ASYMPTOTIC DISTRIBUTION OF RESONANCES FOR CONVEX OBSTACLES.

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ABSTRACT. Let $\mathcal{O}$ be a convex $\mathcal{C}^\infty$ obstacle in $\mathbb{R}^n$ satisfying

$$\frac{\sup_{S\partial\mathcal{O}} Q}{\inf_{S\partial\mathcal{O}} Q} < (\zeta_2/\zeta_1)^{\frac{3}{2}} = 2.31186\cdots ,$$

where $S\partial\mathcal{O}$ is the sphere bundle of the boundary and Q is its second fundamental form. We show that the number of resonances of modulus less than r in a cubic neighbourhood of the continuous spectrum is given by

$$(1+o(1))\frac{\mathrm{vol}(\mathbb{B}^{n-1})}{(2\pi)^{n-1}}\mathrm{vol}(\partial\mathcal{O})r^{n-1} .$$

The size of the cubic neighbourhood is determined by the curvature and we show the existence of pole free regions above and below that neighbourhood. In other words we prove an asymptotic formula for the number of resonances in the first band corresponding to the first zero of the Airy function. Under a stronger pinched curvature hypothesis we also give the same asymptotics and pole free regions for cubic bands corresponding to any fixed number of zeros of the Airy function.

The proof is based on a second microlocal reduction to the boundary and on a trace formula for the reduced problem.

1. Introduction and statement of results

The purpose of this paper is to give asymptotics for the counting function of resonances for scattering by strictly convex $\mathcal{C}^\infty$ obstacles satisfying a pinched curvature condition. We show that the resonances lie in cubic bands and that they have Weyl asymptotics in each band. The asymptotics are in fact the same as those for eigenvalues of the Laplacian on the surface of the obstacle. The closer the pinching condition brings the obstacle to the ball the larger is the number of bands to which our result can be applied. Figure 1 illustrates the main theorem for the first band.

Heuristically, the resonances for convex bodies are created by waves creeping along the geodesics on the boundary and losing energy at a rate depending on the curvature. Consequently, the precise distribution depends in a subtle way on the dynamics of the geodesic flow of the surface and its relation to the curvature. A rigorous indication of that (for the case of analytic obstacles) was given by the first author in [25]. However, those subtle effects are mostly present in the distribution of imaginary parts of the resonances. The crude heuristic picture suggests that as far as the real parts are concerned the distribution should be governed by the same rules as those for eigenvalues of the surface and our result justifies

