). For the right side, the result expresses the fact from Bourbaki that $W(\Phi)$ acts simple transitivity on the set of Weyl chambers. Hence, we get the desired equality of Weyl groups!

Interesting uses of Δ. For inductive arguments, it is useful to have:

Proposition 9.5.6. *Assume Φ is reduced. For $a \in \Delta$,*

$$r_a(\Phi^+ - \{a\}) = \Phi^+ - \{a\}.$$

Proof. We have $\Phi^+ \subset \mathbf{Z}_{\geq 0}\Delta$ and $r_a(x) \in x + \mathbf{Z}a$. All nonzero Δ-coordinate of $r_a(x)$ have the same sign, and only the "a-component" has changed. If $x \neq a$ then x has positive coordinate along another element of Δ (since Φ is reduced), so the nonzero Δ-coordinates of $r_a(x)$ must *all* be positive. □

Also, we have the following:

Proposition 9.5.7. [Bou, VI, §1.6, Cor. ,1] *For $a \in \Phi^+ - \Delta$, there exist $a', a'' \in \Phi^+$ such that $a = a' + a''$.*

This gives an "additive characterization" of $\Delta \subset \Phi^+$. It will underlie the proof (in Proposition 9.6.6) for reduced Φ that the Dynkin diagram $\mathrm{Dyn}(\Phi)$ (to be defined in §9.6) is connected if (and only if) Φ^+ is irreducible.

Some useful results. Let (V,Φ) be a root system. We have seen that the Weyl group W is generated by reflections through root hyperplanes associated to a root basis Δ (i.e., reflections in the hyperplanes giving the walls of a Weyl chamber), underlying a description of W as a (finite) Coxeter group with a presentation given in terms of the Coxeter matrix. Moreover, for irreducible Φ we noted earlier that the W-action on Φ is transitive on the set of roots of a given length.

Here's an important result that encodes a root system in terms of the minimal possible amount of information:

Proposition 9.5.8. [Bou, VI, §1.5, Prop. 15 and Corollary] *For reduced Φ, $W.\Delta = \Phi$ (i.e. every root lies in a root basis), and Φ is determined up to isomorphism by the Cartan matrix $(\langle a, b^\vee\rangle)_{a,b\in\Delta}$ in the following strong sense: if (V',Φ') is a reduced root system with* *root basis Δ' then for any bijection of sets $f : \Delta \to \Delta'$ respecting the Cartan matrices, the resulting $\mathbf{Q}$-linear isomorphism*

$$V = \mathbf{Q}^{\Delta} \simeq \mathbf{Q}^{\Delta'} = V'$$

carries Φ onto Φ'.

Remark 9.5.9. The Cartan matrix is generally not symmetric away from the case when Φ is simply laced (all roots with the same length). The Dynkin diagram $\mathrm{Dyn}(\Phi)$ will be a useful visual way to encode the same information as in the Cartan matrix or Coxeter matrix.

Here is a natural question, leading us to the notion of "associated coroot basis" (in the reduced case). First, some setup. Suppose Δ is a root basis of Φ, with associated positive system of roots Φ^+, so we can write $\Phi^+ = \Phi_{\lambda>0}$ for some regular $\lambda \in V^*$. (Of course λ is not unique.) Choose a W-equivariant identification $V \simeq V^*$ via a symmetric positive-definite W-invariant bilinear form B as usual.

The linear form $\lambda \in V^*$ corresponds to some $v \in V = (V^*)^*$. This v is regular for $\Phi^\vee$ since $\langle v, a^\vee\rangle = \langle a, \lambda\rangle \cdot (2/B(a,a))$. Thus, we can form a positive system of coroots $(\Phi^\vee)^+ := (\Phi^\vee)_{v>0}$. Our question is this: can the root basis corresponding to $(\Phi^\vee)^+$ be described directly in terms of Δ (without reference to the auxiliary non-canonical choice of λ)?

The answer is *yes* when Φ is reduced: the root basis (usually called a "coroot basis") is $\Delta^\vee := \{a^\vee\}_{a\in\Delta}$. Detailed hints for how to prove this are given in [C1, Exer. 1.6.17(i)-(iii)]. If you think about it, even the fact that $\Delta^\vee$ is a root basis for $\Phi^\vee$ at all is not so clear, since the association from roots to coroots is generally not additive. The Cartan matrix of $\Phi^\vee$ relative to $\Delta^\vee$ is the transpose of that for (Φ, Δ). (The reducedness hypothesis cannot be dropped: if Φ is *not* reduced, say $b = 2a$ then $a^\vee = 2b^\vee$, so $a^\vee$ cannot be in a root basis since elements of root bases are non-divisible.)

The root lattice and weight lattices can be explicitly described in terms of Δ and $\Delta^\vee$ respectively:

$$Q = \mathbf{Z}\Phi = \bigoplus_{a\in\Delta} \mathbf{Z}a$$

and the dual lattice

$$P = (\mathbf{Z}\Phi^\vee)^* = \bigoplus_{b\in\Delta} \mathbf{Z}(b^\vee)^*,$$

where $\{(b^\vee)^*\}$ is the dual basis in V to the $\mathbf{Q}$-basis $\Delta^\vee$ of V^*. Clearly the matrix of the inclusion $Q \hookrightarrow P$ with respect to these bases is the Cartan matrix. The group structure of P/Q (hence of the center in the simply connected case, and of the fundamental group in the adjoint case) is given in the extensive Tables at the end of [Bou].

Definition 9.5.10. We call $(b^\vee)^*$ the *fundamental weights* (with respect to the choice of Δ). These are the "highest weights" for the fundamental representations.

For split semisimple pair (G, T) we have $Q \subset \mathrm{X}(T) \subset P$. By the Existence Theorem, we saw just above Definition 9.3.1 that every group X between P and Q arises as $\mathrm{X}(T)$ for some central isogenous quotient of such the pair (G', T') satisfying $\mathrm{X}(T') = P$, which is to say for simply connected G' (by definition). In particular, the following conditions on G are equivalent:

- G is simply connected,
- $\Delta^\vee$ spans $X_*(T)$ (equivalently, $X_*(T) = \mathbf{Z}\Phi^\vee$),
- the natural map $\prod_{b\in\Delta} \mathbf{G}_m \to T$ defined by $(y_b) \mapsto \prod b^\vee(y_b)$ is an *isomorphism.*

Corollary 9.5.11. *Let G be a connected semisimple group that is simply connected. For any k-torus $T' \subset G$, the connected semisimple groups $\mathscr{D}(Z_G(T'))$ is simply connected.*

In particular, Levi factors of parabolic k-subgroups of G have simply connected derived groups, and if G has a split maximal k-torus T then for each $a \in \Phi(G,T)$ and codimension-1 subtorus $T_a := (\ker a)^0_{\mathrm{red}} \subset T$ the subgroup $\mathscr{D}(Z_G(T_a)) = \langle U_a, U_{-a}\rangle$ is SL_2 *rather than* PGL_2.

Remark 9.5.12. The analogue for "adjoint type" (i.e. trivial center) is false. For a counterexample, in the split case with $T' = (\ker a)^0_{\mathrm{red}}$ the k-group $\mathscr{D}(Z_G(T'))$ that is isomorphic to SL_2 or PGL_2 is always SL_2 when G is the adjoint-type PGL_n with $n \geq 3$.

The details of the proof of Corollary 9.5.11 are given in [C1, Exer. 6.5.2(iv)]. Here we just sketch the main ideas.

Idea of proof. Let T be a maximal k-torus containing T'. Without loss of generality $k = k_s$, so T is split. Hence, $T = \prod_{b\in\Delta} b^\vee(\mathbf{G}_m)$. For any "generic" $\lambda' \in X_*(T')$ (i.e., pairing non-trivially with all non-trivial T'-weights on $\mathrm{Lie}(G)$) we see by comparison of Lie algebras that the inclusion of smooth connected k-subgroups $Z_G(T') \subset Z_G(\lambda')$.

The intersection $S := T \cap (\mathscr{D}(Z_G(T')))$ is a maximal k-torus in $\mathscr{D}(Z_G(T'))$ because rather generally if H is a connected linear algebraic k-group with smooth connected normal k-subgroup N (such as $H = Z_G(T')$ and $N = \mathscr{D}(H)$) then for every maximal k-torus $T \subset H$ the (scheme-theoretic) intersection $T \cap N$ is a maximal k-torus of N; see [CGP, Cor. A.2.7] for a (simple and self-contained) proof after first trying to make your own proof as an exercise.

The torus intersection $S \subset T = \prod_{b\in\Delta} \mathbf{G}_m$ then turns out to be a subproduct

$$T \cap (\mathscr{D}(Z_G(T'))) = \prod_{\substack{b\in\Delta \\ \langle\lambda', b\rangle = 0}} \mathbf{G}_m.$$

One deduces that the elements of Δ orthogonal to λ' constitute a root basis for the connected semisimple group $\mathscr{D}(Z_G(T'))$ (with its split maximal torus S). Hence, $X_*(S)$ is spanned by the coroots, so $\mathscr{D}(Z_G(T'))$ is simply connected. $\square$

The point is that we're building things *inside* a torus T via intersection, and the structure of a k-subgroup of T can be probed with cocharacters of T. This situation cannot be similarly probed by characters, which is informally the reason that the proof does not adapt to the opposite extreme in which the property of being simply connected is replaced with the property of having trivial center (and we know it really cannot adapt to that property, due to Remark 9.5.12).

Remark 9.5.13. We say that a connected reductive group G over a field k is *of adjoint type* when $Z_G = 1$. The reason for the terminology is that the inclusion $Z_G \subset \ker \mathrm{Ad}_G$ is an equality of group schemes for any connected reductive k-group G.

This is easy to prove in characteristic 0 using the faithfulness of the Lie-algebra functor for connected algebraic groups in characteristic 0 (and Cartier's theorem on the smoothness of algebraic groups over fields of characteristic 0). But in characteristic $p > 0$ this equality of group schemes is rather specific to the reductive case: there are smooth non-commutative 2-dimensional unipotent groups in characteristic p with trivial adjoint representation! These matters are discussed fully in the handout on the adjoint kernel.

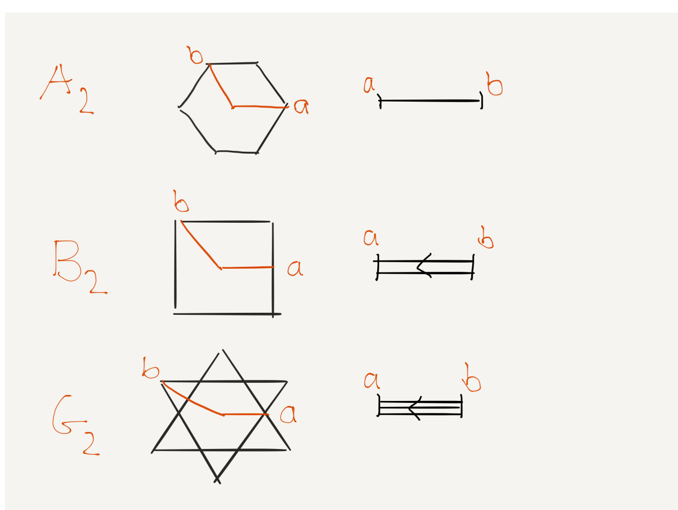
The hard part of the proof that $Z_G = \ker \mathrm{Ad}_G$ is to show that the conjugation action on $\ker \mathrm{Ad}_G$ by a maximal torus is trivial. That would imply $\ker \mathrm{Ad}_G$ is contained in the schematic centralizer of T, but $Z_G(T) = T$ and the kernel of $\mathrm{Ad}_G|_T$ is easily described in terms of roots by the very definition of roots.

9.6. **Dynkin diagrams.**

Definition 9.6.1. Let (V, Φ) be a reduced root system with root basis Δ. The *Dynkin diagram* of Φ is an "oriented weighted graph" (some edges having multiplicity and direction):

- the vertex set is Δ,
- an edge joins distinct $a, b \in \Delta$ precisely when $\langle a, b^\vee \rangle \neq 0$ (equivalently, $\langle b, a^\vee \rangle \neq 0$; i.e. a and b are not orthogonal),
- if an edge joins a and b (so by non-orthogonality they lie in the same irreducible component and hence have an intrinsic ratio between their lengths), and we relabel so that $\|a\| < \|b\|$ in case of distinct root lengths, then the edge multiplicity is
$$\frac{\langle b, a^\vee \rangle}{\langle a, b^\vee \rangle} = \frac{||b||^2}{||a||^2} \in \{1, 2, 3\},$$
with the direction pointing towards the shorter root when there are distinct root lengths.

Example 9.6.2. The Dynkin diagrams for A_2, B_2, and G_2 are depicted below.



Remark 9.6.3. The ratio of squared root lengths as above is the number of times a can be added to b to obtain another root, as is apparent in the figures above.

For distinct $a, b \in \Delta$ we have

$$\langle a, b^\vee \rangle = 2\frac{||a||}{||b||}\cos(\measuredangle(a,b))$$

(vanishing precisely when a and b are not adjacent vertices). If a and b are adjacent then $(\mathbf{Z}a+\mathbf{Z}b)\cap\Phi$ is a reduced rank-2 root system with root basis $\{a, b\}$ consisting of non-orthogonal roots, so this rank-2 root system is irreducible. Indeed, if it would be reducible then there are two irreducible components each of rank 1, giving a root basis consisting of two orthogonal roots, yet all root bases are "created equal" through the action of the Weyl group, so they cannot consist of orthogonal roots since $\{a, b\}$ is a root basis.

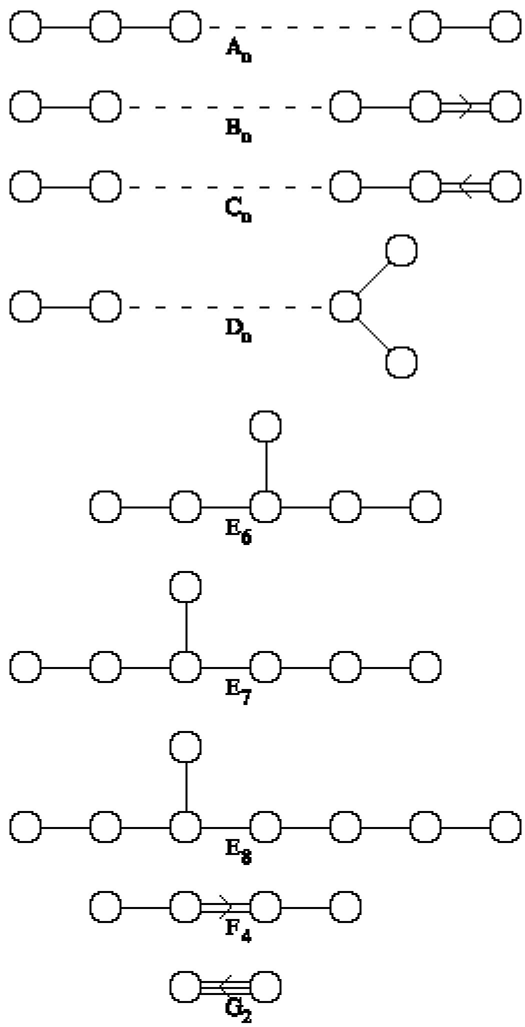
By Euclidean plane geometry, the reduced and irreducible root systems of rank 2 are completely determined in [Bou, VI, §3]. From inspection of the possibilities we see for non-orthogonal $a, b \in \Delta$ that

$$\boxed{\measuredangle(a,b) = \pi - \frac{\pi}{m_{a,b}} = \begin{cases} 2\pi/3 & \cdot - \cdot \\ 3\pi/4 & \cdot \Rightarrow \cdot \\ 5\pi/6 & \cdot \Rrightarrow \cdot \end{cases}}$$

where $m_{a,b}$ is the order of $r_a r_b$. Thus, Dyn(Φ) encodes the Coxeter and Cartan matrices, so it determines (V,Φ) up to isomorphism by Proposition 9.5.8.

The Coxeter matrix determines the presentation for $W(\Phi)$ as a finite Coxeter group. These are classified in [Bou, VI, §4.1] when Dyn(Φ) is *connected* (permitting inductive arguments with subdiagrams for making a determination of all possibilities). By Euclidean geometry [Bou, VI, §4.2] finds a list containing all *possibilities* for connected

Dyn(Φ).



One can directly construct all of these root systems inside suitable $\mathbf{R}^n$'s [Bou, VI, §4.4-4.14] (determining much related information along the way, such as the fundamental weights, the structure of P/Q, etc.), so all of these possibilities really occur.

In Proposition 9.6.6 we will prove the crucial fact that Dyn(Φ) is *connected* whenever the reduced root system Φ is irreducible (the converse is obvious: if Φ is a direct sum of non-empty root systems Φ' and Φ'' then by orthogonality considerations we see that Dyn(Φ) is the disjoint union of Dyn(Φ') and Dyn(Φ''). Hence, the above is a list of *all* reduced irreducible root systems (up to isomorphism; note that $\mathrm{B}_2 = \mathrm{C}_2$).

In the handout on classical groups, it is worked out that the root systems of types A, B, C, and D arise from specific split classical groups: SL_{n+1} is type A_n $(n \geq 1)$, SO_{2n+1} is type B_n $(n \geq 2)$, Sp_{2n} is type C_n $(n \geq 2)$, and SO_{2n} is type D_n $(n \geq 3)$. The equality of root systems $\mathrm{B}_2 = \mathrm{C}_2$ corresponds to a low-rank isomorphism $\mathrm{Sp}_4/\mu_2 \simeq \mathrm{SO}_5$ explained in linear algebraic terms in [C1, Ex. C.6.5].

Convenient low-rank conventions for some special diagram names are inspired by isomorphisms or *central* isogenies among low-rank members of the classical families (and also reasonable in terms of the pictures of the graphs). See [C1, Ex. C.6.2, C.6.3, C.6.5] for justification of the following isomorphism or central isogeny claims via calculations with $\mathbf{Z}$-groups (to make characteristic-free arguments,, especially to avoid special treatment of characteristic 2):

- define B_1 to be A_1 because $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$ through "PGL_2-conjugation" on the 3-dimensional space $\mathfrak{sl}_2 = \mathfrak{gl}_2^{\mathrm{Tr}=0}$ leaving invariant the determinant as a non-degenerate split quadratic form in 3 variables;
- define C_1 to be A_1 because $\mathrm{Sp}_2 = \mathrm{SL}_2$ (determinant on the space Mat_2 is a symplectic form in the ordered pair of columns in k^2);
- define D_2 to be $\mathrm{A}_1 \times \mathrm{A}_1$ because the action of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on the 4-dimensional space Mat_2 through left and right multiplication (i.e., $(g, g')M = gMg'^{-1}$) visibly leaves invariant the non-degenerate split quadratic form $\det : \mathrm{Mat}_2 \to k$ in 4 variables, thereby defining a homomorphism $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}(\mathrm{Mat}_2, \det) = \mathrm{SO}_4$ that is the quotient by the central diagonally embedded μ_2;
- define D_3 to be A_3 because by definition of the determinant in terms of top exterior powers, the natural action of SL_4 on the 6-dimensional $V = \wedge^2(k^4)$ leaves *invariant* the natural non-degenerate split quadratic form $V \to \wedge^4(k^4)$ defined by $q(v) = v \wedge v$, yielding a homomorphism $\mathrm{SL}_4 \to \mathrm{O}_6^0 = \mathrm{SO}_6$ that is the quotient by the central μ_2.

The non-reduced irreducible root systems should not be overlooked! To describe the possibilities, we first need to record a few facts concerning reduced root systems. A reduced irreducible root system such that (i) some ratio of squared root lengths is equal to 2 and (ii) all roots of some length are pairwise orthogonal (ruling out F_4, whose diagram has a pair of adjacent short roots and a pair of adjacent long roots) must be either B_n and C_n for some $n \geq 1$, and these types do satisfy both conditions (using inspection of the construction to verify (ii)). Moreover, an inspection of the Plates at the end of [Bou] shows the curious fact that the root systems C_n ($n \geq 1$, where C_1 means A_1) are precisely the reduced irreducible root systems for which there is a root that is non-trivially divisible inside P: its long roots are twice primitive vectors in the weight lattice P.

Feeding this information into [Bou, VI, §1.4, Prop. 13, Prop. 14], which relates non-reduced root systems to reduced root systems (using the non-divisible roots, or the non-multipliable roots), it follows that for each $n \geq 1$ up to isomorphism there is exactly one rank-n non-reduced irreducible root system: it is "$\mathrm{B}_n \cup \mathrm{C}_n$" where the long roots of B_n are the short roots of C_n and double the short roots of B_n are the long roots of C_n, so it is denoted BC_n.

The elementary divisors of the Cartan matrix describe the group structure of P/Q, and this group is the Cartier dual of Z_G for split connected semisimple G in the simply connected case with an irreducible (and reduced) root system. Let us elaborate on this to explicitly describe Z_G in each such case (granting the Existence and Isomorphism Theorems). Consider the parametrization of a split maximal torus T via cocharacters given by the coroots associated to a root basis:

$$\prod_{a\in\Delta^\vee} \mathbf{G}_m \simeq T \supset Z_G$$
$$(y_a) \mapsto \prod a^\vee(y_a).$$

How do we describe $Z_G \subset T$ in terms of such a cocharacter parameterization? Let's first work out the type-A case via matrices, and then turn to the unified approach through the structure of $P/Q = (\mathbf{Z}\Delta^\vee)^*/(\mathbf{Z}\Delta)$ (as tabulated in Plates I through IX at the end of [Bou]).

Example 9.6.4. Consider $G = \mathrm{SL}_n$ ($n \geq 2$) with split diagonal torus T. This is type A_{n-1}, with P/Q cyclic of order n (dually $Z_G = \mu_n$, as we know). Let's use Δ corresponding to the positive system of roots associated to the upper triangular Borel subgroup $B \supset T$, so $\Delta^\vee$ consists of the cocharacters $a_i^\vee = e_i^* - e_{i+1}^* : \mathbf{G}_m \to T$ given by $y \mapsto \mathrm{diag}(1,\ldots,1,y,1/y,1,\ldots,1)$ with y in the ii-entry.

