ENUMERATING CURVES IN CALABI-YAU THREEFOLDS

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CONTENTS

1. Introduction

This course is about the enumerative geometry of curves in Calabi-Yau threefolds.

Definition 1.0.1. A *Calabi-Yau threefold* is a 3-dimensional projective variety X over $\mathbb{C}$ such that $K_X = \bigwedge^3 T_X^\vee \cong \mathscr{O}_X$ is trivial and $h^1(\mathscr{O}_X) = 0$.

Today we'll explain three examples that illustrate the type of questions that we're interested in.

1.1. Counting rational curves.

Let $X = Q_5$ be a quintic hypersurface in $\mathbb{P}^4$. This is a Calabi-Yau threefold, by the adjunction formula for triviality of K and the Lefschetz hyperplane theorem for triviality of $H^1(\mathscr{O}_X)$.

We are interested in the space $\mathrm{Rat}_d(Q_5)$ of rational curves of degree d in Q_5. We can think of such a curve as a map $F\colon \mathbb{P}^1 \to \mathbb{P}^4$ given by degree d polynomials and factoring through Q_5:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[dr, "F"'] & Q_5 \arrow[d] \\ & \mathbb{P}^4 \end{tikzcd}$$

Now, a map $\mathbb{P}^1 \to \mathbb{P}^4$ is given by $[f_0,\dots,f_4]$ with $f_i \in H^0(\mathscr{O}_{\mathbb{P}^1}(d)) \cong \mathbb{C}^{d+1}$, so the F are parametrized by $\tilde{f} = (f_0,\dots,f_4) \in \mathbb{C}^{5d+5}$ which do not have common zeros (which is what happens in general). The map F factors through Q_5 if $F(f_0,\dots,f_4) = 0 \in H^0(\mathscr{O}_{\mathbb{P}^1}(5d)) \cong \mathbb{C}^{5d+1}$ (by Riemann-Roch). We have constructed a map

$$H^0(\mathscr{O}_{\mathbb{P}^1}(d))^{\oplus 5} \xrightarrow{\widetilde{F}} H^0(\mathscr{O}_{\mathbb{P}^1}(5d))$$

sending $f \mapsto F(f)$. Then $\widetilde{F}^{-1}(0)$ "should" be a union of 4-dimensional cones. Note that if $f\colon \mathbb{P}^1 \to Q_5$, then for any $\alpha \in \mathrm{Aut}(\mathbb{P}^1)$ we also have that $f \circ \alpha$ factors through Q_5. So each $f\colon \mathbb{P}^1 \to Q_5$ contributes to a 4-dimensional subvariety in $\widetilde{F}^{-1}(0)$ (three from automorphisms and one from scaling), which leads to the following conjecture.

Conjecture 1.1.1 (Clemens)**.** *For very general $Q_5 \subset \mathbb{P}^4$, the space $\mathrm{Rat}_d(Q_5)$ is discrete for all $d \geq 1$.*

The classical method is to study the incidence scheme

$$\Phi_d = \{(C,F) \mid C \subset F^{-1}(0), C \in \mathrm{Rat}_d(\mathbb{P}^4)\}$$

and more precisely to study its irreducibility and fibers of the obvious projection

$$\Phi_d \xrightarrow{\pi} \mathrm{Rat}_d(\mathbb{P}^4).$$

Theorem 1.1.2. *If $d \leq 11$, then Φ_d is irreducible. For $d \geq 12$, then Φ_d is reducible.*

Irreducibility implies Clemens' conjecture ♠♠♠ TONY: [why?], but reducibility doesn't disprove it.

1.2. **Discreteness of curves in CY 3-folds.** Curves in a CY 3-fold are "expected" to be discrete. Let X be a CY 3-fold and $C \subset X$ a smooth curve. We can form the Hilbert scheme Hilb_X^P, with respect to the Hilbert polynomial

$$P(m) = \chi(\mathscr{O}_C(m)) = n \deg H|_C + (1-g)$$

(there is always an ample line bundle H on X implicitly fixed). Then Hilb_X^P parametrizes curves in X with Hilbert polynomial P. Assume $g \geq 2$ for simplicity.

We want to calculate $\dim T_{[C]} \mathrm{Hilb}_X^P$. A first-order deformation of a smooth subcurve C *in* X gives (tautologically) an abstract first-order deformation of C in moduli, so we get a map

$$\dim T_{[C]} \mathrm{Hilb}_X^P \to \mathrm{Def}(C).$$

What is the kernel? If the complex structure of C doesn't change, then the deformation must be given by a vector field, so the kernel of this map is $H^0(TX|_C)$. Also, there exists $\mathrm{Def}(C) \xrightarrow{\delta} H^1(T_X|_C)$ (which requires some thinking to see!). We claim that we have an exact sequence

$$0 \to H^0(TX|_C) \to \dim T_{[C]} \mathrm{Hilb}_X^P \to \mathrm{Def}(C) \xrightarrow{\delta} H^1(TX|_C)$$

Exercise 1.2.1. Construct the arrows in this exact sequence, and prove that it is exact.

It is "expected" that δ is surjective, so we can read off

$$\dim T_{[C]} \mathrm{Hilb}_X^P = \dim \mathrm{Def}(C) + h^0(TX|_C) - h^1(TX|_C).$$

By Riemann Roch and well-known fact, this is $(3g-3) + \deg TX|_C + \operatorname{rank} TX|_C(1-g)$, and $\deg TX|_C = 0$ since X is Calabi-Yau, so we see that this is 0.

Remark 1.2.2. If the curves in X are discrete, then we can enumerate the number of curves of genus g and degree d in X.

This hinges on δ being surjective. To get that to be the case, you can try to get this by varying the quintic in $H^0(\mathbb{P}^4, \mathscr{O}(5))$. This works for some small choices of g and d, but it is hard to arrange uniformly.

1.3. **Enumerating rational curves in a K3 surface.**

Definition 1.3.1. A *K3 surface* is a 2-dimensional Calabi-Yau manifold.

Let $X = S$ be a K3 surface. Pick $L \in \mathrm{Pic}(S)$ and $C \in |L|$ a smooth curve. Then the adjunction formula (and triviality of K_S) tells us that

$$g_a(C) = \frac{1}{2} L^2 + 1$$

Applying Riemann-Roch for surfaces and using that $T_C \otimes N_C|_S \cong \wedge^2 TX \cong \mathscr{O}_X$, and $\deg N_C|_X = C \cdot C = L^2$ we also find that

$$\dim |L| = \frac{1}{2} L^2 + 1.$$

We expect that {curves with at least k nodes in $|L|$} has codimension k, hence (expected) dimension $\dim |L| - k$. So the number of *rational* nodal $C \in |L|$, which have $\frac{1}{2}L^2 + 1$ nodes, should be finite. (There's a better argument, showing that $|L|$ must contain a rational curve, and rational curves cannot form a family.)

Theorem 1.3.2 (Chen)**.** *For general S, we have* $\mathrm{Pic}(S) \cong \mathbb{Z}[L]$ *and rational* $C \in |L|$ *are nodal.*

Yau-Zaslow found a way to count this finite number. For smooth $C \in |L|$ and $\mathscr{F} \to C$ an invertible sheaf, we can form the 1-dimensional sheaf $\iota_* \mathscr{F}$ on S.