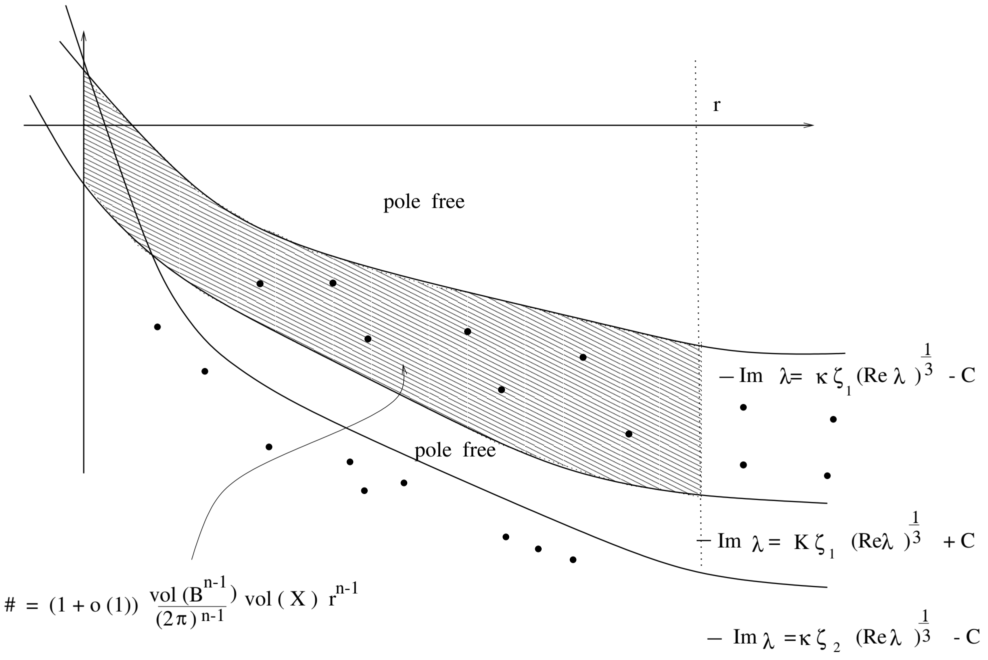

FIGURE 1. Weyl law for the density of resonances in the first band, $X = \partial\mathcal{O}$.

this claim. For our proof the pinching condition for the curvature needs to be imposed to eliminate interference between different bands.

The subject of locating and estimating resonances for convex bodies has a very long tradition. Its origins lie with the study of diffraction by spherical obstacles. The resonances of a ball are given by zeros of Hankel functions and the study of the distribution of those zeros was conducted by Watson [36], Olver [19], Nussenzweig [18] and others. For more general convex obstacles they were then studied by Fock, Buslaev, Babich-Grigoreva [1] and more recently by Bardos-Lebeau-Rauch [2], Popov [20], Filipov-Zayev [5], Hargé-Lebeau [7] and B. and R. Lascar [12]. Also, numerous general results about resonances discovered after the work of Lax-Phillips [10] apply particularly nicely to the case of scattering by convex obstacles.

This paper continues our previous work [31],[32] on upper bounds on the number of resonances in neighbourhoods of the real axis. The starting point is the same as before: we apply an exterior complex scaling argument to turn resonances into eigenvalues of a non-selfadjoint operator. We then proceed to a second microlocal reduction of the scaled problem to the boundary and establish a trace formula for the reduced problem. That trace formula is inspired by recent progress on global and local trace formulæ for resonances [6],[26],[27],[38]. It is in fact very close in spirit to the local trace formula developed by the first author in [26]. The novelty here lies in working with the reduced problem alone and in exploiting the pole free regions implied by the pinching condition on the curvature.

We recall that if $-\Delta_{\mathbb{R}^n\setminus\mathcal{O}}$ is the Dirichlet Laplacian on a connected exterior domain $\mathbb{R}^n\setminus\mathcal{O}$, where $\mathcal{O}$ is compact with a $\mathcal{C}^\infty$ boundary, then the resolvent

$$R_{\mathcal{O}}(\lambda) \stackrel{\rm def}{=} (-\Delta_{\mathbb{R}^n\setminus\mathcal{O}}-\lambda^2)^{-1} \; : \; L^2(\mathbb{R}^n\setminus\mathcal{O}) \longrightarrow H^2(\mathbb{R}^n\setminus\mathcal{O})\cap H^1_0(\mathbb{R}^n\setminus\mathcal{O})\,, \quad \operatorname{Im}\lambda>0\,,$$

continues meromorphically accross continuous spectrum $\operatorname{Im}\lambda = 0$, to an operator

$$R_{\mathcal{O}}(\lambda) \; : \; L^2_{\rm comp}(\mathbb{R}^n\setminus\mathcal{O}) \longrightarrow H^2_{\rm loc}(\mathbb{R}^n\setminus\mathcal{O})\cap H^1_{0,\rm loc}(\mathbb{R}^n\setminus\mathcal{O})\,.$$

Here $H^2(\mathbb{R}^n\setminus\mathcal{O})$ is the standard Sobolev space, $H^1_0(\mathbb{R}^n\setminus\mathcal{O})$ is the closure of $\mathcal{C}^\infty_{\rm comp}(\mathbb{R}^n\setminus\mathcal{O})$ in the H^1-norm and $L^2_{\rm com}$, $H^2_{\rm loc}$, $H^1_{0,\rm loc}$ are defined from these spaces in the usual way. We recall that $R_{\mathcal{O}}$ is globally meromorphic in $\lambda\in\mathbb{C}$ when n is odd and in $\lambda\in\Lambda$, the logarithmic plane, when n is even. In this paper we will be considering *strictly convex* obstacles.

