AUTOMORPHIC FORMS ON SHIMURA VARIETIES

ZHIYUAN LI
LECTURES SCRIBED BY TONY FENG

Contents

1. Overview

Let $f\colon G \to V$ be a function from a topological group to a vector space, which is "nice." We'll want f to be invariant on the right by some compact subgroup $K \subset G$, and on the left by some discrete subgroup of G. In this course, $G = \mathrm{SL}_2$ or $\mathrm{SO}(2,n)$. The theory works more generally for any symplectic or orthogonal varieties. (In those cases, one gets arithmetic manifolds instead of Shimura varieties, but the theta lifting theory still works.)

1.1. **Theta lifting.**

We'll begin with the *theta lifting theory.* From a representation-theoretic perspective, this gives a correspondence between the representation theory of SL_2 and of $\mathrm{SO}(2,n)$.

- Howe's classical theta lifting theory gives a map

  $$\{\text{cusp forms of } \mathrm{SL}_2\} \xrightarrow{\theta-\text{integral}} \{\text{automorphic forms on } SO(2,n)\}.$$

  This works more generally whenever one has a "Howe pair."
- Borcherds' singular theta lifting gives a map

  $$\left\{\begin{matrix}\text{singular modular forms}\\ \text{on } \mathrm{SL}_2\end{matrix}\right\} \xrightarrow{\text{regularized } \theta\text{-integral}} \left\{\begin{matrix}\text{singular automorphic forms}\\ \text{on } \mathrm{SO}(2,n)\end{matrix}\right\}$$
  $$\xrightarrow{\exp} \left\{\begin{matrix}\text{automorphic forms}\\ \text{on } \mathrm{SO}(2,n)\end{matrix}\right\}$$

  In fact, the target has known singularities. One can then exponentiate to get automorphic forms on $\mathrm{SO}(2,n)$.

Under the Langlands decomposition, the representation theory of orthogonal groups should decompose into a cuspidal part and a residue part. Howe's theory gives the cuspidal part, and Borcherd's gives the residue part.

1.2. **Applications in Number Theory: Kudla's Program.**

Let $\mathbb{H}$ be the upper half plane. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup such that $X = \Gamma\backslash\mathbb{H}^*$ an elliptic modular curve. Recall that *Heegner points* on X are the image of quadratic imaginary points of $\mathbb{H}$.

Theorem 1.2.1 (Gross-Kohnen-Zagier)**.** *The generating series of the height of Heegner points is a modular form of weight* $3/2$.

More generally, let $D = G/K$ where $G = \mathrm{SO}(2,n)$ (when $n = 1$, we recover the previous case). Let $X = \Gamma\backslash D$. In favorable circumstances, this is a Shimura variety.

Theorem 1.2.2 (Borcherds, generalizing GKZ)**.** *The generating series of Heegner divisors is a (vector-valued) modular form of weight* $\frac{n+2}{2}$ *with coefficients in the Picard group of* X.

Heegner divisors are "special cycles" on Shimura varieties of orthogonal type. This relates to Kudla's program, which predicts that the generating series of special cycles of arithmetic manifolds is a Siegel modular form. Roughly speaking, in this setting special cycles are special linear combinations of sub-Shimura varieties of the same type.

- This is known if $G = \mathrm{SO}(V)$ or $U(n)$ by Kudla-Millson. However, their method does not give the level of the modular form. In Borcherds' case, it is known that the modular forms are of full level.
- In his thesis, Wei Zhang obtained modularity for $\mathrm{SO}(2,n)$, with coefficients in $CH^*(X)$.

1.3. **Applications to algebraic geometry.**

1.3.1. *Picard groups of moduli problems.*

Definition 1.3.1. A *primitively quasi-polarized K3 surface* is a pair (S,L) where S is a K3 surface and L is a quasi-polarization, i.e. a lind bundle with $L^2 > 0$ and $L \cdot C \geq 0$ for any curve C, and $c_1(L)$ is primitive (not a multiple of some other class). We say that $g := \frac{L^2}{2} + 1$ is the *genus* of (S,L).

Let $\mathscr{K}_g$ be the moduli space of primitively quasi-polarized $K3$ surfaces of genus g,

$$\mathscr{K}_g = \{(S,L) \mid L^2 = 2g-2\}.$$

Think of pairs (S,L) of genus g as being analogous to curves of genus g, so $\mathscr{K}_g$ is analogous to $\mathscr{M}_g$. (If L is effective, then the general element of $|L|$ is a genus g curve.)

Theorem 1.3.2. *We have*

$$\operatorname{rank}\operatorname{Pic}(\mathscr{K}_g) = \frac{31g+24}{24} - \frac{\alpha_g}{4} - \frac{\beta_g}{6} - \sum_{k=0}^{g-1} \frac{k^2}{4g-4} - \#\left\{k \mid \frac{k^2}{4g-4} \in \mathbb{Z}, 0 \leq k \leq g-1\right\}.$$

where

$$\alpha_g = \begin{cases} 0 & 2 \mid g \\ \left(\frac{2g-2}{2g-3}\right) & 2 \nmid g \end{cases} \quad \textit{and} \quad \beta_g = \begin{cases} \left(\frac{g-1}{4g-5}\right) - 1 & 3 \mid g-1, \\ \left(\frac{g-1}{4g-5}\right) + \left(\frac{g-1}{3}\right) & 3 \nmid g-1. \end{cases}$$

and $\left(\frac{a}{b}\right)$ *is the Jacobi symbol.*

This is an analogue of Mumford's "Picard groups of Moduli Problems" results in the setting of $\mathscr{K}_g$ rather than $\mathscr{M}_g$.

1.3.2. *Enumerative geometry.* There is a correspondence between

$$\left\{\begin{matrix}\text{intersection numbers}\\ \text{of special cycles}\end{matrix}\right\} = \left\{\begin{matrix}\text{reduced Gromov-Witten}\\ \text{invariants on hyper-Kählers}\end{matrix}\right\}.$$

This comes through an interpretation of the left hand side as coefficients of a modular form.

1.4. **Kodaira dimension of Shimura varieties.** Let X be a projective variety and K_X its canonical bundle.

Definition 1.4.1. We define the *Kodaira* dimension of X to be

$$\kappa(X) := \dim \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(X, nK_X)\right).$$

We say that X is of *general type* if $\kappa(X) = \dim X$.

The Kodaira dimension is a birational invariant.

Theorem 1.4.2 (Gritsenko-Hulek-Sankaran)**.** *Let $G = \mathrm{SO}(V)$ and $D = G/K$.*

(1) *$X := \Gamma\backslash D$ is of general type if there exists a character χ of finite order and a non-zero cusp form f with weight less than $\dim X$ vanishing along the branch divisor of $\pi: D \to X$.*
(2) *K_g is of general type when $g > 62$.*

The proof of the second assertion is our goal. We will deduce it from the first part by using Borcherds' theory to construct automorphic forms.

2. Modular Forms

We give a very short review of modular forms.

2.1. **Classical modular forms.**

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a congruence subgroup.

Definition 2.1.1. A holomorphic function $f:\mathbb{H}\to\mathbb{C}$ is called a *weakly holomorphic modular form* of weight k for Γ if:

$$f(A\tau)=(c\tau+d)^k f(\tau) \text{ for all } A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma.$$

It is called a *holomorphic modular form* if $f(\tau)$ is holomorphic at all cusps of Γ.

Example 2.1.2. If $\Gamma=\mathrm{SL}_2(\mathbb{Z})$, then the cusps of Γ are $\mathbb{Q}\cup\{\infty\}$.

Example 2.1.3. We give some examples of modular forms.

(1) *Eisenstein series*: for $q=e^{2\pi i\tau}$,

$$E_{2k}(q)=\frac{1}{2\zeta(2k)}\sum_{(m,n)\neq(0,0)}\frac{1}{(m+n\tau)^{2k}}.$$

This is a holomorphic modular form of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$, when $k\geq 2$. The intuition is that if we want to make a function that behaves well under the action, then we should just average. You can compute that this is equal to

$$1-\frac{4k}{B_{2k}}\sum_{d,n}n^{2k-1}q^{nd}.$$

(2) The *discriminant*:

$$\Delta(q)=q\prod_n(1-q^n)^{24}.$$

We define $\mathscr{M}_k(\Gamma)$ to be the space of holomorphic modular forms of weight k for Γ, and $\mathscr{M}^*(\Gamma)=\bigoplus_k\mathscr{M}_k(\Gamma)$. If $\Gamma=\mathrm{SL}_2(\mathbb{Z})$, then $M^*(\Gamma)$ is generated by E_4 and E_6.

If $\Gamma<\mathrm{SL}_2(\mathbb{Z})$, then we need the "Poincaré series" as well.

Definition 2.1.4. We define the *Poincaré series*

$$P_m^k(\tau)=\sum_{\gamma\in\Gamma_\infty\backslash\widetilde{\Gamma}}\frac{e(m\gamma(\tau)/b)}{(c\tau+d)^k}$$

where $e(\cdot)=\exp(2\pi i\cdot)$, $\widetilde{\Gamma}=\Gamma/\pm I$, and $\Gamma_\infty=\langle\pm\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\rangle$.

Theorem 2.1.5. *$P_m^k(\tau)\in\mathscr{M}^k(\Gamma)$. Moreover, $\mathscr{M}^k(\Gamma)$ is spanned by $P_m^k(\tau)$.*

♠♠♠ TONY: [is this true?]

2.2. **Vector-valued modular forms.**

Definition 2.2.1. We define the *metaplectic group*

$$\mathrm{Mp}_2(\mathbb{Z}) = \left\{ (A, \phi(\tau)) \mid A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \phi = \pm\sqrt{c\tau + d} \right\}.$$

There is an obvious 2 : 1 covering map $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$. So by the familiar fact about $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right).$$

(The subgroup generated by these elements obviously surjects onto $\mathrm{SL}_2(\mathbb{Z})$, but must have a kernel since $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ cannot have a section.)

Definition 2.2.2. Let (ρ, V) be a finite-dimensional representation of $\Gamma \subset \mathrm{Mp}_2(\mathbb{Z})$ (we will usually be interested in $\Gamma = \mathrm{Mp}_2(\mathbb{Z})$) such that ρ factors through a finite quotient.

For any $k \in \frac{1}{2}\mathbb{Z}$, a *weakly vector-valued modular form* $f(\tau)$ of weight k is a holomorphic function $f : \mathbb{H} \to V$ such that

$$f(A\tau) = \phi(\tau)^{2k} \rho(g)(f(\tau)) \quad g = (A, \phi(\tau)) \in \Gamma.$$

Example 2.2.3. Let M be an even lattice. Let $M^* = M^\vee / M$, the discriminant group of M. Then $|M^*| = \det(M)$, by which we mean the determinant of the "intersection matrix." Assume that M has signature (b^+, b^-).

Definition 2.2.4 (Weil representation). We define the *Weil representation* ρ_M on the group ring $\mathbb{C}[M^*]$ by

$$\rho_M(T) e_\gamma = e(\langle \gamma, \gamma \rangle) e_\gamma$$
$$\rho_M(S) e_\gamma = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|M^*|}} \sum_{\delta \in M^*} e(-2\langle \gamma, \delta \rangle) e_\delta.$$

Basic properties.

(1) ρ_M factors through a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.
(2) $\rho_M = \rho_{M_1} \otimes \rho_{M_2}$ if $M = M_1 \oplus M_2$.
(3) If M is unimodular then ρ_M is trivial (because $M^* = 0$ and $b^- - b^+$ will be divisible by 8).

Exercise 2.2.5. Using this, prove Milgram's formula

$$\sum_{\gamma \in M^*} e(\gamma^2/2) = \sqrt{|M^*|}\, e\left(\frac{\mathrm{sign}(M)}{8} \right)$$

where $\mathrm{sign}(M) = b^- - b^+$. [Hint: compare the action of $(ST)^3$ and S^2.]

We define $\mathcal{M}_k(\rho)$ to be the space of vector-valued modular forms of weight k and type ρ (of full level). Then the generating series of special cycles of codimension 1 is an element in $M_{\frac{b^-+b^+}{2}}(\rho_M^\vee)$, where M is the even lattice of signature $(b^-, b^+$ and $b^-=2$. We can define the associated Shimura variety in this case, because it's of type $(2,b^+)$.