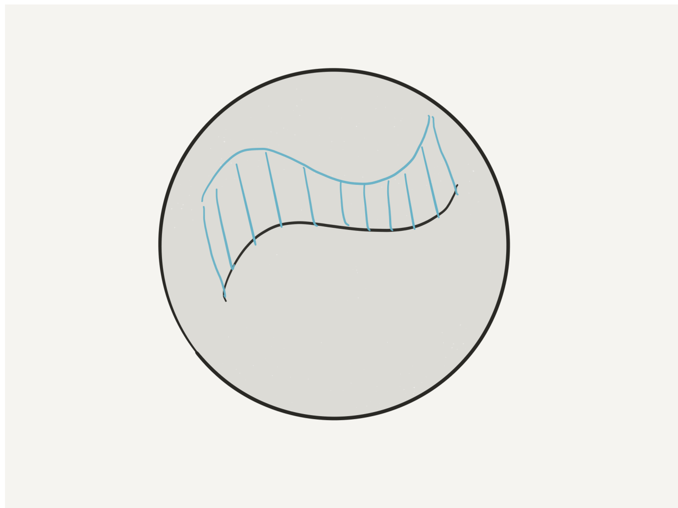

FIGURE 1.3.1. Pushforward of a line bundle from a curve in S.

Then

$$P_{\iota_*\mathscr{F}}(n) = \chi(\iota_*\mathscr{F} \otimes L^{\otimes n}) = \chi(\mathscr{F} \otimes H|_C^{\otimes n}) = n \deg L|_C + \deg \mathscr{F} + (1-g).$$

The point is that this is a linear polynomial. Suppose we assume that $\deg \mathscr{F} + (1-g)$ is an odd number, and let's even assume that it is 1. Define

$$\mathscr{M}_S(P) = \{\text{stable shaves } \mathscr{E} \text{ of } \mathscr{O}_S\text{-modules}, \ \chi(\mathscr{E}(n)) = p(n)\}.$$

There are some "very bad" 1-dimensional sheaves (Figure 1.3), and the "stability" condition is intended to rule them out.

Theorem 1.3.3. *$\mathscr{M}_S(P)$ is smooth (using that $P(0) = 1$).*

There is a map $\mathscr{M}_S(P) \xrightarrow{\pi} \mathrm{Chow}(S)$ (this is the Chow variety of curves in S, not the Chow group!) sending $\mathscr{E} \mapsto \mathrm{supp}(\mathscr{E})$. By the assumptions on the Picard group, this must land in $|L|$.

Yau and Zaslow made the very interesting observation that if $C \subset L$ is a nodal curve with geometric genus > 1, then $e(\pi^{-1}[C]) = 0$.

Exercise 1.3.4. Prove this. [Hint: If C is a smooth curve, then $\mathrm{Pic}^0(C)$ is an abelian variety. Then $\mathrm{Pic}^0(C)$ acts on $\pi^{-1}[C]$ by tensoring a vector bundle with a line bundle on C. Use this to show that $e(\pi^{-1}[C]) = 0$. In fact, we don't need the full abelian variety; it's enough to have a torus action.]

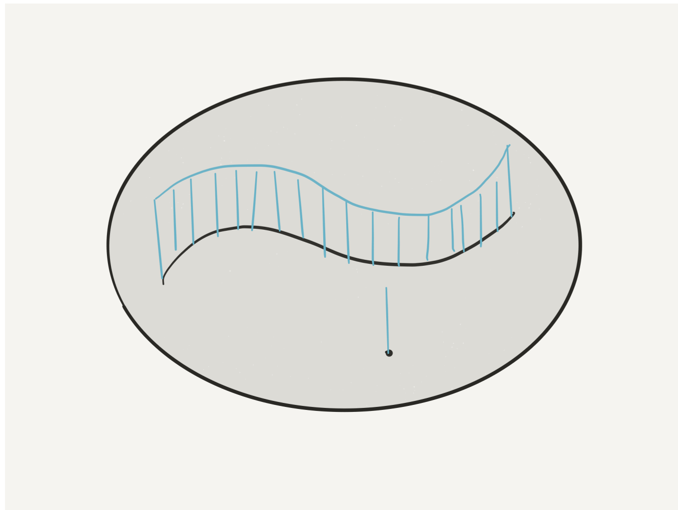

FIGURE 1.3.2. A sheaf on S which is composed of the pushforward of a line bundle curve plus a skyscraper sheaf over a point. The condition of "stability" rules out these sorts of "bad" sheaves.

Therefore,

$$e(M_X(P)) = \sum_{C \in |L| \text{ rational}} e(\pi^{-1}([C])).$$

Assume that S is general. Then any rational C in $|L|$ are necessarily nodal.

Exercise 1.3.5. It is an exercise to show that $e(\pi^{-1}[C]) = 1$ in such cases. [Hint: $\pi^{-1}([C])$ consists of $\mathscr{E}$ on $\mathbb{P}^1$ plus identifications over two pairs of fibers, say $P \longleftrightarrow P'$ and $Q \longleftrightarrow Q'$. This is the same as giving two separate identifications, hence two $\mathbb{P}^1$ each with two points glued, which have Euler number 1.]

Theorem 1.3.6. *$e(\mathscr{M}_S(P))$ can be calculated.*

Theorem 1.3.7 (Yau-Zaslow)**.** *If $n_g = \#\mathrm{Rat}_{|L|}(S)$, where $\frac{1}{2}L^2 + 1 = g$, then*

$$\sum n_g q^g = \prod_{n=1}^{\infty} (1-q^n)^{-24}.$$

These ideas actually come from string theory.

1.4. **Course plan.** The plan of the course is to study curves in Calabi-Yau threefolds. One could first attempt to do this by classical algebraic geometry. It turns out that this leads to a lot of tricky issues about the space of curves.

However, we shall see that if certain genericity is met (as in the above examples), then we can get good answers. The genericity amounts to saying that the counting is a topological enumeration. To achieve this in algebraic geometry, we use *virtual cycles* of certain moduli spaces. For this we need the moduli spaces to be proper.

Here is a sampler of the moduli spaces we're interested in:

- Moduli of stable maps:
$$\mathcal{M}_g(X,\alpha) = \{C_g^{\mathrm{sm}} \xrightarrow{f} X \mid f_*[C] = \alpha \in H_2(X,\mathbb{Z})\} \hookrightarrow \overline{\mathcal{M}_g}(X,\alpha).$$
- $C \subset X$ as subscheme
$$\{C_g^{\mathrm{sm}} \subset X \mid [C] \sim \alpha \in H_2(X,\mathbb{Z})\} \subset \mathrm{Hilb}_X^P$$
- Another compactification of curves as subschemes: any such curve gives a structure exact sequence
$$0 \to \mathcal{I}_{C\subset X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$
and we can consider the data $[\mathcal{O}_X \to \mathcal{O}_C]$ in the derived category and compactify.

The first moduli leads to GW invariants, the second leads to DT invariants, and the third leads to PT invariants.

2. Moduli spaces

We're going to sketch constructions of three types of moduli spaces.

- One is a moduli space of *framed* objects, i.e. objects equipped with an embedding in an ambient space. This includes, for instance, the Hilbert schemes. Slightly more generally, we can consider moduli of *sheaves*, which leads to the Quot schemes.
- Another kind of moduli space is that of *unframed* objects, e.g. the moduli space of curves $\mathcal{M}_g$. This was first studied by Mumford, using the strategy of quotienting a moduli space of framed objects by automorphisms.
- The third kind of moduli spaces are the *stacks*, which go beyond the realm of schemes.

2.1. **Moduli problems.**

Example 2.1.1. Consider all quotient vector spaces $\mathbb{C}^m \to P$ where $\dim P = k < m$. The space of such is parametrized by $\mathrm{Gr}(k,\mathbb{C}^m)$. This is one of the first examples of moduli spaces.

Example 2.1.2. (Hilbert schemes) Let $X \subset \mathbb{P}^n$ be projective and h a polynomial. Let $H = \mathcal{O}_{\mathbb{P}^n}(n)|_X$. We consider
$$\{Z \subset X \mid P_Z = h\}$$
where $P_z(n) = \chi(\mathcal{O}_Z \otimes H^{\otimes n})$ is the Hilbert polynomial. Usually there is no dispute about what the closed points are, but it is not necessarily clear how to define the scheme structure. Sometimes, it is appropriate to define a *nonreduced* scheme structure.

