APPENDIX: A REMARK ON INVERSE PROBLEMS FOR RESONANCES.

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The purpose of this appendix is to indicate how, in great generality, inverse spectral results based on *wave invariants associated to closed orbits* translate immediately to inverse results for resonances. The results presented in [5] are the strongest specific results of this type. However, weaker results, such as the determination of eigenvalues of the Poincaré map of a non-degenerate closed orbit from eigenvalues, belong to this general category – see [2] and references given there.

By *wave invariants*, or more generally speaking, semi-classical invariants, we mean coefficients appearing in expansions of traces of operators: $\operatorname{tr}\cos t\sqrt{\Delta_g}$, $\operatorname{tr} f((h^2\Delta_g + V - E)/h)$, $\operatorname{tr}(\Delta_g - k^2)^{-1} =$ (where in the first and last case, the traces are meant in suitable distributional senses, and in our convention, $\Delta_g \geq 0$). It received a lot of new attention recently, see [2],[4],[3].

For compact manifolds, or for obstacle problems, the invariants obtained using different traces are equivalent: the equivalence of last and second comes from considering $h = 1/k$ as a semi-classical parameter, and the regularization of the last trace; the first is equivalent to the second, essentially via Fourier transform in t – see [3, §5]. For simplicity we will concentrate on the wave trace here and will not discuss the more general semi-classical problem.

Thus suppose that for a compact manifold with boundary, (M, g), we have a method of extracting some infinitesimal information in a neighbourhood of a closed orbit, γ, of the (billiard) geodesic flow from the singularity of $\operatorname{tr}\cos t\sqrt{\Delta_g}$ at the multiples of the the period of γ. That only infinitesimal information can be obtained is clear from the "direct" approach to the trace formula – see references in [3]. Here Δ_g denotes a self-adjoint realization of the Laplacian on M.

The spectral theorem immediately shows that

$$2\operatorname{tr}\cos t\sqrt{\Delta_g} = \sum_{\lambda^2\in\sigma(\Delta_g)} e^{i\lambda t}\,, \tag{A.1}$$

in the sense of distributions on $\mathbb{R}$. Hence, provided that no other singularities are created at the multiples of the period of γ, the information we obtain is determined by the spectrum.

Now, let (X, h) be a Riemannian manifold with a (possibly empty) boundary, such that for some compact set, $K \Subset X$, and $R > 0$, $X \setminus K \simeq \mathbb{R}^n \setminus B(0, R)$. The resonances of X are defined as the poles of the meromorphic continuation of the resolvent: $(\Delta_h - z^2)^{-1} : L^2_{\rm comp}(X) \to L^2_{\rm loc}(X)$. Let $m_R(z)$ denote the multiplicity of the resonance at $z \in \mathbb{C}$. We adopt the convention that $m_R(z) \neq 0$ implies $\operatorname{Im} z > 0$.

The analogue of the trace formula (A.1) for resonances is non-trivial and was developed by Lax-Phillips, Bardos-Guillot-Ralston, Melrose, Sjöstrand, and Zworski – see [6] for a statement relevant here and references. We quote it, somewhat informally, as

$$2\operatorname{tr}(\cos t\sqrt{\Delta_h} - \cos t\sqrt{\Delta_{\mathbb{R}^n}}) - = \sum_{\substack{\operatorname{Im} z \leq N(z)\log|z| \\ N(z)\to\infty, =|z|\to\infty}} m_R(z)e^{i|t|z} \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\})\,, \quad t =\neq 0\,. \tag{A.2}$$

Strictly speaking, we should introduce some cut-offs to have an agreement = of domains for the two wave operators (on X and $\mathbb{R}^n$ respectively). Any choice will work and produce only smooth errors away from $t = 0$. As in the compact case, the Poisson formula (A.2) shows that under suitable assumptions wave invariants are *resonant invariants*.

The inverse problem is now reduced to calculating the wave invariants and to determining the domain from them. The main point of this note is to point out that the calculation of wave trace invariants at a periodic orbit in the non-compact case is not only similar to that in the compact case, but can actually be reduced to it. Hence inverse results in the compact case immediately imply those in the non-compact case.

Indeed, suppose that we wish to caclulate wave trace invariants at a periodic orbit γ that lies in the compact ball $K \subset X$. = We compactify as follows: for $T \gg 1$ we put

$$X_T = K \sqcup (\mathbb{R}^2/(T\mathbb{Z})^n \setminus B(0,R)) \,,$$

and denote the corresponding Laplacian by Δ_T. Then the finite speed of propagation for the wave equation shows that

$$\operatorname{tr}(\cos t\sqrt{\Delta}_h - \cos t\sqrt{\Delta}_{\mathbb{R}^n}) - \operatorname{tr}\cos t\sqrt{\Delta}_T \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\}) \,, \quad |t| < T - 2R \,. \tag{A.3}$$

To obtain an inverse result for resonances we apply the inverse procedure established for compact manifolds, increasing T as higher multiples of the period of the closed orbit are needed. Hence under the same assumptions as in the compact case, we obtain the same infinitesimal information near the orbit from resonances.

We conclude with the comment that this straightforward analogy does not hold when the singularity at $t = 0$ is used: we note that (A.2) does not hold at $t = 0$, but only in the sense of distributions on $\mathbb{R} \setminus \{0\}$. Sometimes a modified argument can be used however, see [1] and [7] – both papers use the singularity at $t = 0$ (in a non-explicit way in case of the latter).

References

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