The poles of $R_{\mathcal{O}}$ are called *resonances* or *scattering poles.* We will use the following notation

$$\begin{aligned} M_{\mathcal{O}}(\lambda_0) = & \text{ the multiplicity of the pole of } R_{\mathcal{O}}(\lambda) \text{ at } \lambda=\lambda_0\neq 0 \\ = & \operatorname{rank} \oint_{|\lambda-\lambda_0|=\epsilon} R(\lambda)d\lambda\,, \quad 0<\epsilon\ll 1\,. \end{aligned} \tag{1.1}$$

Let us also recall the Airy function, $Ai(t)$, which solves $(D_t^2+t)Ai(t)=0$ and is bounded for $t>0$ (that determines it up to a multiple). We will denote the zeros of $Ai(-t)$ by

$$0<\zeta_1<\zeta_2<\cdots<\zeta_j<\cdots\,. \tag{1.2}$$

We will first state our result under the weakest hypothesis:

Theorem 1.1. *Let ζ_j be as in* (1.2)*, Q be the second fundamental form of $\partial\mathcal{O}$ and $S\partial\mathcal{O}$ the sphere bundle of $\partial\mathcal{O}$. Assume that*

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < (\zeta_2/\zeta_1)^{\frac{3}{2}} = 2.31186\cdots\,. \tag{1.3}$$

Let us put

$$\kappa = 2^{-\frac{1}{3}}\cos\frac{\pi}{6}\min_{S\partial\mathcal{O}} Q^{\frac{2}{3}}\,, \quad K = 2^{-\frac{1}{3}}\cos\frac{\pi}{6}\max_{S\partial\mathcal{O}} Q^{\frac{2}{3}}\,.$$

Then for every sufficiently large $C>0$

$$\begin{aligned} \sum\Big\{M_{\mathcal{O}}(\lambda) \; : \; 0\leq \operatorname{Re}\lambda\leq r\,,\; \kappa\zeta_1(\operatorname{Re}\lambda)^{\frac{1}{3}}-C < -\operatorname{Im}\lambda < K\zeta_1(\operatorname{Re}\lambda)^{\frac{1}{3}}+C\Big\} \\ = (1+o(1))\frac{\operatorname{vol}(\mathbb{B}^{n-1}(0,1))}{(2\pi)^{n-1}}\operatorname{vol}(\partial\mathcal{O})r^{n-1}\,. \end{aligned}$$

The only previous lower bound was based on the Gevrey 3 trace formula from [2] (in the analytic case and in odd dimensions) and it was of the order $r^{\frac{2}{3}-\epsilon}$ – see [29]. It applies however to a generic class of obstacles which do not necessarily satisfy the pinched curvature assumption.

Theorem 1.1 is a special case of Theorem 1.2 below but we stated it separately for the clarity of exposition. To put it in perspective we recall now some general facts about counting of resonances for obstacles. The only general lower bound is that of Lax-Phillips [11]:

$$\sum\{M_{\mathcal{O}}(\lambda) \; : \; C\leq|\operatorname{Im}\lambda|\leq r\,, \quad \operatorname{Re}\lambda=0\}\geq c_n(r_I(\mathcal{O}))^{n-1}r^{n-1}\,,$$

where $\mathcal{O} \subset \mathbb{R}^n$ is any obstacle with a smooth boundary, n is odd and $r_I(\mathcal{O})$ is the inscribed radius of $\mathcal{O}$. As far as the resonances on the imaginary axis are concerned this bound is optimal as shown by the case of star shaped obstacles where one has the same upper bound but with the inscribed radius replaced by the superscribed one. Only a weak lower bound for resonances off the imaginary axis seems to be known. It is shown in [30] that

$$\sum \{M_{\mathcal{O}}(\lambda) \; : \; C \leq |\lambda| \leq r \,, \quad \operatorname{Re}\lambda \neq 0\} \geq r^{n-1-\epsilon}/C_\epsilon \,, \quad \epsilon > 0 \,,$$

for the Dirichlet boundary condition when $n = 4k+1$ and for the Neumann boundary condition when $n = 4k-1$.