Grothendieck taught us that scheme structure is captured by considering not just closed points but *all* scheme-valued points. This leads us to consider the *moduli functor*
$$\mathcal{F} : \mathrm{Schemes} \to \mathrm{Sets}^{\mathrm{opp}}$$
sending
$$S \mapsto \left\{ Z \to S \mid \begin{smallmatrix} Z \text{ closed } \subset Z\times S \\ \text{flat over } S \\ P_{Z_s} = h \text{ for all } s\in S \end{smallmatrix} \right\}.$$
This is functorial: if $S \to S'$, then a family over S' can be pulled back to a family over S.

Exercise 2.1.3. Carefully write down a moduli functor for other moduli problems.

Definition 2.1.4. We say that M is the *fine moduli space* of $\mathscr{F}$ if there exists a natural transformation

$$\mathscr{F} \cong \mathrm{Hom}(\cdot, M)\colon \mathrm{Schemes} \to \mathrm{Sets}^{\mathrm{opp}}.$$

Here $\mathrm{Hom}(\cdot, M)(S) = \mathrm{Hom}(S, M)$.

In particular this implies that $\mathscr{F}(M) = \mathrm{Hom}(M, M)$, so there should be a family $\mathscr{X} \to M$ corresponding to $1_M \in \mathrm{Hom}(M, M)$. This $\mathscr{X} \to M$ is called the *universal family*.

By definition, any family $\mathscr{Y} \to T \in \mathscr{F}(T) = \mathrm{Hom}(T, M)$ corresponds to some $\rho\colon T \to M$.

$$\begin{tikzcd} \mathscr{Y} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ T \arrow[r, "\rho"'] & M \end{tikzcd}$$

By the definition of $\mathscr{X} \longleftrightarrow 1_M$, we have that $\mathscr{Y}$ is *canonically* isomorphic to the pullback of $\mathscr{X}$ via ρ.

Example 2.1.5. The fine moduli space for Example 2.1.1 is $\mathrm{Gr}(k, \mathbb{C}^m)$.

2.2. **Quot schemes.** Let Z be a closed subscheme of a projective variety X (with a given ample line bundle $\mathscr{O}(1)$). Then Z is defined by its ideal sheaf $\mathscr{I}_{Z\subset X}$, which fits into a "fundamental exact sequence"

$$0 \to \mathscr{I}_Z \to \mathscr{O}_X \to \mathscr{O}_Z \to 0.$$

The subsheaf is completely determined by the quotient sheaf, and it turns out to be better to think about quotients. So we are going to construct a moduli space for quotient sheaves $\mathscr{O}_X \to \mathscr{F}$ with $\chi_{\mathscr{F}} = h$, where

$$\chi_{\mathscr{F}}(n) = \chi(\mathscr{F}(n)).$$

Definition 2.2.1. A *quotient sheaf* of a fixed vector bundle $\mathscr{E}$ over X is a surjection of sheaves $\mathscr{E} \to \mathscr{F}$.

If $\mathscr{E} \to \mathscr{F}$ is a surjection over X, we can push it forward via a closed embedding $\iota\colon X \hookrightarrow \mathbb{P}^n$. Any quotient sheaf $\iota_*\mathscr{E} \twoheadrightarrow \mathscr{F}'$ over $\mathbb{P}^n$ is obtained from $\mathscr{F}' = \iota_*\mathscr{F}$. Using this fact we can punt our problems to projective space.

Definition 2.2.2. Fix a projective variety $(X, \mathscr{O}(1))$, a polynomial h, and $\mathscr{E}$ a sheaf of $\mathscr{O}_X$-modules. We define a moduli functor $\underline{\mathrm{Quot}}^h_{\mathscr{E}}\colon \mathrm{Schemes} \to \mathrm{Sets}^{\mathrm{opp}}$ sending

$$S \mapsto \left\{ p_X^*\mathscr{E} \twoheadrightarrow \mathscr{F} \mid {}^{\mathscr{F} \text{ flat over } S}_{P_{\mathscr{F}_s} = h \text{ for all } s\in S} \right\}.$$

Example 2.2.3. For $X = \mathrm{pt}$, we are consider quotients as in Example 2.1.1.

Example 2.2.4. If X is a projective scheme and $\mathscr{E} = \mathscr{O}_X$, then we are parametrizing quotients $[\mathscr{O}_X \twoheadrightarrow \mathscr{F}]$.

Theorem 2.2.5. *The moduli functor* $\underline{\mathrm{Quot}}^h_{\mathscr{E}}$ *has a projective fine moduli space, called the Quot scheme.*

Outline of proof. Let T be a scheme. Suppose we have an object $[p_X^*\mathscr{E} \twoheadrightarrow \mathscr{F}] \in \underline{\mathrm{Quot}}_{\mathscr{E}}^h(T)$. We're going to apply a standard trick of twisting by a higher power of $\mathscr{O}(1)$.

First some notation: we have a line bundle $\mathscr{O}(1)$ on X, and we denote

$$p_X^*\mathscr{E}(\mu) := p_X^*\mathscr{E} \otimes p_X^*\mathscr{O}(\mu).$$

Consider the exact sequence

$$0 \to \mathscr{H} \to p_X^*\mathscr{E} \to \mathscr{F} \to 0.$$

We can twist by μ to obtain

$$0 \to \mathscr{H}(\mu) \to p_X^*\mathscr{E}(\mu) \to \mathscr{F}(\mu) \to 0.$$

We can then push forward to T to get a long exact sequence

$$0 \to p_{T*}\mathscr{H}(\mu) \to p_{T*}p_X^*\mathscr{E}(\mu) \to p_{T*}\mathscr{F}(\mu) \to R^1 p_{T*}\mathscr{H}(\mu) \to \ldots.$$

For a given $\mathscr{H}$, we have that $R^1 p_{T*}\mathscr{H}(\mu) = 0$ for large enough choice of μ. That this choice can be made *uniformly* in $\mathscr{H}$ is highly non-obvious, and is the key ingredient in the proof. Also note that $p_{T*}p_X^*\mathscr{E}(\mu) = H^0(X, \mathscr{E}(\mu)) \otimes \mathscr{O}_T$.

Now we use the (crucial!) flatness assumption. One characterization of $\mathscr{F}$ being flatover T is that $\mathscr{F} \to X \times T$ is flat over T if for $\mu \gg 0$, $p_{T^*}\mathscr{F}(u)$ is a locally free sheaf of $\mathscr{O}_T$-modules. (This uses Serre's vanishing theorem for ample line bundles plus cohomology and base change.)

Therefore, from the original quotient $[p_X^*\mathscr{E} \twoheadrightarrow \mathscr{F}] \in \underline{\mathrm{Quot}}_{\mathscr{E}}^h(T)$ we get a point $[H^0(X, \mathscr{E}(\mu)) \otimes \mathscr{O}_T \twoheadrightarrow p_{T*}\mathscr{F}]$ in a certain Grassmannian, namely $\mathrm{Gr}(H^0(X, \mathscr{E}(\mu)), h(\mu))(T)$. This globalizes to a map

$$\xi \mapsto \psi_\mu(\xi) \colon \underline{\mathrm{Quot}}_{\mathscr{E}}^h \to \mathrm{Gr}_\mu := \mathrm{Gr}(H^0(X, \mathscr{E}(\mu)), h(\mu)).$$

This embeds the moduli functor in a Grassmannian, and we want to show that Quot schemes are cut out as a closed subscheme of the Grassmannian.