We have $Z_G = \mu_n$ as scalar matrices:

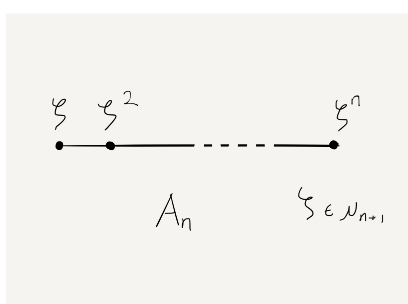
$$\begin{pmatrix} \zeta & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta \end{pmatrix} \hookrightarrow (\mathbf{G}_m^n)^{\det=1}.$$

But these n matrix entries certainly aren't the coroot coordinates (there are $n-1$ elements of $\Delta^\vee$ since this is type A_{n-1}).

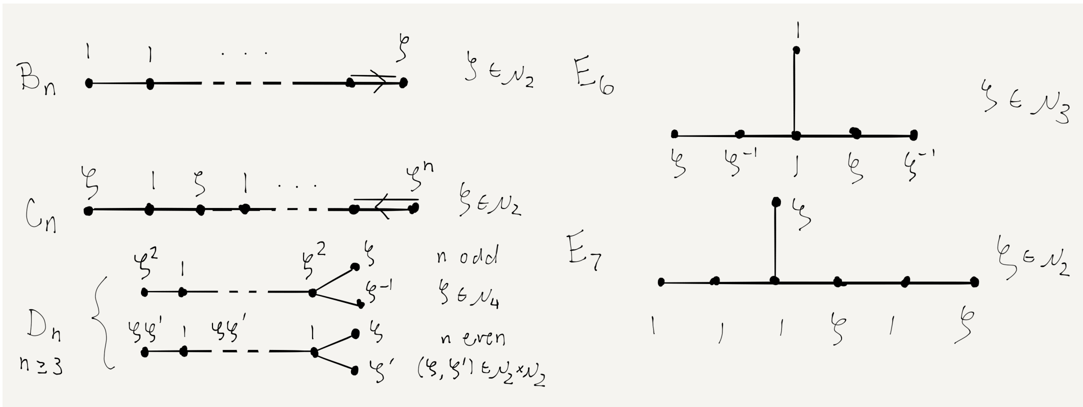
The point

$$(\zeta,\zeta^2,\ldots,\zeta^{(n-1)}) \in \mathbf{G}_m^{\Delta^\vee},$$

is the coroot coordinatization of $a_1^\vee(\zeta^1)a_2^\vee(\zeta^2)\ldots = \mathrm{diag}(\zeta,\ldots,\zeta) \in T$ for $\zeta \in \mu_n$. The way we illustrate this coordinatization of Z_G via the Dynkin diagram is:



For the additional classical types and exceptional cases E_6 and E_7 the centers are described as follows:



For E_8, F_4, and G_2 the root lattice and weight lattice coincide, so the center is trivial (i.e., the simply connected group with each of these root systems is also of adjoint type).

Remark 9.6.5. The strongest formulation of the Isomorphism Theorem gives an interpretation of diagram automorphisms: these are the automorphisms of the triple (G, B, T) when G is simply connected, and the group of such automorphisms maps isomorphically onto $\mathrm{Aut}_k(G)/(G/Z_G)(k)$. In particular, as automorphisms these do not arise from the group G/Z_G of "geometrically inner" automorphisms.

For example, the evident involution of the A_{n-1}-diagram ($n \geq 2$) has the effect of inversion on the coroot parameterization of the center μ_n, so an automorphism of SL_n inducing this could not arise from inner automorphisms when $n \geq 3$ (as inner automorphisms are always trivial on the center!). In contrast, $\mathrm{Aut}_k(\mathrm{SL}_2) = \mathrm{PGL}_2(k) = (\mathrm{SL}_2/\mu_2)(k)$, so all automorphisms of SL_2 are geometrically inner.

Likewise, the involutions of the E_6-diagram (with center μ_3) and the D_n-diagram (with center μ_4 or $\mu_2 \times \mu_2$, depending on the parity of n) for $n \neq 4$, as well as any nontrivial diagram automorphism in the "triality" case D_4, are nontrivial on the coroot parameterizations of their centers by inspection in each case.

We now take care of a loose end in our preceding discussion of the classification of reduced root systems, showing that connectedness of the diagram encodes irreducibility of the root system.

Proposition 9.6.6. *Assume that* Φ *is reduced. Then* $\mathrm{Dyn}(\Phi)$ *is connected if and only if* Φ *is irreducible.*

Proof. If Φ is reducible then $\mathrm{Dyn}(\Phi)$ is obviously disconnected. The converse is the hard part. Choose a root basis Δ for Φ^+. Suppose $\mathrm{Dyn}(\Phi)$ is disconnected; then we can partition $\Delta = \bigsqcup \Delta_i$ via connected components (so each Δ_i is non-empty and there are at least two i's). We have

$$\mathbf{Q}\Phi = \bigoplus_{a \in \Delta} \mathbf{Q}a = \bigoplus V_i \quad \text{where } V_i = \bigoplus_{a \in \Delta_i} \mathbf{Q}a.$$

Let $\Phi_i = \Phi \cap (\mathbf{Z}\Delta_i) \supset \Delta_i$. Each (V_i, Φ_i) is a root system, by inspection of the reflection formula

$$r_a(b) = b - \langle b, a^\vee \rangle a$$

with $\langle b, a^\vee \rangle \in \mathbf{Z}$. Also, the roots in Δ_i are orthogonal to those in Δ_j for any $j \neq i$ by the meaning of edges (or lack thereof!) in the diagram. Clearly Δ_i is a root basis of Φ_i, so it defines a positive system of roots Φ_i^+. We have $\bigsqcup \Phi_i^+ \subset \Phi^+$, and will show that equality occurs, so taking into account negation would give that Φ is the direct sum of the Φ_i's, contradicting the assumption that Φ is irreducible.

So we assume that there exists $a \in \Phi^+$ not in any Φ_i^+ and seek a contradiction. Certainly $a \notin \Delta$, so we can write $a = a' + a''$ for $a', a'' \in \Phi^+$ (Proposition 9.5.7). Choose such an a with minimal total sum of Δ-coordinates. By this minimality we must have $a' \in \Phi_i^+$ and $a'' \in \Phi_j^+$ for some i, j, and necessarily $i \neq j$ (or else a would be in Φ_i^+). Then

$$\Phi \ni r_{a''}(a) = a - \langle a, (a'')^\vee \rangle a'' = a' + a'' - \langle a' + a'', (a'')^\vee \rangle a''.$$

Since a' and a'' are in distinct components we have $\langle a', (a'')^\vee \rangle = 0$. Thus,

$$\Phi \ni r_{a''}(a) = a' + a'' - \langle a'', (a'')^\vee \rangle a'' = a' - a'',$$

so we have found a root whose Δ-coordinates of are mixed signs (a positive coordinate along Δ_i, and a negative coordinate along Δ_j), an absurdity. $\square$

10. Structure of reductive groups II

10.1. **Central isogeny decomposition.**

As an application of the theory of root systems, we are ready to prove Theorem 8.2.5, reproduced below.

Theorem 10.1.1. *Let G be a connected semisimple k-group G, and $\{G_i\}$ its set of minimal non-trivial smooth connected normal k-subgroups (all semisimple by normality, since for any smooth connected normal k-subgroup $N \subset G$ the solvable radical of $N_{\overline{k}}$ is stable under all $G(\overline{k})$-conjugations and hence is contained in $\mathscr{R}_u(G_{\overline{k}}) = 1$). The following hold:*

1. *the set $\{G_i\}$ is finite and these pairwise commute, with the multiplication homomorphism $\prod G_i \to G$ a central k-isogeny;*
2. *each G_i is k-simple;*
3. *the smooth connected normal k-subgroups of G are exactly*

$$N_J = \langle G_i \rangle_{i \in J}$$

for subsets $J \subset I$. Moreover, $N_J \subset N_{J'}$ if and only if $J \subset J'$ (so $G_i \subset N_J$ if and only if $i \in J$).

To provide more motivation, first we shall deduce two striking corollaries and then we will take up the proof of the theorem.

Corollary 10.1.2. *Let G be a connected reductive group over a field k, and let N be a smooth connected normal k-subgroup of G.*

(i) *If N' is a smooth connected normal k-subgroup of N then N' is normal in G.*

(ii) *If G is semisimple then there is a unique smooth connected normal k-subgroup $\mathscr{N} \subset G$ commuting with N such that the natural homomorphism $N \times \mathscr{N} \to G$ is an isogeny* (*in fact, it is a central isogeny*).

Proof. First consider connected semisimple G. By Theorem 8.2.5, we have $N = \langle G_i \rangle_{i \in J}$ for a unique subset $J \subset I$. The multiplication map $\prod_{i \in J} G_i \to N$ is surjective and it has finite central kernel since even $\prod_{i \in I} G_i \to G$ has finite central kernel. In particular, $\sum_{i \in J} \dim G_i = \dim N$. The G_i's for $i \in J$ are certainly minimal nontrivial smooth connected normal k-subgroups of N (as they are minimal as such in G), and Theorem 8.2.5 applied to N implies by dimension considerations that there are no others. Hence, $\{G_i\}_{i \in J}$ is the output of Theorem 8.2.5 applied to N.

In the setting of (i) for semisimple G, we can apply the same conclusion with N' in the role of N and with N in the role of G, so N' must be generated by G_i's for i varying through some subset of J. This implies that N' is normal in G, so (i) is proved when G is semisimple.

In general, if G is reductive then G is a central isogenous quotient of its maximal central k-torus Z and its semisimple derived group $\mathscr{D}(G)$. Likewise, N inherits reductivity from G (as we may check over $\overline{k}$), so it is a central isogenous quotient of its maximal central k-torus S and its semisimple derived group $\mathscr{D}(N)$. By normality of N in G and the compatibility of the formation of S with respect to any extension of the ground field it follows that the torus S inherits normality in G from that of N. Thus, the torus S must be central in G (as G is connected), so $S \subset Z$. Likewise N' is a commuting quotient of its maximal central torus S' that must be contained in S (so $S' \subset Z$) and its semisimple derived group $\mathscr{D}(N')$. Since $\mathscr{D}(N)$ is normal in $\mathscr{D}(G)$ (due to normality of N in G) and $\mathscr{D}(N')$ is likewise normal in $\mathscr{D}(N)$, the settled semisimple case implies that $\mathscr{D}(N')$ is normal in $\mathscr{D}(G)$. But $G = Z \cdot \mathscr{D}(G)$ with Z central, and $S' \subset Z$, so $N' = S'\mathscr{D}(N')$ is normal in G as desired. This proves (i) in general.

In the setting of (ii), our description of the possibilities for N applies to any possibility for $\mathscr{N}$, so it follows that the only possibility for $\mathscr{N}$ is $\langle G_i \rangle_{i \in I-J}$ and this clearly does work (and gives a central isogeny since $\prod_{i \in I} G_i \to G$ is a central isogeny). This proves (ii). $\square$

For connected semisimple G, the k-subgroups $G_i \subset G$ are called the *k-simple factors* of G. In general these are not direct factors, but in two important cases they are:

Corollary 10.1.3. *If G is connected semisimple then the central isogeny $f : \prod G_i \to G$ is an isomorphism if G is either simply connected or of adjoint type* (*in which case each G_i inherits the same property*).

Proof. If G is simply connected then we know that there is no nontrivial central isogeny onto G from another connected semisimple group. This settles the simply connected case.

Suppose instead that $Z_G = 1$. To prove f is an isomorphism, by the centrality of $\ker f$ note that $\ker f \subset \prod Z_{G_i}$. Hence, to prove $\ker f = 1$ (so f is an isomorphism) it suffices to directly prove that $Z_{G_i} = 1$ for each i. But G_i commutes with G_j for all $j \neq i$, so Z_{G_i} commutes with G_j for *all* j. The faithful flatness of $\prod_{j \in I} G_j \to G$ then implies that Z_{G_i} centralizes G, so $Z_{G_i} \subset Z_G = 1$ as desired. $\square$

Remark 10.1.4. The significance of Corollary 10.1.3 is addressed in the handout on simple isogeny factors: in both the simply connected and adjoint-type cases, to each class of which a general connected semisimple k-group is canonically related via a central k-isogeny, the group G admits an isomorphism to $\mathrm{R}_{k'/k}(G')$ for a *canonically associated* pair $(k'/k, G')$ consisting of a finite étale k-algebra k' and a smooth affine k'-group G' whose fiber G'_i over each factor field k' of k is a connected semisimple k_i-group that is *absolutely simple*.

This explains the essential role of the absolutely simple case (over finite separable extensions of k) in the study of general connected semisimple groups over k. More specifically, constructions as in Example 8.1.5 (but without the split hypothesis imposed there) are more ubiquitous than one might have initially expected.

10.2. **Proof of Theorem 8.2.5.** The general case can be deduced from the split case; this is explained in the handout on simple isogeny factors via arguments with Galois descent. Here we will focus on the split case, so suppose G admits a split maximal k-torus $T \subset G$. The key tool in the proof is the irreducible decomposition of root systems.

The idea is to build the G_i's directly from the irreducible components Φ_i of $\Phi(G,T)$, and they will be k-split with root system Φ_i. Thus, the k-simplicity of each G_i will reduce to proving the general fact that when Φ is irreducible then G is k-simple (and so absolutely simple, since Φ never changes under field extension in the split case). Such k-simplicity will ultimately rest on a nontrivial fact in the theory of root systems: if (V,Φ) is an irreducible root system then $W(\Phi)$ acts irreducibly (even absolutely irreducibly) on V.

Let $\Phi = \Phi(G,T)$, and $V = X_{\mathbf{Q}}$ for $X = \mathrm{X}(T)$, so V is spanned by Φ (as G is semisimple). Let Δ be the base for a positive system of roots $\Phi^+ \subset \Phi$, and consider the irreducible decomposition

$$(V,\Phi) = \bigoplus_{i \in I} (V_i, \Phi_i),$$

under which Δ decomposes as $\coprod \Delta_i$ for a base Δ_i of $\Phi_i^+ = \Phi_i \cap \Phi^+$.

We want to define $G_i = \langle U_a \rangle_{a \in \Phi_i}$. The problem is that it is hard to prove much using this definition, so we're going to take a different approach, using torus centralizers and various commutators instead. Informally, we want $\prod G_i \to G$ to be a central isogeny, so we know that the tori should match up:

$$\begin{array}{ccc} \prod G_i & \supset & \prod T_i \\ \downarrow & & \downarrow \\ G & \supset & T \end{array}$$

We'll first build the T_i's and define $G_i = \mathscr{D}(Z_G(T'_i))$ where $T'_i = \langle T_j \rangle_{j \neq i}$.

One could try to define these tori $T_i \subset T$ using quotients of X, but this leads to confusion because the character lattice is contravariant (and may lead to some mild headaches since $X/\sum_{j \neq i} \mathbf{Z}\Psi_j$ may have nonzero torsion. Hence, it is more useful to use *cocharacters*, so we'll build $T_i \subset T$ using coroots for Φ_i.

Consider the isogeny

$$\mathbf{G}_m^{\Delta} \to T$$
$$(y_a) \mapsto \prod a^{\vee}(y_a)$$

Letting $T_i = \mathrm{Im}(\mathbf{G}_m^{\Delta_i} \to T)$, we have a factorization

$$\mathbf{G}_m^{\Delta} \twoheadrightarrow \prod T_i \twoheadrightarrow T$$

of an isogeny into a composition of surjections, so the second map $\prod T_i \to T$ induced by multiplication must be an isogeny, with $\mathrm{X}_*(T_i)_{\mathbf{Q}} = \mathbf{Q}\Delta_i^{\vee} = \mathbf{Q}\Phi_i^{\vee}$.

Define $T_i' = \langle T_j \rangle_{j \neq i} \subset T$, so

$$T_i' \times T_i \to T$$

is an isogeny. Set $G_i = \mathscr{D}(Z_G(T_i'))$. It is not yet clear that $T_i \subset G_i$, nor that T_i' is the maximal central torus in $Z_G(T_i')$ (in general, a torus need not be the maximal torus in its centralizer: this fails for the centralizer of a maximal split torus in any quasi-split group that is not split).

The roots of $G_i = \mathscr{D}(Z_G(T_i'))$ relative to its *split* maximal torus intersection with T are the same as the T-roots of $Z_G(T_i')$. (Note that G_i is normalized by T, since it is a characteristic subgroup of $Z_G(T_i) \supset T$.) The T-roots for $Z_G(T_i')$ are the T-roots for G that are *trivial* on $T_i' = \langle T_j \rangle_{j\neq}$, so the T-roots for $Z_G(T_i')$ are exactly the roots $a \in \Phi$ such that $\langle a, b^{\vee} \rangle = 0$ for all $b \in \coprod_{j \neq i} \Phi_j$. Via the decomposition $X_{\mathbf{Q}} = \mathrm{X}(T_i')_{\mathbf{Q}} \oplus \mathrm{X}(T_i)_{\mathbf{Q}}$ we have shows that $\Phi_i \subset \mathrm{X}(T_i)_{\mathbf{Q}}$, so $\mathrm{X}(T_i)_{\mathbf{Q}} \supset \mathbf{Q}\Phi_i = V_i$. Passing to the direct sum over all i yields an equality, so $\mathrm{X}(T_i)_{\mathbf{Q}} = V_i$ inside $X_{\mathbf{Q}} = V$.

It is clear from the irreducible decomposition that the set of such a is exactly Φ_i. Hence, the T-root groups for $Z_G(T_i)$ are the U_a's for $a \in \Phi_i$ (in view of the unique characterization for root groups, those of $Z_G(T_i)$ and G for a common T-root must coincide). Thus, the split derived group G_i is generated by the U_a's for $a \in \Phi_i$, and its associated coroots must coincide with those for (G, T) relative to a (as each is determined by $\langle U_a, U_{-a} \rangle$ and its intersection with T). It follows that G_i contains $\langle a^{\vee}(\mathbf{G}_m) \rangle_{a \in \Phi_i} = T_i$, and as such T_i must be a split *maximal* torus of G_i. By dimension reasons with tori, it follows that T_i' must exhaust the maximal central torus in $Z_G(T_i')$.

Lemma 10.2.1. *If $i \neq j$ then G_i commutes with G_j.*

Proof. It is enough to show that $[U_a, U_b] = 1$ for $a \in \Phi_i$ and $b \in \Phi_j$. We can pick Φ^+ such that $a \in \Phi_i^+$ and $b \in \Phi_j^+$. Then

$$(U_a, U_b) \subset \prod_{c \in (a,b)} U_c$$

where $(a, b) = \Phi \cap (\mathbf{Z}_{>0} a + \mathbf{Z}_{>0} b) = \emptyset$ because $\Phi = \coprod \Phi_i \subset \bigoplus V_i$. $\square$

This shows that we have a multiplication homomorphism

$$\begin{array}{ccc} \prod G_i & \supset & \prod T_i \\ {\scriptstyle f}\downarrow & & \downarrow \\ G & \supset & T \end{array}$$

so $(\ker f_{\overline{k}})^0_{\mathrm{red}}$ has no non-trivial torus, hence is trivial (because it is normal in a reductive group). Therefore, $\ker f$ is finite; i.e. f is an isogeny.

Lemma 10.2.2. *The subgroup scheme* $\ker f \subset \prod G_i$ *is central.*

Proof. For a k-algebra R, consider $(g_i) \in (\ker f)(R)$ so $g_i^{-1} = \prod_{j \neq i} g_j$ inside $G(R)$. Then g_i commutes with $(G_i)_R$ (as each $(G_j)_R$ does) and also with $(G_j)_R$ for all $j \neq i$, so by the faithful flatness of $\prod_{i' \in I} G_{i'} \twoheadrightarrow G$, we have $g_i \in Z_G(R)$. $\square$

It remains to show:

Proposition 10.2.3. *We have:*

1. *each G_i is k-simple (even absolutely simple),*
2. *each smooth connected $N \triangleleft G$ is of the form $\langle G_i \rangle_{i \in J}$ for $J = \{i \in I \mid G_i \subset N\}$, and $N \cap G_i$ is finite for all $i \notin J$.*

Remark 10.2.4. The second part implies that the G_i are the minimal nontrivial smooth connected normal k-subgroups (as we wanted them to be). This finally provides a description of the G_i that doesn't mention T!

Proof. Let's first show that (1) $\implies$ (2). Choose $N \subset G$ as in (2), with $N \neq 1$ without loss of generality. It suffices to show that some $G_i \subset N$, since we can then induct, by passing to the quotient $G/G_i \triangleright N/G_i$ with minimal normal subgroups $\overline{G}_j := G_j/G_j \cap G_i$ (central isogenous quotients of G_j, hence with the *same* root system as G_j relative to the image of T_j). The main point, left to the reader to check by going back to the definitions, is that $\Phi(G/G_i, T/T_i) = \coprod_{j \neq i} \Phi_j$ and that $\{\overline{G}_j\}_{j \neq i}$ is the output of our main construction applied to G/G_i relative to its split maximal k-torus T/T_i.

Now we find some G_i contained in N, granting (1). Certainly N is not central in G since Z_G is finite and the smooth connected N is nontrivial, so there exists some G_{i_0} which doesn't commute with N. Consider $[G_{i_0}, N]$, which is non-trivial by assumption. It is a smooth connected normal subgroup of G, and it is contained in both N and G_{i_0} by the normality of each in G. By (1), the containment $[N, G_{i_0}] \subseteq G_{i_0}$ is therefore an equality, so $G_{i_0} = [N, G_{i_0}] \subset N$.

It remains to show (1). Renaming G_i as G, we reduce to the following lemma. $\square$

Lemma 10.2.5. *If Φ is irreducible then G is absolutely simple over k.*

Proof. Without loss of generality assume $k = \overline{k}$. Consider $N \triangleleft G$ a nontrivial smooth connected normal k-subgroup. Note that $\mathscr{R}(N)$ is solvable connected normal in G, which is connected semisimple, so $\mathscr{R}(N) = 1$; i.e. N is also semisimple.

The goal is to show that $N = G$. We'll do this by showing that a maximal torus for N is one for G as well. This will conclude the proof, because then the maximal torus image of T in the connected semisimple group G/N is trivial, forcing $G/N = 1$; i.e., $N = G$.