2.3. **Poincaré series.** For $k\in\frac{1}{2}\mathbb{Z}$ and $f\colon\mathbb{H}\to\mathbb{C}[M^*]$, we define the Peterson *slash operator* $|_k^*$ given by

$$(f|_k^*(g))(\tau)=\phi(\tau)^{-2k}\rho_M(g)^{-1}f(A\tau)$$

where $g=(A,\phi(\tau))$.

For all $\beta\in M^*$, $n\in\mathbb{Z}-\langle\beta,\beta\rangle$ we define the ($\mathbb{C}[M^*]$-valued) *Poincaré series*

$$P_{n,\beta}(\tau)=\frac{1}{2}-\sum_{g\in\Gamma_\infty\backslash\mathrm{Mp}_2(\mathbb{Z})}e_\beta(n\tau)|_k^*(g)\in\mathcal{M}_k(\rho_M).$$

where $\Gamma_\infty=\langle T\rangle$. It is a fact that $P_{n,\beta}\in\mathcal{M}_k(\rho_M)$ and in fact the collection of Poincaré series span $\mathcal{M}_k(\rho_M)$.

Why is this interesting? To $P_{n,\beta}(\tau)$ one can associate a "Noether-Lefschetz divisor" $NL_{n,\beta}\in K_g$. This induces an isomorphism of $M_k(\rho_M)$ with $\mathrm{Pic}(K_g)$ modulo the Hodge line bundle.

2.4. **Properties of modular forms.**

Proposition 2.4.1. *Suppose $f(\tau)=\sum c_nq^n$ is a cusp form (i.e. $c_0=0$) of weight k for some Γ. Then*

$$|c_n|=O(n^{k/2}).$$

Proof. Let $g(\tau)=|f(\tau)|\cdot|\mathrm{Im}(\tau)|^{k/2}$. It is easy to check that this is *invariant* under Γ. So it extends to a continuous function on the compact Riemann surface $\Gamma\backslash\mathbb{H}^*$, and is therefore bounded. That implies $|f(\tau)|\le C(\mathrm{Im}\,\tau)^{-k/2}$ for some C, hence

$$\begin{aligned}|c_n|&=\frac{1}{2h}\int_0^{2h}f(x+iy)e^{-n\pi i(x+iy)/h}\,dx\\&\le Cy^{-k/2}e^{n\pi y/h}.\end{aligned}$$

Taking $y=1/n$, we get the result. □

Remark 2.4.2. You can use this to prove any K3 surface has an infinite family of elliptic curves! More precisely, if$\mathscr{X}\to B$ is a family of K3 surfaces, then $\#\{\mathscr{X}_b \text{ is elliptic}\}=\infty$ if $\dim B\ge 1$. The idea is that the generating series $\sum$(NL-divisor)q^n is modular.

Proposition 2.4.3. *$\mathscr{S}_k(\Gamma)$ admits a Hermitian form.*

Proof. We define the *Petersson inner product* of $f, g \in \mathscr{S}_k(\Gamma)$ to be

$$\langle f, g\rangle = \int\int_D f(z)\overline{g(z)} y^{k-1} dx dy$$

where D is a fundamental domain for Γ. It is easily checked that this defines a positive-definite Hermitian form on $\mathscr{S}_k(\Gamma)$. □

2.5. **Dimension of spaces of modular forms.**

2.6. **Hecke operators.** There is an importantly family of operators $\{T_m\}_{m\geq 1}$ on spaces of modular forms, called the *Hecke operators*. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ for simplicity. The Hecke operators come from considering the double coset

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma.$$

Let $\Gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$. Then there is a bijection

$$\Gamma\backslash\Gamma \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma \longleftrightarrow \Gamma_m\backslash\Gamma.$$

Now Γ_m has finite index in Γ, so we may write

$$\Gamma\backslash\Gamma \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma = \bigcup_i \Gamma\beta_i \text{ for some } \beta_i.$$

Recall the *Petersson slash operator*

$$f|_k(\gamma) := (\det\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau).$$

Definition 2.6.1. For $f \in \mathscr{M}_k(\Gamma)$, we define the mth *Hecke operator* T_m on $\mathscr{M}_k$ by

$$T_m(f) = \sum_i f|_k(\beta_i) \in \mathscr{M}_k(\Gamma).$$

It easy to check that T_m preserves $\mathscr{S}_k(\Gamma)$. We denote by $\mathrm{Tr}(T_m)$ the corresponding action.

Theorem 2.6.2 (Eichler-Selberg trace formula)**.** *We have*

$$\mathrm{Tr}(T_m) = -\frac{1}{2}\sum_{t=-\infty}^{\infty} P_k(t,m)H(4m - t^2) - \frac{1}{2}\sum_{dd'=m} \min(d,d')^{k-1}$$

where $H(n)$ is a weighted class number for positive definite binary quadratic forms of discriminant $-n$ (hence $H(0) = -1/12$ and $H(n) = 0$ if $n < 0$), and $P_k(t,m) = \frac{\rho^{k-1} - \overline{\rho}^{k-1}}{\rho - \overline{\rho}}$ for ρ satisfying $|\rho| = m$ and $\mathrm{Re}\,\rho = t/2$.

Example 2.6.3. Since T_1 is the identity operator, $\dim \mathscr{S}_k = \mathrm{Tr}(T_1)$. By the Eichler-Selberg trace formula, that

$$\begin{aligned}\dim \mathscr{S}_k &= -\frac{1}{4}P_k(0,1) - \frac{1}{3}P_k(1,1) + \frac{1}{12}P_k(2,1) - \frac{1}{2} \\ &= -\frac{1}{2} + \frac{1}{12}(k-1) - \frac{\sin(\pi(k-1)/3)}{3\sin(\pi/3)} - \frac{1}{4}\sin(\pi(k-1)/2).\end{aligned}$$

3. Borcherds' singular theta lift

3.1. **Overview.** Let M be an even lattice of signature $(2,n)$ and $M^* = |M^\vee/M|$. (The theory works for more general signature, but the results are not as nice.) Let $(\cdot,\cdot)$ denote the interserction form on M.

Definition 3.1.1. We define D_M to be the Grassmannian of positive definite 2-planes in $M\otimes\mathbb{R}$.

The goal is to give a correspondence

$$\{\text{singular modular forms}\} \xrightarrow{\theta\text{-lift}} \{\text{automorphic functions on } D_M\}.$$

The great thing about this correspondence is that it gives some very explicit formulas: you can write down an infinite product expression for the automorphic forms in terms of the Fourier coefficients of the singular modular forms,

$$\sum_n c_n q^n \mapsto \prod_{\lambda\in M^\vee} (1-\exp\langle\lambda, v\rangle))^{c_n}.$$

Here $n = \frac{(\lambda,\lambda)}{2}$. This makes the zeros of these automorphic forms easy to recognize.

Example 3.1.2. If M has signature $(2,1)$ then an automorphic form on D_M is $\eta(q) = q^{1/24}\prod_n(1-q^{n^2})$. This is a θ-lift from the theta series $\theta(q) = 1/2+\sum q^{n^2}$.

If M has signature $(2,3)$, the automorphic forms are Siegel modular forms of genus 2.

$$\sum_{m,n}(-1)^{m+n}p^{m^2}q^{n^2}r^{mn} = \sum_{a+b+c>0}\left(\frac{1-p^aq^cr^b}{1+p^aq^cr^b}\right)^{C_{(ac-b^2}}$$

where C_k is the coefficients of $\frac{1}{\sum(-1)^n q^{n^2}} = 1+2q+4q^2+8q^3+\dots$.

There are three steps.

1. (1) Construct Siegel theta functions, and "regularized" theta-integrals.
2. (2) Analyze singularities of θ-lifts and compute their Fourier coefficients.
3. (3) Proof of the infinite product formula.

3.2. Siegel theta functions.

Definition 3.2.1. Let (ρ, V) be a representation of $\mathrm{Mp}_2(\mathbb{Z})$. We say that a real-analytic function $f\colon\mathbb{H}\to V$ is a (non-holomorphic) *modular function* of weight (m_1,m_2) if

$$f(A\tau) = (c\tau+d)^{m_1}(c\overline{\tau}+d)^{m_2}\rho(A)f(\tau).$$

3.2.1. *Fourier transform.* Let $(V,(\cdot,\cdot))$ be a real quadratic space of signature (b^+,b^-).

If $f\colon V\to\mathbb{R}$ is a function, we define

$$\mathscr{F}f(y):=\widehat{f}(y):=\int_V f(x)e((x,y))\,dx.$$

$\mathscr{F}(e^{-\pi x^2})=e^{-\pi x^2}$ ♠♠♠ TONY: [for mixed signature? constant factors?] This implies that

$$\mathscr{F}(e(\tau x_+^2/2+\overline{\tau}x_-^2/2))=(\tau/i)^{-b^+/2}(i\overline{\tau})^{-b^-/2}e(-x_+^2/2\tau-x_-^2/2\tau).$$

3.2.2. *Poisson summation formula.* If $V=M\otimes\mathbb{R}$ where M is a lattice, then

$$\sqrt{|M^*|}\sum_{\lambda\in M}f(\lambda)=\sum_{\delta\in M^\vee}\widehat{f}(\delta).$$

Definition 3.2.2. For $\gamma\in M^*$, we define the *Siegel theta function*

$$\theta_{M+\gamma}(\tau,v)=\sum_{\lambda\in M+\gamma}e(\tau\lambda_v^2/2+\overline{\tau}\lambda_{v^\perp}^2/2)$$

where $\tau\in\mathbb{H}$, $v\in D_M$.

Notation: $\lambda_+=\lambda_v$ is the orthogonal projection to v, and $\lambda_-=\lambda_{v^\perp}$ is the orthogonal projection to $v^\perp$.

Definition 3.2.3. Let $\{e_\gamma]$ be the standard basis of $\mathbb{C}[M^*]$. Define

$$\Theta_M(\tau,v)=\sum_{\gamma\in M^*}\theta_{M+\gamma}(\tau,v)\cdot e_\gamma.$$

Theorem 3.2.4. *We have*

$$\Theta_M(A\tau,v)=(c\tau+d)(c\overline{\tau}+d)^{n/2}\rho_M(g)\Theta_M(\tau,v)$$

where $g=(A,\sqrt{c\tau+d})\in\mathrm{Mp}_2(\mathbb{Z})$. *Therefore,* Θ_M *is a modular function of weight* $(1,n/2)$.

Proof. We just have to check this for $A=T$ and $A=S$. For $A=T$, you can easily check this by hand.

Let's check the case $A=S$. Then the left hand side is

$$\sum_{\gamma\in M^*}e_\gamma\theta_{M+\gamma}(-\tau^{-1},v)=\sum_{\gamma\in M^*}e_\gamma\sum_{\lambda\in M+v}e(-\lambda_+^2/2\tau-\lambda_-^2/2\overline{\tau})$$

while the right hand side is

$$\tau\cdot\overline{\tau}^{n/2}\frac{(\sqrt{i})^{n-2}}{\sqrt{|M^*|}}\sum_{\gamma\in M^*}e_\gamma\sum_{\delta\in M^*}e(-(\delta,\gamma))\theta_{M+\delta}.$$

Comparing, it suffices to show that

$$\sqrt{|M^*|}\theta_{M+\gamma}(-1/\tau,v)=-\tau\cdot\overline{\tau}^{n/2}(\sqrt{i})^n\sum_{\delta\in M^*}e(-(\gamma,\delta))\theta_{M+\delta}(\tau,v).$$

We're just going to check the case $\gamma=0$.

Write $f(\lambda)=-\tau^{-1}(\sqrt{\overline{\tau}/i})^{-n}e(-\lambda_+^2/2\tau-\lambda_-^2/2\overline{\tau})$. Then by definition

$$\begin{aligned}\sqrt{|M^*|}\mathcal{O}_M(-1/\tau,v)&=-\tau\overline{\tau}^{n/2}(\sqrt{i})^n\sqrt{|M|^*}\sum_{\lambda\in M}f(\lambda)\\&=-\tau\overline{\tau}^{n/2}(\sqrt{i})^n\sum_{\delta\in M^\vee}\widehat{f}(\delta)\\&=-\tau\overline{\tau}^{n/2}(\sqrt{i})^n\sum_{\delta\in M^*}\sum_{\lambda\in M}\widehat{f}(\lambda+\delta)\\&=-\tau\overline{\tau}^{n/2}(\sqrt{i})^n\sum_{\delta\in M^*}\theta_{M+\delta}.\end{aligned}$$

□

3.3. **Borcherds' theta lift.** We will use Borcherds' theta functions to define the theta lift, which is a map

$$\left\{f\in\mathcal{M}_{\rho_M}(1-\frac{m}{2})\right\}\to\left\{\Theta_f:\text{singular aut. forms on }D_m\right\}$$

defined as follows. Given $f\in\mathcal{M}_{1-n/2}(\mathrm{SL}_2)$, we define

$$\Phi_f(v)=\int_D f(\tau)\cdot\overline{\Theta_M(\tau,v)}\,\frac{dx\,dy}{y}$$

where D is a fundamental domain for $\mathrm{SL}_2\backslash\mathbb{H}$.