There is a significant issue outstanding here, which is that we need to argue that we can find a single μ that works for all $\xi \in \underline{\mathrm{Quot}}_{\mathscr{E}}^h(T)$. For dimension 1, this is clear by Riemann-Roch. In general, one tries to slice the dimension down and use induction. We will not give the proof.

The problem is equivalent to showng that $\underline{\mathrm{Quot}}_{\mathscr{E}}^h(\mathbb{C})$ is "bounded" in an appropriate sense. To be precise, this should mean that it is parametrized by a finite type scheme.

Lemma 2.2.6. *The map* $\psi_\mu \colon \underline{\mathrm{Quot}}_{\mathscr{E}}^h \to \mathrm{Gr}_\mu$ *is injective.*

(It doesn't follow immediately that ψ_μ is actually an embedding, but we'll show that later.)

Proof. Suppose that $\xi_1 = [\mathscr{E} \twoheadrightarrow \mathscr{F}_1]$ and $\xi_2 = [\mathscr{E} \twoheadrightarrow \mathscr{F}_2]$ are two elements of $\underline{\mathrm{Quot}}_{\mathscr{E}}^h(\mathbb{C})$ such that $\psi_\mu(\mathscr{E}_1) = \psi_\mu(\mathscr{E}_2)$. Then we want to show that $\xi_1 = \xi_2$.

In other words, we have that $H^0(\mathscr{F}_1(\mu)) = H^0(\mathscr{F}_2(\mu))$ as elements of Gr_μ. Then we want to argue that $\mathscr{F}_1 = \mathscr{F}_2$. Well, we have the exact sequences

$$0 \to \mathscr{H}_1 \to \mathscr{E} \to \mathscr{F}_1 \to 0$$

and

$$0 \to \mathscr{H}_2 \to \mathscr{E} \to \mathscr{F}_2 \to 0.$$

We have that $\mathscr{E}(\mu)$ is generated by global sections for μ sufficiently large, i.e. $H^0(\mathscr{E}(\mu))\otimes\mathscr{O}_X \twoheadrightarrow \mathscr{E}(\mu)$. The key point is that we can *also* assume for μ large that $\mathscr{K}_1(\mu)$ is generated by global sections, and that this can be arranged *uniformly in* $\mathscr{K}_1$. This implies that $\mathscr{K}_1(\mu)$ is the image of $H^0(\mathscr{K}_1(\mu))\otimes\mathscr{O}_X \to H^0(\mathscr{E}(\mu))\otimes\mathscr{O}_X \to \mathscr{E}(\mu)$, and similarly for $\mathscr{K}_2$.

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathscr{K}_1(\mu)) \arrow[r] \arrow[d] & H^0(X,\mathscr{E}(\mu)) \arrow[r] \arrow[d, equal] & H^0(\mathscr{F}_1(\mu)) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & H^0(\mathscr{K}_2(\mu)) \arrow[r] & H^0(X,\mathscr{E}(\mu)) \arrow[r] & H^0(\mathscr{F}_2(\mu)) \arrow[r] & 0 \end{tikzcd}$$

So we have shown that the equality of global section implies (for large enough μ) that $\mathscr{K}_1(\mu)=\mathscr{K}_2(\mu)$, which implies that $\mathscr{F}_1=\mathscr{F}_2$. This shows (granting μ exists with all desired vanishing) that $\mathscr{F} \xrightarrow{\psi_h} \mathrm{Hom}(-,\mathrm{Gr}_\mu)$ is injective.

□

Next, we define closed subscheme $Q_\mu \subset \mathrm{Gr}_\mu$ such that for all $\rho\colon T\to \mathrm{Gr}_\mu$, ρ factors through

$$\begin{tikzcd} T \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Gr}_\mu \\ & Q_\mu \arrow[ur] & \end{tikzcd}$$

if and only if there is a $\xi \in \underline{\mathrm{Quot}}^h_{\mathscr{E}}(T)$ such that $\rho=\psi_\mu(\xi)$, i.e.

$$\begin{tikzcd} \underline{\mathrm{Quot}}^h_{\mathscr{E}} \arrow[rr] \arrow[dr, "\cong"'] & & \mathrm{Hom}(-,\mathrm{Gr}_\mu) \\ & \mathrm{Hom}(-,Q_\mu) \arrow[ur] & \end{tikzcd}$$

Recall that the map ψ_μ was constructed by starting with

$$\xi=[0\to\mathscr{K}\to p_X^*\mathscr{E}\to\mathscr{F}\to 0]\in\underline{\mathrm{Quot}}^h_{\mathscr{E}}(T)$$

and twisting and pushing it forward to

$$0\to p_{T*}\mathscr{K}(\mu)\to p_{T*}p_X^*\mathscr{E}(\mu)\to p_{T*}\mathscr{F}(\mu)\to 0$$

The key trick is to *apply this to the universal family on* $\mathrm{Gr}_\mu =: \mathrm{Gr}$. That is a vector bundle $\mathscr{P}$ equipped with a natural surjection

$$0\to\mathscr{K}\to W\otimes\mathscr{O}_{X\times\mathrm{Gr}}\to\mathscr{P}\to 0.$$

where $W=H^0(X,\mathscr{E}(\mu))$. Taking global sections, we get (for large enough μ) an exact sequence

$$0\to K=H^0(\mathscr{K}(\mu))\to W\to H^0(\mathscr{P}(\mu))\to 0.$$

A map $X\to\mathrm{Gr}$ is determined by the information of a surjection of bundles over X

$$W\otimes\mathscr{O}_X\twoheadrightarrow\mathscr{E},$$

which we can suppose has kernel $\mathscr{K}$. Then for our choice of μ, the image of $K\to W\otimes\mathscr{O}_X\to\mathscr{E}(\mu)$ is $\mathscr{K}(\mu)\subset\mathscr{E}(\mu)$. The point is that this recovers the subsheaf $\mathscr{K}(\mu)\subset\mathscr{K}(\mu)$.

Proposition 2.2.7. *There exists a locally closed $Q \subset \mathrm{Gr}_\mu$ such that*

$$\begin{tikzcd} \underline{\mathrm{Quot}}^P_{\mathscr{E}} \arrow[rr] \arrow[dr, "\cong"'] & & \mathrm{Hom}(-, \mathrm{Gr}_\mu) \\ & \mathrm{Hom}(-, Q) \arrow[ur] & \end{tikzcd}$$

Proof. A T-point of $\underline{\mathrm{Quot}}^P_{\mathscr{E}} \to \mathrm{Hom}(-, \mathrm{Gr}_\mu)$ is given by a surjection

$$p_X^* \mathscr{E} \twoheadrightarrow \mathscr{F}.$$

We have shown how to choose a uniform μ so that this gives a T-point of Gr_μ:

$$W \otimes \mathscr{O}_{X \times \mathrm{Gr}} \twoheadrightarrow p_X^* \mathscr{E}(\mu).$$

Let $\mathscr{H}(\mu)$ be the image of

$$\mathscr{K} \otimes \mathscr{O}_{X \times \mathrm{Gr}} \to W \otimes \mathscr{O}_{X \times \mathrm{Gr}} \to p_X^* \mathscr{E}(\mu).$$

The quotient sheaf $P(\mu) := p_X^* \mathscr{E}(\mu)/\mathscr{H}(\mu)$ is not flat over *all* of Gr (as expected). What we expect is that for some closed subset $Q \subset \mathrm{Gr}$, all closed points $\omega \in Q \subset \mathrm{Gr}$ satisfy $H^0(P|_{X \times \omega}(\mu)) = P(\mu)$ and $H^{i>0}(P|_{X \times \omega}(\mu)) = 0$.