A global optimal upper bound was provided by Melrose [15]:

$$\sum \{M_{\mathcal{O}}(\lambda) \; : \; |\lambda| \leq r\} \leq C r^n + C \,, \; n \text{ odd} \,.$$

This was generalized, in a suitable form, to even dimensions by Vodev [35] while a different proof in the odd dimensional case, more in the spirit of the present paper, was given by the authors in [28]. Obtaining a lower bound r^n in general consitutes an outstanding problem. Obstacles for which there is a lot of trapping are perhaps the most likely to yield first. In that direction we have shown [29] that if there exists a non-degenerate closed trajectory of the broken geodesic flow of $\mathbb{R}^n \setminus \mathcal{O}$ such that no essentially different closed trajectory has the same period, then

$$\sum \{M_{\mathcal{O}}(\lambda) \; : \; |\lambda| \leq r + C \,, \; |\operatorname{Im}\lambda| < C \log|\lambda|)\} \geq r/C \,.$$

When the trajectory is non-degenerate and elliptic then there exist $x \mapsto T(x)$ and $C > 0$ such that

$$\sum \{M_{\mathcal{O}}(\lambda) \; : \; |\lambda| \leq r + C \,, \; |\operatorname{Im}\lambda| < T(|\lambda|)\} \geq r^n/C \,, \quad T(x) = \mathcal{O}(x^{-\infty}) \,,$$

as shown recently by Stefanov [33] who improved the linear lower bound of Tang and the second author [34].

For more information on resonances in obstacle scattering and in many other settings we refer to recent surveys [16],[26] and [37].

Before stating our full result let us recall the result of [7] and [32] on the pole free region for an arbitrary strictly convex obstacle. In the notation of Theorem 1.1, it says that for some constant C the region

$$C \leq \operatorname{Re}\lambda \,, \quad 0 \leq -\operatorname{Im}\lambda \leq \kappa\zeta_1(\operatorname{Re}\lambda)^{\frac{1}{3}} - C \,,$$

is *free* of resonances. When a pinching condition on the curvature is assumed we now obtain alternating cubic bands free and full of resonances:

Theorem 1.2. *With the notation of Theorem 1.1 we assume that for some $j_0 \geq 1$*

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < \left(\frac{\zeta_{j_0+1}}{\zeta_{j_0}}\right)^{\frac{3}{2}} \,. \tag{1.4}$$

Then for some $C > 0$ and for all $0 \leq j \leq j_0$
i) We have no resonances in

$$C \leq \operatorname{Re}\lambda \,, \quad K\zeta_j(\operatorname{Re}\lambda)^{\frac{1}{3}} + C < -\operatorname{Im}\lambda < \kappa\zeta_{j+1}(\operatorname{Re}\lambda)^{\frac{1}{3}} - C$$

ii)

$$\sum \Big\{ M_{\mathcal{O}}(\lambda) \ : \ |\lambda| \leq r \, , \ \kappa \zeta_j (\operatorname{Re} \lambda)^{\frac{1}{3}} - C < - \operatorname{Im} \lambda < K \zeta_j (\operatorname{Re} \lambda)^{\frac{1}{3}} + C \Big\}$$
$$= (1 + o(1)) \frac{\operatorname{vol}(\mathbb{B}^{n-1}(0,1))}{(2\pi)^{n-1}} \operatorname{vol}(\partial \mathcal{O}) r^{n-1} \, .$$