Remark 3.3.1. Conjugation on $\mathbb{C}[M^*]$ is $\overline{e_\gamma}=e_{-\gamma}$. The product of vectors is given by $(e_\alpha,e_{-\beta})=\delta_{\alpha\beta}$.

The final goal is to show that the generating series of special cycles is an element in $\mathcal{M}_{\rho_M^\vee}(1+\frac{m}{2})$, which by Serre duality related to the left hand side because weight two cusp forms are the canonical bundle.

Informal discussion. Our problem is that the integral diverges if one of the two integrand forms is not cuspidal. Using the definition of $\Theta_M(\tau,v)$, the expansion of Φ_f looks approximately like a sum of things of the form

$$\int_{|x|\le 1/2,y\ge C}\exp(2\pi ikx+2\pi|k|y-Ly)\,dx\,dy$$

If $2\pi|k|-L<0$, then the integral converges. But if this quantity is nonnegative, then it diverges.

If $k \neq 0$, then we can handle this by (integrating over x first, and then y) defining it to be 0. If $k = 0$ and $L > 0$, then everything is fine, as discussed. The problem is if $k = 0, L = 0$. Here we use the *Harvey-Moore* method to define it as follows.

Definition 3.3.2. Let $F_w = \{\tau \in \mathbb{H} \colon |\tau| \geq 1, |x| \leq 1/2, y \leq w\}$. Suppose

$$\lim_{w \to \infty} \int_{F_w} F(\tau) y^{-s} \frac{dx\,dy}{y}$$

exists for $\operatorname{Re} s \gg 0$, (in terms of earlier notation, $F = f\overline{\Theta}$) and can be continued to a meromorphic function $G(s)$ for all $s \in \mathbb{C}$. Then

$$\Phi_f(v) := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} F(\tau) \frac{dx\,dy}{y}$$

is the constant term of $G(s)$ at $s = 0$.

So we have to study the singularities of Φ_f.

Definition 3.3.3. A function f has a *singularity of type* g if $f - g$ can be redefined on a set of codimension ≥ 1 so that $f - g$ is a real analytic near that point.

Theorem 3.3.4. *Near a point $v_0 \in D_M$, $\Phi_f(v)$ has a singularity of type*

$$\sum_{\lambda \in M^\vee \cap v_0^\perp} -c_\lambda(\lambda^2/2) \log(\lambda_+^2).$$

Here $f(\tau) = \sum_{\gamma \in M^*} c_\gamma(n) q^n$, λ_+ is the projection of λ to v and λ_- is the projection of λ to $v^\perp$.

The singular locus is the locus where $\lambda_+ = 0$, which is a locally finite set of codimension 2 sub Grassmannian of D_M of the form $\lambda^\perp$, i.e.

$$\operatorname{Sing}(\Phi_f) = \bigcup_{\lambda \in M^\vee \cap v_0^\perp} \lambda^\perp$$

where $\lambda^\perp = \{(w, \lambda) = 0 \mid w \in D_M\}$. (Codimension is in the real sense.)

Proof. We have

$$\Phi_f(v) = \int_{y>0} \int_{|x| \leq 1/2, x^2+y^2 \geq 1} \overline{\Theta}(\tau, v) f(\tau) \frac{dx\,dy}{y}$$

Then substitute $\overline{\theta}_{M+\beta} = \sum_{\lambda\in M+\beta} q^{-\lambda^2/2}|q|^{\lambda_+^2}$ (where as usual $q = e(\tau)$) above, and the Fourier expansion $f(\tau) = \sum_{\gamma\in M^*} c_\gamma(n)q^n$. So we get (after some work)

$$\Phi_f \approx \sum_{\gamma\in M^*}\sum_{\lambda\in M+\gamma} c_\gamma(n)\int_{y\geq 1,|x|\leq 1/2} q^{n-\lambda^2/2}|q|^{\lambda_+^2}\frac{dx\,dy}{y}.$$

First we carry out the x-integral. It's 0 unless $n = \lambda^2/2$, so we get

$$\int_{y\geq 1} c_0(0)\frac{dy}{y} + \sum_{\lambda\in M^\vee,\lambda\neq 0} c_\lambda(\lambda^2/2)\int_{y\geq 1}\exp(-2\pi y\lambda^2)\,dy.$$

We can throw away the first term, because it doesn't depend on v. So we are interested in

$$\sum_{\lambda\in M^\vee,\lambda\neq 0} c_\lambda(\lambda^2/2)\int_{y\geq 1}\exp(-2\pi y\lambda_+^2)\,dy.$$

The assertion then follows from the following result.

Lemma 3.3.5. *The function*

$$f(r) = \int_1^\infty e^{-r^2 y}y^{s-1}\,dy = |r|^{-2s}\Gamma(s,r^2)\quad s>0$$

has a singularity at $r = 0$ of type $|r|^{-2s}\Gamma(s)$, and type $(-1)^{s+1}r^{-2s}\log(r^2)/(-s)$ if $s\leq 0$.

Apply this with $r = \lambda_+$ (we are in the second case). The difficult with working more general stuff is that you get polynomial singularity instead of log singularity.

□

3.4. **Exponentiation.** If M is an even lattice of signature (b^+,b^-), then the definition of Θ_M remains valid: for $\tau\in\mathbb{H}$ and $v\in\mathrm{Gr}(M)$, the set of b^+-dimensional positive-definite subspaces of $M\otimes\mathbb{R}$, we define

$$\Theta_M(\tau,v) = \sum_{\lambda\in M}\exp(\tau(\lambda_v)^2/2+\overline{\tau}(\lambda_{v^\perp}^2)/2)$$

if $M^* = M^\vee/M = \{0\}$. This is modular of weight $(\frac{b_+}{2},\frac{b_-}{2})$. So

$$\Theta_M(A\tau,v) = (c\tau+d)^{b_+/2}(c\overline{\tau}+d)^{b_-/2}\Theta_M(\tau,v)$$

(maybe with a character too.)

If $f\in\mathcal{M}_{(b^+-b^-)/2}(\rho_M)$, the set of modular forms of weight $\frac{b^+-b^-}{2}$ and type ρ_M (the Weil representation), we can define

$$\Phi_f = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(\tau)\overline{\Theta_M(\tau,v)}\frac{dx\,dy}{y^{b_+/2}}.$$

This is an "automorphic function" on D_M minues some hyperplanes.

Remark 3.4.1. When $b^+ = 2$, we define Heegner divisors as follows. For $n \in \mathbb{Q}, \beta \in M^*$, we define

$$\mathscr{H}_{n,\beta} = \coprod_{\substack{\lambda \equiv \beta \pmod M \\ (\lambda,\lambda)/2 = n}} (\lambda^\perp)$$

where $\lambda^\perp = \{v \in D_M \mid (v,\lambda) = 0\}$. This has (real) codimension 2, and in general the codimension is b^+.

Fact: if one takes $f(\tau) = P_{n,\beta}$ (the Poincaré series defined earlier),

$$\Phi_{n,\beta} := \Phi_{P_{n,\beta}}$$

is real-analytic on $D_M - \mathscr{H}_{n,\beta}$. Since these span the space of modular forms, their singularities determine the singularities of everything.

The main result on singularities of Φ_f, which we discussed last time, is the following: if $f = \sum_{\gamma \in M^*} \sum_{n \in \mathbb{Z} - \gamma^2/2} c_\gamma(n) q^n e_\gamma$ has a singularity of type $\sum_{\lambda \in M^\vee} c_\gamma(\gamma^2/2) \log(2\pi\lambda_v^2)$ for some λ if

Actually, Φ_f is of type $-\sum c_\gamma(\gamma^2/2)(2\pi\lambda_v^2)^{1-b^+/2}$ times some constant.

Remark 3.4.2. If $b^+ = 1$, then Φ_f is a polynomial (actually, a polyomial on each "Weyl chamber," and there is a "wall-crossing formula" to get between these.

Ideas. (for $b^+ = 2$) $\Phi_f = -\log(\Psi_f) +$ (analytic stuff) where Ψ_f is a meromorphic automorphic function on D_M. This implies that Ψ_f has an infinite product expression. The singularities of Φ_f, which are the Heegner divisors, are the zeros/poles of Ψ_f.

Why do we fail to get a theorem when $b^+ > 2$? Using this machinery you always get Φ_f, whose singularities have codimension b^+, which can't come from a single function.

Theorem 3.4.3. *Let M be a lattice of signature $(2, m)$. Let $f \in \mathscr{M}_{1-m/2}(\rho_M)$ have Fourier expansion*

$$f(\tau) = \sum_{\gamma \in M^*} \sum_n c_\gamma(n) q^n e_\gamma.$$

Assume that $c_\gamma(n) \in \mathbb{Z}$ when $n \leq 0$. Then there exists a meromorphic automorphic function Ψ_f on D_M, satisfying:

1. *The zeros or poles of Ψ_f lie on $\lambda^\perp$ for $\lambda \in M, \lambda^2 < 0$, and has order $\sum_{x>0, x\lambda \in M^\vee} e_{x\lambda}(x^2\lambda^2/2)$.*
2. $\log \Psi_f = -\frac{\Phi_f}{4} - \frac{c_0(0)}{2}(\log(y_v) + const)$ *(where y_v is the "imaginary part" of $v = x + iy$)*
3. $\Psi_f = e((\rho, v)) \prod_{\substack{\gamma \in M \\ (\gamma, v_0) > 0}} (1 - e((\gamma, v)))^{c_\gamma((\gamma,\gamma)/2)}$.

This last part describes the zeros/poles and their multiplicities.

The idea is that

$$\Phi_f = \widetilde{\Phi_K} + (\text{Integral part})$$

where $K \subset M$ is a sublattice of signature $(1, m-1)$. Think of K as coming from a parabolic subgroup. Why is this true? Θ_M can be expressed in terms of Θ_K (they are on different spaces, but their Fourier coefficients are related) and $\widetilde{\Phi_K}$ is some kind of pullback of Φ_K.

Example 3.4.4. If $z^2 = 0$, then $K = (M \cap z^\perp)/z$ so any $\lambda \in M$ can be written as $\lambda = \lambda_k + z + z'$.

3.5. **Coefficients.** Recall that given $f = \sum_{\gamma \in M^*} \sum_{n \in \mathbb{Z} - \frac{\gamma^2}{2}} c_\gamma(n) q^n e_\gamma \in \mathscr{M}_{1-\frac{m}{2}}(\rho_M)$ (ρ_M the Weil representation), we defined Φ_f as the constant term of

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(\tau) \cdot \overline{\Theta_M(\tau)} \frac{dx\,dy}{y^{1+s}}$$

at $s = 0$. This is a meromorphic function in s, so it has a Laurent expansion at 0. It is called a *regularized integral.*

Φ_f is real analytic on D_M − sub-Grassmannians. The goal is to show that

$$\Phi_f = -4\log(\Psi_f) + (\text{analytic functions}).$$

Borcherds showed this by giving a very detailed computation of the Fourier coefficients of Φ_f as integrals. We'll give as simplified version of his result/computation.

Theorem 3.5.1. *Let*

- *z be a primitive norm 0 vector, i.e. $z^2 = 0$ and $z' \in M^\vee$, i.e. $(z, z') = 1$. (So the intersection matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$).*
- *$z_\pm$ be the projection of z onto $v, v^\perp$.*
- *$w^\pm$ be the orthogonal complement of $z_\pm$ in v and $v^\perp$.*
- *$K = (M^\vee \cap z^\perp)/z$, a lattice of signature $(1, m-1)$.*
- *$\mu = \ldots$*

Then $\Phi_f(v)$ is the constant term of the following integral

$$\frac{1}{\sqrt{2|z_+|}}\phi_K + \frac{\sqrt{2}}{|z_+|} \cdot \sum_{n>0} \sum_{\lambda \in K^\vee} e((n\lambda, \mu)) \cdot \sum_{\substack{\sigma \in M^* \\ \sigma|_L = \lambda}} e(n(\sigma, z')) \int_{y>0} c_\sigma(\lambda^2/2) \exp\left(-\frac{\pi n^2}{2y z_+^2} - 2\pi y \lambda_{w^+}^2\right) y^{-s-3/2}\, dy.$$

What does this mean? ϕ_K is the theta lift of f to K. This is piecewise-linear, since for signature $(1, ?)$ you get locally polynomial plus some wall-crossing formula. So this part is analytic, which is fine.