Exercise 2.2.8. Consider the map of sheaves

$$\mathscr{O}_{\mathbb{A}^2}^{\oplus 3} \to \mathscr{O}_{\mathbb{A}^2}^{\oplus 2}$$

given by

$$(a_1, a_2, a_3) \mapsto (x a_1 + y^2 a_2, x a_3).$$

Check that the loci where the rank of the quotient is equal to $0, 1, 2$ are locally closed subschemes.

Lemma 2.2.9. *Let Z be a scheme. For any vector bundles V_1, V_2 and $\varphi \colon \mathscr{O}_Z(V_1) \to \mathscr{O}_Z(V_2)$ and any r, there exists a unique maximal locally closed $Z_r \subset Z$ such that the cokernel of $O_Z(V_1)_{Z_r} \xrightarrow{\phi|_{Z_r}} \mathscr{O}_Z(V_2)|_{Z_r}$ is a rank r locally free sheaf of $\mathscr{O}_{Z_r}$-modules.*

This is a determinantal construction. This lemma plus a technical argument implies what we want. We're going to skip the details. The idea is that you push all calculations to the Grassmannian.

$\square$

It only remains to show that $Q \subset \mathrm{Gr}_\mu$ is proper. This will imply that Q is projective and since $\mathrm{Quot}^P_{\mathscr{E}} \to \mathrm{Hom}(-, Q)$ is an equivalence. This proves that $\mathrm{Quot}^P_{\mathscr{E}}$ has a fine moduli space.

Proof. We'll use the valuative criterion for properness, which says that if $C^* = C - \{\mathrm{pt}\}$, then a map $C^* \to Q$ can be extended to $C \to Q$:

$$\begin{tikzcd} C \arrow[d] \arrow[dr, dotted, "\exists"] & \\ C^* \arrow[r] & Q \end{tikzcd}$$

A map $C^* \to Q$ is equivalent to an element of $\underline{\mathrm{Quot}}^P_{\mathscr{E}}(C^*)$, which is the data of a surjection $p_X^*\mathscr{E} \twoheadrightarrow \mathscr{F}^*$ over $X \times C^*$. The game is to extend this to a flat surjection of sheaves $p_x^*\mathscr{E} \twoheadrightarrow \mathscr{F}$ over $X \times C$. Let $\mathscr{K}^* = \ker p_X^*\mathscr{E} \to \mathscr{F}^*$ over $X \times C^*$, which fits into a short exact sequence.

$$0 \to \mathscr{K}^* \to p_X^*\mathscr{E} \to \mathscr{F}^* \to 0.$$

Let $\mathscr{K} = \{s \in p_X^*\mathscr{E} \mid s|_{X\times C^*} \in \mathscr{K}^*\}$. The key point is that defining this as a subsheaf of $p_X^*\mathscr{E}$ on $X \times C$ makes it coherent and flat.

Exercise 2.2.10. Prove the flatness.

□

Example 2.2.11. Here is an example of what can go wrong if you extend too naïvely. If $\mathscr{F}^* = 0$, then you could take $\mathscr{O}_C \xrightarrow{t} \mathscr{O}_C \to k_O \to 0$. This is not a flat extension.

We have finally established (modulo some details):

Theorem 2.2.12. $\underline{\mathrm{Hilb}}^P_X$ *is represented by a projective scheme.*

3. Moduli of sheaves via GIT

3.1. **Stable sheaves.** Suppose (X, H) is a projective scheme. Let V be a holomorphic vector bundle on X. Then its sheaf of sections $\mathscr{O}_X(V)$ is locally free. This construction induces an equivalence

$$\{\text{Vector Bundles}/X\} \cong \{\text{locally free } \mathscr{O}_X\text{-modules}\}.$$

If $\iota\colon Z \hookrightarrow X$ and $\mathscr{E}$ is locally free over Z, then $\iota_*\mathscr{E}$ is a sheaf of $\mathscr{O}_X$-modules. We have $\mathscr{O}_X/\mathscr{I}_Z \cong \iota_*\mathscr{O}_Z$.

Dimension of a sheaf. We want to quantify a notion of dimension for sheaves.

Example 3.1.1. $\mathscr{O}_X/\mathscr{I}_Z$ is supported on $Z \subset X$, and so should have the same dimension as Z.

Definition 3.1.2. We say that $\mathscr{E}$ is *supported on* Z if for all $f \in I_{Z\subset X}$, multiplication by f on $\mathscr{E}$ is 0.

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "\times f"] \arrow[dr] & & \mathscr{E} \\ & 0 \arrow[ur] & \end{tikzcd}$$

Example 3.1.3. If $\mathscr{E} = k[x,y]/(x,y^2)$ then $\mathscr{E}$ is supported on $V(x,y^2)$, which has dimension 0.

Definition 3.1.4. Let $\mathscr{E}$ be a sheaf and $s \in \mathscr{E}$ a non-zero section. Then we define

$$\begin{aligned} \dim s &= \dim \mathrm{Im}(\mathscr{O}_X \xrightarrow{\times s} \mathscr{E}\} \\ &= \dim \mathrm{supp(s)}. \end{aligned}$$

Example 3.1.5. Let X be a smooth variety and $\mathscr{E}$ a sheaf. The *torsion subsheaf* is $\mathscr{E}^{\mathrm{tor}} := \{s \in \mathscr{E} \mid \dim s < \dim X\}$. The quotient $\mathscr{E}/\mathscr{E}^{\mathrm{tor}}$ is torsion-free.

Definition 3.1.6. A sheaf $\mathscr{E}$ is of *pure dimension* r if for any non-zero $s \in \mathscr{E}$, we have $\dim s = r$.

Definition 3.1.7. If $\mathscr{E}$ has pure dimension r, then its *Poincaré polynomial* is

$$P_{\mathscr{E}}(n) := \chi(\mathscr{E} \otimes H^{\otimes n}) = a_r n^r + \dots$$

which has degree r. We also define the monic normalization

$$p_{\mathscr{E}}(n) := \frac{P_{\mathscr{E}}(n)}{a_r} = n^r + \dots.$$

Example 3.1.8. If $\dim X = 1$, then

$$P_{\mathscr{E}}(n) = n(\operatorname{rank} \mathscr{E}) \deg H + \chi(\mathscr{E})$$

by Riemann-Roch, and

$$p_{\mathscr{E}}(n) = n + \frac{\chi(\mathscr{E})}{\operatorname{rank} \mathscr{E}} \frac{1}{\deg H}.$$

We can finally give the definition of stable sheaves.

Definition 3.1.9 (Simpson). A $\mathscr{E}/X$ is *stable* if and only if it is of pure dimension r and for any subsheaf $\mathscr{F} \subset \mathscr{E}$, we have

$$p_{\mathscr{F}}(n) \ll p_{\mathscr{E}}(n)$$

i.e. $p_{\mathscr{F}}(n) < p_{\mathscr{E}}(n)$ for sufficiently large n.

Proposition 3.1.10. *(i) The collection of all stable sheaves which are pure of fixed Poincaré polynomial P is bounded.*

(ii) If $\mathscr{E}$ is stable, then $\operatorname{Hom}(\mathscr{E}, \mathscr{E}) \cong \mathbb{C}$.