Except for the review of exterior complex scaling in Sect.3 and for the references to some applications of the FBI transform in Sect.7 and 10, we attempted to make the paper essentially self-contained. It is organized as follows. In Sect.2 we present a general outline of the proof. Sect.4 is devoted to general theory of semiclassical second microlocalization with respect to a hypersurfaces. It is developed in a slightly more general form but with applications to further sections in mind. In Sect.5 we discuss Grushin problems for model ordinary differential problems: they indicate how a reduction to the boundary should proceed. Sect.6 introduces symbol classes adapted to the second microlocal calculus and to the ordinary differential model problems. They will give the parametrix valid near the boundary. In Sect.7 we give estimates which will allow treatment away from the boundary and in Sect.8 the methods of Sections 4,6 and 7 will produce a global well posed Grushin problem which will realize the reduction to the boundary. In other words we will reduce the study of resonances to the study of a well understood operator in a second microlocal pseudo-differential class on the boundary. In Sect.9 we prove a trace formula for that operator and then in Sect.10 we use it to prove the local semi-classical version of our result.
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2. Outline of the proof

The purpose of this section is to provide a broad outline of the proof. Some of the ideas presented here come from older works and the detailed references are given in corresponding sections.

The first step of the argument is a deformation of $\mathbb{R}^n \setminus \mathcal{O}$ to a totally real submanifold, Γ, with boundary $\partial\Gamma = \partial\mathcal{O}$ in $\mathbb{C}^n$. The Laplacian $-\Delta_{\mathbb{R}^n \setminus \mathcal{O}}$ on $\mathbb{R}^n \setminus \mathcal{O}$ can be considered as a restriction of the holomorphic Laplacian on $\mathbb{C}^n$ and it in turn restricts to an operator on Γ, $-\Delta_\Gamma$. When Γ is equal to $e^{i\theta_0}\mathbb{R}^n$ near infinity then the resonances of $-\Delta_{\mathbb{R}^n \setminus \mathcal{O}}$ (that is the poles of the meromorphic continuation of $(\Delta_{\mathbb{R}^n \setminus \mathcal{O}} - \zeta)^{-1}$; we take the Dirichlet boundary conditions) coincide with the complex eigenvalues of Δ_Γ in a conic neighbourhood of $\mathbb{R}$. That is the essence of the well known complex scaling method adapted to this setting.

Normal geodesic coordinates are obtained by taking x' as coordinates on $\partial\mathcal{O}$ and x_n as the distance to $\partial\mathcal{O}$. In these coordinates the Laplacian near the boundary is approximated by

$$D_{x_n}^2 - 2x_n Q(x', D_{x'}) + R(x', D_{x'}) \tag{2.1}$$

where R is the induced Laplacian on the boundary and the principal symbol of Q is the second fundamental form of the boundary. The complex deformation near the boundary can be obtained by rotating x_n in the complex plane: $x_n \mapsto e^{i\theta}x_n$ which changes (2.1) to

$$e^{-2i\theta}D_{x_n}^2 - 2e^{i\theta}x_n Q(x', D_{x'}) + R(x', D_{x'})\,. \tag{2.2}$$

The natural choice of θ comes from the homogeneity of the equation: $\theta = \pi/3$. This is discussed in Sect.3.

It is also natural to work in the semi-classical setting, that is, to consider resonances of $-h^2\Delta_{\mathbb{R}^n\setminus\mathcal{O}}$ near a fixed point, say 1. Letting $h \to 0$ gives then asymptotic information about resonances of $-\Delta_{\mathbb{R}^n\setminus\mathcal{O}}$.

Hence we are lead to an operator which near the boundary is approximated by

$$P_0(h) = e^{-2\pi i/3}((hD_{x_n})^2 + 2x_n Q(x', hD_{x'})) + R(x', hD_{x'})\,, \tag{2.3}$$

and we are interested in its eigenvalues close to 1.