The integral is some coefficient which can be expressed in terms of Γ functions and Bessel functions.

Assume that M, K are unimodular. Then the nasty eqquation simplifies. In this case, $f(\tau) = \sum_k C(k)q^k$, and we get that it equals

$$\frac{1}{\sqrt{2|z_+|}}\phi_K + \frac{\sqrt{2}}{|z_+|}\sum_{n>0} \cdot \sum_{\lambda \in K^\vee} e((n\lambda, \mu)) \int_{y>0} c_\sigma(\lambda^* 2/2) \exp(-\frac{\pi n^2}{2yz_+^2} - 2\pi y \lambda_{w^+}^2) y^{-s-3/2}\, dy.$$

Sketch. Write Θ_M in terms of Θ_K. This requires another theorem, which is itself quite involved.

Theorem 3.5.2.

$$\Theta_M = \frac{1}{\sqrt{2y}|z_+|}\sum_{\lambda \in M/z}\sum_{n\in\mathbb{Z}} e(\tau\lambda_{w^+}^2/2 + \overline{\tau}\lambda_{w-}^2/2 - n(\lambda,(z_+ - z_-))/2z_+^2 - \frac{|(\lambda, z)\tau + n|^2}{4iyz_+^2}).$$

Proof. The idea is to write an element of M as a sum of elements in K, z, z' and apply standard lattice theorems. ☐

Then, insert this formula into the definition of Φ_f, and you get that $\sqrt{2}|z_+|\phi_f(v)$ is the constant term of

$$\phi_K + \int_{\mathrm{SL}_2\backslash\mathbb{H}} \frac{1}{\sqrt{y}} \sum_{(c,d)\neq(0,0)} e\left(\frac{-|c\tau + d|^2}{4iyz_+^2}\right) \overline{\Theta}_K(\tau, \mu d, -c\mu) f(\tau)\frac{dx\,dy}{y^{1+s}}.$$

Let's ignore the first term because it's nice. It's the integral that we want to study. We divide the sum into multiples (nc, nd) over (c, d) primitive:

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{(c,d)\text{ coprime}} \sum_{n>0} e\left(\frac{-|c\tau + d|^2}{4iyz_+^2}\right) \overline{\Theta}_K(\tau, n\mu d, -nc\mu) f(\tau)\frac{dx\,dy}{y^{1+s}}.$$

Why is that useful anyway? The idea is to integrate first over x, then y. Now the point is that $f(\tau)$ has a modular transformation property, $f(\tau) = (c\tau + d)^{1-\frac{m}{2}} f(A\tau)$, so we can rewrite the above as

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \sum_{n>0} \sum_{A \in \mathrm{SL}_2(\mathbb{Z})/\mathbb{Z}} \overline{\Theta}_K(A\tau, n\mu, 0) f(A\tau) \operatorname{Im}(A\tau)^{-1/2} \exp(-\frac{-\pi n^2}{2\operatorname{Im}(A\tau) z_+^2}).$$

(Here the $\mathbb{Z}$ action on $\mathrm{SL}_2(\mathbb{Z})$ is by translation). Now the point is to interchange the integral over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and summation over $\mathrm{SL}_2(\mathbb{Z})/\mathbb{Z}$, so we get

$$\int_{\mathbb{Z}\backslash\mathbb{H}} \overline{\Theta}_K(\tau) f(\tau) \exp\left(-\frac{\pi n^2}{2yz_+^2}\right)\frac{dx\,dy}{y^{1+s}}.$$

Now you just plug in the Fourier coefficients of $f(\tau) = \sum c(k)q^k$, and write the integral as $\int_y \int_x$. The point was that the fundamental domain has been changed to something nice. ☐

Lemma 3.5.3. *The integral*

$$\int_{y>0} \exp\left(-\frac{\pi n^2}{2yz_+^2} - 2\pi y \lambda_{w^+}^2\right) c(\lambda^2/2) y^{-z/2-s}\, dy$$

at $s = 0$ is equal to:

$$c(\lambda^2/2)\frac{|z_+|}{n}\exp(-2\pi n|\lambda_{w^+}|/|z_+|) \text{ if } \lambda_{w^+} \neq 0,$$

and

$$c(\lambda^2/2)(\frac{\pi n^2}{2|z_+|^2})^{-1/2}\Gamma(1/2) \text{ if } \lambda_{w^+} = 0.$$

3.6. **Borcherds infinite products.** For $v \in D_M$, let X_M, Y_M be an orthogonal basis for v. Since M has signature $(2, m)$, the a priori real Grassmannian D_M has a complex structure, and we may set $Z_M = X_M + iY_M$.

Another perspective on the complex structure is that $D_M = \{v \in \mathbb{P}(M \otimes \mathbb{C}) \mid (v, \overline{v}) = 0, (v, v) > 0\}$. The complex structure on D_M comes from that of $\mathbb{P}(M \otimes \mathbb{C})$.

D_M parametrizes Hodge structures of type $(1, m, 1)$.

Let K, z, z' be as before: $K = (M \cap z^\perp)/z$, i.e. $K := M-$ hyperbolic. The relations are $z^2 = 0, (z, z') = 1$.

Parametrize $(\lambda, k, \ell) := \lambda + kz' + \ell z \in M$ where $\lambda \in k$.

Definition 3.6.1. (Weyl vector) Let $f \in \mathscr{M}_{1-\frac{m}{2}}(\rho_M)$, $\phi_k(f)$ be the lift to functions on D_K. This is a piecewise-linear function, linear on the Weyl chambers.

Let W be a Weyl chamber of $\phi_K(f)$, i.e. $\phi_K(f)$ is linear on W. There is a unique vector $\rho(W)$ with the property that

$$|w|\phi_K(f)|_W(\frac{W}{|W|}) = 8\sqrt{2}(\rho(W), W).$$

Weyl chambers are *defined* by this linearity.

For different Weyl chambers, you get a different vector $\rho(W)$.

Recall that in the theory of automorphic forms, one usually considers $\Gamma \backslash D_M$ for $\Gamma \subset \mathrm{Aut}(M \otimes \mathbb{Q})$.

Theorem 3.6.2. *Let $f \in \mathscr{M}_{1-\frac{m}{2}}(\rho_M)$. Then there exist a meromorphic function ψ_M on D_M satisfying the following properties:*

(1) ψ_M is automorphic of weight $c_0(0)/2$ for $\mathrm{Aut}(M)$.
(2) $\log|\psi(Z_M)| = -\frac{\phi_M(f)}{4} - \frac{c_0(0)}{2}(\log|Y_M| + ? + \log(\sqrt{2\pi})$,
(3) For each Weyl chamber of $\phi_K(f)$, ψ_M has an infinite product expression. When M is unimodular,

$$\psi_M = e((\rho(W), Z_M)) \prod_{\lambda \in K, (\lambda, W) > 0} (1 - (e(\lambda, Z_M))^{c(\lambda^2/2)}$$

if $f = \sum c(n)q^n$.

Remark 3.6.3. The original theta lift was not defined on all of D_M (it was defined away from complex codimension 1 Heegner divisors), but ψ_M is.

Proof Sketch. We assume the simpler case where M, K are unimodular, which implies that there exists a norm 0 vector, which is not necessarily true for some even latttices, so we can choose $z^2 = (z')^2 = 0$. We will perform the following steps.

(1) $\phi_M(f)$ is the constant term of the following integral at $s = 0$:

$$\frac{1}{\sqrt{2z_+}}\phi_K(f)+\frac{\sqrt{2}}{|z_+|}\sum\sum e(\dots)s\int_{y>0} c(\frac{\lambda^2}{2})\exp(-\frac{\pi n^2}{2yz_+^2}-2\pi y\lambda_+^2)y^{-s-3/2}\,dy.$$

(2) Similarly use Γ functions: the integral equals

$$8\pi(Y_M,\rho(W))+2\sum_{n>0}\left(\frac{\pi n^2}{2z_+^2}\right)^{-s-1/2}c(0)\Gamma(s+\frac{1}{2})+2\sum_{\lambda\neq 0\in K,n>0}e(\dots)\frac{c(\lambda^2/2}{n}\exp(-2\pi n|(\lambda,Y_M)|).$$

Using a Taylor series expansion, this is

$$4\sum_{\lambda\neq 0\in K}-c(\lambda^2/2)\log(1-e(\lambda,X_M)+i|(\lambda,Y_M)|).$$

$\mathrm{Div}(\psi_M) = \sum_{\lambda^2/2\leq 0} c(\lambda^2/2)\mathscr{H}_{\lambda^2/2}$ where $\mathscr{H}_{\lambda^2/2}$ is the *Heegner divisor* $\bigcup_{\ell\in M,\ell^2/2=\lambda^2/2}\ell^\perp$.

Remark 3.6.4. This is an infinite union of hyperplanes, but actually we should have been talking about $\mathrm{Pic}(D_M/\mathrm{Aut}(M))$ to make it algebraic (D_M is not), and so the infinite things occupy only finitely many orbits here, so we're good.

Using this theorem gives a map $\mathscr{M}_{1-m/2}(\rho_M)$ to Heegner divisors on D_M, by

$$f \mapsto \sum_{\lambda^2/2} c(\lambda^2/2)\mathscr{H}_{\lambda^2/2}.$$

If M has no norm 0 vector (which never happens if $\operatorname{rank} M \geq 5$), then this strategy doesn't work. Borcherds uses a trick to handle this case.

Remark 3.6.5. This ≥ 5 result implies that if $\mathrm{Pic}(S)$ has $\rho(S) \geq 5$, then S is an elliptic fibration, S a K3 surface. Basically, if the Picard lattice has a norm zero vector, then it must be an elliptic curve.

Lemma 3.6.6. *$\phi_M(f)$ can be written as a linear combination of functions, each the restriction to D_M of a function of the form $\phi_{M\oplus M_j}(F)-$singularities where M_j is unimodular.*

Here F is related to f, and is also obtained by a theta lift. The idea here is that we're just adding a unimodular lattice to get int he situation we want.

Concretely, $\phi_M(f) = \phi_{M\oplus A_3^{\oplus 8}}|_{D_M} - \phi_{M\oplus A_2^2}|_{D_M}$. □

Next time, we can prove: if $\beta \in M^*$, $e_\beta \in \mathbb{C}[M^*]$ then

$$\sum_n \sum_\beta e_\beta q^n \mathscr{H}_{n,\beta} \in \mathrm{Pic}(D_M/\mathrm{Aut}(M)) \otimes \mathbb{C}[M^*][[q]].$$

4. Generalized GKZ Theorem

4.1. **Heegner divisors.** Let M be a lattice of signature $(2,m)$. We identify

$$D_M \cong \{w \in \mathbb{P}(M\otimes\mathbb{C}) \mid \langle w,w\rangle = 0, \langle w,\overline{w}\rangle > 0\}.$$

[It parametrizes the Hodge Structures on M of type $(1,m,1)$.] To a 2-plane, you form the $X_M + iY_M$ from last time.

$\Gamma_M := \{g \in \mathrm{Aut}(M) \mid g$ acts trivially on $M^* = M^\vee/M\}$.

Then $\mathscr{X}_M := \Gamma_M\backslash D_M$ is an irreducible, quasiprojective variety of dimension m, with at worst quotient singularities. This means in particular that $\mathrm{Pic}(\mathscr{X}_M)\otimes\mathbb{Q} = \mathrm{Cl}(\mathscr{X}_M)\otimes\mathbb{Q}$, i.e. $\mathscr{X}_M$ is a $\mathbb{Q}$-factorial variety.

Definition 4.1.1. Given a pair $n \in \mathbb{Q}^{<0}$ and $\gamma \in M^*$, we define

$$\mathscr{Y}_{n,\gamma} = \Gamma_M\backslash H_{n,\gamma} = \left(\sum_{(v,v)/2=n,\, v\equiv\gamma \pmod M} v^\perp\right)/\Gamma_M$$

where $v^\perp = \{w \in D_m \mid \langle v,w\rangle = 0\}$.

In general, $\mathscr{Y}_{n,\gamma}$ is not irreducible. It is called a *Heegner divisor* on $\mathscr{X}_M$.