Pick a maximal torus $S \subset N$ (so $S \neq 1$, since $N \neq 1$) and extend it to a maximal torus $T \subset G$; note that $S \subset T \cap N \subset Z_S(N) = S$ so $T \cap N = S$. Thus, $N_G(T)(k)$-conjugation preserves $T \cap N = S$, and so $N_G(T)(k)$ naturally acts on $\mathrm{X}(S)$. The quotient map

$$\mathrm{X}(T)_{\mathbf{Q}} \twoheadrightarrow \mathrm{X}(S)_{\mathbf{Q}} \neq 0.$$

is clearly equivariant for the natural action of $W(G,T) = W(\Phi)$ on $\mathrm{X}(T)_{\mathbf{Q}}$. But as a representation space for $W(\Phi)$, we know that $\mathrm{X}(T)_{\mathbf{Q}}$ is (absolutely) irreducible since Φ is irreducible (Proposition 9.1.5), so this forces $S = T$. □

10.3. **Bruhat decomposition.** Let G be a connected reductive group over k, $S \subset G$ a maximal split k-torus, and $P \subset G$ a minimal parabolic k-subgroup such that $S \subset P$. Then $N := N_G(S) \supset Z_G(S) =: Z$ and $P = Z \ltimes U$ for $U = \mathscr{R}_{u,k}(P)$, which is k-split (as the smooth connected unipotent k-groups $U_G(\lambda)$ built by the dynamic method for smooth connected affine G and nontrivial cocharacters λ are always k-split). We consider the relative Weyl group ${}_kW := N(k)/Z(k)$ that is always a finite group (being contained in $(N/Z)(k)$ with N/Z a finite type k-subgroup of the étale automorphism scheme of S).

We want to show:

Theorem 10.3.1 (Relative Bruhat decomposition)**.** *We have that*

(1) the natural map ${}_kW \to P(k)\backslash G(k)/P(k)$ is bijective,
(2) the finite k-group N/Z is constant and the natural map $N(k)/Z(k) \to (N/Z)(k)$ is bijective.

Remark 10.3.2. The relative Bruhat decomposition is not a "geometric" result: this equality at the level of rational points does not correspond to a stratification of G *except* in the split case. However, we make the following remarks.

(i) In (1) one has the stronger disjointness property that the locally closed subvarieties $\{Pn_wP\}_{w\in {}_kW}$ inside G *are* pairwise disjoint; this is proved in §3 of the handout on the relative Bruhat decomposition.
(ii) The constancy of N/Z in (2) is easy: this finite type k-group scheme is a k-subgroup of the étale (locally finite type) automorphism scheme $\mathrm{Aut}_{S/k}$ that is constant since S is split (it is represented by $\mathrm{GL}_n(\mathbf{Z})$ if $S \simeq \mathbf{G}_m^n$).

But the equality in (2) is less clear. The proof uses that $N_G(S) \cap P = Z_G(S)$ to show that $N(k)/Z(k)$ and $(N/Z)(k)$ compatibly act simply transitively on the set of minimal parabolic k-subgroups of G containing S. (Note the analogy with our proof of such an equality in the split case, though at present we have not yet proved that $\Phi(G,S)$ is a root system inside its $\mathbf{Q}$-span in $\mathrm{X}(S)_{\mathbf{Q}}$.) This argument is given in §4 of the handout on the relative Bruhat decomposition.
(iii) In the split case (2) is immediate from Hilbert 90 (since Z is a split torus in such cases), but its validity beyond the split case is remarkable because often $\mathrm{H}^1(k,Z) \neq 1$ when G is not split. Examples with non-split quasi-split G (so Z even a torus), including some absolutely simple cases, are given in §5–§7 of the handout on the relative Bruhat decomposition.

The proof of the Bruhat decomposition in the general case involves passage to problems over $\overline{k}$ that are solved using the Bruhat decomposition over $\overline{k}$ (with Borel subgroup). Hence, we'll treat the split case first (aside from some gritty group-theoretic calculations that are presented in full in the handout on the geometric Bruhat decomposition), and then turn to the general case.

Proof of split case. Now assume that S is a maximal k-torus of G, so we denote it as T. Let $\Phi = \Phi(G,T)$ and $W := W(\Phi) = N(k)/Z(k)$. For $w \in W$, let $n_w \in N(k)$ be a representative (the choice of which will be easily seen not to matter in what follows). Let $\Phi^+ = \Phi(B,T)$, and let Δ be the corresponding root basis. We need to show that the $B(k)$-double cosets $B(k)n_w B(k)$ for $w \in W$ (which clearly do not depend on the choice of n_w) form a pairwise disjoint cover of $G(k)$. Note that for disjointness it is enough to check over $\overline{k}$.

Define $C(w) = Bn_w B$, the $B \times B$-orbit of n_w in G under the action $(b,b').g = bgb'^{-1}$, so $C(w)$ is naturally a smooth locally closed subvariety of G. We call $C(w)$ the *Bruhat cell* for w. It suffices to prove:

(i) the $C(w)$'s are pairwise disjoint (as locally closed subschemes of G),
(ii) the subschemes $C(w)$ cover G (which is sufficient to check on $\overline{k}$-points, as each $C(w)$ is locally closed), so $G(k)$ is the disjoint union of its subsets $\{C(w)(k)\}_{w\in W}$,
(iii) the natural inclusion $B(k)n_w B(k) \subset C(w)(k)$ is an equality.

We first dispose of (iii) using the product structure of open cells for G relative to the borus (B,T). The point is to remove as much of the redundancy as possible in the description of points in BnB by moving parts of the left B into the right one. Recall that

$$B = T \ltimes U$$

for $U := \mathscr{R}_{u,k}(B) = \prod_{a\in\Phi^+} U_a$, with multiplication of the positive root groups taken in *any* order (Theorem 5.3.6). Since n_w normalizes T, we have.

$$B(k)n_w B(k) = U(k)n_w B(k)$$

Next observe that $U_a n_w = n_w U_{w^{-1}(a)}$ for any $a \in \Phi$, so if $w^{-1}(a)$ is positive then we can move it across as well. This motivates us to define

$$\Phi_w^+ = \{a \in \Phi^+ \mid w^{-1}(a) \in \Phi^+\}$$
$$\Phi_w' = \{a \in \Phi^+ \mid w^{-1}(a) \in -\Phi^+\}$$

(We write Φ_w' rather than Φ_w^- to avoid notational confusion: one can make the analogue of Φ_w^+ for $\Phi^- := -\Phi^+$ in place of Φ^+, but this does not agree with Φ_w'.) These are *closed* subsets of Φ^+, so by Theorem 5.3.6 we have smooth connected k-subgroups $U_{\Phi_w'}, U_{\Phi_w^+} \subset U$ that are respectively *directly spanned* (in any order) by the root groups for Φ_w', Φ_w^+.

Clearly we have $U = U_{\Phi_w'} \times U_{\Phi_w^+}$ via multiplication, so

$$B(k)n_w B(k) = U_{\Phi_w'}(k)n_w B(k).$$

This has *all* redundancy removed, by applying the following proposition on k-points.

Proposition 10.3.3. *The multiplication map*

$$U_{\Phi_w'} n_w \times B \to C(w)$$

is an isomorphism of k-schemes.

This implies that $C(w)/B \simeq U_{\Phi_w'} n_w \simeq \mathbf{A}_k^{\#\Phi_w'}$ is an *affine space*, so the Bruhat decomposition thereby provides a covering of G/B by affine spaces. A result in the theory of root systems (see Remark 1.1 in the handout on the geometric Bruhat decomposition for the Bourbaki reference) gives that $\#\Phi'_w$ coincides with the minimal length $\ell(w)$ of w as a word in the generating set $\{r_a\}_{a\in\Delta}$ of the Coxeter group W.

Proof. Running the preceding calculation on k-points with $\overline{k}$ in place of k gives an equality

$$C(w) = Bn_wB = U_{\Phi'_w}n_wB$$

of smooth locally closed subschemes of G (as such an equality holds if it does on $\overline{k}$-points), so it suffices to show that the multiplication morphism

$$U_{\Phi'_w}n_w \times B \to G$$

(whose image is $C(w)$) is a locally closed immersion.

It is harmless to first apply left multiplication by n_w^{-1}, which turns this into the multiplication map

$$U_{w^{-1}(\Phi'_w)} \times B \xrightarrow{\text{mult}} G.$$

Since $w^{-1}(\Phi'_w) \subset -\Phi^+$, this is subsumed by the direct product structure of the open cell:

$$\begin{tikzcd}
U_{w^{-1}\Phi'_w} \times B \arrow[r] \arrow[d, hook] & G \arrow[d, equal] \\
U_{-\Phi^+} \times T \ltimes U_{\Phi^+} \arrow[r] & G
\end{tikzcd}$$

with the bottom side an open immersion and the left side a closed immersion. □

Using Proposition 10.3.3, to complete the proof of the Bruhat decomposition in the split case it remains to prove that the $C(w)$'s are pairwise disjoint and cover G at the level of $\overline{k}$-points. Hence, we may and do now assume $k = \overline{k}$. We give some highlights, and refer to the handout on the geometric Bruhat decomposition for the omitted details (especially for certain intricate group-theoretic manipulations). In the following discussion, we work throughout with k-points.

Since $C(w)$ and $C(w')$ are B-double cosets, if they intersect non-trivially at all then they are equal. So let's first address why $C(w) = C(w')$ are not disjoint, then $w' = w$. Since $n_{w'} \in C(w') = C(w) = U_{\Phi'_w}n_wB$, there exist $u \in U_{\Phi'_w}$ and $b \in B$ such that $n_{w'} = un_wb$. Recall that $B = T \ltimes U$ with $U = U_{\Phi'_w} \times U_{\Phi^+_w}$ via multiplication. It is therefore not surprising that

$$U_{\Phi^+_w} = U \cap n_wBn_w^{-1},$$

and the relation $n_w = u^{-1}n_{w'}b^{-1}$ then implies

$$U_{\Phi^+_w} = u^{-1}U_{\Phi^+_{w'}}u$$

inside U.

The T-weights on $\mathrm{Lie}(U)$ are nontrivial and linearly independent with 1-dimensional weight spaces, so an inductive argument using the descending *central* series of the nilpotent U ensures that if two smooth connected T-stable subgroups of U are conjugate then they are actually *equal* inside U (see Lemma 2.2 in the handout on the geometric Bruhat decomposition). Hence, $U_{\Phi^+_w} = U_{\Phi^+_{w'}}$. Comparing T-weights then gives

$\Phi^+_w = \Phi^+_{w'}$ inside Φ^+, so $\Phi'_w = \Phi'_{w'}$ by passing to complements inside Φ^+. But then

$$w^{-1}(\Phi^+) = \Phi^+_w \coprod -\Phi'_w = \Phi^+_{w'} \coprod -\Phi'_{w'} = w'^{-1}(\Phi^+),$$

so $w' = w$ by the freeness of the W-action on the set of positive systems of roots (equivalently, on the set of Weyl chambers).

Finally, we check that the inclusion $\coprod C(w) \subset G$ just established (say on k-points with $k = \overline{k}$) is actually an equality. The idea is to show that $\coprod C(w)$ is stable under $G(k)$-stable inside G, by checking this for enough subgroups of $G(k)$.

In the special case that the connected reductive G has rank 1 (i.e., its derived group is SL_2 or PGL_2), so W has order 2 and the maximal central torus lies in every Borel subgroup, it is an elementary calculation with SL_2 that the Bruhat decomposition holds for G. This applies in general to the rank-1 connected reductive subgroups $Z_G(T_a)$ for $a \in \Phi^+$. Combining the settled rank-1 case with some clever but long group theory calculations (given in the proof of Proposition 2.4 in the handout on the geometric Bruhat decomposition) resting on

(i) root group commutation formulas,
(ii) the equality $r_a(\Phi^+ - \{a\}) = \Phi^+ - \{a\}$ for all $a \in \Delta$,

one finds that $Z_G(T_a) \cdot C(w) \subset C(w) \cup C(r_a w)$ for all $w \in W$ and $a \in \Delta$.

It follows that $\bigcup_{w \in W} C(w)$ is stable under left multiplication by $\langle Z_G(T_a)\rangle_{a \in \Delta}$. But this latter group coincides with $\langle Z_G(T_a)\rangle_{a \in \Phi^+}$ because $W(\Delta) = \Phi^+$ and W is generated by the reflections $\{r_a\}_{a \in \Delta}$ admitting "Weyl element" representatives in $\mathscr{D}(Z_G(T_a))$ for $a \in \Delta$. Consideration of Lie algebras shows that the subgroups $Z_G(T_a) \subset G$ for $a \in \Phi^+$ generate G, so we conclude that $\bigcup_{w \in W} C(w)$ is stable under left multiplication by G, forcing this union to be equal to G. $\square$

Having finished our discussion of the proof of the Bruhat decomposition in the split case, before we turn to the general case we record a few consequences of wide interest in the split case:

I. (BRUHAT STRATIFICATION) The geometrically reduced Zariski closure $\overline{C(w)}$ is $B \times B$-stable (check on geometric points), so this closure is a *union* of Bruhat cells (as we may check on geometric points, using the Bruhat decomposition!). Thus, $\{C(w)\}_{w \in W}$ is a stratification of G, and likewise for the affine spaces $C(w)/B$ inside G/B.

The Bruhat cells $C(w')$ appearing in the closure of $C(w)$ are described by the "Bruhat order" on W (with respect to $\{r_a\}_{a \in \Delta}$, where Δ is the root basis for $\Phi^+ = \Phi(B, T)$). Remark 1.2 in the handout on the geometric Bruhat decomposition provides further discussion of this and relevant literature references (in [Bou] and [Spr]).

II. (KNESER–TITS PROPERTY) Suppose G is connected semisimple, split, and simply connected. The k-group isomorphism $\mathbf{G}_m^\Delta \simeq T$ defined by $(y_a) \mapsto \prod a^\vee(y_a)$ gives $T(k) = \langle a^\vee(k^\times)\rangle_{a \in \Delta}$ upon passage to k-points. But $\langle U_a, U_{-a}\rangle = \mathscr{D}(Z_G(T_a)) \simeq \mathrm{SL}_2$ (not PGL_2) since we proved the derived group of any torus centralizer is a *simply connected* semisimple group is always simply connected (see Corollary 9.5.11). Consequently, $a^\vee : \mathbf{G}_m \to \langle U_a, U_{-a}\rangle$ is identified with $t \mapsto \mathrm{diag}(t, 1/t)$, and the classical fact that $\mathrm{SL}_2(k)$

is generated by $U^{\pm}(k)$ for any field k implies that $U_a(k)$ and $U_{-a}(k)$ generate the group $\mathscr{D}(Z_G(T_a))$ that contains $a^\vee(k^\times)$. Hence, it follows that $T(k) \subset \langle U_a(k)\rangle_{a\in\Phi}$.

It now follows from the Bruhat decomposition and the equality $B(k) = T(k)U(k)$ with $U = \prod_{a\in\Phi^+} U_a$ via multiplication in any fixed ordering of Φ^+ that $G(k)$ is generated by the subgroups $U_a(k)$ for $a \in \Phi$ and a choice of representative n_w for each $w \in W$. But W is *generated* by the r_a's for $a \in \Delta$, so we can take such n_w's to be words in choices of representatives for these r_a's. By design we can pick the representative for r_a to correspond to the standard Weyl element in $\mathrm{SL}_2(k)$ via an identification of SL_2 with the group $\mathscr{D}(Z_G(T_a))$ whose k-points are generated by $U_a(k)$ and $U_{-a}(k)$. That is, we can choose the $n_{r_a} \in \langle U_a(k), U_{-a}(k)\rangle$ for each $a \in \Delta$. Summarizing we have proved:

Theorem 10.3.4 (Chevalley)**.** *For split simply connected G, as above, $G(k) = \langle U_a(k)\rangle_{a\in\Phi}$.*

It follows that $G(k)$ coincides with its subgroup $G(k)^+$ generated by the subgroups $\mathscr{R}_{u,k}(B)(k)$ for B varying through all Borel k-subgroups $B \subset G$ (or even just two such B's, namely one Borel and its "opposite" relative to a fixed split maximal torus in the initial choice of Borel k-subgroup). This latter property makes sense to contemplate more widely for possibly non-split simply connected groups over fields, using minimal parabolic k-subgroups. The equality of $G(k)$ and $G(k)^+$ is the *Kneser–Tits Conjecture*, which we address more fully in §3 of the handout on Tits systems.

III. (TITS SYSTEMS AND SIMPLICITY RESULTS) Using the gritty group-theoretic calculations in the proof of the Bruhat decomposition, one can verify that $(B(k), N(k), \{r_a\}_{a\in\Delta})$ is a "Tits system" (or "BN-pair") for $G(k)$ (a concept that is defined and explored in the handout on Tits systems). The interest in this is that Tits developed a *uniform* method to prove essentially all known simplicity results for matrix groups (including for *all* finite simple groups of Lie type), based on a short list of group-theoretic axioms, and his method is applicable whenever one has a Tits system (but determining the specific subquotient proved to be simple via his method generally entails some additional work; e.g., one needs to address a version of the Kneser–Tits property).

The original proofs of simplicity results for $G(k)/Z_G(k)$ with split simply connected and k-simple connected semisimple k-groups G used tedious case-by-case analysis, and Tits united these results into a common formalism. The proof that $G(k)/Z_G(k)$ is a simple abstract group for split simply connected and k-simple connected semisimple groups G is given in §2 of the handout on Tits systems, with the *exceptions* that such simplicity fails for $\mathrm{SL}_2(\mathbf{F}_2)$, $\mathrm{SL}_2(\mathbf{F}_3)/\mathbf{F}_3^\times$, and $\mathrm{Sp}_4(\mathbf{F}_2) \simeq S_6$ (so the proofs require extra care over fields of sizes 2 and 3 to navigate around the counterexamples). The method of proof involves knowledge of the full classification of connected Dynkin diagrams (especially the nature of the diagrams that are not simply laced).

Let us now take up the proof of Theorem 10.3.1 in general:

Proof. We shall address (1), with (2) treated in the handout on the relative Bruhat decomposition (see Remark 10.3.2).

First we discuss injectivity for (1). That is, given $n, n' \in N(k)$ such that

$$n' = (p')^{-1} n p \text{ for some } p, p' \in P(k),$$

we need to show that $n^{-1}n' \in Z(k)$.

Since $Z = N \cap P$ (Corollary 7.4.5), it is enough to show that $n^{-1}n' \in P(k)$. We have the maximal split torus $S \subset P$, and $S' := p'S(p')^{-1} \subset P$ is another maximal torus in P. The idea is to establish a refined conjugacy statement for maximal split tori. Let

$$H = \langle S, S' \rangle \subset P = Z \ltimes U.$$

Observe that under the projection to the connected reductive k-group $Z = Z_G(S)$, the subgroup S maps to a *central* maximal split torus, so this is the only maximal split k-torus in S due to centrality (and the known k-rational conjugacy for maximal split k-tori in connected reductive k-groups); this just recovers the proof that $Z_G(S)/S$ is k-anisotropic. It follows that the k-split torus S' also maps into S under this projection. Hence, $H \to Z$ has image equal to S.

Lemma 10.3.5. *We have $S' = hSh^{-1}$ for some $h \in H(k)$.*

Proof. We know that $S \subset H \subset S \ltimes U$, so $H = S \ltimes U'$ for $U' = H \cap U$ (a smooth connected unipotent group since it is a direct factor scheme of the smooth connected H and is a k-subgroup of U). In particular, S is a maximal k-torus of H (so S' is as well, for dimension reasons).

By definition $S' = p'S(p')^{-1}$ with $p' \in P(k) = Z(k) \ltimes U(k)$, so S' is $U(k)$-conjugate to S (because the $Z(k)$-component of p' commutes with S, so conjugation by it doesn't do anything to S).

On the other hand, $S', S \subset H$ are maximal k-tori, so $S_{\overline{k}}$ and $S'_{\overline{k}}$ are conjugate under $H(\overline{k})$, and hence under $U'(\overline{k})$ (because conjugation by the $S(\overline{k})$-factor doesn't do anything to $S_{\overline{k}}$).

We have thus produced $u \in U(k)$ and $u' \in U'(\overline{k})$ such that

$$uSu^{-1} = S', \quad u'S_{\overline{k}}(u')^{-1} = S'_{\overline{k}}.$$

These points of $U(k)$ and $U'(\overline{k})$ are off by an element of $N_G(S)(\overline{k})$, and since everything is happening in P they are even off by something in $(N_G(S) \cap P)(\overline{k}) = Z_G(S)(\overline{k})$. But $Z_G(S) \cap U = 1$ (as $P = Z_G(S) \ltimes U$), so $u = u'$. Thus, we get an element $u = U(k) \cap U'(\overline{k}) = U'(k) \subset H(k)$ that conjugates S to S'. ☐

We now return to the proof of the theorem. Thanks to the Lemma, we have

$$hSh^{-1} = S' = p'S(p')^{-1}$$

for some $h \in H(k)$. Then $h^{-1}p' \in N(k) \cap P(k) = Z(k)$, so $p' = hz$ for some $z \in Z(k)$.

Also, since n' normalizes S we have by the definition of S' and the choices of p and p' that

$$S' := p'S(p')^{-1} = p'n'S(n')^{-1}(p')^{-1} = npSp^{-1}n^{-1} \subset nPn^{-1}.$$

which implies that $H \subset nPn^{-1}$ (obviously $S \subset nPn^{-1}$, since S is normalized by n and $S \subset P$, and $S' \subset nPn^{-1}$ by the preceding calculation). Thus, $h = np''n^{-1}$ for some $p'' \in P(k)$.

We may now write

$$np(n')^{-1} = p' = hz = np''n^{-1}z,$$

so

$$P(k) \ni (p'')^{-1}p = n^{-1}zn' = (n^{-1}n')((n')^{-1}zn')$$

with $((n')^{-1}zn') \in Z(k) \subset P(k)$. We may finally conclude that $n^{-1}n' \in P(k)$. This completes the proof that the map in Theorem 10.3.1(1) is injective.

For surjectivity, the crucial step is the following important general fact:

Theorem 10.3.6. *For parabolic k-subgroups $Q, Q' \subset G$, the intersection $Q \cap Q'$ is smooth and connected, it contains a maximal split k-torus of G.*

This Theorem is remarkable even for Borel subgroups when k is algebraically closed, and that special case will be a consequence of the *Bruhat decomposition* over algebraically closed fields (part of the settled split case). It is at this step that the proof of the relative Bruhat decomposition uses the split case.