As in many diffraction problems the natural homogeneity of the leading part near the boundary plays a crucial rôle. To indicate it here let us take the principal symbol of (2.3) in the tangential variables. That gives

$$p_0(h) = e^{-2\pi i/3}((hD_{x_n})^2 + 2x_n Q(x', \xi')) + R(x', \xi')\,.$$

We are interested in the invertibility of $P_0(h) - \zeta$ for ζ close to 1 and that should be related to invertibility of the operator valued symbol $p_0(h) - \zeta$. We rewrite it as

$$\begin{gathered} p_0(h) - \zeta = h^{\frac{2}{3}}\left(e^{-2\pi i/3}(D_t^2 + t\mu) + \lambda - z\right)\,, \\ t = h^{-\frac{2}{3}}x_n\,,\ \lambda = h^{-\frac{2}{3}}(R(x', \xi') - 1)\,,\ z = h^{-\frac{2}{3}}(\zeta - 1)\,,\ \mu = 2Q(x', \xi')\,, \end{gathered} \tag{2.4}$$

that is, we rescale the variables using the natural homogeneity of $p_0(h) - \zeta$. This gives us the model operator which will be studied in Sect.5.

On the symbolic level the operator (2.3) can be analyzed rather easily. We can describe $(p_0(h) - \zeta)^{-1}$ using the Airy function:

$$(D_t^2 + t)Ai(t) = 0\,,\quad Ai(-\zeta_j) = 0\,,\quad Ai \in L^2([0, \infty))\,.$$

The resulting separation of variables is described in terms of a *Grushin problem* in Sect.5 so that it becomes "stable under perturbations". Because of the rescaling, the symbol class of the inverse is very bad in the original coordinates: we lose $h^{-\frac{2}{3}}$ when differentiating in the direction transversal to the hypersurface $R - 1 = 0$. To remedy that we develop, in Sect.4, a form of a semi-classical second microlocal calculus with respect to a hypersurface. That allows operators with symbols given in the rescaled coordinates.

From separation of variables (or more correctly from the Grushin problem), the invertibility of $(P_0(h) - (1 + h^{\frac{2}{3}}z))^{-1}$ for $|\operatorname{Im} z| \leq C$ is controlled by invertibility of an operator on the boundary with the principal symbol given by

$$\begin{gathered} E_{-+}^0 \in \operatorname{Hom}(\mathbb{C}^N, \mathbb{C}^N)\,,\quad (E_{-+}^0)_{1\leq i,j\leq N} = -(\lambda - z + \mu^{\frac{2}{3}}e^{-2\pi i/3}\zeta_j)\delta_{ij}\,, \\ \lambda = h^{-\frac{2}{3}}(R(x', \xi') - 1)\,,\ \mu = 2Q(x', \xi')\,. \end{gathered} \tag{2.5}$$

Here N depends on C which controls the range of $\operatorname{Im} z$. The pinching condition on the curvature (1.3) implies that the values of z's for which $\lambda - z + \mu^{\frac{2}{3}} e^{-2\pi i/3}\zeta_1$ is not invertible are separated from the values for which $\lambda - z + \mu^{\frac{2}{3}} e^{-2\pi i/3}\zeta_j$, $j > 1$ are not invertible. The stronger condition (1.4) provides more bands separated from each other and corresponding to different j's – see Fig.2. This separation is eventually responsible for the pole free strips between the bands. After rescaling to the original coordinates the strips become the cubic regions in *i)* of Theorem 1.2

The passage to a global operator on the boundary, $E_{-+}(z)$, with poles of $E_{-+}(z)^{-1}$ corresponding to the rescaled resonances is rather delicate. In Sect.6 we develop a symbolic calculus which takes into account lower order terms near the boundary and in Sect.7 we give the needed estimates away from the boundary, where nothing interesting is happening. These are put together in Sect.8 and the operator $E_{-+}(z)$ is described in Theorem 8.1. In a suitable sense it is close to the model operator E^0_{-+} described above. It has to be stressed that a restriction on the range of $\operatorname{Re} z$ has to be made: for every large constant L we construct a different $E_{-+}(z)$ which works for $|\operatorname{Re} z| \leq L$. The properties of the leading symbol remain unchanged but the lower order terms and the symbolic estimates depend on L.