Example 4.1.2. (Degenerate case). We take $\mathscr{Y}_{0,0}$ as a $\mathbb{Q}$-Cartier divisor to be $\mathscr{O}(1)/\Gamma_M$. (Equivalently, it's the Hodge line bundle on $\mathscr{X}_M$). This is the "constant term" of a modular form. If $\gamma \neq 0$, then $n > 0$ by convention.

Theorem 4.1.3 (GKZ)**.** *The generating series*

$$\vec{\Phi}(q) = \sum_{\gamma\in M^*}\sum_{n\in\mathbb{Q}^{\geq 0}} y_{-n,\gamma}e_\gamma q^n$$

is an element of $\mathrm{Pic}_{\mathbb{Q}}(\mathscr{X}_M)\otimes_{\mathbb{Q}}\mathscr{M}_{1+m/2}(\rho_M^*)$ *where* $\mathscr{M}_{1+m/2}(\rho_M^*)$ *is the space of vector-valued modular forms of weight* $1+m/2$ *and type* ρ_M^*.

4.2. **Serre duality on modular curves.** The idea of the proof is an application of Serre duality.

Let $\mathscr{M}_k(\rho)$ be the space of global sections of the vector bundle

$$E_{k,\rho} = \Gamma\backslash\mathrm{Mp}_2(\mathbb{R})\times V/K$$

where (ρ,V) is a representation of $\mathrm{Mp}_2(\mathbb{R})$ and K is the pre-image of $\mathrm{SO}(2)$ in $\mathrm{Mp}_2(\mathbb{R})$, so $\mathrm{Mp}_2(\mathbb{R})/K = \mathbb{H}$.

In other words, $\mathscr{M}_k(\rho)$ is H^0 of some vector bundle on a modular curve, and Serre duality relates it to some H^1 group.

Suppose Γ has only one cusp at ∞ (for us, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$).

Set:

- q to be the uniformizing parameter of Γ at ∞,
- $\mathrm{Pow}(\Gamma,\rho) = \mathbb{C}[[q]]\otimes V$,
- $\mathrm{Laur}(\Gamma,\rho) = \mathbb{C}[[q]][q^{-1}]\otimes V$

- $\mathrm{Sing}(\Gamma,\rho) = \mathrm{Laur}(\Gamma,\rho)/q\cdot\mathrm{Pow}(\Gamma,\rho)$, the space of singularities and constant terms of terms of Laurent series at ∞.

There is a natural pairing

$$\langle -,-\rangle\colon \mathrm{Pow}(\Gamma,\rho^\vee)\times\mathrm{Sing}(\Gamma,\rho)\to\mathbb{C}$$

where $\langle f,\phi\rangle = \mathrm{Res}_{q=0}(f\phi q^{-1}dq)$ (using the pairing of ρ and $\rho^\vee$). This is the residue of $f\phi$ at ∞, also the constant term of $f\phi$ about $q=0$.

Recall that we have a map $\alpha\colon \mathscr{M}_k(\rho)\to\mathrm{Pow}(\Gamma,\rho)$.

Theorem 4.2.1. *Let* $\mathrm{Obs}_k(\Gamma,\rho) = \mathrm{Sing}(\Gamma,\rho)/\alpha(\mathrm{Mod}_k(\Gamma,\rho))$. *Then* $\mathrm{Obs}_{2-k}(\Gamma,\rho)$ *is finite-dimensional and dual to* $\mathscr{M}_k(\Gamma,\rho^\vee)$ *under the pairing* $\langle -,-\rangle$. *In other words,*

$$\alpha(\mathrm{Mod}_k(\Gamma,\rho)) = \alpha(\mathscr{M}_k(\Gamma,\rho^\vee))^\perp.$$

Proof. First, let us assume that ρ is 1-dimensional and acts trivially. Then Γ acts freely on $\mathbb{H}$. In this case, $\mathscr{L}_k = E_{k,\rho}$ is the line bundle with $H^0(\mathscr{L}_k) = \mathscr{M}_k(\Gamma)$.

Let $\mathscr{L}_{\text{cusp}}$ be the union of the cusps of X, which we think of as a line bundle or element of $\mathrm{Pic}(X)$.

Then $\omega_X := \mathscr{L}_2\otimes\mathscr{L}_{\text{cusp}}^*$, since holomorphic 1-forms correspond to cusp forms of weight 2. By Serre duality,

$$H^0(\mathscr{L}_k) = H^1(\omega_X\otimes\mathscr{L}_k^*) = H^1(\mathscr{L}_{2-k}\otimes\mathscr{L}_{\text{cusp}}^*).$$

The pairing here is just the pairing we defined previously.

Now, $H^1(\mathscr{L})$ on a Riemann surface is precisely the obstruction of finding a meromorphic section of $\mathscr{L}$ with given singularities at some fixed point, and holomorphic elsewhere. Applying this to $\mathscr{L} = \mathscr{L}_{2-k}\otimes\mathscr{L}_{\text{cusp}}^*$, we find that

$$H^1(\mathscr{L}_{2-k}\otimes\mathscr{L}_{\text{cusp}}^*) = \mathrm{Obs}_{2-k}(\Gamma,k) = \mathrm{Sing}(\Gamma,\rho)/\alpha(\mathrm{Mod}_{2-k}(\Gamma,\rho)).$$

That was the case of the trivial representation. In general, we can choose a finite index subgroup $\Gamma'\subset\Gamma$ such that ρ is trivial on Γ' and Γ' acts freely on $\mathbb{H}$. To get back, the quotient is a finite group so you can quotient nicely. □

Now, in order to prove that $\vec{\Phi}(q)\in\mathscr{M}_{1+m/2}(\rho_M^\vee)\otimes\mathrm{Pic}(\mathscr{X}_M)_{\mathbb{Q}}$, it suffices to show that $\vec{\Phi}(q)$ is perpendicular to the elements of in the obstruction space $\mathrm{Obs}_{-1-m/2}(\rho_M) = \mathrm{Obs}_{1-m/2}(\rho_M)^\vee$. You can check this explicitly; it is just multiplication of power series.

Proof of GKZ Theorem. There is a map $\xi\colon \mathrm{Mod}(\Gamma_m, 1-\frac{m}{2},\rho_M)\to\mathrm{Heegner}(\mathscr{X}_M)$ sending $q^n e_\gamma\mapsto\mathscr{Y}_{n,\gamma}$ if $n\le 0$, and crushing all holomorphic ($q\ge 0$) terms.

Given $f = \sum_\gamma \sum_n c_n(\gamma) q^n e_\gamma \in \mathrm{Mod}(\Gamma_M, 1-\frac{m}{2}, \rho_M)$, Borcherds' Infinite Product Theorem gives a singular lifting: there exists Ψ_f on $\mathscr{X}_M$ such that $\mathrm{Div}(\Psi_f) = \sum_{n\le 0} c_n(\gamma)\mathscr{Y}_{n,\gamma}$. This gives a relation in $\mathrm{Pic}(\mathscr{X}_M)$.

Using the pairing $\langle -,-\rangle$ we have for any $f \in \mathrm{Mod}(\Gamma_M, 1-\frac{m}{2}, \rho_M)$:

$$\langle f, \vec{\Phi}\rangle = \text{constant term of } (f\cdot\vec{\Phi}). \qquad = \sum_{\gamma,n} c_n(\gamma) e_\gamma q^n \cdot \sum_{\gamma,n} \mathscr{Y}_{-n,\gamma} q^n e_\gamma$$

$$= \sum_{\gamma,n} c_n(\gamma)\mathscr{Y}_{n,\gamma}$$

which is 0 as we just saw. Therefore, $\vec{\Phi}$ is orthogonal to $\mathrm{Mod}(\Gamma, 1-\frac{m}{2}, \rho_M)$, so it lies in $\mathrm{Pic}(\mathscr{X}_M)\otimes \mathscr{M}_{1+\frac{m}{2}}(\rho_M^\vee)$.

As we saw, the key input was the explicit expression for $\mathrm{Div}(\Psi_f)$. $\square$

5. The theta correspondence

The goal is to prove the Kudla-Millson theorem. Recall that the Kudla program predicts that the generating series of special cycles on arithmetic manifolds is an automorphic form.

Let G be a reductive Lie group, K a maximal compact subgroup, and $D = G/K$, a symmetric space. Let $\Gamma \subset G$ be a discrete subgroup, and $X_\Gamma = \Gamma \backslash D$.

In the case $G = \mathrm{SO}(V)$ for some quadratic space V, Kudla-Millson prove that the generating series of special cycles is a Siegel modular form. Note that this applies for *arbitrary* signature, and in that sense is stronger than Borcherds' theorem.

Example 5.0.2. (Appliction to enumerative geometry)

Suppose V has signature $(2,m)$ (i.e. we are in the Shimura case). The result can be applied to reduced Gromov-Witten invariants on K_3 surfaces (if $m = 19$). The reason is that the Hodge structure is $(1,19,1)$, so D is the period domain.

If V has sign (p,q) with $p > 2$, e.g. $(p,q) = (4,28)$ then it can be applied to Noether-Lefschetz theory on elliptic surfaces. Here the Hodge structure o fan elliptic surface is $(2,28,2)$.

5.1. Heisenberg algebra and Weil representation.

Let V be a quadratic space over F (we have in mind $F = \mathbb{Q}, \mathbb{Q}_p, \mathbb{A}$) or some totally real extension) of dimension m. Let W be a symplectic space over F of dimension $2n$.

The goal is to construct a unitary representation on $O(V) \times \widetilde{\mathrm{Sp}(W)}$ (here $\widetilde{\mathrm{Sp}(W)}$ is the double cover of $\mathrm{Sp}(W)$. This is called the *Weil representation.*

Local Weil representation. Here we take $F = \mathbb{Q}_p$ or $\mathbb{R}$ (though the discussion applies to any local field or finite field of characteristic not equal to 2.) Let W be a symplectic space over F of dimension $2n$.

Definition 5.1.1. The *Heisenberg group* associated to W is

$$H(W) = W \oplus F$$

with multiplication using the symplectic form on W:

$$(w_1, t_1)(w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle).$$

Then $\mathrm{Sp}(W)$ acts on $H(W)$ by $g \cdot (w,t) = (wg, t)$, with the action on the center $Z(H(W)) \cong F$ being trivial.

Remark 5.1.2. $H(W)$ is a central extension of W by F:

$$0 \to F \to H(W) \to W \to 0$$

corresponding to the cocycle $(w_1, w_2) \mapsto \frac{1}{2}\langle w_1, w_2\rangle$. ♠♠♠ TONY: [what if you get rid of the 1/2?]

Here is the most important result in the representation theory of the Heisenberg group.

Theorem 5.1.3 (Stone, von Neumann)**.** *Let $\psi: Z(H(W)) \cong F \to \mathbb{C}$ be an additive character. Then there exists a unique irreducible representation (ρ_ψ, S) of $H(W)$ with central character ψ, i.e.*

$$\rho_\psi((0,t)) = \psi(t)\cdot \mathrm{Id}_S.$$

Example 5.1.4. If $\dim W = 2$ (so $\mathrm{Sp}(W) \cong \mathrm{SL}_2$), then we have

$$H(W) \cong \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in F \right\}.$$

Under this isomorphism, a and b are coordinates for a choice of isotropic subspaces of W. Indeed, identifying $W \cong F \oplus F$ with the pairing matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have

$$((a,b),t) \longleftrightarrow \begin{pmatrix} 1 & a & \frac{1}{2}ab \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

♠♠♠ TONY: [this doesn't seem to be correct] For an additive character $\psi: F \to \mathbb{C}^\times$, we can realize the representation (ρ_ψ, S) with $S = L^2(F)$ and ρ_ψ acting on $f \in S$ by

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \cdot f(x) = \psi(-bx + c) f(x-a).$$

Schrödinger model. In general, (ρ_ψ, S) can be realized as follows. Write $W = X \oplus Y$ where X, Y are maximal isotropic subspaces of W. Let $S = \mathscr{S}(X)$ be the space of $\mathbb{C}$-valued Schwartz functions. This means more concretely

$$\mathscr{S}(X) = \begin{cases} \mathbb{C}\text{-valued locally constant compactly supported functions} & F = \mathbb{Q}_p \\ \text{Schwartz functions} & F = \mathbb{R} \end{cases}$$

For $\varphi \in \mathscr{S}(X)$, the action is defined by

$$\rho_\psi(x+y,t)\varphi(x') = \psi(t + \langle x', y\rangle + \frac{1}{2}\langle x, y\rangle)\varphi(x+x').$$

Projective representation on $\mathrm{Sp}(W)$**.** For all $g \in \mathrm{Sp}(W)$, we can form a *new* representation $\rho_\psi^g(h) = \rho_\psi(g\cdot h)$, which also has central character

ψ. By the theorem of Stone and von Neumann, there exists $A(g) \in \mathrm{Aut}(S)$ such that $A(g)^{-1}\rho(h)A(g) = \rho(g \cdot h)$ for all $h \in H(W)$.