Proof. First assume k is algebraically closed, and consider Borel subgroups $B, B' \subset G$, so $B' = gBg^{-1}$ for some $g \in G$. It is an instructive exercise using the Bruhat decomposition for g relative to B (and a choice of maximal torus of B) to prove $B \cap B'$ contains a maximal torus T' of G; see the proof of Proposition 2.2 of the handout on the relative Bruhat decomposition for this calculation. Once that is done, for any two parabolic subgroups $Q, Q' \subset G$ we choose Borel subgroups $B \subset Q$ and $B' \subset Q'$ to obtain a maximal torus $T \subset G$ contained in B and B', hence also contained in Q and Q'. But then $Q = P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(T)$, so $Q \cap Q' = P_{Q'}(\lambda)$ as schemes, and this inherits smoothness and connectedness from Q'! That settles the case over algebraically closed fields.

Now consider general k. Since $(Q \cap Q')_{\overline{k}} = Q_{\overline{k}} \cap Q'_{\overline{k}}$ is $\overline{k}$-smooth and connected by the preceding conclusions, $Q \cap Q'$ is k-smooth and connected. It remains to find a maximal split k-torus of G contained in $Q \cap Q'$, and for that purpose we may pass to *minimal* Q and Q'. But the Levi decomposition $Z_G(S) \ltimes U$ for minimal parabolic k-subgroups implies that *every* maximal k-torus T in such a k-subgroup automatically contains a maximal split k-torus in G (as any such T maps isomorphically onto its maximal torus image in the connected reductive quotient $Z_G(S)$, so the maximal split subtorus of T has the same dimension as S and hence is maximal in G). Consequently, we can *forget about* seeking a maximal split k-torus (a notion very sensitive to extension on k) and instead aim to show that the smooth connected affine group $Q \cap Q'$ contains a maximal k-torus of G, or in other words that its maximal k-tori have the same dimension as those of G. Grothendieck's theorem on maximal tori in *smooth* affine groups (such as $Q \cap Q'$!) thereby reduces our task to the settled case of ground field $\overline{k}$ (over which $Q_{\overline{k}}$ and $Q'_{\overline{k}}$ are generally not minimal, but that doesn't matter). $\square$

Using Theorem 10.3.6, we now prove the surjectivity of the map in Theorem 10.3.1(1). That is, for $g \in G(k)$ we seek $n \in N(k)$ such that $n \in P(k)gP(k)$. We apply Theorem 10.3.6 to P and gPg^{-1} to get a maximal split k-torus $S' \subset G$ contained in both P and gPg^{-1}. Consequently, the maximal split k-tori S, S', $g^{-1}S'g$ of G are contained in P.

We know that maximal split k-tori in a parabolic k-subgroup of a connected reductive k-group are k-rationally conjugate (Proposition 7.4.1), so there exist $p_1, p_2 \in P(k)$ such that

$$p_1^{-1}S'p_1 = S = p_2(g^{-1}S'g)p_2^{-1}.$$

Hence, $n := p_1^{-1}(gp_2^{-1})$ conjugates S into S' into S. It follows that $n \in N(k)$, yet clearly $n \in P(k)gP(k)$. $\square$

11. Relative root systems and applications

11.1. **Goals.** Let G be a connected reductive k-group and $S \subset G$ a maximal split k-torus. Let ${}_k\Phi$ denote the set $\Phi(G,S)$ of non-trivial S-weights on $\mathfrak{g}$. We call ${}_k\Phi$ the *relative root system* of S, although we have yet to prove that it really is a root system inside its $\mathbf{Q}$-span in $\mathrm{X}(S)_{\mathbf{Q}}$.

Remark 11.1.1. Note that

$$\begin{aligned} {}_k\Phi = \emptyset &\iff Z_G(S) \subset G \text{ has full Lie algebra} \\ &\iff S \subset G \text{ central (e.g., this means } S = 1 \text{ when } G \text{ is semisimple),} \\ &\iff G \text{ has no proper parabolic } k\text{-subgroups.} \end{aligned}$$

Consequently, to keep the story interesting, we assume below that S is not central (and the reader who dislikes that is welcome to keep track of $\emptyset$ and $\{0\}$ in various places). We want to:

(i) show that ${}_k\Phi$ is a root system (possibly non-reduced) in its $\mathbf{Q}$-span inside $\mathrm{X}(S)_{\mathbf{Q}}$;

(ii) relate ${}_k\Phi$ to the "absolute roots" $\Phi(G_{k_s}, T_{k_s}) \subset \mathrm{X}(T_{k_s}) \twoheadrightarrow \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ for a maximal k-torus $T \supset S$ (e.g., can we relate a basis ${}_k\Delta \subset {}_k\Phi$ to a basis Δ of Φ?);

(iii) define ${}_k\Phi^\vee \subset \mathrm{X}_*(S) - \{0\}$ making $(\mathrm{X}(S), {}_k\Phi, \mathrm{X}_*(S), {}_k\Phi^\vee)$ a root datum.

This is a challenge since there is no SL_2-crutch! For instance, the weight spaces $\mathfrak{g}_a$ for $a \in {}_k\Phi$ can be huge. We'll use dynamics to make root groups U_a, which can be non-commutative for multipliable $a \in {}_k\Phi$. We preview some applications:

(1) prove *Cartan's Theorem* that $G(\mathbf{R})$ is connected in the analytic topology for G a connected semisimple $\mathbf{R}$-group that is simply connected in the sense of algebraic groups (in contrast, $\mathrm{PGL}_{2m}(\mathbf{R})$ is disconnected since the determinant carries it continuously onto $\mathbf{R}^\times/(\mathbf{R}^\times)^2 = \{\pm 1\}$).

(2) a natural bijection

$$\{\text{parabolic } k\text{-subgroups} \supset S\} \longleftrightarrow \{\text{parabolic subsets of } {}_k\Phi\},$$

and more generally

$$\{\text{parabolic } k\text{-subgroups}\}/G(k)\text{-conj.} \longleftrightarrow \{\text{subsets of } {}_k\Delta\},$$

and $P = Q$ if (and only if) $P(k) = Q(k)$; i.e. parabolic k-subgroups are determined by their k-points inside $G(k)$ (remarkable for finite k, as we cannot use Zariski-density),

(3) establish the Tits–Selbach classification of k-forms in the semisimple case.

Example 11.1.2. For type A_{n-1}, two classes are:

- $\mathrm{SL}_d(D)$ for a central division algebra D over k with $\dim_k D = m^2$ where $dm = n$,

- $\mathrm{SU}(h)$ for n-dimensional hermitian spaces (V', h) over quadratic Galois extensions k'/k.

Are there more possibilities? How do we know if we have found an exhaustive list of constructions over a given field?

11.2. **Examples.** We'll first look at three classes of examples for which ${}_k\Phi$ can be seen directly (all details are in the handout on "Relative roots".)

Example 11.2.1 (Weil restriction). Let k'/k be finite separable and G' a connected reductive group over k', $G = R_{k'/k}(G')$. The main case to keep in mind is where G' is split. It is proved in §6 of the Relative Bruhat handout that we have a bijection

$$\{\text{maximal } k'\text{-tori of } G'\} \longleftrightarrow \{\text{maximal } k\text{-tori of } G\}$$
$$T' \mapsto R_{k'/k}(T')$$

from which is obtained a correspondence at the level of maximal split tori

$$\{\text{maximal split } k'\text{-tori of } G'\} \longleftrightarrow \{\text{maximal split } k\text{-tori of } G\}$$
$$S' \mapsto S \subset R_{k'/k}(S')$$

(with S defined to be the maximal split k-subtorus of $R_{k'/k}(S')$, do $\dim S = \dim S'$).

The bijection of Lie algebras

$$\mathfrak{g}' = \ker(G'(k'[\epsilon]) \to G'(k')) = \ker(G(k[\epsilon]) \to G(k)) = \mathfrak{g}$$

is a k-linear isomorphism, and it is an instructive exercise to check that this is compatible with Lie brackets.

Also, we have a homomorphism $\mathrm{X}(S') \to \mathrm{X}(S)$ defined as follows. For a character $a' : S' \to \mathbf{G}_m$, the Weil restriction $R_{k'/k}(a')$ is valued in $R_{k'/k}(\mathbf{G}_m)$, so its restriction to S is valued in the maximal split k-subtorus $\mathbf{G}_m \subset R_{k'/k}(\mathbf{G}_m)$; i.e., this restriction is a character of S. The resulting map $\mathrm{X}(S') \xrightarrow{\sim} \mathrm{X}(S)$ defined by

$$a' \mapsto (R_{k'/k}(a')|_S : S \xrightarrow{a} \mathbf{G}_m \subset R_{k'/k}(\mathbf{G}_m))$$

is easily checked to be bijective (hint: reduce to the case $S' = \mathbf{G}_m$).

Proposition 11.2.2. *The isomorphism* $\mathrm{X}(S') \simeq \mathrm{X}(S)$ *identifies* $\Phi(G', S')$ *with* $\Phi(G, S)$ *under which* $\mathfrak{g}'_{a'} = \mathfrak{g}_a$ *for* $a' \mapsto a$. *In particular,* $\dim_k \mathfrak{g}_a = [k' : k] \dim_{k'} \mathfrak{g}'_{a'}$ *(so* $\dim_k \mathfrak{g}_a = [k' : k]$ *for all* $a \in \Phi(G, S)$ *if* G' *is split).*

The large root spaces (over k) in the preceding example are not impressive since the root spaces are secretly 1-dimensional over an extension field.

Example 11.2.3. Let $G = \mathrm{SL}_n(D)$ for D a finite-dimensional central division algebra over k. This is the algebraic group of units of reduced norm 1 in $\mathrm{Mat}_n(D) = \mathrm{Mat}_n(k) \otimes_k D$. The tensor decomposition (or more canonically the inclusion of k into the center of D) provides a natural k-subgroup $\mathrm{SL}_n \subset G$.

Letting $S = (\mathbf{G}_m^n)^{\det=1}$ be the split diagonal k-torus in SL_n, we then have $S \subset \mathrm{SL}_n(D)$. One finds that $Z_G(S)$ is the "diagonal" subgroup

$$\{(d_1, \dots, d_n) \in (D^\times)^n \mid \prod \mathrm{Nrd}(d_j) = 1\}$$

whose quotient modulo S is k-anisotropic (since $D^\times/\mathbf{G}_m$ is k-anisotropic, as D is a central *division algebra*). This shows that S really is a maximal k-split torus, and it is easy to verify that ${}_k\Phi = \Phi(\mathrm{SL}_n, S) = \mathrm{A}_{n-1}$ with $\mathfrak{g}_a = D$ as a k-group.

The preceding large root spaces might still be considered unimpressive, since the root spaces are naturally 1-dimensional over the division algebra D.

Example 11.2.4. Let $G = \mathrm{SU}(h)$ for an n-dimensional hermitian space (V', h) over quadratic Galois extensions k'/k with h having its isotropic part of k'-dimension $2q$, so anisotropic part of k'-dimension $n - 2q \geq 0$. In this case the absolute root system Φ is of type A_{n-1}, but the system of relative roots is

$$ {}_k\Phi = \begin{cases} \mathrm{C}_q & n = 2q, \\ \mathrm{BC}_q & n > 2q. \end{cases} $$

Note in particular that the absolute and relative root systems have a rather huge difference in their ranks (especially for q near $n/2$).

For $n = 2q$, the root spaces are k'-lines. For $n > 2q$, root spaces for non-multipliable roots are k'-lines whereas $\mathfrak{g}_a$ is a k'-vector space of dimension $n - 2q$ (not naturally a line over an extension field when $n > 2q + 1$) for *multipliable* a.

This completes our warm-up, and it is time to dive into setting up the general theory.

11.3. **Basic properties.**

Lemma 11.3.1. *The subset ${}_k\Phi \subset \mathrm{X}(S)$ is stable under negation.*

Proof. Pick a maximal k-torus $T \supset S$, so we can consider the restriction to S of the adjoint action of T on $\mathfrak{g}$, compatible with the natural quotient map

$$ \mathrm{X}(T_{k_s}) \twoheadrightarrow \mathrm{X}(S_{k_s}) = \mathrm{X}(S). $$

This restriction carries Φ into ${}_k\Phi \cup \{0\}$ by coarsening the decomposition

$$ \mathfrak{g}_{k_s} = \mathfrak{g}_{k_s}^{T_{k_s}} \oplus \left(\bigoplus_{b \in \Phi} (\mathfrak{g}_{k_s})_b \right) $$

to a weight space decomposition relative to S_{k_s} (under which some absolute root lines are lumped together and some may fall into $\mathfrak{g}_{k_s}^{S_{k_s}} = (\mathfrak{g}^S)_{k_s}$). This shows that every root in ${}_k\Phi$ is the restriction of an absolute root.

For $a \in {}_k\Phi$, pick $a' \in \Phi$ such that $a'|_{S_{k_s}} = a_{k_s}$. Then $-a'|_{S_{k_s}} = (-a)_{k_s}$, so $-a \in {}_k\Phi$ since $(\mathfrak{g}_{k_s})_{-a'} \neq 0$ (as $-a' \in \Phi$). □

We next seek $n_a \in N_G(S)(k)$ whose effect on $\mathrm{X}(S)_{\mathbf{Q}}$ (necessarily preserving ${}_k\Phi$) is a reflection negating a (among other desirable properties). Recall that for the absolute root system, we found such n_a by explicitly computing (as SL_2 or PGL_2) the derived group of the centralizer of the codimension-1 torus killed by a. We'll adopt a similar strategy here, but replacing the explicit computation of a derived group with a study of proper parabolic k-subgroups.

Define

$$ G_a = Z_G(S_a) \supset S $$

where $S_a := (\ker a)^0_{\mathrm{red}} \subset S$ is the codimension-1 subtorus of S killed by the nontrivial character a. Note that

$$\mathrm{Lie}(G_a) = \mathfrak{g}^{S_a} = \mathfrak{g}^S \oplus \left(\bigoplus_{b \in {}_k\Phi \cap \mathbf{Q}a} \mathfrak{g}_b \right).$$

Where is the reflection of $\mathrm{X}(S)$ negating a going to come from? Another way of describing the reflection arising in the SL_2 and PGL_2 cases is that it swaps the two Borels containing the diagonal torus. We know in general that $N_G(S)(k)/Z_G(S)(k)$ acts simply transitively on the set of minimal parabolic k-subgroups containing S (this was part of the proof that ${}_kW = (N/Z)(k)$ in the handout on Relative Bruhat Decomposition). We shall apply this general fact to the pair (G_a, S) using:

Lemma 11.3.2. *There exist exactly two proper parabolic k-subgroups $P_{\pm a} \subset G_a$ containing S, with*

$$\mathrm{Lie}(P_{\pm a}) = \mathfrak{g}^S \oplus \left(\bigoplus_{b \in {}_k\Phi \cap \mathbf{Q}_{>0}(\pm a)} \mathfrak{g}_b \right).$$

Proof. Consider $P \subset G$ a proper parabolic k-subgroup containing S. Choose a maximal k-torus $T \subset P$ containing S. Since we know that the dynamic method produces all parabolics (Theorem 6.1.1), we have $P = P_{G_a}(\lambda)$ for some $\lambda \colon \mathbf{G}_m \to T$ over k, which necessarily factors through S because S is maximal split:

$$\begin{tikzcd} \mathbf{G}_m \arrow[r, "\lambda"] \arrow[dr, dotted] & T \\ & S \arrow[u, hook] \end{tikzcd}$$

Let $S'_a \simeq \mathbf{G}_m \subset S$ be an isogeny complement to $S_a \subset S$. At the level of character groups, this induces a finite-index inclusion

$$\mathrm{X}(S) \hookrightarrow \mathrm{X}(S'_a) \oplus \mathrm{X}(S_a). \tag{11.3.1}$$

Passing to $\mathbf{Z}$-duals of (11.3.1) gives a finite-index inclusion

$$\mathrm{X}_*(S'_a) \oplus \mathrm{X}_*(S_a) \hookrightarrow \mathrm{X}_*(S)$$

Replacing λ by λ^n for positive n doesn't affect $P_G(\lambda)$, so doing this with sufficiently divisible n allows us to arrange that $\lambda \in \mathrm{X}_*(S'_a) \oplus \mathrm{X}_*(S_a)$. But the $\mathrm{X}_*(S_a)$-component of such a cocharacter is irrelevant for the dynamic method since $S_a \subset G_a$ is central (so it doesn't affect the outcome of conjugation against λ). Hence, we may further arrange that $\lambda \in \mathrm{X}_*(S'_a) = \mathbf{Z}$.

Certainly λ is nonzero (as $P \neq G$), so there are at most *two* different possibilities of $P_{G_a}(\lambda)$ (because replacing λ with λ^m for any positive integer m has no effect), namely $P_{G_a}(\lambda_\pm)$ for $\lambda_\pm = \pm 1 \in \mathbf{Z}$. The pairings $\langle a, \lambda_\pm \rangle$ are nonzero since $\lambda_\pm(\mathbf{G}_m) = S'_a$ is not killed by a, and these have opposite signs, so the corresponding Lie algebras for $P_{G_a}(\lambda_\pm)$ are as expected. $\square$

We can now produce a "reflection in a" inside $W(G,S)$ are follows. Applying to (G_a,S) the result from the relative Bruhat decomposition that ${}_kW$ acts simply transitively on the set of minimal parabolic k-subgroups containing S, we see via Lemma 11.3.2 that $N_{G_a}(S)(k)/Z_{G_a}(S)(k)$ has size 2. Labelling its elements as $\{1,r_a\}$, we see that r_a has order 2 and that for a representative $n_a\in N_{G_a}(S)(k)$ the effect of n_a-conjugation on S is trivial on the central $S_a\subset G_a$ and induces inversion on the 1-dimensional quotient $S/S_a\simeq \mathbf{G}_m$ (because it swaps P_a and P_{-a}). Thus, r_a really is a reflection on $\mathrm{X}(S)$ and its negates a.

This reflection of $\mathrm{X}(S)$ in a comes from $W(G_a,S)$, but we really want one coming from $W(G,S)$. Note that $Z_{G_a}(S)=Z_G(S)$ since anything centralizing S certainly centralizes S_a and hence lies in G_a, so $\{1,r_a\}=W(G_a,S)\hookrightarrow W(G,S)$. (Later we'll show that the group inclusion $\langle r_a\rangle_{a\in{}_k\Phi}\subset W(G,S)$ is an equality, relating algebraic groups to Coxeter groups beyond the split case.)

To prove that ${}_k\Phi$ is a root system in its own $\mathbf{Q}$-span inside $\mathrm{X}(S)_{\mathbf{Q}}$, it will be useful to first describe this $\mathbf{Q}$-span in a manner reminiscent of the split case. Let $S_0\subset G$ be the maximal *split* central k-torus, so $S_0\subset S$ by maximality of S. Let $S'=(S\cap\mathscr{D}G)^0_{\mathrm{red}}$ be the maximal k-subtorus of S inside $\mathscr{D}G=:G'$.

Lemma 11.3.3. *We have:*

1. *the k-torus S' is a maximal split k-torus in G';*
2. *the natural map $S_0\times S'\to S$ is an isogeny, and the resulting equality* $\mathrm{X}(S)_{\mathbf{Q}}=\mathrm{X}(S_0)_{\mathbf{Q}}\oplus\mathrm{X}(S')_{\mathbf{Q}}$ *puts ${}_k\Phi$ inside* $\mathrm{X}(S')_{\mathbf{Q}}$*;*
3. $\mathbf{Q}\cdot{}_k\Phi=\mathrm{X}(S')_{\mathbf{Q}}$.

Example 11.3.4. Before proving this lemma, we give an example to show that $S\cap\mathscr{D}G$ can be disconnected (or non-reduced), so it is really necessary to pass to $(S\cap\mathscr{D}G)^0_{\mathrm{red}}$. The key issue is that a maximal *split* k-torus S is generally not its own centralizer and correspondingly can fail to contain Z_G.

Let (V',h) be a hermitian space of even dimension $n\geq 4$ over k relative to a quadratic Galois extension k'/k, and assume it is isotropic but not "split" (i.e. $n>2q>0$ in the notation of §4 of the handout on relative root systems). Consider the description of $\mathrm{SU}(h)$ and a suitable "diagonal" maximal split k-subtorus S' of dimension q in terms of matrices in §4 of the handout on relative root systems, with the upper-left corner stabilizing the $2q\times 2q$ isotropic part of the hermitian form and the lower-right corner stabilizing the k-anisotropic part.

We thereby see that the central $\mu_2\subset\mathrm{SU}(h)$ (recall n is even) is not contained in S' (since S' is trivial in the lower-right part), so for $G:=\mathbf{G}_m\times^{\mu_2}\mathrm{SU}(h)$ we have $S=\mathbf{G}_m\times S'$ with S' the maximal split in $\mathrm{SU}(h)$, but $S\cap\mathscr{D}G=\mu_2\cdot S'\subset\mathrm{SU}(h)$ is a direct product of μ_2 and S'.

Proof. Pick a maximal k-torus $T\subset G$ containing S. For $Z\subset G$ the maximal central torus, we have a central isogeny

$$\pi\colon Z\times G'\to G,$$

and a maximal k-torus of $Z\times G'$ is given by $\pi^{-1}(T)=Z\times T'$ for $T':=T\cap\mathscr{D}G$ a maximal torus of $\mathscr{D}G$.

The isogeny $Z \times T' \to T$ induces an isogeny between between maximal split k-subtori:

$$S_0 \times T'_0 \to S$$

where T'_0 is the maximal split k-subtorus of T'. We have $S' \subset T \cap \mathscr{D}G = T'$ as a split k-subtorus, so we certainly have $S' \subset T'_0$. Also, T'_0 is contained inside the maximal k-split subtorus of T, which is S, so $T'_0 \subset (S \cap \mathscr{D}G)^0_{\mathrm{red}} = S'$; i.e. $T'_0 = S'$. This establishing (2).

Now by dimension reasons we get $S' \subset \mathscr{D}G$ is maximal split (otherwise since $S_0 \cap \mathscr{D}G$ is finite we could use dimension considerations to see that combining a bigger split k-torus with S_0 gives a split k-torus inside G properly containing S, a contradiction). Hence, (1) is established.

Finally we show (3). Now we have $\mathrm{X}(S)_{\mathbf{Q}} = \mathrm{X}(S_0)_{\mathbf{Q}} \oplus \mathrm{X}(S')_{\mathbf{Q}}$ and ${}_k\Phi|_{S_0} = 1$ by centrality of S_0 inside G. Since the projection from $\mathrm{X}(S)_{\mathbf{Q}}$ into the two direct summands arise from restriction of characters on S to characters on subtori of S, we have ${}_k\Phi \subset \mathrm{X}(S')$. We want this inclusion to be an equality after tensoring with $\mathbf{Q}$.