Proof. We give only the proof of (ii). Suppose you have $\mathscr{E} \xrightarrow{\alpha} \mathscr{E} \to \mathscr{F}$ with $\mathscr{F}$ non-zero. (Replacing α be $\alpha - c\,\mathrm{Id}$, we can ensure that the map is not surjective.) Then there is a kernel $\mathscr{K}$, which fits into the short exact sequence

$$0 \to \mathscr{K} \to \mathscr{E} \xrightarrow{\alpha} \mathscr{E} \to \mathscr{F} \to 0.$$

Since the Hilbert polynomials are equal, we must have $\mathscr{K}$ non-zero. Since $\mathscr{E}$ is stable, $p_{\mathscr{K}} \ll p_{\mathscr{E}}$. Let $\mathscr{P}$ be the cokernel of $\mathscr{K} \to \mathscr{E}$, which fits into

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{K} \arrow[r] & \mathscr{E} \arrow[rr] \arrow[dr] & & \mathscr{E} \arrow[r] & \mathscr{F} \arrow[r] & 0 \\
& & & \mathscr{P} \arrow[ur] & & &
\end{tikzcd}$$

Then by the definition of stability we also have

$$p_{\mathscr{P}} \ll p_{\mathscr{E}},$$
$$P_{\mathscr{K}} + P_{\mathscr{P}} = P_{\mathscr{E}}.$$

It is an exercise to show that these equations are incompatible. □

A consequence of the first part is:

Corollary 3.1.11. *There is a uniform μ_0 such that for $\mathscr{E}$ as in the hypotheses of the proposition, $H^0(\mathscr{E}(\mu)) \otimes \mathscr{O}_X \twoheadrightarrow \mathscr{E}(\mu)$ for all $\mu \geq \mu_0$.*

We want to make a moduli space of sheaves. You always run into trouble when making a moduli space of objects with jumping automorphisms. For stable bundles we at least understand the automorphisms, which are as small as possible.

3.2. Moduli of stable sheaves. We want to construct a moduli space for sheaves $\mathscr{E}$ with $P_{\mathscr{E}} = P$ for a fixed polynomial P. Pick a large μ such that $H^0(\mathscr{E}(\mu)) \otimes \mathscr{O}_X \twoheadrightarrow \mathscr{E}(\mu)$ for all stable $\mathscr{E}$. Then can try to construct a moduli space of stable sheaves by embedding it in a quot scheme via

$$[\mathscr{E}] \rightsquigarrow [H^0(\mathscr{E}(\mu)) \otimes \mathscr{O}_X \twoheadrightarrow \mathscr{E}(\mu)]. \tag{1}$$

Here the Quot scheme in question is $\mathrm{Quot}_{\mathscr{O}_X}^{P(\mu)}(\mathbb{C})$. For $N = P(\mu)$, let

$$\mathscr{O}_{X\times Q}^{\oplus N} \twoheadrightarrow \mathscr{P}$$

be the tautological surjection onto the universal bundle, corresponding to the map $\mathrm{Gr} \to \mathrm{Gr}$.

The "map" (1) depends on a choice of isomorphism $H^0(\mathscr{E}(\mu)) \cong \mathbb{C}^{\oplus P(\mu)}$. The ambiguity in the choice of basis is measured by $\mathrm{GL}(N)$. In fact we can shrink the automorphisms to $\mathrm{SL}(N)$. Therefore, the map takes points to SL_N-orbits:

$$\{\mathscr{E} \mid \mathscr{E} \text{ stable}, P_{\mathscr{E}} = P\} \xrightarrow{\text{one to } \mathrm{SL}(N) \text{ orbit}} Q.$$

The action of SL_n on $[\mathscr{O}^{\oplus N} \twoheadrightarrow \mathscr{E}]$ is by pre-composition. The image of φ inside Q is an $\mathrm{SL}(N)$-invariant open subscheme. The idea is to define a moduli space of stable sheaves by taking the "quotient" with respect to the $\mathrm{SL}(N)$-action.

3.3. Geometric invariant theory. We're going to discuss a simple version of GIT for $G = \mathrm{SL}(n, \mathbb{C})$.

Let V be a G-representation over $\mathbb{C}$, which we can think of in terms of a map

$$G \to \mathrm{GL}(V).$$

Then G acts on $\mathbb{P}(V) = (V - 0)/\mathbb{C}^*$. The goal is to construct a quotient scheme $\mathbb{P}V/G$, which is morally the "space of G-orbits".

Here is the key observation. We have an embedding $\mathbb{P}V \hookrightarrow \mathbb{P}H^0(\mathscr{O}(\ell))^\vee$ for $\ell > 0$, and action of G on $H^0(\mathscr{O}_{\mathbb{P}V}(\ell))^\vee$. The invariant functions on this space are $H^0(\mathscr{O}_{\mathbb{P}V}(\ell))^\vee)^G$. If ℓ is large, then $H^0(\mathscr{O}_{\mathbb{P}V}(\ell))^\vee$ becomes large and we might hope to find enough G-invariant functions to find an embedding.

$$\begin{tikzcd}
& \mathbb{P}(H^0(\mathscr{O}_{\mathbb{P}V}(\ell))^\vee)^G \arrow[d] \\
\mathbb{P}V \arrow[r] \arrow[ur, dotted] & \mathbb{P}H^0(\mathscr{O}_{\mathbb{P}V}(\ell))^\vee
\end{tikzcd}$$

The crux of the matter is then to study G-invariant sections in $H^0(\mathscr{O}(\ell))$.

Definition 3.3.1. Let $v \in V - 0$ represent $[v] \in \mathbb{P}V$. We say that

(1) $[v]$ is *semistable* if $\overline{Gv}$ does not contain 0,
(2) $[v]$ is *stable* if Gv is closed in V and $\mathrm{Stab}_G(v) < \infty$.
(3) $[v]$ is *weakly stable* if Gv is closed.

Lemma 3.3.2. *Let $X_1, X_2 \subset V$ be two disjoint G-invariant closed subsets. Then there is $s \in (\operatorname{Sym} V^\vee)^G$ such that $s|_{X_1} = 0$ and $s|_{X_2} = 1$.*

Proof. Let $I(X_1)$ be the ideal of polynomials vanishing on X_1 and define $I(X_2)$ similarly. We know that $I(X_1) + I(X_2)$ contains the element 1, so we can find

$$1 = f_1 + f_2 \tag{2}$$

with $f_1 \in I_1$ and $f_2 \in I_2$, which implies that $f_1|_{X_1} = 0$ and $f_1|_{X_2} = 1$, and $f_2|_{X_2} = 0$ and $f_2|_{X_1} = 1$. Then we apply a Reynolds operator

$$R \colon \{\text{polynomials}\} \to \{G\text{-invariant polynomials}\}.$$

At least over characteristic 0, this exists for any reductive group. (For reductive groups, GIT always amounts to careful applications of the Reynolds operators.)

Definition 3.3.3. A *Reynolds operator* is a G-equivariant map $R \colon W_i \to W_i^G$ which is

(1) functorial:

$$\begin{tikzcd} W_1 \arrow[r, "\phi"] \arrow[d, "R"'] & W_2 \arrow[d, "R"] \\ W_1^G \arrow[r] & W_2^G \end{tikzcd}$$

(2) $R(\alpha f) = \alpha R(f)$ for G-invariant α.

What is the construction of the Reynolds operator? You take the maximal compact subgroup K of G, whose tangent space complexifies to $\mathfrak{g}$. Then there is a Haar measure μ on K, and you can apply Weyl's trick to:

$$R(f)(x) = \int_K f(gx)\, dg.$$

This is obviously K-invariant. It follows that it is even G-invariant, since this is equialent to being $\mathfrak{g}$-invariant, and $\mathfrak{g}$ is the complexification of $\operatorname{Lie}(K)$.

Applying this to (2) above, we obtain

$$R(1) = R(f_1) + R(f_2)$$

where $R(f_1)|_{X_1} = 0$ and $R(f_1)|_{X_2} = 1$, and $R(f_2)|_{X_2} = 0$ and $R(f_2)_{X_1} = 1$. □

Lemma 3.3.4. *Let $v \in V$ be semistable. Then there exists ℓ and an $s \in H^0(\mathscr{O}(\ell))^G$ such that $s(v) \neq 0$.*

Let $V_\ell = H^0(\mathscr{O}(\ell))$ be the space of homogeneous polynomials of degreen ℓ on V.