This defines a *projective* representation $\omega_\psi \colon \mathrm{Sp}(W) \to \mathrm{GL}(S)/\mathbb{C}^\times$ sending $g \mapsto A(g)$, because $A(g)$ is only defined up to scalars.

This isn't quite what we wanted - we wanted a *linear* representation. We basically accomplish this by lifting to the universal cover:

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}(W)}_\psi & \longrightarrow & \mathrm{GL}(S) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(W) & \longrightarrow & \mathrm{GL}(S)/\mathbb{C}^\times \end{array}$$

Here $\widetilde{\mathrm{Sp}(W)}_\psi$ is a central extension:

$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}(W)}_\psi \to \mathrm{Sp}(W) \to 1$$

We emphasize that $\widetilde{\mathrm{Sp}(W)}_\psi$ depends on a choice of ψ, but they are all canonically isomorphic.

Then we can lift ω to a linear representation $\omega_{\psi,W}$ of $\widetilde{\mathrm{Sp}(W)}_\psi$.

Fact: take $\widetilde{\mathrm{Sp}(W)}$ to be the double cover of $\mathrm{Sp}(W)$, corresponding to

$$1 \to \mu_2 \to \widetilde{\mathrm{Sp}}(W) \to \mathrm{Sp}(W) \to 1.$$

Then $\widetilde{\mathrm{Sp}_\psi}(W) = \widetilde{\mathrm{Sp}}(W) \times_{\mu_2} \mathbb{C}^\times$. Now restrict $\omega_{\psi,W}$ to $\widetilde{\mathrm{Sp}}(W)$. Then $\omega_{\varphi,W}$ is given by

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \varphi(v) = |\det A| \varphi({}^t A v)$$

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \varphi(v) = \psi(\frac{t_v B_v}{2})\varphi(v)$$

$$\begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \varphi(v) = \nu \widehat{\varphi}(v)$$

where $\widehat{\varphi}$ is the Fourier transform.

Now we define a representation $O(V) \times \widetilde{\mathrm{Sp}}(W)$. We have a map

$$O(V) \times \widetilde{\mathrm{Sp}}(W) \to \widetilde{\mathrm{Sp}}(W \otimes V).$$

In this case, $S = \mathscr{S}(V^n)$ where $2n = \dim W$.

Pull back the Weil representation to $O(V) \times \widetilde{\mathrm{Sp}}(W)$ and call it $\widetilde{\omega}^\vee_{\psi,W}$, so

$$\widetilde{\omega}^\vee_{\psi,W}(g,1)(\varphi(x)) = \varphi(g^{-1}x) \text{ for } g \in O(V).$$

and$\widetilde{\omega}^\vee_{\psi,W}(1,g')(\varphi(x))$ acts as the pullback of $\omega_{\psi,W\otimes V}$ of $\widetilde{\mathrm{Sp}}(W \otimes V)$ via the inclusion map $\widetilde{\mathrm{Sp}}(W \otimes V)$.

On $\widetilde{\mathrm{Sp}}(W)$, the restriction of $\widetilde{\omega}^{\vee}_{\psi,W}$ to μ_2 acts as $z^m \cdot \mathrm{Id}$ for $z \in \mu_2$, where $m = \dim V$. In particular, if m is even then it factors through $\mathrm{Sp}(W)$. In summary, $\widetilde{\omega}^{\vee}_{\psi,W}$ factors through $O(V) \times \mathrm{Sp}(W)$ if m is even. So let

$$\mathrm{Mp}(W) := \begin{cases} \mathrm{Sp}(W) & m \text{ odd}, \\ \mathrm{Sp}(W) & m \text{ even}. \end{cases}$$

Anyway, we've constructed a representation $\widetilde{\omega}^{W}_{\psi,V}$ on $O(V) \times \mathrm{Mp}(W)$.

Now that we have a representation, we can define theta functions.

$$\theta_{\psi,\varphi}(g,g') = \sum_{\xi \in V^n(F)} \widetilde{\omega}^{W}_{\psi,V}(g,g')(\varphi)(\xi)$$

where $\varphi \in \mathscr{S}(V^n)$ and F is a number field.

The Kudla-Millson theorem syas that there exists a very special $\varphi =: \varphi_{KM} \mathscr{S}(V^n) \otimes \mathscr{A}(\mathscr{X}_M)$, such that $\theta_{\psi,\varphi_{KM}}$ is the generating series of special cycles. That gives modularity, since θ is evidently invariant. What we have to do is compute the Fourier coefficients of this theta function.

5.2. **Theta correspondence.** Anyway, using the representation $\omega := \omega^{W}_{\psi,V}$ on $\mathrm{O}(V) \times \mathrm{Mp}(W)$, we define a *theta correspondence* between automorphic representations. The idea is to lift a modular form on $\mathrm{Mp}(W)$ to an automorphic form on the product.

Notation: $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$, G is reductive group over $\mathbb{Q}$, and $G(\mathbb{A})$ its adelic points. For us, $G = \mathrm{O}(V)$ or $\mathrm{Mp}(W)$, so for instance $G(\mathbb{Q}_p) = SO(V \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

Recall that ω was a representation on the sapce $\mathscr{S}(V^n)$.

Definition 5.2.1. On $\mathrm{O}(V) \times \mathrm{Mp}(W)$,

$$\theta_{\phi}(g,g') = \sum_{\xi \in V^n(\mathbb{Q})} \omega(g,g')(\phi)(\xi)$$

This is invariant under $O(V)$ and $\mathrm{Mp}(W)$ (the latter by Poisson summation).

Notation: $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ is the space of square-integrable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})$. [Assume for now that G has no center.] Then by Langlands, $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})) = L^2_{\mathrm{disc}} \oplus L^2_{\mathrm{cont}}$. There is a further decomposition of the discrete part into $L^2_{\mathrm{cusp}} \oplus L^2_{\mathrm{residue}}$. The cuspidal part is "filled out" by the cuspidal automorphic forms, and the residue part is filled out by Eisenstein series.

These correspond to automorphic forms on $\Gamma \backslash G(\mathbb{R})/K(\mathbb{R})$. If $K_f \subset G(\mathbb{A}_f)$ and $\Gamma = K \cap G(\mathbb{Q})$, then $\Gamma \backslash G(\mathbb{R})/K(\mathbb{R}) \cdot K(\mathbb{R})$.

Now for the theta correspondence. Let $\mathscr{A}_{\text{cusp}}(G) = \{\text{irreducible representations in } L^2_{\text{cusp}}\}$. These are called *cuspidal automorphic representations*. We define a map

$$\Theta_W^V : \mathscr{A}_{\text{cusp}}(\text{Mp}(W)) \to \mathscr{A}_{\text{cusp}}(\text{O}(V)).$$

For $(\tau, H) \in \mathscr{A}_{\text{cusp}}(\text{Mp}(W))$, we denote $\theta(\tau) \in \mathscr{A}_{\text{cusp}}(\text{O}(V))$ the representation

$$\Theta(\tau) = \{\theta^f(g) := \int_{\text{Mp}(\mathbb{Q})\backslash Mp(\mathbb{A})} \theta_\phi(g, g') \cdot f(g') dg' \forall f \in H\}.$$

Notice the similarity to Borcherds' theta lifting. Here, convergence is ok because we are working with *cusp* forms, and $\text{Mp}(\mathbb{Q})\backslash\text{Mp}(\mathbb{A})$ has finite volume.

Remark: $\theta(\tau)$ is not necessarily cuspidal.

Similarly, there is $\Theta_V^W : \mathscr{A}_{\text{cusp}}(O(V)) \to \mathscr{A}_{\text{cusp}}(\text{Mp}(W))$.

Example 5.2.2. $\phi(v) = \prod_i e^{-(v_i, v_i)}$ gives the classical theta functions.

Remark 5.2.3.
(1) When does θ^f exist and $\theta(\tau) \neq 0$? [Moeglon, Wee Teck Gan, Takeda] It only depends on the pole of the L-function of τ.
(2) If $\theta(\tau) \neq 0$ and it is cuspidal (which also depends on the L-function), then $\theta(\tau)$ is irreducible.
(3) If $\theta(\tau)$ is cuspidal, then $\Theta_V^W \circ \Theta_W^V(\tau) = \tau$ (up to a character).
(4) If $\tau \in \mathscr{A}_{\text{cusp}}(\text{Mp}(W))$, for any V we have $\Theta_W^V(\tau)$. Then we have the following Ralli Tower property. Suppose for some V_c, $\Theta_W^{V_c}(\tau)$ is cuspidal. Then choosing subquadratic space $V' \subset V_c$, we have $\Theta_W^{V'}(\tau) = 0$, and $\Theta_W^{V_c \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}(\tau)$. In other words, the first occurence of non-zero theta lifting is cuspidal.

Note that this is again similar to Borcherds' proof, lifting from sublattices.

Next time, we'll show that if $\theta_\phi(g, g')$ is a theta function, for all ϕ Schwartz, there exists a special Schwartz form, $\phi \in \mathscr{S}(V^n) \otimes C^\infty(X)$ then $\theta_\phi(g, g')$ is a theta form, and the Fourier expansion is a generating series for special cycles.

6. Geometry and cohomology on arithmetic manifolds

Let G be a Lie group and K a maximal compact subgroup of G. Then $D = G/K$ is a symmetric space. Let $\Gamma \subset G$ is a discrete subgroup.

(1) Connection between $H^*(\Gamma\backslash D, \mathbb{C})$ and relative Lie algebra cohomology. (Matsushima formula)
(2) $(\mathfrak{g}, K)$-cohomology.

Because D is contractible, $H^*(\Gamma;\mathbb{C}) \cong H^*(\Gamma\backslash D;\mathbb{C})$ as $\Gamma\backslash D$ is contractible.

6.1. $(\mathfrak{g}, K)$-modules.

Let $\mathfrak{g} = \mathrm{Lie}(G)$. Given a representation (π, V) of G, we associate a representation of $\mathfrak{g}$ as follows.

Definition 6.1.1. An element $v \in V$ is *smooth* if for $X \in \mathfrak{g}$,

$$X \cdot v := \lim_{t\to 0} \frac{\exp(tX)\cdot v - v}{t}$$

exists.

Remark 6.1.2. This is only interesting when $\dim V = \infty$; when $\dim V < \infty$ then all vectors are smooth.

Definition 6.1.3. $v \in V$ is *K-finite* if $\dim K \cdot v$ is finite.

The idea is that if we view (π, V) as a representation of K, we have $V = \bigoplus V_i$ as a K-representation (because K is compact). Decomposing into irreducible classes, we have $V \cong \bigoplus V_i^{\oplus m_i}$. Then K-finite is equivalent to m_i being finite for all i.

Definition 6.1.4. A $(\mathfrak{g}, K)$-*module* is a $\mathbb{C}$-vector space V together with a representation of $\mathfrak{g}$ on V and a continuous action of K on V such that

(1) every vector in V is K-finite,
(2) $\frac{d}{dt}|_{t=0}(\exp tY)\cdot v) = Y \cdot v$, for $v \in V$ and $Y \in \mathrm{Lie}(K)$.
(3) $k\cdot(X\cdot v) = \mathrm{Ad}(k)X\cdot(k\cdot v)$ for all $v \in V, k \in K, X \in \mathfrak{g}$.

Facts: for all unitary representation (π, V) of G, one can associate a $(\mathfrak{g}, K)$-module by taking

$$V_f^\infty := \{k-\text{finite smooth vectors on } V\} \subset V.$$

Theorem: if $(\pi, V) \cong (\pi', V')$ is an isomorphism of *unitary representations*, $V_f^\infty \cong (V_f')^\infty$.