The quotient $\mathrm{X}(S')_{\mathbf{Q}}/\mathbf{Q}\cdot{}_k\Phi$ corresponds to a torsion-free quotient of the corresponding lattice $\mathrm{X}(S')$ (namely, the quotient of $\mathrm{X}(S')/\mathbf{Z}\cdot{}_k\Phi$ modulo its torsion subgroup). That in turn corresponds to the maximal k-subtorus $S'' \subset S'$ killed by ${}_k\Phi$. We want to show that $S'' = 1$.

Note that $Z_G(S'')$ has Lie algebra $\mathfrak{g}^{S''} = \mathfrak{g}$ since by design ${}_k\Phi|_{S''} = 1$. Hence, the smooth connected k-group $Z_G(S'')$ must exhaust G. This shows that $S'' \subset Z_G$, so $S'' \subset S_0$. But we chose $S'' \subset S'$, so $S'' \subset S_0 \cap S'$ yet this intersection is finite by (1), so $S'' = 1$. □

Our aim is to show that $(\mathrm{X}(S'))_{\mathbf{Q}}, {}_k\Phi)$ is a root system. For this we need to exhibit a cocharacter $a^\vee \in \mathrm{X}_*(S') \subset \mathrm{X}_*(S)$ such that $r_a : \mathrm{X}(S) \simeq \mathrm{X}(S)$ is given by

$$r_a(x) = x - \langle x, a^\vee \rangle a \text{ for all } a \in {}_k\Phi.$$

The reason to seek $a^\vee \in \mathrm{X}_*(S')$ is to guarantee the integrality $\langle b, a^\vee \rangle \in \mathbf{Z}$ (and to ensure that the $\mathbf{Q}$-span of the coroots will coincide with $\mathrm{X}_*(S')_{\mathbf{Q}}$). The formula for r_a on $\mathrm{X}(S)$ says that at the level of S we have

$$r_a(s) = s/a^\vee(a(s))$$

or in other words that the map $s \mapsto s/r_a(s)$ factors through a:

$$\begin{tikzcd} S \arrow[r, "a"] \arrow[dr] & \mathbf{G}_m \arrow[r, "a^\vee"] & G \\ & S/\ker a \arrow[ur] & \end{tikzcd}$$

Hence, we want to show that $r_a|_{\ker a}$ is the identity automorphism. By design r_a comes from $N_{G_a}(S)(k)$, so it is trivial on the torus $S_a = (\ker a)^0_{\mathrm{red}}$ that is central in G_a. But we need to show that $r_a|_{\ker a}$ is the identity automorphism of $\ker a$, and that is much stronger than on S_a because $\ker a$ could be disconnected or non-reduced.

Warning 11.3.5. In the proof of 5.8 of the big paper on reductive groups in IHES 27 by Borel and Tits, they built $a^\vee$ merely in $\mathrm{X}_*(S)_{\mathbf{Q}}$ and to prove $\langle b, a^\vee \rangle \in \mathbf{Z}$ for all $b \in {}_k\Phi$ their argument has a gap when $\mathbf{Q}a \cap {}_k\Phi \not\subset \mathbf{Z}a$ (e.g., it can happen that $a/2 \in {}_k\Phi$, and before establishing the root system property we can't rule out even worse divisibilities).

To prove r_a restricts to the identity on $\ker a$ (allowing that $\ker a$ might be non-smooth in positive characteristic), we will need to form schematic centralizers against $\ker a$. The Galois-theoretic method of constructing schematic centralizers against smooth subgroups in the first course does not help with the subgroup scheme $\ker a$ that might be non-reduced. (A priori $\ker a$ might be non-reduced in any positive characteristic, though after the root system property is established we know this only happens in characteristic 2 since by inspection of the reduced irreducible root systems every root is primitive in the weight lattice except that long roots are divisible by 2 in the weight lattice for type C.)

Remark 11.3.6. The fact that r_a is a reflection on $\mathrm{X}(S)_{\mathbf{Q}}$ that negates a, acts trivially on the subspace $\mathrm{X}(S_0)_{\mathbf{Q}}$, and preserves the finite spanning set ${}_k\Phi$ of the complementary space $\mathrm{X}(S')_{\mathbf{Q}}$ *determines* r_a. Indeed, if r'_a is another such reflection then $r_a r'_a$ is unipotent inside $\mathrm{GL}(\mathrm{X}(S)_{\mathbf{Q}})$ yet also of finite order (since only finitely many linear automorphisms of $\mathrm{X}(S)_{\mathbf{Q}}$ act trivially on $\mathrm{X}(S_0)_{\mathbf{Q}}$ and preserve the finite set ${}_k\Phi$), so it is trivial; i.e., $r'_a = r_a$.

Proposition 11.3.7. *We have*

$$r_a|_{\ker a} = \mathrm{Id}.$$

We'll prove this by building the reflection r_a using a finer centralizer than G_a, namely

$$\mathscr{G}_a := Z_G(\ker a)^0.$$

Making sense of such a centralizer is subtle since $\ker a$ could be non-reduced. Thus, we now digress to discuss some general facts concerning scheme-theoretic centralizers.

Digression on centralizers. For affine k-group schemes G of finite type and closed k-subgroup schemes $H \subset G$, the *centralizer functor* is

$$\underline{Z}_G(H) \colon R \mapsto \{g \in G(R) \mid g \text{ centralizes } H_R\}.$$

The handout "Reductive centralizer" gives several non-trivial results concerning this functor:

(1) The construction of a representing closed k-subgroup scheme $Z_G(H) \subset G$ with

$$\mathrm{Lie}(Z_G(H)) = \mathfrak{g}^H$$

is provided by Proposition 1.1. The idea of the construction is to consider the conjugation-action map

$$\begin{aligned} \alpha \colon H \times G &\to G \times G \\ (h,g) &\mapsto (hgh^{-1}, g), \end{aligned}$$

a closed immersion. For each element of the ideal in $k[H] \otimes_k k[G] = k[H \times G]$ for $\alpha^{-1}(\Delta_{G/k})$, consider its $k[G]$-coefficients with respect to a k-basis of $k[H]$ (viewed as a $k[H]$-basis of $k[H \times G]$). The ideal in $k[G]$ generated by such coefficients gives a closed subscheme of G that does the job.

(2) Assume G is *smooth*. For closed k-subgroup schemes $M \subset G$ of multiplicative type (such as a torus, or μ_n with $n > 0$ possibly divisible by $\mathrm{char}(k)$), the centralizer $Z_G(M)$ is smooth due to the infinitesimal criterion (which is sufficient to check over $\overline{k}$). The proof comes down to the fact that, as for tori, finite-dimensional representations of M are completely reducible over an algebraically closed field; see Exercise 3 in HW8 of the previous course for the method.

If $\mathrm{char}(k) = p > 0$ and $M = \mu_p$ then $\mathrm{Lie}(M)$ is a line in $\mathfrak{g}$ whose nonzero elements X are semisimple (as we can realize G inside some GL_n, and then $M = \mu_p$ is contained in a torus of that GL_n). The k-group scheme $Z_G(M)$ coincides with the construction $Z_G(X)$ considered in classical treatments (we do not need this fact; see [CGP, Prop. A.8.10(3)] for a proof).

(3) Finally, if G is connected reductive and M is inside a torus of G then $Z_G(M)^0$ is reductive. (This reduces to an SL_2-calculation, given in §2 of the handout "Reductive centralizer".) The hypothesis that M is contained in a torus of G can fail (see Example 1.2 of the handout "Reductive centralizer" for counterexamples with special orthogonal groups), and the reductivity conclusion holds without that torus hypothesis but the proof is much more difficult (see [CGP, Prop. A.8.12], which rests on hard input from étale cohomology or geometric invariant theory characterizing reductivity in an entirely novel manner).

The upshot is that $\mathscr{G}_a = Z_G(\ker a)^0$ makes sense and is reductive (and contains S). Then we have

$$\Phi(\mathscr{G}_a, S) = \mathbf{Z}a \bigcap {}_k\Phi$$

since

$$\mathrm{Lie}(\mathscr{G}_a) = \mathfrak{g}^{\ker a} = \mathfrak{g}^S \oplus \left(\bigoplus_{b \in {}_k\Phi \cap \mathbf{Z}a} \mathfrak{g}_b \right),$$

the point being that a character of S lies in $\mathbf{Z}a$ if and only if it kills the group scheme $\ker a$; note that in contrast, $\mathrm{Lie}(G_a)$ involves all elements of ${}_k\Phi$ that are *rational* multiples of a.

Our earlier considerations for rank-1 groups apply to $\mathscr{G}_a$ just as they do to G_a. Hence, we obtain an inclusion of relative Weyl groups

$$\{1, \widetilde{r_a}\} = W(\mathscr{G}_a, S) \hookrightarrow W(G_a, S) = \{1, r_a\}$$

since $\mathscr{G}_a \subset G_a$ and obviously $Z_{\mathscr{G}_a}(S) = Z_{G_a}(S) = Z_G(S)$ (since anything centralizing S also centralizes $\ker a$ and thus lies in $\mathscr{G}_a$. It follows that $r_a = \widetilde{r_a}$ comes from $N_{\mathscr{G}_a}(S)(k) \subset N_G(S)(k)$, so $r_a|_{\ker a} = \widetilde{r_a}|_{\ker a} = \mathrm{Id}$. This completes the proof of Proposition 11.3.7.

As an application of the cocharacters $a^\vee$, we can prove that $(\mathrm{X}(S')_{\mathbf{Q}}, {}_k\Phi)$ is a root system. The only thing remaining to be checked is that $a^\vee$ comes from the $\mathrm{X}_*(S')_{\mathbf{Q}}$-component under the decomposition

$$\mathrm{X}_*(S)_{\mathbf{Q}} = \mathrm{X}_*(S')_{\mathbf{Q}} \oplus \mathrm{X}_*(S_0)_{\mathbf{Q}}$$

for each $a \in {}_k\Phi$. For $a' := a|_{S'}$, under the inclusion $W(\mathscr{D}G, S') \hookrightarrow W(G, S)$ (which makes sense because S_0 is central, so everything normalizing/centralizing S' trivially also normalizes/centralizes all of $S_0 \cdot S' = S$) we get $r_{a'} \mapsto r_a$ by the unique characterization of

r_a (Remark 11.3.6). Thus, the cocharacter $(a')^\vee \in \mathrm{X}_*(S') \subset \mathrm{X}_*(S)$ also computes r_a, so $a^\vee = (a')^\vee \in \mathrm{X}_*(S')$ as desired.

For the root system $(\mathrm{X}(S')_{\mathbf{Q}}, {}_k\Phi)$, the associated reflections

$$r_a \in \mathrm{GL}(\mathrm{X}(S')_{\mathbf{Q}}) \times \mathrm{GL}(\mathrm{X}(S_0)_{\mathbf{Q}}) \subset \mathrm{GL}(\mathrm{X}(S)_{\mathbf{Q}})$$

and coroots $a^\vee \in \mathrm{X}_*(S') \subset \mathrm{X}(S)$ are as above. By [Bou, VI, §1.1, Lemma 2] (a general lemma concerning root systems), each $(a')^\vee \in \mathrm{X}_*(S')_{\mathbf{Q}}$ determines a' hence also a (since the triviality of $a|_{S_0}$ implies that a is determined by $a|_{S'} = a'$). Hence we have an injection ${}_k\Phi \hookrightarrow \mathrm{X}_*(S') \subset \mathrm{X}_*(S)$ onto a subset of non-zero elements denoted ${}_k\Phi^\vee$.

Proposition 11.3.8. *The 4-tuple $(\mathrm{X}(S), {}_k\Phi, \mathrm{X}_*(S), {}_k\Phi^\vee)$ is a (possibly non-reduced) root datum.*

Proof. It only remains to check that the dual reflections $(r_a)^\vee \in \mathrm{GL}(\mathrm{X}_*(S))$ actually preserve the coroots ${}_k\Phi^\vee$. (The reflections r_a on $\mathrm{X}(S)$ preserve ${}_k\Phi$ by construction, since the reflections were constructed from $N_G(S)(k)$ and hence preserve anything intrinsic to the pair (G,S).)

More generally, we claim that for any $n \in N_G(S)(k)$, the map

$$n.(a^\vee)\colon t \mapsto n a^\vee(t) n^{-1}$$

coincides with $(n \cdot a)^\vee$, using the $N_G(S)(k)$-action on $\mathrm{X}(S)$, which preserves ${}_k\Phi$. The key point is to show that

$$n r_a n^{-1} = r_{n\cdot a}.$$

To establish this equality, we can use either the construction or the unique characterization of these reflections (see Remark 11.3.6). $\square$

11.4. **Parametrization of parabolics.** Next we will prove two important results:

Theorem 11.4.1. *Let G be a connected reductive group over k and $S \subset G$ a maximal split torus. The map $P \mapsto {}_k\Phi_P := \Phi(P,S)$ is an inclusion-preserving bijection*

$$\{\textit{parabolic } k\textit{-subgroups} \supset S\} \longleftrightarrow \{\textit{parabolic subsets of } {}_k\Phi\}.$$

We will also (later) describe the inverse map in terms of an "explicit" dynamic description upon fixing a minimal parabolic $P_0 \subset P$ containing S.

Remark 11.4.2. Recall that for a root system (V,Φ), a *parabolic* subset is a subset $\Psi \subset \Phi$ that is closed and satisfies $\Psi \cup (-\Psi) = \Phi$, and early in §6.3 we saw that it is equivalent to say $\Psi = \Phi_{\lambda \geq 0}$ for some $\lambda \in V^*$. Thus, comparing minimal objects on both sides of the bijection relates minimal $P \supset S$ to positive systems of roots in ${}_k\Phi$.

The next result we state only informally at the moment.

Theorem 11.4.3. *We can describe an explicit roots basis ${}_k\Delta \subset {}_k\Phi$ using a choice of root basis $\Delta \subset \Phi := \Phi(G_{k_s}, T_{k_s})$ for a maximal k-torus T of G containing S.*

Definition 11.4.4. We call Φ the set of *absolute roots*, relative to the choice of T.

This has two spectacular consequences. First, by using Theorem 11.4.3 and some additional ideas, one recovers a classical result of Cartan originally proved by Riemannian geometry:

Theorem 11.4.5 (Cartan)**.** *Let G be a connected semisimple $\mathbf{R}$-group. If G is simply connected then $G(\mathbf{R})$ is connected for the analytic topology.*

The Borel–Tits proof is in the "Cartan connectedness" handout. It is very easy in the split case (using Chevalley's Proposition 2.5 in the handout on the geometric Bruhat decomposition), so the real content is beyond the split case. The argument uses highest roots in a clever manner to establish that in the simply connected case $\mathrm{X}_*(S)$ is spanned over $\mathbf{Z}$ by ${}_k\Phi^\vee$ (not obvious beyond the split case!). This in turn allows one to eventually (after some group-theoretic calculations) reduce to the case of the split group SL_2.

Theorem 11.4.6. *The natural inclusion $W({}_k\Phi) \subset W(G,S)$ is an equality.*

Proof. The group $W(G,S)$ and its subgroup $W({}_k\Phi)$ act compatibly and simply transitively on the *minimal* members of the two respective sides of the "parabolic bijection" in Theorem 11.4.1 above. (The "simply transitive" property for $W({}_k\Phi)$ is a general fact in the theory of root systems, and for $W(G,S)$ it was shown in the proof of Proposition 4.2 in the handout on the relative Bruhat decomposition.) $\square$

Proof of Theorem 11.4.1. We already know that any such P has the form

$$P = P_G(\lambda) \text{ for } \lambda \in \mathrm{X}_*(S)$$

(i.e. arises from the dynamic method, which allows us to arrange λ to be valued in any desired maximal k-torus T of P, such as one containing S, and then would be valued in the maximal split subtorus S of T), so

$${}_k\Phi_P := \Phi(P,S) = \Phi(G,S)_{\lambda\geq 0} \text{ is parabolic in } {}_k\Phi.$$

This shows that the map $P \mapsto {}_k\Phi_P$ makes sense into the intended target and is surjective.

For the injectivity and the inclusion-preserving property, it is enough to show that for any two parabolic k-subgroups $Q, Q' \supset S$ we have

$$Q \subset Q' \iff \Phi(Q,S) \subset \Phi(Q',S). \tag{11.4.1}$$

The direction $\Longrightarrow$ is obvious (as in such cases $\mathrm{Lie}(Q) \subset \mathrm{Lie}(Q')$). For the other direction, suppose $\Phi(Q,S) \subset \Phi(Q',S)$. We'll show that $Q_{k_s} \subset Q'_{k_s}$ by using that (11.4.1) is already known in the split case (such as over k_s), so then $Q \subset Q'$ as desired.

Pick a maximal k-torus T of G containing S. We claim that $T \subset Q, Q'$, so it makes sense to try to show that

$$\Phi(Q_{k_s}, T_{k_s}) \subset \Phi(Q'_{k_s}, T_{k_s})$$

(from which the settled split case, over k_s, then gives $Q_{k_s} \subset Q'_{k_s}$). Indeed, we can write $Q = P_G(\mu)$ for $\mu \in \mathrm{X}(S)$, so $Q \supset Z_G(\mu) \supset Z_G(S) \supset T$, and likewise $Q' \supset T$.

We need a way to construct $\Phi(Q_{k_s}, T_{k_s})$ from $\Phi = \Phi(G_{k_s}, T_{k_s})$ and $\Phi(Q,S)$, so then we can hope to bootstrap the hypothesis $\Phi(Q,S) \subset \Phi(Q',S)$ into the analogous containment for absolute roots. To give the recipe, note that since $Q = P_G(\mu)$ for some $\mu \in X_*(S) \subset X_*(T_{k_s})$, we have

$$\Phi(Q_{k_s}, T_{k_s}) = \{a \in \Phi \mid \langle a, \mu \rangle \geq 0\}.$$

Now comes the main point: since μ is a cocharacter valued in S, the pairing $\langle a, \mu \rangle$ only involves a through its restriction $a|_{S_{k_s}} \in X(S_{k_s}) = X(S)$ that lies in $\Phi(G, S) \cup \{0\}$. Recall also that the restriction-to-S map $\Phi(G_{K_s}, T_{k_s}) \to \Phi(G, S) \cup \{0\}$ hits everything in $\Phi(G, S)$. Therefore

$$\Phi(Q_{k_s}, T_{k_s}) = \{a \in \Phi : a|_{S_{k_s}} \in \Phi(G, S)_{\mu \geq 0} \cup \{0\}\} = \{a \in \Phi \ : a|_{S_{k_s}} \in \Phi(Q, S) \cup \{0\}\}.$$

□

Remark 11.4.7. Since minimal parabolic sets of roots in a root system are *exactly* the positive systems of roots (why?), we obtain a bijection

$$\{\text{minimal parabolic } k\text{-subgroups } \supset S\} \longleftrightarrow \{\text{positive systems of roots in } {}_k\Phi\}.$$

Corollary 11.4.8. *Fix a minimal parabolic k-subgroup $P_0 \subset G$. Every $G(k)$-conjugacy class of parabolic k-subgroups of G contains a unique member $Q \supset P_0$.* (*Such Q are called "standard" with respect to P_0.*)

Proof. Any parabolic k-subgroup contains a minimal one, so by $G(k)$-conjugacy of the minimal parabolic k-subgroups every $G(k)$-conjugacy class contains a standard member. It remains to show that if $Q, Q' \supset P_0$ are parabolic k-subgroups and $Q' = gQg^{-1}$ for some $g \in G(k)$ then $Q' = Q$.

We have $P_0 \subset Q$ and also $gP_0g^{-1} \subset Q'$. These are two minimal parabolics in Q', containing $\mathscr{R}_u(Q')$ (since any parabolic subgroup of Q' contains the unipotent radical, as we may check over $\overline{k}$ by reasoning with Borel subgroups), and $\mathscr{R}_u(Q')$ is k-split by the dynamical description of Q'. (Here we are abusing notation by writing $\mathscr{R}_u(Q)$ rather than $\mathscr{R}_{u,k}(Q)$, but this is harmless since $\mathscr{R}_{u,k}(Q)_{\overline{k}} = \mathscr{R}_u(Q_{\overline{k}})$ due to the dynamical description of parabolic k-subgroups of connected reductive k-groups.) Hence, by working in the connected reductive k-group $Q'/\mathscr{R}_u(Q')$ (which has the expected k-points, by the split property for $\mathscr{R}_u(Q')$) we see that that gP_0g^{-1} is $Q'(k)$-conjugate to P_0.

If we modify g on the left by an element of $Q'(k)$ then no harm is done, so by adjusting g in that we we can arrange that $gP_0g^{-1} = P_0$. But then $g \in P_0(k)$ by Chevalley's self-normalizing theorem for parabolic k-subgroups of connected linear algebraic groups, so $g \in Q(k)$. Thus, $Q' = gQg^{-1} = Q$. □

Fix $S \subset P_0$, so ${}_k\Phi^+ := \Phi(P_0, S) \subset {}_k\Phi$ is a *positive system of roots* for ${}_k\Phi$, and let ${}_k\Delta \subset {}_k\Phi^+$ be the associated root basis. Then $Q \mapsto \Phi(Q, S)$ is an inclusion-preserving bijection

$$\{Q \supset P_0\} \longleftrightarrow \{\text{parabolic subsets } \Psi \subset {}_k\Phi \text{ containing } {}_k\Phi^+\}.$$

By [Bou, VI, §1.7, Prop. 20, Lemma 3], such Ψ are *exactly* the subsets ${}_k\Phi^+ \cup [I]$ for a subset $I \subset {}_k\Delta$ (that is moreover unique), where $[I] := (\mathbf{Z} \cdot I) \cap {}_k\Phi$. Informally, this says that standard parabolic k-subgroups are obtained from the minimal P_0 by permitting negative relative roots supported only in specific directions relative to the relative root basis attached to P_0. (We cannot yet express this in terms of root groups as we would do in the split case since we haven't yet defined the notion of "root group" for a relative root! That concept will be treated soon.)