Proof. Take $X_1 = \overline{G \cdot v}$ and $X_2 = \{0\}$. By the definition of semistability, they are disjoint. Apply the preceding lemma to find $s \in \operatorname{Sym}(V^\vee)^G$ such that $s|_{\overline{G \cdot v}} = 1$ and $s(0) = 0$. Write

$$s = f_1 + f_2 + \ldots + f_\ell + \ldots$$

with $f_\ell \in V_\ell^G$. So there exists ℓ such that $f_\ell(v) \neq 0$. □

Let V_{ss} be the set of semistable points in $V-\{0\}$ and U_{ss} be the image of V_{ss} in $\mathbb{P}V$. Then we know that for all $[v]\in U_{ss}$, there exists $f\in V_\ell^G$ with $f(v)\neq 0$.

Lemma 3.3.5. *There exists ℓ_0 such that when ℓ_0 divides non-zero ℓ, for any $v\in V_{ss}$ there exists $s\in V_\ell^G$ such that $s(v)\neq 0$.*

Proof. Take $R=\bigoplus_{\ell>0}H^0(\mathcal{O}(\ell))=\bigoplus V_\ell$. Then $R^G=\bigoplus_{\ell>0}V_\ell^G$. It suffices to show that R is finitely generated as an algebra over $\mathbb{C}$.

Let S^+ be the subset of G-invariant homogeneous polynomials of positive degree. We consider $S^+\cdot \mathrm{Sym}(V^\vee)$, the ideal generated by G-invariant polynomials of positive degree. It is finitely generated, say be $f_1g_1,\dots,f_ng_n$ where the f_i are homogeneous and G-invariant.

We claim that $f_1,\dots,f_N$ generate R^G. To see this, consider $h\in R^G$ with vanishing constant term. Then $h\in S^+\mathcal{O}[V]$, and we can write

$$h=\sum_{i=1}^N f_i(g_ih_i).$$

We can assume that $\deg(g_ih_i)<\deg h$. Using that f_i are homogeneous of positive degree, we have

$$h=R(h)=\sum_{i=1}^N f_iR(g_ih_i)$$

where the $R(g_ih_i)$ are G-invariant. The result follows by induction. $\square$

Now we have constructed

$$U_{ss}\xrightarrow{\Phi_\ell}\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}V}(\ell))^\vee)^G.$$

We claim that the maximal domain of Φ_ℓ is U_{ss}. Suppose $[v]\in U_{ss}$ and suppose that Φ_ℓ can be extended to $[v]$. Therefore, $Gv\subset s_\ell^{-1}(1)$ for some $s_\ell\in(H^0(\mathcal{O}_{\mathbb{P}V}(\ell))^\vee)^G$, since we can find a homogeneous polynomial taking value 1 on the image of v, and then by invariance it takes that value everywhere. But 0 is in the closure of the orbit by definition of not being semistable.

The morphism

$$\Phi_\ell\colon U_{ss}\to\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}V}(\ell))^\vee)^G$$

factors through $\mathrm{Proj}\,R^G$, where $R^G=\bigoplus_{d\geq 0}H^0(\mathcal{O}(\ell))^G$.

$$\begin{tikzcd}
U_{ss} \arrow[rr, "\Phi_\ell"] \arrow[dr] & & \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}V}(\ell))^\vee)^G \\
& \mathrm{Proj}\,R^G \arrow[ur] &
\end{tikzcd}$$

An important issue is how we know that $U_{ss}\to U_{ss}//G$ is surjective. This had better be a property of a quotient map. It is easy to show that its image is dense, so we should check the valuative criterion for properness. This *cannot* be checked at the level of U_{ss}, since that is not proper! A "sequence" of points can go off to ∞ in the G-direction, so e somehow need to use the G-action to "renormalize" it.

Proposition 3.3.6. *We have*

(1) Φ_ℓ *is surjective.*
(2) Φ_ℓ *induces a bijection from weakly stable* G*-orbits to closed points in* $\operatorname{Proj} R^G$.

Proof. We need to quote the semistable replacement theorem, which says:

Let $p \in C$ be a pointed smooth curve (C is really the spectrum of a DVR). Let $\phi : C - p \to U_{ss}$ be a morphism. Then there exists a finite base change $f : C' \to C$ and $\sigma : C' - p' \to G$ such that

$$(\phi \circ f)^\sigma = \sigma(x) \cdot \phi(x) \colon C' - p \to U_{ss}$$

extends to $C' \to U_{ss}$.

□

3.4. **The numerical criterion.** There is an effective method to characterize (semi)stable points. Again, we're going to study the special case $G = \mathrm{SL}_n$.

Definition 3.4.1. A *one-parameter subgroup* is a map $\lambda \colon \mathbb{C}^* \to G$.

There is a basis $w_1, \dots, w_n$ of V such that λ can be diagonalized as $w_i^{\lambda(t)} = t^{r_i} w_i$ for $r_1 \le r_2 \le \dots \le r_n$ and $\sum r_i = 0$.

Theorem 3.4.2 (Numerical criterion)**.** *For all* $v \in V \setminus 0$,

(1) v *is stable if and only if for all one-parameter subgroups* λ,

$$\lim_{t \to 0} \lambda(t) \cdot v = \infty.$$

(2) v *is semistable if and only if for all* λ,

$$\lim_{t \to 0} \lambda(t) \cdot v \neq 0.$$

The point is that $\lim_{t \to 0} \lambda(t) \cdot v$ is in $\overline{\mathbb{C}^* \cdot V}$ and not in $G \cdot v$ if it exists. So if $G \cdot v$ is closed (as it must be in the stable case), then this limit cannot exist. In the semistable case, it must not be 0.

Proof. Suppose that v is not semistable. Then $\overline{G \cdot v}$ contains 0.

We're going to unravel the valuative criterion for properness to see that there is a one-parameter subgroup with limit 0. The point is that if 0 is the limit of some curve, then it is the limit of the orbit of a one-parameter subgroup.

For R a DVR with fraction field K, and t a uniformizing parameter, we consider diagrams

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] & \overline{Gv} \\ \operatorname{Spec} K \arrow[r] \arrow[u] & Gv \arrow[u] \end{tikzcd}$$

Assume that $\operatorname{Spec} K \to Gv$ is represented by $\rho \colon \operatorname{Spec} K \to G$, say

$$\rho(t) = \left(a_{ij}(t)\right) \in \mathrm{SL}(N, K).$$

Then there exist $A_1, A_2 \in \mathrm{SL}(R)$ such that

$$A_1 \rho A_2^{-1} = \begin{pmatrix} t^{a_1} c_1(t) & & \\ & \ddots & \\ & & t^{a_N} c_N(t) \end{pmatrix}$$

with the $c_i(t) \in R^\times$. We can then further modify to assume that all the c_i are 1.

Suppose $\lim \rho(t) v = 0$. Since $\rho(t)V = \rho(t) A_2^{-1} A_2 v$, if we pick λ to be the one-parameter subgroup $\lambda = A_1 \rho(t) A_2^{-1}$ then

$$\lim_{t\to 0} \lambda(t) v = 0.$$

□

Apply the numerical criterion to $[\mathbb{C}^n \otimes V \xrightarrow{\phi} P] \in \mathrm{Gr}(\mathbb{C}^n \otimes V, m)$. For λ a one-parameter subgroup in $\mathrm{SL}(n, \mathbb{C})$ we can find a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$ such that $e_i^{\lambda(t)} = t^{a_i} e_i$ for $a_1 \leq \dots \leq a_n$. We want to show that $\lim_{t\to 0} \xi^{\lambda(t)} \neq 0$.