In Sect.9 we give a trace formula for $E_{-+}(z)$. For that we start with the obvious observation that the trace of the integral of $E_{-+}(z)^{-1}(d/dz)E_{-+}(z)$ against a holomorphic function f over a closed curve, $\sum_{i=1}^4 \gamma_i$, shown in Fig.2 gives the sum of values of f at resonances enclosed by the curve. Assuming that $|f(z)| \leq 1$ near the vertical sides and adding a normalizing term to guarantee the trace class property we can reduce the integration to the horizontal sides, at the expense of a controllable error – see Theorem 9.1 for the precise statement. That argument involves a further Grushin reduction, a local lower modulus theorem and a good choice of contours. The gain is in obtaining an integral in the region where the operator $E_{-+}(z)$ is elliptic (roughly speaking in the pole free region). In Sect.10 we use the second microlocal functional calculus of Sect.4 to understand those integrals. Finally a good choice of f, small near the vertical sides of the contour and large near the horizontal ones yields the asymptotic formula for resonances in the bands when we let $L \to \infty$.

3. **Preliminaries**

We recall here the facts about the structure of the scaled operator for the strictly convex exterior problem – we refer to [31] and [32] for more details.

We start by recalling the form of the Laplacian in normal geodesic coordinates with respect to the boundary: if we introduce the following coordinates on $\mathbb{R}^n \setminus \mathcal{O}$ then

$$\begin{gathered} x = (x', x_n) \mapsto x' + x_n \vec{n}(x'), \quad x' \in \partial\mathcal{O}, \quad x_n = d(x, \partial\mathcal{O}), \\ \vec{n}(x') \in N_{x'}\partial\mathcal{O}, \quad \|\vec{n}(x')\| = 1, \end{gathered} \tag{3.1}$$

where $\vec{n}(x')$ is the exterior unit normal to $\mathcal{O}$ at x', then

$$-\Delta = D_{x_n}^2 + R(x', D_{x'}) - 2x_n Q(x_n, x', D_{x'}) + G(x_n, x')D_{x_n}, \tag{3.2}$$

where $R(x', D_{x'}), Q(x_n, x', D_{x'})$ are second order operators on $\partial\mathcal{O}$ and

$$\begin{gathered} R(x', D_{x'}) = -\Delta_{\partial\mathcal{O}} = \bar{g}^{-\frac{1}{2}} \sum_{i,j=1}^{n-1} D_{y_i} \bar{g}^{\frac{1}{2}} g^{ij} D_{y_j}\,, \quad \bar{g} = (\det(g^{ij})^{-1}\,, \\ \text{is the Laplacian with respect to the induced metric on } \partial\mathcal{O}\,. \end{gathered} \tag{3.3}$$

We also note that if we put $Q(x', D_{x'}) = Q(0, x', D_{x'})$ then $Q(x', D_{x'})$ is of the form

$$\bar{g}^{-\frac{1}{2}} \sum_{i,j=1}^{n-1} D_{y'_j} \bar{g}^{\frac{1}{2}} a_{ij} D_{y'_i} \tag{3.4}$$

in any local coordinates. We recall that $\bar{g}$ gives the Riemannian density, $\bar{g}^{\frac{1}{2}} dx$ on $\partial\mathcal{O}$. We observe that demanding that $Q(x', D_{x'})$ is of the form (3.4) and self-adjoint with respect to the induced Riemannian measure determines $Q(x', D_{x'})$ uniquely from the quadratic form $Q(x', \xi') = \sum_{i,j=1}^{n-1} a_{ij}(x) \xi_i \xi_j$. We have

$$Q(x', \xi') \ \text{ is the second fundamental form of } \partial\mathcal{O} \text{ lifted by duality to } T^*\partial\mathcal{O}\,, \tag{3.5}$$

and the principal curvatures of $\partial\mathcal{O}$ are the eigenvalues of the quadratic form $Q(x', \xi')$ with respect to the quadratic form $R(x', \xi')$ – see Lemma 2.1 of [32] for a direct argument.