Since ${}_k\Phi \cup [I] \subset {}_k\Phi \cup [I']$ if and only if $I \subset I'$ (for subsets $I, I' \subset {}_k\Delta$, we conclude that

$$ {}_kP_I \subset {}_kP_{I'} \iff I \subset I', $$

so we have an inclusion-preserving bijection

$$ \{\text{standard } Q\} \longleftrightarrow \{\text{subsets of } {}_k\Delta\}. \tag{11.4.2} $$

(The uninteresting cases $I = \emptyset$ and $I = {}_k\Delta$ correspond to $Q = P_0$ and $Q = G$ respectively.) In particular, the number of standard parabolic k-subgroups, and hence the number of $G(k)$-conjugacy classes of parabolic k-subgroups, is $2^{\#_k\Delta} = 2^{r_k(\mathscr{D}(G)}$ where $r_k(\mathscr{D}(G))$ is the k-rank of $\mathscr{D}(G)$ (as we have shown that $S' := (S \cap \mathscr{D}(G))^0_{\mathrm{red}}$ is a maximal split k-torus in $\mathscr{D}(G)$).

Example 11.4.9. Let $G = \mathrm{GL}_n$, and take S to be the split diagonal torus, so $\mathrm{X}(S) = \mathbf{Z}^n$ via diagonal matrix entries (indexed by $1, \dots, n$ moving from upper-left to lower-right). Take Φ^+ corresponding to the upper triangular Borel subgroup B containing S, so its associated root basis is

$$ (\mathbf{Z}^n)^{\Sigma=0} \supset \Delta = \{e_i - e_{i+1}\}_{1 \le i \le n-1} = \{1, \dots, n-1\} $$

for the standard basis $\{e_i\}$ of $\mathbf{Z}^n$. There are 2^{n-1} parabolic k-subgroups containing B. What are they?

Consider the "staircase" subgroups whose stairs begin and end at positions on the diagonal (corresponding to preservation of a flag relative to the standard basis), such as this:

$$ \begin{pmatrix} * & * & * & * & * & * & \dots & * & * \\ * & * & * & * & * & * & \dots & * & * \\ & & * & * & * & * & \dots & * & * \\ & & * & * & * & * & \dots & * & * \\ & & * & * & * & * & \dots & * & * \\ & & & & & * & \dots & * & * \\ & & & & & & \ddots & * & * \\ & & & & & & & * & * \\ & & & & & & & * & * \end{pmatrix} $$

(Staircases whose stairs do not begin and end on the diagonal don't correspond to preserving a standard flag and correspondingly aren't stable under matrix multiplication!)

The negative root groups contained in such a standard parabolic correspond to sums of negative simple roots $-(e_j - e_{j+1})$ $(1 \le j \le n-1)$ for consecutive sequences of indices j *strictly between* the indices $i \in \{1, \dots, n-1\}$ for which positions i and $i+1$ on the diagonal straddle opposite sides of a corner in the staircase at vertex $(i, i+1)$. In other words, if $\{b_1, \dots, b_r\}$ is the sequence of column positions *just before* vertical drops in the staircase (so $\{2, 5, \dots, n-2\}$ in the picture above, and $b_r < n$ in general) then I is the *complement* in $\{1, \dots, n-1\}$ since the negative root groups correspond to the vertices of the stairs strictly "between" the vertical drops). The above construction yields 2^{n-1} distinct standard parabolic subgroups, so that exhausts the all possibilities.

It follows from the preceding example that for a vector space V of dimension $n \geq 2$, parabolic subgroups of $\mathrm{GL}(V)$ are in bijection with increasing flags of nonzero proper subspaces of V via the formation of stabilizers of flags. (It is clear that we can reconstruct the flag from its stabilizer by identifying the subspaces stable under such a flag, via inspection in the standard cases. The empty flag corresponds to $\mathrm{GL}(V)$ itself, and a full flag corresponds to Borel subgroups.) The maximal (proper) parabolic subgroups correspond to maximal proper subsets of the root basis Δ, or equivalently complements of subsets of Δ whose complement is minimal non-empty, so these correspond to flags with exactly one step: stabilizers of nonzero proper subspaces.

The handout "Standard parabolic subgroups" gives a thorough description (building on the earlier handout on root systems for classical groups) of parabolic k-subgroups of $\mathrm{SO}(q)$ for non-degenerate quadratic spaces (V,q) of dimension ≥ 3 and of the symplectic group Sp_{2n} for $n \geq 1$. These are the stabilizers of flags of *isotropic* nonzero subspaces (and maximal parabolic k-subgroups are related to stabilizers of nonzero isotropic subspaces) Here, "isotropic" means that the symplectic form vanishes on the subspace in the case of Sp_{2n} and that the quadratic form vanishing on the subspace for $\mathrm{SO}(q)$.

12. THE $*$-ACTION AND TITS–SELBACH CLASSIFICATION

12.1. **Link between absolute and relative roots.** Choose a maximal split k-torus $S \subset G$ and minimal parabolic k-subgroup $P \supset S$. Pick a maximal k-torus $S \subset T \subset P$ and a Borel k_s-subgroup $T_{k_s} \subset B \subset P_{k_s}$. Inside the absolute root system $\Phi = \Phi(G_{k_s}, T_{k_s})$ we have a positive system of roots $\Phi^+ = \Phi(B, T_{k_s})$. Let Δ be the basis of Φ^+. We are going to use Δ to a construct the basis of ${}_k\Phi = \Phi(G,S)$ corresponding to its positive system of roots $\Phi(P,S) =: {}_k\Phi^+$. (Keep in mind that we chose B inside P_{k_s}.)

Since $S_{k_s} \subset T_{k_s}$, we have a surjective restriction map

$$\mathrm{X}(T_{k_s}) \to \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$$

carrying Φ into ${}_k\Phi \cup \{0\}$, hitting all of ${}_k\Phi$. This carries Φ^+ into ${}_k\Phi^+ \cup \{0\}$ since $B \subset P_{k_s}$. Let $\Delta_0 = \{a \in \Delta \colon a|_{S_{k_s}} = 1\}$. Let ${}_k\Delta$ be the image of $\Delta - \Delta_0$ in ${}_k\Phi$, so ${}_k\Delta \subset {}_k\Phi^+$ since $B \subset P_{k_s}$ (and ${}_k\Phi^+ = \Phi(P,S)$, $\Phi^+ = \Phi(B, T_{k_s})$).

Lemma 12.1.1. *The parabolic subset* $\Phi(P_{k_s}, T_{k_s}) \subset \Phi$ *coincides with* $\Phi^+ \cup [\Delta_0]$.

Proof. Any parabolic subset containing Φ^+ has the form $\Phi^+ \cup [I]$ for a unique subset $I \subset \Delta$ (by the general description of parabolic sets of roots containing a positive system of roots), and $\Phi(P_{k_s}, T_{k_s})$ is such a subset since P is parabolic and $B \subset P_{k_s}$. Our task is to show that the unique such I corresponding to $\Phi(P_{k_s}, T_{k_s})$ is Δ_0.

For any subset J of Δ, the set J is characterized in terms of $\Phi^+ \cup [J]$ as those elements of Δ whose negative also lies in $\Phi^+ \cup [J]$. Hence, I is the set of elements $a \in \Delta$ such that $-a \in \Phi(P_{k_s}, T_{k_s})$. Since $P = Z_G(S) \ltimes U$ with $U_{k_s} \subset \mathscr{R}_u(B)$ (as B is a Borel k_s-subgroup of P_{k_s}), we have $\Phi(U_{k_s}, T_{k_s}) \subset \Phi(B, T_{k_s}) = \Phi^+$. Clearly $-\Delta$ is disjoint from Φ^+, so I is the set of $a \in \Delta$ such that

$$-a \in \Phi(Z_G(S)_{k_s}, T_{k_s}) = \{b \in \Phi : b|_{S_{k_s}} = 1\},$$

which is to say $a \in \Delta_0$. $\square$

Remark 12.1.2. A refinement of the preceding argument shows that Δ_0 is a basis of the root system $\Psi := \Phi(Z_G(S)_{k_s}, T_{k_s})$, as follows.

First, we claim that $\Psi = [\Delta_0]$ inside $\mathrm{X}(T_{k_s})$. Clearly $[\Delta_0] \subset \Psi \subset \Phi(P_{k_s}, T_{k_s})$, so any $a \in \Psi - [\Delta_0]$ lies in $\Phi(U_{k_s}, T_{k_s}) \subset \Psi^+$ and thus has a positive Δ-coefficient outside Δ_0. But then $-a \notin \Phi^+ \cup [\Delta_0] = \Phi(P_{k_s}, T_{k_s})$, contradicting that $-a \in \Psi$. This proves that $\Psi = [\Delta_0]$.

But Δ_0 is contained in the positive system of roots $\Psi \cap \Phi^+$, so clearly $\Psi \subset \mathbf{Z}_{\geq 0}\Delta_0 \cup \mathbf{Z}_{\leq 0}\Delta_0$. Hence, since Δ_0 is also linearly independent, it is a basis of $\Phi(Z_G(S)_{k_s}, T_{k_s})$ by [Bou, VI, §1.7, Cor. 3].

Proposition 12.1.3. *The set ${}_k\Delta$ defined above is the basis of ${}_k\Phi^+$.*

To prove this result, we may assume that G is semisimple since ${}_k\Phi = \Phi(\mathscr{D}(G), S')$ under the identification $\mathrm{X}(S)_{\mathbf{Q}} = \mathrm{X}(S_0)_{\mathbf{Q}} \oplus \mathrm{X}(S')_{\mathbf{Q}}$. Indeed, by commutativity of $G/\mathscr{D}(G)$ it follows that the action of $S = S' \cdot S_0$ on the Lie algebra $\mathrm{Lie}(G/\mathscr{D}(G)) = \mathrm{Lie}(G)/\mathrm{Lie}(\mathscr{D}(G))$ is trivial. Thus, the S-root spaces inside $\mathrm{Lie}(G)$ all lie inside $\mathrm{Lie}(\mathscr{D}(G))$, where they coincide with the S'-root spaces.

Since $\Phi \subset \mathbf{Z}_{\geq 0}\Delta \cup \mathbf{Z}_{\leq 0}\Delta$, applying restriction gives

$$ {}_k\Phi \subset \mathbf{Z}_{\geq 0}({}_k\Delta) \cup \mathbf{Z}_{\leq 0}({}_k\Delta). \tag{12.1.1} $$

But we arranged G to be semisimple, so ${}_k\Phi$ spans $\mathrm{X}(S)_{\mathbf{Q}}$ and hence ${}_k\Delta$ spans $\mathrm{X}(S)_{\mathbf{Q}}$. In particular, $\#{}_k\Delta \geq \dim S$ with equality if and only if ${}_k\Delta$ is $\mathbf{Q}$-linearly independent.

By [Bou, VI, §1.7, Cor. 3], it follows from (12.1.1) and the containment ${}_k\Delta \subset {}_k\Phi^+$ that ${}_k\Delta$ is a basis if it is linearly independent. Thus, if we can prove $\dim S \geq \#{}_k\Delta$ then we will be done. This reverse inequality will be establishing by using a remarkable construction, the so-called (continuous) $*$-action of $\Gamma_k := \mathrm{Gal}(k_s/k)$ on the basis Δ of $\Phi(B, T_{k_s}) = \Phi^+ \subset \Phi$. (This is a surprising notion because beyond the quasi-split case the subset $\Phi(B, T_{k_s})$ is *never* stable under the natural action of Γ_k on Φ! Why not?) We now digress to define the $*$-action, then use it to prove that $\dim S \geq {}_k\Delta$, and finally explore the deeper significance of the $*$-action in the context of the Tits–Selbach classification for connected semisimple groups over general fields.

12.2. **The $*$-action.** We define the $*$-action of $\Gamma = \mathrm{Gal}(k_s/k)$ on Δ (as a diagram, not just as a set of vertices) satisfying the following two key properties.

(*1) The restriction map $\Delta \xrightarrow{\text{res}} {}_k\Delta \cup \{0\}$ is Γ-invariant (so all fibers, and in particular Δ_0, are Γ-stable).

(*2) The action "detects fields of definition for parabolics". More precisely, a parabolic k_s-subgroup $Q \supset P_{k_s} \supset B$ corresponds to a unique subset of Δ containing Δ_0 via (11.4.2) and Lemma 12.1.1, so that subset has the form $\Delta_0 \sqcup \Delta'$ for some $\Delta' \subset \Delta - \Delta_0$. The property of interest is that Q is defined over k (i.e. come from parabolic k-subgroups of G, necessarily containing P) if and only if $\Delta_0 \sqcup \Delta' \subset \Delta$ is Γ-stable, or equivalently the subset $\Delta' \subset \Delta - \Delta_0$ is Γ-stable.

Granting these two properties, we establish the reverse inequality

$$ \dim S \geq \#{}_k\Delta $$

(and hence finish the proof of Proposition 12.1.3) as follows. Since $\dim S$ is the size of a basis of the relative root system (as we have arranged G to be semisimple), by (11.4.2) we have

$$2^{\dim S} \geq \#\{Q \supset P \text{ over } k\}.$$

Condition (*2) above gives another way to describe the right side: it is the number of Γ-stable subsets of $\Delta - \Delta_0$. But a Γ-stable subset is exactly a (possibly empty!) union of Γ-orbits, so the number of Γ-stable subsets is $2^{\#\{\Gamma\text{-orbits}\}}$, so

$$2^{\dim S} = \#\{Q \supset P \text{ over } k\} = 2^{\#\{\Gamma\text{-orbits}\}}.$$

Now by (*1) the number of Γ-orbits is at least the number of fibers of the Γ-invariant surjection $\Delta - \Delta_0 \twoheadrightarrow {}_k\Delta$, and the number of such fibers is obviously $\#_k\Delta$. So we conclude that

$$2^{\dim S} = \#\{Q \supset P \text{ over } k\} = 2^{\#\{\Gamma\text{-orbits}\}} \geq 2^{\#_k\Delta}.$$

This gives the reverse inequality $\dim S \geq \#_k\Delta$, forcing all inequalities to be equalities throughout, so in addition to completing the proof of Proposition 12.1.3 we obtain:

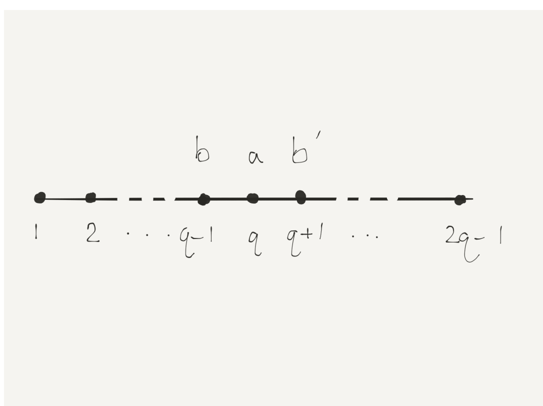
Corollary 12.2.1. *The fibers of $\Delta - \Delta_0 \twoheadrightarrow {}_k\Delta$ are exactly the Γ-orbits away from Δ_0.*

Before we define the $*$-action, let's show how it works in a special case (to be revisited from a broader point of view in our discussion of the Tits–Selbach classification):

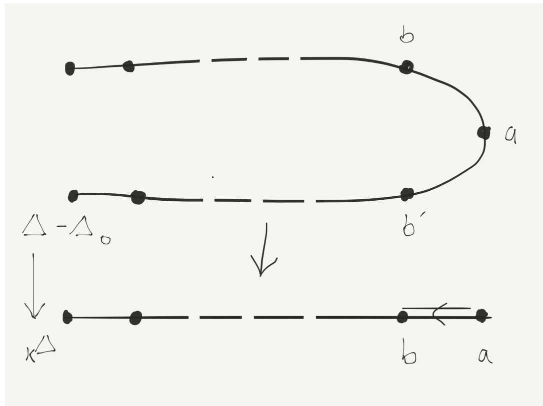
Example 12.2.2. Let $G = \mathrm{SU}(h)$ for a "maximally split" hermitian (V', h) of dimension $n > 2$ over a quadratic Galois extension k'/k. That is, $n = 2q$ is even and

$$h = \sum_{i=1}^{q} (x_i \overline{y}_{q+i} + x_{q+i} \overline{y}_i).$$

Such G is quasi-split, and $n - 1 = 2q - 1 \geq 3$. The absolute diagram (for Δ) is A_{n-1}:



The $*$-action of Γ goes through the quotient $\mathrm{Gal}(k'/k)$ acting through flipping around the central vertex a, and the quotient $\Delta - \Delta_0 \twoheadrightarrow {}_k\Delta$ modulo Γ is depicted below.



(The emptiness of Δ_0 expresses that G is quasi-split.)

How do we determine the edges (with multiplicity) in ${}_k\Delta$; e.g., that there is a double edge between the respective images $\overline{a}$ of a and $\overline{b}$ of b and b', with $\overline{a}$ long and $\overline{b}$ short? Rather than get involved with a general method, we explain a hands-on argument in this case for analyzing the relationship between $\overline{a}$ and $\overline{b}$ (with the others easier to analyze by a similar method). The quotient map $\Delta = \Delta - \Delta_0 \to {}_k\Delta$ corresponds to restriction of characters along the inclusion $S_{k_s} \hookrightarrow T$, so b and b' restrict to a common character $\overline{b}$ distinct from the restriction $\overline{a}$ of a, The subset $\{a, b, b'\}$ clearly spans an A_3 root system. The roots appearing are $a, b, b', a+b+b'$, as evidenced by the concrete model with SL_4:

$$\begin{pmatrix} & b & a+b & a+b+b' \\ & & a & a+b' \\ & & & b' \\ & & & \end{pmatrix}$$

The images in ${}_k\Phi$ of the roots shown in the above SL_4 picture are $\overline{a}, \overline{b}, \overline{a}+\overline{b}, \overline{a}+2\overline{b}$.

Recall that in any irreducible (possibly non-reduced) root system, for any two roots c and c' the pairing $\langle c, c'^{\vee}\rangle$ is negative to the largest integer $j \geq 0$ such that $c + jc'$ is a root. (This was noted long ago, as an immediate consequence of inspection of the rank-2 root systems $\mathrm{A}_1 \times \mathrm{A}_1$, A_2, B_2, and G_2.) Hence, $\langle \overline{a}, \overline{b}^{\vee}\rangle = -2$. Analyzing the other edges similarly, we obtain that ${}_k\Phi$ is of type C_q (seen in a more explicit manner by arguments with big matrices in the handout on relative root systems), for which the Dynkin diagram is as shown above with the vertex set ${}_k\Delta$.

Remark 12.2.3. Any finite set with a continuous action of Γ corresponds canonically to a finite étale k-scheme. The finite set Δ equipped with the $*$-action has associated finite étale k-scheme with an intrinsic meaning in terms of G without reference to a preferred choice of tori or Borel k_s-subgroup or minimal parabolic k-subgroup: it is the "scheme of Dynkin diagrams" in the sense of SGA3 (see the end of the $*$-action handout for a reference). This is a fancy way of expressing an alternative and more widely-known

Kottwitz method (described near the start of §12.4) for removing the apparent dependence on auxiliary choices.

The group Γ acts on $\mathrm{X}(T_{k_s})$ via scalar extension along k-automorphisms of k_s, and using inversion on Γ we likewise get an action (on the left!) on $\mathrm{X}_*(T_{k_s})$ that makes the pairing of characters and cocharacters Γ-invariant. For $T' = T \cap \mathscr{D}(G)$ and the maximal central k-torus Z, these actions preserve the decompositions $\mathrm{X}(T_{k_s})_{\mathbf{Q}} = \mathrm{X}(T'_{k_s})_{\mathbf{Q}} \oplus \mathrm{X}(Z_{k_s})_{\mathbf{Q}}$ and the analogue for cocharacters. Moreover, these actions factor through $\mathrm{Gal}(K/k)$ for any finite Galois extension K/k splitting T, so they are continuous.

Note that Γ preserves the finite subset Φ and is compatible with the natural action on $W(\Phi) = N_G(T)(k_s)/T(k_s) = N_{\mathscr{D}(G)}(T')(k_s)/T'(k_s)$ and the $W(\Phi)$-action on $\mathrm{X}(T_{k_s})_{\mathbf{Q}}$, so by consideration of reflections in $\mathrm{X}(T'_{k_s}) = \mathbf{Q}\cdot\Phi$ the action on $\mathrm{X}_*(T'_{k_s})$ preserves $\Phi^\vee$ via the relation $(\gamma.a)^\vee = \gamma.a^\vee$ for any $a \in \Phi$.

Easy case of $*$-action. Suppose G is quasi-split, so P is a Borel k-subgroup (and $B = P_{k_s}$). The action on Φ certainly preserves $\Phi^+ = \Phi(B = P_{k_s}, T_{k_s})$, and so must preserve its basis Δ. That resulting action of Γ on Δ is easily checked to be through diagram automorphisms (since it respects pairings of roots and coroots), and to satisfy the desired properties.

In general, for $\gamma \in \Gamma$ clearly $\gamma(\Phi^+) \subset \Phi$ is a positive system of roots, so there exists a unique $w_\gamma \in W(\Phi) = N_G(T)(k_s)/T(k_s)$ such that $w_\gamma(\gamma(\Phi^+)) = \Phi^+$, and so

$$w_\gamma(\gamma(\Delta)) = \Delta.$$

Beware that generally w_γ does not arise from $N_G(T)(k)$. Note also that away from the quasi-split case some w_γ must be non-trivial. Indeed, $w_\gamma = 1$ for all γ if and only if Δ is Γ-stable, or equivalently Φ^+ is Γ-stable, yet $\gamma(\Phi^+) = \Phi(\gamma^*(B), T_{k_s})$ for any $\gamma \in \Gamma$, so $w_\gamma = 1$ precisely when $\gamma^*(B) = B$ inside G_{k_s}. Hence, that holds for all γ precisely when B is defined over k, which is just another way to say that G is quasi-split.

Lemma 12.2.4. *The map*

$$\Gamma \times \mathrm{X}(T_{k_s}) \to \mathrm{X}(T_{k_s})$$

defined by $\gamma * a = w_\gamma(\gamma(a))$ preserving Δ is a left action. The induced left action (*via the dual representation*) *on $\mathrm{X}_*(T_{k_s})$ preserves $\Delta^\vee$ via the relation $(\gamma * a)^\vee = \gamma * a^\vee$.*

By compatibility with the pairings of characters and cocharacters, this gives a left Γ-action on the Dynkin diagram.