The homogeneous coordinates of ξ in terms of a basis $w_1, \dots, w_l$ of V are

$$[\phi(e_{i_1} \otimes \omega_{j_1}) \wedge \phi(e_{i_2} \otimes \omega_{j_2}) \wedge \dots \wedge \phi(e_{i_d} \wedge \omega_{j_d})]^{\lambda(t)} \neq 0 \in \wedge^d P \cong \mathbb{C}.$$

The one-parameter subgroup scales this by $t^{a_1 + \dots + a_d}$. We need to verify that for all 1-parameter subgroups λ, we have $\xi^\lambda(t) \neq 0$. So we need to find bases $e_{i_1}, \dots, e_{i_d}$ and $\omega_{j_1}, \dots, \omega_{j_d}$ such that

$$\phi(e_{i_1} \otimes \omega_{j_1}) \wedge \phi(e_{i_2} \otimes \omega_{j_2}) \wedge \dots \wedge \phi(e_{i_d} \wedge \omega_{j_d}) \neq 0$$

and $a_{i_1} + \dots + a_{i_d} \leq 0$. If you think about this problem, you'll realize that it the key is to consider the subgroups $\mathrm{Im}\, \phi(\mathrm{Span}\{e_1, \dots, e_i\} \otimes V) \subset P$.

Theorem 3.4.3 (Mumford)**.** *Let λ be a one-parameter subgroup with $e_1, \dots, e_n$ a choice of diagonalizing basis. Let $E_i = \mathrm{Span}(e_1, \dots, e_i)$. Let $H_i = \phi(E_i \otimes V) \subset P$. Then*

$$\lim_{t\to 0} \xi^{\lambda(t)} = \infty$$

if and only if for any $k < n$,

$$\frac{k}{\dim H_k} < \frac{m}{\dim P}.$$

3.5. **Construction of the moduli space.** Let $\mathscr{E} \to X$ be a vector bundle of rank 2 on a smooth curve.

Given $[H^0(\mathscr{E}(\mu)) \otimes \mathscr{O}_X \twoheadrightarrow \mathscr{E}]$, we get

$$H^0(\mathscr{E}(\mu)) \otimes \mathscr{O}_X(k) \twoheadrightarrow \mathscr{E}(k)$$

Taking global sections, we get a map *of vector spaces*

$$[H^0(\mathscr{E}(\mu)) \otimes H^0(\mathscr{O}_X(k)) \to H^0(\mathscr{E}(k+\mu))]$$

such that the $\mathrm{SL}(N)$ orbit of $[\mathbb{C}^{\oplus n} \otimes V \to P]$ is well-defined. So we want to make sense of an embedding to the quotient of Gr by $\mathrm{SL}(N)$. To do this, we need to check that the point $[\mathbb{C}^n \otimes V \twoheadrightarrow P]$ of Gr is $\mathrm{SL}(n)$-stable. Mumford's criterion shows that this is the case provided that: for any basis $e_1, \dots, e_n \in H^0(\mathscr{E}(\mu)) \cong \mathbb{C}^n$ and $E_i = \mathrm{Span}(e_1, \dots, e_i)$, $H_i = \phi(E_i \otimes V)$, we have for any $1 \le k < m$

$$\frac{k}{\dim H_k} < \frac{n}{\dim P}.$$

Okay, so fix $e_1, \dots, e_n$ and $k \in [1, n-1]$. We need to estimate

$$h_i := \dim \Phi(\mathrm{Span}(e_1, \dots, e_i) \otimes V).$$

Let $\mathscr{F}_i(\mu) = \mathrm{Span}(e_1, \dots, e_i) \subset \Gamma(\mathscr{E}(\mu))$. The point is that

$$\mathrm{Im}\, \Phi(\{e_1, \dots, e_k\} \otimes V) \subset H^0(\mathscr{F}_i(k+\mu)) \subset H^0(\mathscr{E}(k+\mu)).$$

Since $H_i \subset H^0(\mathscr{F}(k+\mu))$, Riemann-Roch shows

$$h_i \le (\text{Riemann-Roch number of } \mathscr{F}_i(k+\mu)) + h^1(\mathscr{F}_i(k+\mu)).$$

There are two cases.

(1) If $h^1(\mathscr{F}_i(\mu)) = 0$, then we get that $i \le h_i$ is at most the Riemann-Roch number of $\mathscr{F}_i(\mu)$, so

$$\begin{aligned} \frac{i}{\dim H_i} &\le \frac{RR(\mathscr{F}_i(\mu))}{\dim H_i} \\ &= \frac{RR(\mathscr{F}_i(\mu))}{RR(\mathscr{F}_i(\mu+k))} \\ &= \frac{\mathrm{rank}\, \mathscr{F}_i(\mu + \deg \mathscr{F}_i) + \chi(\mathscr{O}_X)}{\mathrm{rank}\, \mathscr{F}_i(\mu + k + \deg \mathscr{F}_i) + \chi(\mathscr{O}_X)}. \end{aligned}$$

When $\mathscr{E}$ is stable we have $\frac{\deg \mathscr{F}_i}{\mathrm{rank}\, \mathscr{F}_i} < \frac{\deg \mathscr{E}}{\mathrm{rank}\, \mathscr{E}}$, so this is $< \frac{n}{p}$. We have used that for $k \gg 0$, $\dim H_i = h^0(\mathscr{F}_i(\mu+k))$.

(2) If $H^1(\mathscr{F}_i(\mu)) \neq 0$ and $\mathrm{rank}\, \mathscr{E} = 2$, then this computation requires some H^1 inserted in. The essential case is $\mathrm{rank}\, \mathscr{F}_i = 1$. This is only possible if the degree is negative. The point is that this makes a huge gap in the inequality

$$\frac{\deg \mathscr{F}_i}{\mathrm{rank}\, \mathscr{F}_i} < \frac{\deg \mathscr{E}}{\mathrm{rank}\, \mathscr{E}}.$$

The point is that rank one subsheaves are not boundd, but the subsheaves of degree bounded from below is a bounded set.

Theorem 3.5.1. *If $\mathscr{E}$ is stable, then for $k \gg \mu \gg 0$,*

$$\begin{tikzcd} H^0(\mathscr{E}(\mu)) \otimes H^0(\mathscr{O}_X(k)) \arrow[r, two heads] \arrow[d, equal] & H^0(\mathscr{E}(k+\mu)) \arrow[d, equal] \\ \mathbb{C}^n \otimes V \arrow[r] & P \end{tikzcd}$$

is $\mathrm{SL}(n)$*-stable in the Quot scheme.*

3.6. **Coarse moduli spaces.**

Definition 3.6.1. A scheme M is a *coarse moduli space* of $\mathscr{F} : \mathrm{Sch} \to \mathrm{Set}$ if there is a transformation $\mathscr{F} \to \mathrm{Hom}(-, M)$ such that

(1) $\mathrm{Hom}(k, M) \cong \mathscr{F}(k)$ for algebraically closed k,
(2) For all N and $\mathscr{F} \to \mathrm{Hom}(-, N)$ there is a unique $\rho : M \to N$ such that the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[rr] \arrow[dr] & & \mathrm{Hom}(-, N) \\ & \mathrm{Hom}(-, M) \arrow[ur, dotted, "\exists!"] & \end{tikzcd}$$

commutes.

The uniqueness of the coarse moduli space is clear from the fact that it possesses a universal property. The point is that in constructing moduli spaces, you sometimes have to shrink into order to get something separated. So M is the space involving the least shrinkage.

We want to define $M := \mathrm{Quot}//\mathrm{SL_N}$ and call this the coarse moduli space. The issue is how to construct the transformation $\mathscr{F} \to \mathrm{Hom}(-, M)$. From a family $\mathscr{E} \to X \times S$, how do you get a map to M? If S is affine, you can twist by a large ample line bundle to make things generated by global sections, then map to Grassmannian, and prove that the map factors through the relevant locus.