Example 6.1.5. The Weil representation $(\omega, \mathscr{S}(X))$. Its $(\mathfrak{g}, K) = (\mathfrak{sp}, \widetilde{\mathrm{U}(n)})$ is on $S_f^\infty = \{\varphi_0(x)p\}$ where p is a polynomial on X and $\varphi_0(x) = \exp(-(x,x))$ [note that this is G-invariant]. ♠♠♠ TONY: [There was some confusion why not to allow other Gaussians - not totally satisfied. Perhaps involves the discrete guy Γ]

6.2. **Relative Lie algebra cohomology.** Given a $(\mathfrak{g}, K)$-module, let (π, V) we can define the cohomology $H^*(\mathfrak{g}, K; V)$ to be the cohomology of the complex $\mathrm{Hom}_{\mathfrak{g}}(\bigwedge^\bullet \mathfrak{p}, V)$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition. The differentials: if $f : \bigwedge^k \mathfrak{p} \to V$, then $df(x_1 \wedge \ldots \wedge x_{k+1}) = \sum_k (-1)^k x_i f(\widehat{x_i})$. ♠♠♠ TONY: [check this]

We say that (π, V) is *cohomological* if $H^*(\mathfrak{g}, K; V) \neq 0$.

Example 6.2.1. If $G = \mathrm{O}(p, q)$, then $\mathrm{Lie}(G) = \{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} + \begin{pmatrix} 0 & X \\ {}^t X & 0 \end{pmatrix})\}$.

Example 6.2.2. If $G = K$ is compact, then $H^*(G, \mathbb{C}) = H^*(\mathfrak{G}, C^\infty(G))$. This is de Rham's Theorem. Why is it true?

$\{k-\text{forms on } G\} = \mathrm{Hom}(\bigwedge^k \mathfrak{g}, C^\infty(G))$. For ω, and $X \in \mathfrak{g}$ viewed as a left-invariant vector field, we get $\omega(X_1, \ldots, X_k) \in C^\infty(G)$.

Recall that: if (π, V) is a representation of a reductive Lie group G, we can associate a $(\mathfrak{g}, K)$-module

$$V_f^\infty = \{\text{ smooth, } K\text{-finite vectors in } V\}.$$

We introduced this in order to connect the cohomology of arithmetic groups with Lie algebra cohomology.

Suppose (ρ, V^∞) is a $(\mathfrak{g}, K)$-module. Then $H^*(\mathfrak{g}, K; V^\infty) := H^*(\mathrm{Hom}(\bigwedge^\bullet P, V^\infty))$ where $\mathfrak{g} = \mathfrak{k} \oplus P$ is the Cartan decomposition.

Example 6.2.3. When G is compact, we $\mathfrak{g} = \mathfrak{k} = P$. Then

$$H^*(\mathfrak{g}, C^\infty(G)) \cong H^*_{\mathrm{dR}}(G, \mathbb{C}).$$

This is essentially because there is a bijection

$$\{k-\text{forms on } G\} \longleftrightarrow \{\mathrm{Hom}(\bigwedge^k \mathfrak{g}, C^\infty(G))\}$$

sending $\omega \mapsto \omega(X_1, \ldots, X_k) \subset C^\infty(G)$ where the X_i are left-invariant vector fields.

6.3. **L^2-cohomology.** Let $Y = \Gamma \backslash D$ and $D = G/K$, Γ a torsion-free arithmetic subgroup of D. We produce now a non-smooth analogue of de Rham cohomology which will be better suited for *noncompact* spaces.

Definition 6.3.1. Let $\Omega^\bullet(Y)$ be the de Rham complex on Y. Define the *L^2-complex on Y*

$$\Omega^i_{(2)}(Y) = \{\mathbb{C}\text{−valued smooth square-integrable } i\text{-forms, whose } d \text{ is still } L^2\}.$$

This forms a complex under exterior derivative, and we define

$$H^*_{(2)}(Y, \mathbb{C}) := H^*(\Omega^i_{(2)}(Y), d).$$

The point of this is that it allows us to imitate Hodge theory for compact manifolds. By Hodge theory,

$$H^*_{(2)}(Y,\mathbb{C}) \cong \mathscr{H}(Y) = \{L^2 - \text{harmonic forms on } Y\}$$

if $H^*_{(2)}(Y,\mathbb{C})$ is finite-dimensional (it often is not).

Remark 6.3.2. This isomorphism actually factors through $\overline{H}^*_2(Y,\mathbb{C})$, called the "reduced L^2-cohomology," which is *always* isomorphic to $\mathscr{H}^*(Y)$.

In general, the map $\mathscr{H}(Y) \to H^*_{(2)}(Y,\mathbb{C})$ is neither surjective nor injective.

$H^*_{(2)}(Y,\mathbb{C})$ is *not* preserved by homeomorphisms, unlike de Rham cohomology. So it's really not a topological invariant.

When Y is complete (every geodesic is global), $\overline{H}^*_{(2)}(Y,\mathbb{C}) \cong H^*_{(2)}(Y,\mathbb{C})$ if the latter is finite-dimensional. Fortunately, all arithmetic manifolds are complete because there is a group action.

By work of Zueker and Borel: for $G = \mathrm{SO}(2,n)$ and $Y = \Gamma\backslash D$,

$$H^*_{(2)}(Y,\mathbb{C}) \cong H^*(Y,\mathbb{C}) \text{ when } i < n.$$

Corollary 6.3.3. *$H^i(Y,\mathbb{C})$ has a pure Hodge structure when $i < n$.*

In general, for noncompact manifolds (even "nice" ones like hypersurfaces) you can only expect a mixed Hodge structure.

6.4. **Matsushima formula.**

Theorem 6.4.1 (Borel, Casselman)**.** *If the spaces are finite-dimensional, then*

$$H^*(\mathfrak{g},K;L^2_{disc}(\Gamma\backslash G)) \cong H^*_{(2)}(Y,\mathbb{C}).$$

This implies that

$$H^*_{(2)}(Y,\mathbb{C}) = \bigoplus_{\pi \text{ irred.}} m_\pi H^*(\mathfrak{g},K,\pi^\infty).$$

There is a map

$$H^*(\mathfrak{g},K;L^2_{\mathrm{disc}}(\Gamma\backslash G) \to H^*(Y,\mathbb{C}).$$

Remark 6.4.2. We have $H^*_{(2)}(\Gamma,\mathbb{C}) \cong H^*(\mathfrak{g},K,L^2_{\mathrm{disc}}(\Gamma\backslash G)$

For example, if $G = \mathrm{SO}(2,n)$ the Vogan-Zvekerman says $H^i(\mathfrak{g},K;\pi) = 0$ if $i < n/2$ and i is odd. Thus we get this vanishing for $H^i_{(2)}(\Gamma,\mathbb{C})$.

Application: $H^1(Y,\mathbb{Q}) = 0$ if Y is a connected Shimura variety of (real) dimension at least 3, hence Y has trivial Albanese.

(2) Consider {special cycles of codimension i} $\subset H^{2i}(Y,\mathbb{C})$. For i small enough, this decomposes as

$$\bigoplus_\pi m_\pi H^{2i}(\mathfrak{g},K;\pi^\infty)$$

where π comes from the theta correspondence (recent result of Li, Millson, Reregeron, Moeglin). So checking classes coming from special cycles is equivalent to checking representations coming from the theta correspondence!

7. Kudla-Millson special theta lifting

Observation:

$$H^*(Y,\mathbb{C}) \cong H^*(\mathfrak{g},K;SV^n))$$

where $G = \mathrm{SO}(p,q) = \mathrm{SO}(V)$, $K = S(O(p)\times O(q))$, and $S(V^n)$ are Schwartz functions on V^n (in particular, lying in L^2), and $\mathscr{S}(V^n)$ is the Fock space.

8. Modularity of generating series of special cycles

8.1. **Construction of special cycles.** Let V be a quadratic space over $\mathbb{Q}$ of signature (p,q), $G=\mathrm{SO}(V)^0$, and $\mathfrak{g}=\mathrm{Lie}(G(\mathbb{R}))$. The Cartan decomposition is $\mathfrak{g}=\mathfrak{p}+\mathfrak{k}$, where

$$\mathfrak{p}\cong\left\{\begin{pmatrix}0 & X\\ X^t & 0\end{pmatrix} : X\in M_{p,q}\right\}\cong M_{p,q}.$$

This has a natural basis $X_{\alpha,\nu}$ where $1\le\alpha\le n$ and $p+1\le\mu\le m=p+q$.

As usual, we set $D=G(\mathbb{R})/K(\mathbb{R})$ where $K(\mathbb{R})\cong\mathrm{SO}(p)\times\mathrm{SO}(q)$ is a maximal compact subgroup of $G(\mathbb{R})$. Then D can be identified with the Grassmannian of q-planes in $V\otimes\mathbb{R}$, by the usual presentation of the Grassmannian as a quotient of a Stiefel manifold. Then for $z\in D$, we have

$$T_z^*(D)\cong\mathfrak{p}^*=\{\omega_{\alpha,\mu}:=X_{\alpha,\mu}^*\}$$

Definition 8.1.1. Let k be a positive integer. For $v\in V^{\oplus k}$, we define $U=U(v)$ to be the $\mathbb{Q}$-subspace of V spanned by the components of v. Let

$$D_v=\{z\in D\mid z\perp U\}$$

identifying D as the Grassmannian of q-planes in $V\otimes\mathbb{R}$.

Note that if $r=\operatorname{rank}U$, then $D_v\cong\mathrm{SO}(p-r,q)$ has codimension rq. In particular, generically $\operatorname{rank}U=k$ so $C(U):=\Gamma_U\backslash D_v$ is a codimension kq cycle. Here Γ is a congruence subgroup of G, and Γ_U is the stabilizer of U in Γ, which admits a natural inclusion into $Y=\Gamma\backslash D$.

For any β a $k\times k$ symmetric matrix over $\mathbb{Q}$, we set

$$\Omega_\beta=\left\{v\in V^k\mid {\scriptstyle \frac{1}{2}\langle v,v\rangle=\beta \text{ as matrices} \atop \dim U(v)=\operatorname{rank}\beta}\right\}.$$

♠♠♠ TONY: [is the second condition redundant?]

Definition 8.1.2. Let β be as above. Define the codimension $q\cdot\operatorname{rank}(\beta)$-cycle $Z(\beta)=\sum_{v\in\Gamma\backslash\Omega_\beta}C(U(v))$ in $Y=\Gamma\backslash D$.

♠♠♠ TONY: [why finite?]

However, at this point Y is not a Shimura variety, so we want to put some extra cycle on it. ♠♠♠ TONY: [don't understand this "philosophy"]

Definition 8.1.3. The *Euler form* $e_q\in\Omega^q(D)$ is defined as follows: $e_q=0$ if q is odd, and otherwise

$$e_q=\left(-\frac{1}{4\pi}\right)^\ell\frac{1}{\ell!}\sum_{\sigma\in S_q}\mathrm{sgn}(\sigma)\Omega_{\sigma(1)\sigma(2)}\wedge\Omega_{\sigma(3)\sigma(4)}\ldots\wedge\omega_{\sigma(q-1)\sigma(q)}$$

where $q=2\ell$, $\Omega_{ij}=\sum_\alpha\omega_{\alpha i}\wedge\omega_{\alpha j}$, and $\omega_{\alpha i}\in\mathfrak{p}^*=T^*D$ as before.

Remark 8.1.4. When $q = 2$, e_q is the Chern class of the Hodge line bundle on Y. When $q = 2$, D parametrizes Hodge structures of type $(1, p, 1)$, and the Hodge bundle is the line bundle $L \to D$ whose fiber over $z \in D$ is the $H^{2,0}$ of the corresponding Hodge structure.

Let $t = \text{rank}(\beta)$. We define

$$[z_\beta] := [Z(\beta)] \wedge e_q^{k-t} \in H^{kq}(Y, \mathbb{C}).$$

Here the class $[z(\beta)] \in H^{tq}(Y, \mathbb{C})$ is defined by the usual Poincaré duality:

$$\eta \mapsto \int_{z(\beta)} \eta \quad \text{for } \eta \in H_c^{pq-tq}(Y, \mathbb{C})$$

where we have used that e_q is Γ-invariant, because it comes from the Hodge bundle and Γ doesn't affect the Hodge structure.

Remark 8.1.5. A related fact used often in analysis is that we can just take η a closed $pq - tq$ form rapidly decreasing. This is equivalent because every closed, rapidly decreasing form differs from a compactly supported form by something exact.