Proof. By definition-chasing, this comes down to establishing the equality

$$w_{\gamma'\gamma} = w_{\gamma'} \cdot \gamma'(w_\gamma)$$

in $W(G,T) = W(\Phi)$. That in turn can be verified by applying both sides to the positive system of roots $\gamma'\gamma(\Phi^+)$. $\square$

The quotient map $\mathrm{X}(T_{k_s}) \twoheadrightarrow \mathrm{X}(S_{k_s}) = \mathrm{X}(S)$ is certainly invariant for the natural Γ-action on the source (why?), and it is also visibly invariant by the induced action of $N_{Z_G(S)}(T)(k_s)$ on the source (why?), so to prove invariance of the restriction map $\Delta \to {}_k\Delta \cup \{0\}$ with respect to the $*$-action on the source it suffices to prove:

Proposition 12.2.5. *Each w_γ can be chosen to arise from $N_{Z_G(S)}(T)(k_s)$.*

Proof. The point is that $\gamma(\Phi^+)$ is not an arbitrary positive system of roots inside Φ: it satisfies

$$\gamma(\Phi^+)\subset\gamma(\Phi(P_{k_s},T_{k_s}))=\Phi(P_{k_s},T_{k_s}).$$

Thus, $\gamma(\Phi^+)$ corresponds to a Borel k_s-subgroup of G_{k_s} contained in P_{k_s} and containing T_{k_s}. Since $P=Z_G(S)\ltimes U$, $N_{Z_G(S)}(T)(k_s)$ acts transitively on the set of Borels of P_{k_s} containing T_{k_s} because this can be checked modulo U_{k_s} (as U_{k_s} is contained in every Borel k_s-subgroup of P_{k_s}, and $P_{k_s}/U_{k_s}=Z_G(S)_{k_s}$ is reductive). □

Corollary 12.2.6. *Such w_γ are generated by reflections in Δ_0.*

Proof. The root system $\Phi(Z_G(S)_{k_s},T_{k_s})$ has Δ_0 as a basis by Remark 12.1.2. □

This establishes property (*1) formulated near the start of §12.2. What about property (*2)? Consider $Q\longleftrightarrow\Delta_0\sqcup\Delta'$. What subset of Δ corresponds to $\gamma^*(Q)\supset\gamma^*(P_{k_s})=P_{k_s}$?

Proposition 12.2.7. *We have $\gamma^*(Q)\longleftrightarrow\Delta_0\sqcup(\gamma*\Delta')=\gamma*(\Delta_0\sqcup\Delta')$. In particular, $\gamma^*(Q)=Q$ if and only if $\gamma*\Delta'=\Delta'$, so Q is defined over k if and only if Δ' is Γ-stable under the $*$-action.*

Proof. For the full proof see §2 of the $*$-action handout. The key is to show that $\gamma*(\Delta')\subset\gamma(\Delta')+\mathbf{Z}\Delta_0$, which ultimately follows from Corollary 12.2.6 due to the general reflection formula $r_a(b)=b-\langle b,a^\vee\rangle a\in b+\mathbf{Z}a$ for $a,b\in\Phi$. □

We end our initial discussion of the $*$-action with an overview of its role in the definition of the Langlands dual group. Let $R=(\mathrm{X}(T_{k_s}),\Phi,\mathrm{X}_*(T_{k_s}),\Phi^\vee)$ be the associated root datum over k_s. In the quasi-split case, we have seen that the $*$-action of Γ on the based root datum (R,Δ) is induced by the natural Γ-action on $\mathrm{X}(T_{k_s})$. In general, this action has significance going beyond its role in the Tits–Selbach classification: it also underlies the definition of the *Langlands dual group*, as we now sketch.

Let $B\subset G_{k_s}$ be the unique Borel k_s-subgroup containing T_{k_s} such that $\Phi(B,T_{k_s})$ is the positive system of roots with basis Δ, and let $\{X_a\}_{a\in\Delta}$ be a pinning (i.e., choice of basis of $\mathfrak{g}_a$ for each $a\in\Delta$). The Isomorphism Theorem gives that the natural map

$$\mathrm{Aut}_{k_s}(G_{k_s},T_{k_s},\{X_a\}_{a\in\Delta})\to\mathrm{Aut}(R,\Delta)$$

is bijective. That is, every automorphism of the based root datum uniquely lifts to a k_s-automorphism of the "pinned" split reductive pair.

Since Δ is a basis of $\mathrm{X}((T/Z_G)_{k_s})$, any change in (T,Δ) is attained by composing with the effect of an element of $(G/Z_G)(k_s)$ unique modulo $(T/Z_G)(k_s)$ (exercise!). Thus, the lifting of $\mathrm{Aut}(R,\Delta)$ into $\mathrm{Aut}(G_{k_s})$ via a pinning is well-defined up to the effect of $(G/Z_G)(k_s)$. (This is the main content of the fact that the automorphism scheme $\mathrm{Aut}_{G/k}$ has identity component G/Z_G and component group that is a k_s/k-form of $\mathrm{Aut}(R,\Delta)$.)

Now suppose k is a local or global field, and let W_k be the Weil group of k. Let ${}^LG^0$ be the unique pinned connected reductive $\mathbf{C}$-group whose root datum is equipped with an identification with the dual root datum $R^\vee=(\mathrm{X}_*(T_{k_s}),\Phi^\vee,\mathrm{X}(T_{k_s}),\Phi)$. The $*$-action defines a composite homomorphism

$$\rho:\Gamma\to\mathrm{Aut}(R^\vee,\Delta^\vee)\hookrightarrow\mathrm{Aut}({}^LG^0)$$

whose second step rests on the choice of pinning of ${}^L G^0$ (applying the preceding discussion with $k = \mathbf{C}$), so the ${}^L G^0(\mathbf{C})$-*conjugacy class* of this homomorphism is independent of all choices. If k'/k is a finite Galois subextension of k_s such that $G_{k'}$ is split then $\mathrm{Gal}(k_s/k')$ acts trivially on Δ under the $*$-action as defined above, so ρ factors through $\mathrm{Gal}(k'/k)$. The *Langlands dual* is the disconnected locally algebraic group

$$ {}^L G := \Gamma \ltimes {}^L G^0; $$

this group is only well-defined up to ${}^L G^0(\mathbf{C})$-conjugation. (The main content occurs "at finite level", using the typically disconnected linear algebraic $\mathbf{C}$-group $\mathrm{Gal}(k'/k) \ltimes {}^L G^0$.) Hence, the notion of ${}^L G^0(\mathbf{C})$-*conjugacy class* of continuous homomorphism

$$ \phi : W_k \to {}^L G(\mathbf{C}) $$

over the natural map $W_k \to \Gamma$ is intrinsic to the k-group G. Such conjugacy classes, or variants with W_k replaced by the Weil–Deligne group, are of central importance in the Langlands Program. (If G is a split group then ${}^L G = \Gamma \times {}^L G^0$ and such ϕ are exactly conjugacy classes of continuous homomorphisms $W_k \to {}^L G^0(\mathbf{C})$.)

Example 12.2.8. In the special case that G is a k-torus T, so there are no absolute roots and the $*$-action is just the natural Γ-action on $\mathrm{X}(T_{k_s})$, we have ${}^L T = \Gamma \ltimes \widehat{T}$ where the *dual torus* $\widehat{T}$ is $\underline{\mathrm{Hom}}(\mathrm{X}(T_{k_s}), \mathbf{G}_m)$ on which Γ acts via the natural action on the geometric character lattice. The homomorphisms $\phi : W_k \to {}^L T(\mathbf{C}) = \Gamma \ltimes \widehat{T}(\mathbf{C})$ have second component that is exactly a continuous 1-cocycle $f : W_k \to \widehat{T}(\mathbf{C})$, and the effect of composing ϕ with a $\widehat{T}(\mathbf{C})$-conjugation is exactly to change f by a 1-coboundary.

Note that if T splits over a finite Galois extension k'/k inside k_s/k then the $*$-action on $\mathrm{Gal}(k_s/k')$ is trivial, so $f|_{W_{k'}}$ is just a continuous homomorphism by another name. But the target of f is commutative, so f must kill the commutator subgroup of $W_{k'}$. By the construction of W_k using class formations in [AT, Ch. XV] (the only unified approach that treats all local and global fields on an equal footing and provides the only known definition for W_k when k is a number field), the quotient of W_k modulo the commutator subgroup of $W_{k'}$ is the group $W_{k'/k}$ that is naturally a topological extension of $\mathrm{Gal}(k'/k)$ by $A(k)$, where $A(k) = \mathbf{A}_k^\times / k^\times$ in the global case and $A(k) = k^\times$ in the local case (this extension representing the fundamental class associated to k'/k). That is, such 1-cocycles f arise from 1-cocycles $W_{k'/k} \to \widehat{T}(\mathbf{C})$.

An early verification of Langlands [La, Thm. 2] was that for any local field k there is a "natural" isomorphism from $\mathrm{H}^1_{\mathrm{cont}}(W_k, \widehat{T}(\mathbf{C}))$ onto the group continuous homomorphisms $T(k) \to \mathbf{C}^\times$, and that for any global field k there is a "natural" surjection with finite (explicitly described) kernel from $\mathrm{H}^1_{\mathrm{cont}}(W_k, \widehat{T}(\mathbf{C}))$ onto the group of continuous homomorphisms $T(\mathbf{A}_k)/T(k) \to \mathbf{C}^\times$. These "natural" maps are specific constructions (satisfying good properties), called *Langlands duality for tori*, and for $T = \mathbf{G}_m$ by design this recovers the construction made via Artin reciprocity maps in class field theory.

12.3. **Relative root groups.** As a companion to the use of the relative root system to keep track of parabolic k-subgroups, there is a theory of (high-dimensional, and possibly non-commutative) relative root groups. These satisfy many of the familiar properties of root groups in the split case, including to give a concrete description of Levi factors and unipotent radicals of parabolic k-subgroups. We will record the main results (which are unsurprising once the definitions are given) and *refer to the handout "Tits systems and Root groups" for the proofs.*

But first we address a loose end from our earlier discussion of Levi factors of parabolic subgroups: we know that if P is a parabolic k-subgroup of a connected reductive k-group G and $U := \mathscr{R}_{u,k}(P)$ is the split k-descent of its geometric unipotent radical (by writing $P = P_G(\lambda)$ for a k-homomorphism $\lambda : \mathbf{G}_m \to G$ we have $U = U_G(\lambda)$), there exists a k-subgroup L such that $L \to P/U$ is an isomorphism, or equivalently $L \ltimes U = P$; e.g., for a dynamic description $P_G(\lambda)$ of P we can take L to be $Z_G(\lambda)$. The k-isomorphism class of L is obviously intrinsic (after all, $L \to P/U$ is an isomorphism!), but as a k-subgroup of P how are the different choices related to each other?

In view of the use of Levi factors for calculations in representation theory, it would be best if all such L are related through $P(k)$-conjugacy, or equivalently (!) through $U(k)$-conjugacy. Fortunately, this is true in the following more precise form:

Proposition 12.3.1. *The action of $U(k)$ on the set of Levi factors of P is simply transitive. Moreover, every maximal k-torus T of P is contained in a unique Levi factor of P.*

Some Levi k-subgroups arise by the dynamic method, so the $U(k)$-conjugacy implies that *all* Levi k-subgroups arise by the dynamic process!

Proof. In view of the uniqueness assertion, by Galois descent it suffices to treat the situation over k_s; i.e., we now may and do assume $k = k_s$, so all k-tori are split. We shall first prove that any maximal k-torus $T \subset P$ lies in a unique Levi factor L. Since T is maximal in P, we can write $P = P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(T)$. Hence, $Z_G(\lambda)$ is a Levi factor of P containing T. To prove that there is at most one L containing T, we shall describe any such L directly in terms of T and P.

Consider the set $\Phi(L,T)$ of nontrivial T-weights that occur on $\mathrm{Lie}(L)$. We have $L \simeq P/U$, so $\Phi(L,T) = \Phi(P/U,T)$ is independent of L inside $\mathrm{X}(T)$. Let Ψ denote this subset of $\Phi = \Phi(G,T)$. For each $a \in \Psi$, the a-root group for the connected reductive L with respect to T is *the same* as for G by uniqueness of root groups (as T-stable smooth connected unipotent k-subgroups exhibiting a specific root line as their Lie algebra). But L is is generated by T and the root groups U_a for $a \in \Psi = \Phi(P/U,T)$. The latter is a description of L in terms of just P and T, establishing the uniqueness of L.

Since all maximal (split) k-tori of P are $P(k)$-conjugate, the preceding uniqueness result implies that all Levi factors of P are $P(k)$-conjugate to each other, and so are $U(k)$-conjugate to each other. It remains to show that if $u \in U(k)$ normalizes a Levi factor L then $u = 1$. For any $x \in L(k)$ we have $uxu^{-1} \in L(k)$, so $u(xu^{-1}x^{-1}) \in L(k)$. But U is normal in P, so $u(xu^{-1}x^{-1}) \in U(k)$. Since $L \cap U = 1$ it follows that $uxu^{-1}x^{-1} = 1$, which is to say that u centralizes $L(k)$. Since $L(k)$ is Zariski-dense in L (as $k = k_s$), we have $u \in Z_G(L)(k) \subset Z_G(T)(k)$ for a maximal k-torus $T \subset G$. But $Z_G(T) = T$, so $u \in T \cap U = 1$. $\square$

Example 12.3.2. Consider G containing a split maximal k-torus T, and let I be a subset of a basis Δ of a positive system of roots Φ^+ for $\Phi := \Phi(G,T)$. Let B be the Borel k-subgroup containing T for which $\Phi^+ := \Phi(B,T)$. For the parabolic k-subgroup ${}_kP_I$

containing B we have $\Phi({}_kP_I, T) = \Phi^+ \cup [I]$, so the unique Levi k-subgroup L_I containing T satisfies $\Phi(L_I, T) = [I]$ (argue exactly as in Remark 12.1.2).

We can describe L_I explicitly as follows. Consider the subtorus $T_I = (\cap_{a\in I} \ker a)^0_{\mathrm{red}} \subset T$. The connected reductive k-group $Z_G(T_I)$ containing T is generated by T and the root groups U_a for roots a trivial on T_I, which is to say $a \in [I]$ (as I is part of a basis Δ for Φ). But by the same reasoning L_I is generated by the same subgroups since $\Phi(L_I, T) = [I]$. Hence, $L_I = Z_G(T_I)$.

Writing ${}_kP_I = P_G(\lambda)$ for some $\lambda \in \mathrm{X}_*(T)$, so $U_I := \mathscr{R}u, k({}_kP_I) = U_G(\lambda)$, we have $U_I = \prod_{a\in\Phi(U_I,T)} U_a$ (with multiplication in any order) by Theorem 5.3.6 applied to $A = \Phi_{\lambda>0}$ with A_j its singleton subsets (Φ is reduced!), and the collection of roots $\Phi(U_I, T)$ is exactly $\Phi^+ - [I]$ because ${}_kP_I = \Phi^+ \cup [I]$ and $\Phi(L_I, T) = [I]$.

We would like to push Example 12.3.2 beyond the split setting, to describe ${}_kP_I$ in terms of specific root groups U_a attached to relative roots $a \in {}_k\Phi$ and the centralizer of a specific subtorus $S_I \subset S$. But to do this we first need to *define* what U_a means beyond the split case! Recall that in the split case we use a dynamical construction, namely $U_{\langle a\rangle}(G)$ to define U_a, and then used SL_2-calculations to deduce that U_a is a 1-dimensional vector group. Moreover, dynamical principles (and *not* SL_2-calculations as in standard textbooks) were used to deduce directly spanning results and commutation relations among root groups, sometimes relying on reducedness of Φ to ensure any two distinct roots inside a positive system of roots are linearly independent.

Provided that we are attentive to whether roots are divisible or multipliable (or neither), much of the previous work carries over *unchanged* to the general case, including the definition of U_a: define it to be $U_{\langle a\rangle}(G)$. There are some new features:

(i) without the SL_2-crutch, we don't immediately see whether or not U_a is a vector group (is it even commutative?), nor if so whether it has a preferred linear structure (a genuine issue in higher dimensions since in characteristic $p > 0$ the k-group $\mathbf{G}_{\mathrm{a}}^2$ has non-linear automorphisms such as $(x, y) \mapsto (x + y^p, y)$),
(ii) The set of S-weights on $\mathrm{Lie}(U_a)$ is ${}_k\Phi \cap \mathbf{Z}_{\geq 0}a$ by design, and this is $\{a, 2a\}$ when a is multipliable, so this Lie algebra can be larger than a single weight space.

The groups $\mathrm{SU}(h)$ exhibit all of these issues (including that U_a can be non-commutative for multipliable a).

Fortunately, *dynamical methods* (having nothing to do with reductive groups) in [CGP, §3.3] dispose of all of these problems: U_a is always a vector group when $2a \notin {}_k\Phi$, and in the multipliable case (so U_{2a} is a vector group, as $4a \notin {}_k\Phi$!) the vector group U_{2a} is a *central* k-subgroup of U_a with U_a/U_{2a} also a vector group, and moreover that all of these vector groups admit a unique S-equivariant linear structure. These matters are discussed at length in the handout "Tits systems and root groups", the upshot of which is that (with a bit more work required in a few places) one has reasonable analogues of all of the familiar features of root groups from the split case, so here we just wish to highlight two aspects:

(1) For $I \subset {}_k\Delta$, the parabolic k-subgroup ${}_kP_I$ has a unique Levi k-subgroup $L_I \supset S$ (improving on Proposition 12.3.1!) and both L_I and U_I are described by the analogue of Example 12.3.2 using S and ${}_k\Delta$ and the set of non-divisible elements

of ${}_k\Phi^+ - [I]$. (This "direct spanning" description of U_I is crucial for calculations in the foundations of Bruhat–Tits theory.)

(2) If G is semisimple, k-simple, and isotropic then ${}_k\Phi$ is *irreducible* (often non-reduced!) and G is generated by the relative root groups U_a for $a \in {}_k\Phi$. The proof of this result is *much harder* than its counterpart in the split case (as the structure of $Z_G(S)$ a mystery, and we cannot increase k since that may change the relative root system and ruin the maximality of S).

For k-simple isotropic G, the simply connected central cover of G is also k-simple. Hence, by Corollary 1.2 in the handout on simple isogeny factors, that cover has the form $\mathrm{R}_{k'/k}(G')$ for a canonically determined finite separable extension k'/k and connected semisimple k'-group G' that is *absolutely simple* and simply connected.

The relative root datum of $\mathrm{R}_{k'/k}(G')$ is identified with that of G' (see §2 of the handout on relative root systems), so for the purpose of describing the possibilities for connected semisimple groups over a field in terms of the relative root system and $*$-action on the absolute diagram (and perhaps further information), the essential case is that of absolutely simple groups (which are moreover simply connected).

12.4. **Tits-Selbach classification.** Let G be a connected semisimple k-group, $S \subset G$ a maximal split k-torus.

Theorem 12.4.1. *If S is maximal (as a k-torus) in G then the reduced root datum $R(G,S)$ is a complete isomorphism invariant.*

By "complete invariant" we mean that every reduced root datum actually arises from a split reductive pair over k (the Existence Theorem) and that two split connected reductive k-groups are isomorphic if their root data are isomorphic (the Isomorphism Theorem).

In the general case, we get the following data from G:

(1) The Dynkin diagram $\mathrm{Dyn}(G)$ with $*$-action given by $\Gamma = \mathrm{Gal}(k_s/k)$. A priori this depends upon a choice of $S \subset T \subset P$ and $T_{k_s} \subset B \subset P_{k_s}$. However, we can suppress the auxiliary choices by using either of the following viewpoints:
   - the finite étale k-scheme of Dynkin diagrams, or
   - the "Kottwitz method": if T' and $T'_{k_s} \subset B' \subset P_{k_s}$ are another pair for the same S and P then there exists $g \in G(k_s)$ such that $B' = gBg^{-1}$ and $T'_{k_s} = gT_{k_s}g^{-1}$, with g unique up to right multiplication against $B \cap N_G(T)_{k_s} = T_{k_s}$, so the isomorphism of diagrams $\Delta(B, T_{k_s}) \simeq \Delta(B', T'_{k_s})$ induced by g-conjugation is *independent* of the choice of g. This independence ensures that it is compatible with the $*$-actions on both sides, and likewise if we vary the initial pair (S,P). This canonically identifies all such diagrams (i.e., we can declare $\mathrm{Dyn}(G)$ to be the inverse limit of all diagrams $\Delta(B, T_{k_s})$ along these specified canonical Γ-compatible diagram isomorphisms as we vary (B,T) and then vary (S,P)).

   The significance of a viewpoint that avoids reliance on a specific choice of auxiliary data is that it makes the $*$-action of Γ on $\mathrm{Dyn}(G)$ functorial with respect to *any* k-isomorphism in G. That is, if $f : G' \simeq G$ is a k-isomorphism then we get a canonically associated diagram isomorphism $\mathrm{Dyn}(f)$ that is compatible with $*$-actions on both sides.

(2) $M = \mathscr{D}(Z_G(S)/S)^{\mathrm{ad}}$ is k-anisotropic connected semisimple of adjoint type.

(3) There are canonical identifications

$$\begin{tikzcd} \Delta_0 := \{a \in \Delta \mid a|_{S_{k_s}} = 1\} \arrow[r, equal] \arrow[d, hook] & \mathrm{Dyn}(M) \arrow[d, hook, dotted] \\ \Delta \arrow[r, equal] & \mathrm{Dyn}(G) \end{tikzcd}$$

inducing a canonical inclusion $\mathrm{Dyn}(M) \hookrightarrow \mathrm{Dyn}(G)$ which is Γ-*invariant*. This is not obvious because the $*$-action depends on the Weyl action, but follows from Proposition 12.2.5, which says that w_γ in the definition of the $*$-action on $\mathrm{Dyn}(G)$ comes from $N_{Z_G(S)}(T)(k_s)$.