For η a rapidly decreasing $pq - kq$ form, we write

$$\langle z_\beta, \eta \rangle = \int_{z(\beta)} \eta \wedge e_q^{k-t}.$$

Theorem 8.1.6 (Kudla-Millson)**.** *The generating series*

$$P(\tau, \eta) = \sum_{t=0}^{k} \sum_{\substack{\beta \in M_{n\times n}(\mathbb{Q}) \\ \beta \text{ symmetric, rank } n}} \langle [z_\beta], \eta \rangle \exp(2\pi i \, \text{tr}(\beta\tau))$$

is a Siegel modular form of weight $m/2$ for some congruence subgroup in $\text{Sp}(W)$*. Here we view τ as an element of the Siegel upper half-plane $\mathscr{H}_{2n}$, so $\beta\tau$ is a product of matrices.*

When q is odd, the only non-zero term comes from β with rank k because there is no e_q, hence $P(\tau, \eta)$ is in face a *cusp form*. When q is even, $P(\tau, \eta)$ is Eisenstein series (this case is more interesting because $q = 2$ is the Shimura case. Recall from Borcherds' setting that this case gave the Hodge line bundle.)

Example 8.1.7. For $p = 2$, $Y = \Gamma \backslash D$ is a connected Shimura variety (parametrizing Hodge structures of K3 surfaces). This recovers Borcherds' GKZ Theorem.

For $p=1$, Y is a hyperbolic manifold and the special cycles are "totally geodesic submanifolds," so this tells us that the generating series of totally geodesic submanifolds is a Siegel modular form.

Idea of Proof. $P(\tau,\eta)$ is the Fourier expansion of the theta series defined previously.

8.2. **Theta functions and theta forms.** Recall that we have a Weil representation ω on $G\times G'$, where $G=\mathrm{SO}(V)$ (signature p,q)and $G'=\mathrm{Mp}(W)$ (the symplectic group of W if $p+q$ is even, and a double cover if it's odd).

Definition 8.2.1. View ω as a representation on $G(\mathbb{A})\times G'(\mathbb{A})$ (via work of Weil). $G\times G'$ acts on $S(V^n(\mathbb{A}))$, the space of Schwartz functions. Given $\varphi\in S(V^n(\mathbb{A}))$, we define

$$\theta_\varphi(g,g')=\sum_{x\in V^n(\mathbb{Q})}\omega(g,g')\cdot\varphi(x)$$

Note that $\omega=\omega_\psi$ depends on a choice of additive character, which we suppress.

Example 8.2.2. If $\varphi=e^{-\mathrm{tr}(x,x)}$ then one gets something looking like the classical theta functions.

To remind you of the representation $\omega(g,g')$, the above is

$$\sum_{x\in V^n\mathbb{Q}}\omega(g')(\varphi)(g^{-1}x).$$

The idea is that if we choose a Schwartz function φ which is $K\times K'$-invariant, where K is the maximal compact of $G(\mathbb{R})$ and K' is a maximal compact of $G'(\mathbb{R})$, then θ_φ descends to a function on $G(\mathbb{Q})\backslash G(\mathbb{A})/K$, and if you then quotient by some level you get $\Gamma\backslash D$.

More generally, we can choose some "Schwartz form", i.e. an element of $\mathscr{S}(V^n(\mathbb{A}))\otimes\Omega^i(D)$. Similarly, we define the theta function

$$\theta_\varphi(g,g')=\sum_{x\in V^n(\mathbb{Q})}\omega(g,g')(\varphi)(x).$$

For fixed g', this is a differential form on $\Gamma\backslash D$. Then you view this as a function on G', and show that you get a modular form.

Let us take K-invariant Schwartz functions/forms:

$$S(V^n(\mathbb{R}))^K\xleftarrow{\sim}[S(V^n(\mathbb{R}))\otimes C^\infty(D)]^{G(\mathbb{R})}$$

by evaluation at a base point of D. Then for $\varphi=\varphi_f\otimes\varphi_\infty$, where $\varphi_\infty\in[S(V^n(\mathbb{R}))]^K$, $\theta_\varphi(g,g')$ descends to a function on D.

Similarly,

$$[S(V^n(\mathbb{R}))\otimes\Omega^i(D)]^{G(\mathbb{R})}\cong[S(V^n(\mathbb{R}))\otimes\wedge^i\mathfrak{p}^*]^K$$

where $\mathrm{Lie}(G) = \mathrm{Lie}(K) \oplus \mathfrak{p}$. For $\varphi = \varphi_f \otimes \varphi_\infty$ where $\varphi_\infty \in [S(V^n(\mathbb{R})) \otimes \bigwedge^i(\mathfrak{p}^*)]^K$, $\theta_\varphi(g, g')$ descends to a differential i-form on D.

We want g' also to descend to some function on G'/K'. More precisely, the hope is that $\theta_\varphi(g, g')$ descends to a (holomorphic) section of some line bundle on $G'(\mathbb{R})/K'$. This is basically the Siegei upper half space $\mathfrak{h}_{2n}$. That is exactly the notion of Siegel modular form! So if we can prove this, then we will have that $\theta_\varphi(g, g')$ is a Siegel modular form as a function of g'.

The point is then that a clever choice of φ will turns $\theta_\varphi(g, g')$ into the generating series for special cycles.

Differentials. From ω, we get a $\mathrm{Lie}(G) \times \mathrm{Lie}(G')$ action on the Fock space $\mathscr{S}(V^n(\mathbb{A})) \subset S(V^n(\mathbb{A}))$ ♠♠♠ TONY: [gah] which was the subspace of functions of the form $\{p(x_1, \dots, x_n)\varphi_0(x)\}$ where p is any polynomial and φ_0 is $\exp(-\mathrm{tr}(x, x))$.

Definition 8.2.3. Let $C^{i,j} = [\bigwedge^i \mathfrak{p} \otimes \bigwedge^j(\mathfrak{l}^-) \otimes \mathscr{S}(V^n(\mathbb{R})) \otimes \mathbb{C}\chi_m]^{K \times K'}$, where $\mathrm{Lie}(G') = \mathrm{Lie}(K') \oplus \mathfrak{l}$ (Cartan decomposition), and $\mathfrak{l} = \mathfrak{l}^+ \oplus \mathfrak{l}^-$ via the complex structure. Here χ_m is the character $g' \mapsto (\det g')^{m/2}$.

So the above is smooth i-forms on D and antiholomorphic J-forms on $D' = G'/K'$. It forms a double complex with d and $\overline{\partial}$. Here d comes from the exterior derivative $\Omega^\bullet(D) \to \Omega^{\bullet+1}(D)$, so $d \colon C^{i,j} \to C^{i+1,j}$. On the other hand, $\overline{\partial}$ comes from the differential operator $\Omega^\bullet(D') \to \Omega^{\bullet+1}(D')$.

Using the theta correspondence, we can construct a correspondence between differential forms on D and holomorphic forms on D'. With $j = 0$, we get holomorphic functions. So this is a correspondence between two Shimura varieties.

Let $S(V^n(\mathbb{A}))$ be the space of adelic Schwartz-Bruhat functions. For $\varphi \in S(V^n(\mathbb{A}))$, we write $\varphi = \varphi_f \otimes \varphi_\infty$ where $\varphi_\infty \in S(V^n(\mathbb{R}))$ and $\varphi_f \in S(V^n(\mathbb{A}_f))$.

By abuse of notation, we consider $\varphi_\infty \in C^{i,j}$ (a Schwartz function tensored with forms). We say that it is *holomorphic* if $\overline{\partial}\varphi_\infty = 0$ in the d-cohomology of the double complex, i.e. this is $d\psi$ for $\psi \in C^{i-1,j+1}$. We say that φ_∞ is closed if $d\varphi_\infty = 0$.

Definition 8.2.4. For a rapidly decreasing closed $pq - i$ form η on Y, we define

$$\theta_\varphi(\eta) = \int_Y \eta \wedge \theta_\varphi(g, g')$$

where $\varphi = \varphi_f \otimes \varphi_\infty$ and $\varphi_\infty \in C^{i,0}$.

The goal is to show that for some φ,

$$\theta_\varphi(\eta) = P(\eta, \tau).$$

Proposition 8.2.5 (Kudla-Millson)**.** *If $\varphi_\infty \in C^{i,0}$ is closed and holomorphic, then $\theta_\varphi(\eta)$ is a holomorphic section of $\mathscr{L}_m := G' \times \mathbb{C}_{\chi_m}/K'$.*

Note that $\theta_\varphi(\eta)$ is a section of $\mathscr{L}_m$ because $\varphi_\infty \in [(\bigwedge^i \mathfrak{p}) \otimes \mathscr{S}(V^n(\mathbb{R})) \otimes \mathbb{C}\chi_m]^{K \times K'}$ because $k' \cdot \varphi_\infty = (\det k')^{m/2} \varphi_\infty$.

To show that $\theta_\varphi(\eta)$ is actually *holomorphic*, we need to check that $\overline{\partial}\theta_\varphi(\eta) = 0$. By definition, and using that $\overline{\partial}$ is only on the metaplectic half,

$$\begin{aligned}
\overline{\partial}\theta_\varphi(\eta) &= \overline{\partial} \int_Y \eta \wedge \theta_\varphi(g, g') \\
&= \int_Y \overline{\partial}(\eta \wedge \theta_\varphi(g, g')) \\
&= \int_Y \eta \wedge \overline{\partial}(\theta_\varphi(g, g')) \\
&= \int_Y \eta \wedge \theta_{\overline{\partial}\varphi}(g, g') \\
&= \int_Y \eta \wedge \theta_{d\psi}(g, g') \\
&= \int_Y \eta \wedge d(\theta_\psi(g, g')) \\
&= \int_Y d(\eta \wedge \theta_\psi(g, g')) \\
&= 0
\end{aligned}$$

because $\eta \wedge \theta_\psi(g, g')$ is rapidly decreasing. We need the following:

(1) find $\varphi_\infty \in C^{nq,0}$ which is holomorphic and closed.
(2) Prove $\theta_\varphi(\tau) = P(\tau, \eta)$.

If (1) is true then $\theta_\varphi(\eta)$ is a holomorphic Siegel modular form of weight m by the previous theorem.

You basically choose the finite part $\varphi_f \in S(V^n(\mathbb{A}_f))^L$ arbitrarily. If you don't get something invariant on $Y = \Gamma\backslash D$ but on $\widetilde{Y} = \widetilde{\Gamma}\backslash D$ where $\widetilde{\Gamma}$ is a finite index subgroup, then you can take invariants to get something on Y.

Construction of (1). (This is called the Kudla-Millson special Schwartz form) I will construct

$$\varphi_{KM} \in [S(V^n(\mathbb{R})) \otimes \Omega^{nq}(D)]^G = (\mathscr{S}(V^n(\mathbb{R})) \otimes \bigwedge^q (\mathfrak{p}^*)]^K.$$

Note that this is not $C^{nq,0}$ because there is no character, but we will prove that it our φ_{KM} does lie there.

(1) Define a *Howe operator*

$$\Delta \colon \mathscr{S}(V(\mathbb{R})) \otimes \bigwedge^{\bullet} \mathfrak{p}^* \to \mathscr{S}(V(\mathbb{R})) \otimes \bigwedge^{\bullet+q} \mathfrak{p}^*$$

defined by

$$\Delta = \frac{1}{2} \cdot \prod_{\mu=p+1}^{p+q} \left[\left(\sum_{\alpha=1}^{p} x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) \otimes A_{\alpha\mu} \right]$$

where $(x_1, \ldots, x_{p+q})$ are the coordinates on $V(\mathbb{R})$, $A_{\alpha,\mu}$ is the left multiplication by $\omega_{\alpha\mu}$, which was dual to $X_{\alpha\mu}$.

(2) $\varphi_q = \Delta(\varphi_0)$, where φ_0 is the standard Gaussian

$$\varphi_0 = \exp(-2\pi i \operatorname{tr}(x, x)).$$

Then $\varphi_q \in [\mathscr{S}(V(\mathbb{R})) \otimes \Omega^q(D)]^G$. Then finally

$$\varphi_{KM} = \varphi_q \wedge \varphi_q \wedge \ldots \wedge \varphi_q \in [\mathscr{S}(V(\mathbb{R})) \otimes \Omega^{nq}(D)]^G.$$

Remark 8.2.6. $\mathscr{S}(V^n(\mathbb{R})) = \{p(v_1, \ldots, v_n)\varphi_0\}$. There's an intertwining operator with $P(\mathbb{C}^{mn}) = P(z_{\alpha\mu})$ taking $\varphi_0 \mapsto 1$. Then

$$L \left(\sum_{\alpha=1}^{p} x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) L^{-1} = \frac{1}{2\pi i} z_{\alpha i}$$