Example 12.4.2. Let D be a central division algebra over k of dimension d^2, and $G = \mathrm{SL}_m(D)$. Then G is of type A_{n-1} where $n = md$. The maximal split torus S is of dimension m, consisting of the diagonal scalars in

$$S = \left\{ \begin{pmatrix} \mathrm{GL}_1 & & & \\ & \mathrm{GL}_1 & & \\ & & \ddots & \\ & & & \mathrm{GL}_1 \end{pmatrix} \right\} \subset \left\{ g \in \begin{pmatrix} \underline{D} & \underline{D} & \dots & \underline{D} \\ \underline{D} & \underline{D} & \dots & \underline{D} \\ \vdots & \ddots & \ddots & \vdots \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} \end{pmatrix} \mid \mathrm{Nrd}(g) = 1 \right\}$$

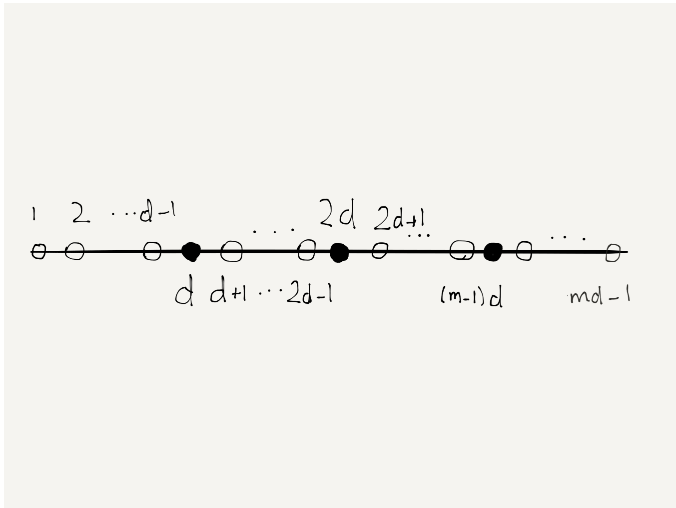
where $\mathrm{Nrd} : \mathrm{Mat}_m(D) \to k$ is the reduced norm and $\underline{D}$ is the affine ring scheme associated to D (i.e., represents the functor on k-algebras given by $A \rightsquigarrow A \otimes_k D$). The centralizer $Z_G(S)$ is then the diagonal matrices:

$$Z_G(S) = \left\{ \begin{pmatrix} \underline{D}^\times & & & \\ & \underline{D}^\times & & \\ & & \ddots & \\ & & & \underline{D}^\times \end{pmatrix} \right\} \subset \left\{ \begin{pmatrix} \underline{D} & \underline{D} & \dots & \underline{D} \\ \underline{D} & \underline{D} & \dots & \underline{D} \\ \vdots & \ddots & \ddots & \vdots \\ \underline{D} & \underline{D} & \underline{D} & \underline{D} \end{pmatrix} \right\}.$$

Therefore $M = (\underline{D}^\times / \mathrm{GL}_1)^m$.

The Dynkin diagram is depicted below, with roots in Δ_0 circled and the roots in $\Delta - \Delta_0$

marked with black dots:



The Γ-action is trivial. We have $\Delta_0 = \mathrm{A}_{d-1}^m$ and ${}_k\Delta = \mathrm{A}_{m-1}$.

From G we get a 4-tuple (R, τ, M, j) where

- R is a semisimple root datum,
- τ is a continuous action of Γ on Dyn(R),
- M is a k-anisotropic group of adjoint type,
- $j : (\mathrm{Dyn}(M), *) \hookrightarrow (\mathrm{Dyn}(\mathscr{R}), \tau)$ is a Γ-invariant inclusion.

There is an evident notion of isomorphism among such 4-tuples, and the isomorphism class of the 4-tuple we obtain from G is easily checked (do it!) to be independent of all choices; i.e., it only depends on the isomorphism class of G.

Remark 12.4.3. The k-group M is a mystery over general fields, usually entailing involvement of central division algebras, anisotropic quadratic or hermitian forms, and so on (e.g., for a general field k, Br(k)[2] may contain classes of central division algebras beyond dimension 4). Over local and global fields this data is understood via class field theory.

The main result announced by Tits [T, 2.7.1] and completed by Selbach [Sel] is:

Theorem 12.4.4 (Relative Isomorphism Theorem)**.** *The association from G to this 4-tuple is injective on isomorphism classes.*

On its own, the statement of this theorem is not as useful as it may seem to be (in contrast with the split case), since to actually determine which 4-tuples can arise over a given k generally entails a lot of effort specific to a given root system. In practice, the importance of this theorem is two-fold:

(i) it provides a useful framework to organize the information that arises in the analysis of possible k-groups with a given absolute semisimple root datum,

(ii) knowledge of the *proof* of the theorem (in the format we will describe it), especially the idea of "reduction of the structure group", helps very much to systematically analyze what G may occur.

I've not been able to understand Tits' original exposition of his proof, but I am certain that his cocycle manipulations should amount to an alternative formulation of what is done in the proof-sketch below. (A difference between Tits' approach and the one sketched below is that Tits uses the idea of the automorphism variety of a split group over k whereas we will work throughout with the automorphism variety of a given connected semisimple k-group G that is typically not split.)

Idea of proof. The slogan is "reduction of the structure group"! We will need to use degree-1 non-abelian Galois cohomology; this is a systematic technique for working with torsors under smooth affine groups over a field, and is set up in [Se, Ch.,I, §5.1–§5.5]. We will explain the basic idea behind this formalism, and then indicate how it is used.

Suppose G and G' give isomorphic 4-tuples. By the Isomorphism Theorem over k_s, the identification of their absolute root data implies that $G'_{k_s} \simeq G_{k_s}$. Thus, there is some finite Galois extension K/k inside k_s/k (such as any that splits choices of maximal k-tori in G and G') such that $G'_K \simeq G_K$ as K-groups. This says that G' can be obtained from G by modifying the Galois descent datum

$$\{\varphi_\gamma : \gamma^*(G_K) \simeq G_K\}_{\gamma \in \mathrm{Gal}(K/k)}$$

which reconstructs G from G_K. (Any K/k-descent datum on an affine scheme is always effective, via Galois descent for the coordinate ring.)

Upon fixing a K-isomorphism $f : G'_K \simeq G_K$, a descent datum $\{\varphi'_\gamma\}$ encoding G' as a K/k-form of G is given by

$$\varphi'_\gamma : \gamma^*(G_K) \simeq \gamma^*(G'_K) \simeq G'_K \simeq G_K$$

(using the k-descent G' of G'_K for the middle isomorphism). We can write $\varphi'_\gamma = c(\gamma)\varphi_\gamma$ for a unique K-automorphisms $c(\gamma) : G_K \simeq G_K$, and the condition that φ' be a descent datum (given that φ is one) is exactly that c is a 1-cocycle:

$$c(\gamma'\gamma) = c(\gamma') \circ \gamma'^*(c(\gamma))$$

for all $\gamma, \gamma' \in \mathrm{Gal}(K/k)$.

The notion of non-commutative *cohomologous* 1-cocycles as defined in [Se, Ch. I, §5] encodes exactly the effect on c of changing φ and φ', so the resulting pointed set $\mathrm{H}^1(K/k, \mathrm{Aut}_K(G_K))$ classifies isomorphism classes of k-groups G' that become isomorphic to G over K (i.e., $G'_K \simeq G_K$). To remove the dependence on the choice of finite Galois subextension $K \subset k_s$ over k, we pass to the limit and impose an appropriate continuity condition on the 1-cocycles to arrive at a pointed set

$$\mathrm{H}^1(k_s/k, \mathrm{Aut}_{k_s}(G_{k_s}))$$

that classifies isomorphism classes of k-groups which become isomorphic to G over k_s (or equivalently, by the Isomorphism Theorem in our connected semisimple setting, have the same absolute root datum as G). Our task is to use an isomorphism of 4-tuples to show that the class in this H^1 corresponding to G' is trivial.

At this point, we need a useful way to describe $\mathrm{Aut}_{k_s}(G_{k_s})$. Remarkably, the automorphisms of a connected semisimple group are classified by a (possibly disconnected) linear algebraic group:

Proposition 12.4.5. *For a connected semisimple k-group G, the functor on k-algebras $A \rightsquigarrow \mathrm{Aut}_A(G_A)$ is represented by a smooth affine k-group $\mathrm{Aut}_{G/k}$ whose identity component is $G^{\mathrm{ad}} = G/Z_G$ via the conjugation action on G.*

In the split case, the finite étale component group is the constant k-group associated to the group of diagram automorphisms that respect the root datum (i.e., automorphisms of the diagram Δ so that the resulting permutation automorphism of $\mathbf{Q}^{\Delta} = \mathrm{X}(T)_{\mathbf{Q}}$ for a split maximal k-torus T preserves $\mathrm{X}(T)$).

What is ultimately needed for our purposes over fields is much less than Proposition 12.4.5; we just need a way to describe all automorphisms in the split case (such as over k_s) in terms of both the action of points of G^{ad} over specific fields and appropriate diagram automorphisms (which are "lifted back" to automorphisms of G_{k_s} by using the notion of a pinning on a split reductive pair), and this must all be done functorially with respect to extension of the ground field (such as through automorphisms of k_s as a ground field). For k-split G this special case of Proposition 12.4.5 can be deduced from the Isomorphism Theorem in [C1, (1.5.2)–Prop. 1.5.5] over algebraically closed fields k (for which $G^{\mathrm{ad}}(k) = G(k)/Z_G(k)$) and then [C1, (7.1.2)–(7.1.3)] over arbitrary fields; the general case can then be deduced via Galois descent.

The proof of the full statement of Proposition 12.4.5 requires the theory of reductive groups over rings (not a surprise, since it concerns a functor on k-algebras), and is given in [C1, Thm. 7.1.9].

Remark 12.4.6. Diagram automorphisms are the same as automorphisms of the root system, by Proposition 9.5.8. In the simply connected and adjoint-type cases the root datum is determined by the root system (i.e., either $X = \mathbf{Z}\Phi$ or $X^{\vee} = \mathbf{Z}\Phi^{\vee}$), so in such cases that are split the component group of $\mathrm{Aut}_{G/k}$ is the constant group associated to the group of diagram automorphisms.

Likewise, in cases where the fundamental group of the root system is cyclic (such as for type A) it is automatic that every lattice between the root lattice and the weight lattice is preserved by every automorphism of the root system (since a subgroup of a cyclic group is uniquely determined by its size). Thus, in such cases that are split again the component group is the group of diagram automorphisms. Cyclicity fails for type D_{2n}, for which it is $(\mathbf{Z}/2\mathbf{Z})^2$; see [C1, Ex. 1.5.2] for further discussion of this case. (Keep in mind that if G is not absolutely simple then the group of diagram automorphisms may involve isomorphisms between different irreducible components of the diagram!)

Example 12.4.7. Consider $G = \mathrm{SL}_n$ with $n \geq 2$. The adjoint quotient $\mathrm{SL}_n/\mu_n = \mathrm{PGL}_n = \mathrm{GL}_n/\mathbf{G}_m$ acts on SL_n in the natural way, and the diagram A_{n-1} has no nontrivial diagram automorphisms when $n = 2$ and has exactly one when $n > 2$. If $n > 2$ then transpose-inverse is an automorphism that does not arise from the adjoint quotient since its effect on the center μ_n is inversion, which is nontrivial when $n > 2$! In contrast, for $n = 2$, transpose-inverse on SL_2 is induced by conjugation against the standard Weyl element. Thus, $\mathrm{Aut}_{\mathrm{SL}_2/k} = \mathrm{PGL}_2$ and if $n > 2$ then $\mathrm{Aut}_{\mathrm{SL}_n/k} = \mathrm{PGL}_n \ltimes (\mathbf{Z}/2\mathbf{Z})$.

In general, if $\mathscr{G}$ is a smooth affine k-group then the pointed set $\mathrm{H}^1(k_s/k, \mathscr{G}(k_s))$ has a useful geometric description that suppresses the mention of k_s, much as étale cohomology over a field can be more convenient than Galois cohomology by avoiding the need to work with a separable closure.

More specifically, there is a natural bijection of pointed sets

$$\theta : \{\text{right } \mathscr{G}\text{-torsors over } k\}/\text{isom.} \simeq \mathrm{H}^1(k_s/k, \mathscr{G}(k_s))$$

by assigning to any right $\mathscr{G}$-torsor X the class of the 1-cocycle c arising from a choice of base point $x_0 \in X(k_s)$: the k_s-isomorphism of right torsors $f : \mathscr{G}_{k_s} \simeq X_{k_s}$ via $f(g) = x_0.g$ yields an automorphism of right torsors

$$\mathscr{G}_{k_s} \simeq \gamma^*(\mathscr{G}_{k_s}) \overset{\gamma^*(f)}{\simeq} \gamma^*(X_{k_s}) \simeq X_{k_s} \overset{f^{-1}}{\simeq} \mathscr{G}_{k_s}$$

that must be *left* multiplication by some unique $c(\gamma) \in \mathscr{G}(k_s)$ for each $\gamma \in \mathrm{Gal}(k_s/k)$. One checks without difficulty that c is a continuous 1-cocycle and that passing to a cohomologous 1-cocycle is exactly the same as changing the choice of $x_0 \in X(k_s)$.

The procedure $X \mapsto c$ defines θ as a map of pointed sets, and Galois descent in the affine setting ensures that θ is bijective. The viewpoint of torsors artfully avoids any mention of k_s, and so it is often convenient to *define* the notation

$$\mathrm{H}^1(k, \mathscr{G}) := \{\text{right } \mathscr{G}\text{-torsors over } k\}/\text{isom.}$$

with functoriality in k via scalar extension and functoriality in $\mathscr{G}$ via a "pushout" construction; see [C1, Ex. 2.4.11],

As a specific case of interest, the set of isomorphism classes of connected semisimple k-groups with the same absolute root datum as G is given by the pointed set

$$\mathrm{H}^1(k, \mathrm{Aut}_{G/k}).$$

In explicit terms, if H is such a k-group then composition of isomorphisms with automorphisms makes the Isom-scheme $\underline{\mathrm{Isom}}(G, H)$ (a Galois-twisted form of the *affine* $\mathrm{Aut}_{G/k}$) into an $\mathrm{Aut}_{G/k}$-torsor, and that is the torsor assigned to H.

Coming back to our situation of interest, here is how we use the assumption of isomorphic 4-tuples. The hypothesis of isomorphic absolute root data for G' and G yields a class $[G'] \in \mathrm{H}^1(k, \mathrm{Aut}_{G/k})$ whose triviality is equivalent to G' being k-isomorphic to G. That is, this puts our task into a cohomological framework. The hypothesis that the isomorphism of absolute root data can be chosen compatibly with Galois actions on the diagrams for G and G' (i.e., "the same τ") implies after some work that $[G']$ arises from $\mathrm{H}^1(k, G^{\mathrm{ad}})$ via the natural pushout map $\mathrm{H}^1(k, G^{\mathrm{ad}}) \to \mathrm{H}^1(k, \mathrm{Aut}_{G/k})$ (that is generally *not* injective). This is a first "reduction of the structure group". The existence of a k-isomorphism $M \simeq M'$ compatible with j and j' allows one to shrink the structure group further, and get that $[G']$ arises from $\mathrm{H}^1(k, Z_{Z_G(S)}/Z_G)$ (with $Z_{Z_G(S)}/Z_G \subset G^{\mathrm{ad}}$).

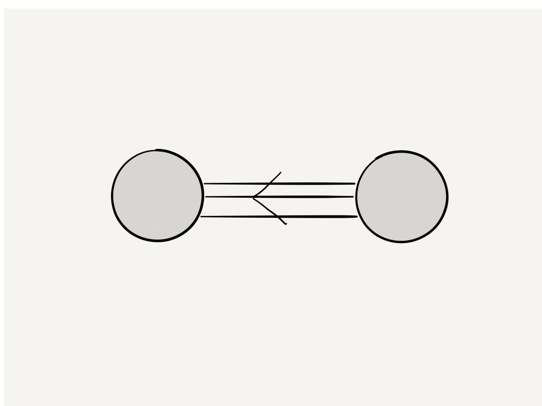
Now a miracle happens: as k-groups we have

$$Z_{Z_G(S)}/Z_G \simeq \prod_{a \in_k \Delta} \mathrm{R}_{k_a/k}(\mathbf{G}_m),$$

for finite separable extensions k_a/k. (Explicitly, k_a can be realized inside k_s as the finite extension of k corresponding to the open stabilizer inside $\mathrm{Gal}(k_s/k)$ for a point in the fiber over a for the quotient map $\Delta - \Delta_0 \twoheadrightarrow {}_k\Delta$.) The key fact underlying this miracle is that the surjective map $\Delta - \Delta_0 \to {}_k\Delta$ is the quotient by the $*$-action that transitively permutes simple absolute roots restricting to a given simple relative root, and that for a finite subextension K/k inside k_s/k, $\mathrm{R}_{K/k}(\mathbf{G}_m)$ corresponds to a geometric character group that is a permutation representation of $\mathrm{Gal}(k_s/k)$. A detailed explanation is given in the proof of [CP, Prop. 6.3.12], which is written in the wider context of pseudo-reductive groups and so simplifies a lot for reductive groups (e.g., the purely inseparable extensions which arise are all trivial and the notation there has the following meaning in our situation: $M = Z_G(S)$, C is a maximal k-torus T in M, $Z_{G,C} = T/Z_G$, and $\ker q = \ker \rho = Z_{Z_G(S)}/Z_G$).

Having reduced the structure group so much that $[G']$ arises from the degree-1 Galois cohomology of a direct product of "induced tori", we are done by Shapiro's Lemma and Hilbert 90! □

Example 12.4.8. Here is an application of the technique of reduction of the structure group: we will show that if G is a non-split form of the group G_2 over a field k then G must be anisotropic. In other words, the irreducible relative root system ${}_k\Phi$ must be empty rather than of rank 1. Here is the absolute diagram Δ:



Suppose to the contrary that ${}_k\Phi = \Phi(G,S)$ has rank 1 (so it is A_1 or BC_1, but we do not need that). There are no nontrivial diagram isomorphisms, so the $*$-action must be trivial. Hence, Δ_0 is one of the two vertices and the other restricts on S to give the unique vertex in ${}_k\Delta$. Since there are no diagram automorphisms and G_2 has trivial center, so G_2 is its own automorphism scheme, the k-group G is classified by an element $[G] \in \mathrm{H}^1(k, \mathrm{G}_2)$. We will prove that $[G]$ is trivial by performing reduction of the structure group (guided by such reduction steps in the proof of the Relative Isomorphism Theorem).

Let $c \in \Delta$ be the vertex not in Δ_0. (If we label the absolute diagram with a as the long root and b as the short root then necessarily $c = a$ because if $c = b$ then the highest absolute root $2a + 3b$ for G_2 would restrict to $3c|_S$, an absurdity since roots are never multipliable by 3 in a root system. But it does not matter for the argument that we can identify c.) The minimal parabolic k-subgroup of G corresponding to ${}_k\Delta$ therefore corresponds to the standard parabolic k-subgroup $P = {}_kP_c \subset \mathrm{G}_2$.

The Galois-twisting by points in $\mathrm{G}_2(k_s)$ that builds G from the split group G_2 therefore can be given by conjugation against points in $\mathrm{G}_2(k_s)$ that normalize P_{k_s}. By Chevalley's Theorem from the first day of the course, $P(k_s)$ is the normalizer of P_{k_s} inside $\mathrm{G}_2(k_s)$. Hence, the 1-cocycle for G can be arranged to be valued in $P_c(k_s)$, which is to say $[G]$ comes from $\mathrm{H}^1(k, P_c)$. That provides a first reduction of the structure group. A second reduction is achieved by:

Lemma 12.4.9. *Let P be a parabolic k-subgroup in a connected reductive k-group, with $U = \mathscr{R}_{u,k}(P)$. For any Levi k-subgroup $L \subset P$, the natural map $\mathrm{H}^1(k, L) \to \mathrm{H}^1(k, P)$ is bijective.*

Proof. Identifying L with P/U provides a natural map $f : \mathrm{H}^1(k, P) \to \mathrm{H}^1(k, L)$ that is surjective (using $\mathrm{H}^1(k, L) \to \mathrm{H}^1(k, P)$ arising from the inclusion of L into P). We will show that f is actually bijective, from which the desired result is then immediate.

The technique of twisting by torsors (cocycle-twisting in more classical language) implies that all elements of the fiber of f through a P-torsor ξ arise from $\mathrm{H}^1(k, U_\xi)$ for a certain Galois-twisted form U_ξ of U. But whether or not a smooth connected unipotent k-group is k-split is *unaffected* by separable extension of the ground field (this is part of Tits' structure theorem for unipotent groups; see the compatibility with separable extension at the end of Theorem 3.7 in the handout on the structure of solvable groups). Thus, since $(U_\xi)_{k_s} \simeq U_{k_s}$ is k_s-split, it follows that U_ξ is k-split. Hence, U_ξ has a composition series whose successive quotients are vector groups over k, so $\mathrm{H}^1(k, U_\xi) = 1$. This implies that the surjective f has singleton fibers and so is bijective as claimed. □

Now we know that $[G]$ arises from $\mathrm{H}^1(k, L_c)$ for a Levi k-subgroup L_c of P_c. Note that L_c is split (we are working inside the split G_2, after all), and it has root system with basis $\{c\}$ of size 1. Since L_c is split, the quotient torus $L_c/\mathscr{D}(L_c)$ is a split torus and so has vanishing H^1 by Hilbert 90. Thus, the natural map $\mathrm{H}^1(k, \mathscr{D}(L_c)) \to \mathrm{H}^1(k, L_c)$ is surjective, so we achieve yet another reduction of the structure group: $[G]$ arises from $\mathrm{H}^1(k, \mathscr{D}(L_c))$.

But what is $\mathscr{D}(L_c)$? This is a split connected semisimple group of rank 1, so it is either SL_2 or PGL_2. In fact, it must be SL_2. This could be seen by a direct argument with root groups, but here is a reason based on general principles applicable more widely: Levi subgroups of parabolic subgroups are always given by torus centralizers (these always arises as $Z_G(\lambda)$), and in a connected semisimple group that is simply connected the derived group of any torus centralizer is always simply connected by Corollary 9.5.11! Since G_2 is simply connected, that rules out PGL_2 and so $\mathscr{D}(L_c) = \mathrm{SL}_2$. But $\mathrm{H}^1(k, \mathrm{SL}_n) = 1$ for any $n \geq 2$ due to the exact sequence of pointed sets

$$\mathrm{GL}_n(k) \xrightarrow{\det} k^\times \xrightarrow{\delta} \mathrm{H}^1(k, \mathrm{SL}_n) \to \mathrm{H}^1(k, \mathrm{GL}_n)$$

along with the surjectivity of the determinant and the vanishing of $\mathrm{H}^1(k, \mathrm{GL}_n)$ (as all vector bundles on $\mathrm{Spec}(k)$ are free!). Thus, $\mathrm{H}^1(k, \mathrm{SL}_2) = 1$, so we are done.

The moral is that the classification of forms involves systematic reduction of the structure group.

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