As in Sect.1 we will also use Q to denote the second fundamental form on $T\partial\mathcal{O}$ identifying it with the quadratic form on $T^*\partial\mathcal{O}$ by duality.

The complex scaling for the exterior problem was considered in [31] and then in [7], [32] and [24]. It is given by

$$\mathbb{R}^n \setminus \mathcal{O} \ni x \longmapsto z = x + i\theta(x) f'(x) \in \Gamma \subset \mathbb{C}^n \setminus \mathcal{O}\,, \tag{3.6}$$

where $f(x) = d(x, \partial\mathcal{O})^2/2$. Following Hargé and Lebeau [7] we scale by the angle $\frac{\pi}{3}$ near the boundary:

$$\frac{1 + i\theta(x)}{|1 + i\theta(x)|} = e^{i\frac{\pi}{3}}\,, \quad d(x, \partial\mathcal{O}) < \frac{1}{C}\,,$$

and then connect to the scaling with a smaller angle $\theta(x) = \theta_0$ – see [32], (2.13).

The image of (3.6) is a totally real submanifold with boundary, $\Gamma \subset \mathbb{C}^n \setminus \mathcal{O}$, and as in our previous works starting with [28], we define $-\Delta_\Gamma$ as the restriction of the holomorphic Laplacian $-\Delta_z = \sum_{j=1}^n D_{x_j}^2$ on $\mathbb{C}^n$ to Γ. As domain we take the natural space corresponding to the Dirichlet boundary condition: $H_0^1(\Gamma) \cap H^2(\Gamma)$. Following the long tradition of complex scaling in mathematical physics we showed in [31] that

> The poles of the meromorphic continuation of $(-\Delta_{\mathbb{R}^n \setminus \mathcal{O}} - \zeta)^{-1}$ in $0 < -\arg\zeta < 2\theta_0$, are given, with the agreement of multiplicities, by the complex eigenvalues of the Dirichlet realization of $-\Delta_\Gamma$ in the same region.

Hence we have reduced the problem to studying the spectrum of $-\Delta_\Gamma$ and it is convenient to work in the semi-classical setting. Thus we introduce

$$\begin{aligned} P(h) &\stackrel{\text{def}}{=} -h^2\Delta_\Gamma \\ &= e^{-\frac{2\pi i}{3}}\left((hD_{x_n})^2 + 2x_n Q(x_n, x', hD_{x'}; h)\right) \\ &\qquad + R(x', hD_{x'}; h) + hF(x_n, x')hD_{x_n}\,, \end{aligned} \tag{3.7}$$

where the second equality is valid only near the boundary. The additional dependence on h in Q and R comes from lower order terms arising in semi-classical quantization of symbols Q and R. We used here (x', x_n) as coordinates on Γ and where no confusion is likely to arise we will in fact identify Γ with $\mathbb{R}^n \setminus \mathcal{O}$ through these coordinates.

4. Second microlocalization with respect to a hypersurface

In our analysis we will encounter symbols with a non-classical behaviour when differentiated transversally to the glancing hypersurface $\Sigma = \{R(x,\xi) = 1\} \subset T^*\partial\mathcal{O}$. Since we lose $h^{-2/3}$ at each differentiation, the standard quantization will not give us a good calculus of the corresponding operators. Roughly speaking we cannot localize to regions of diameter $h^{2/3}$ without contradicting the uncertainty principle. Instead we locally straighten out Σ by means of canonical trasformations to $\Sigma_0 = \{\xi_1 = 0\}$. Then we may localize to rectangles in the (x_1, ξ_1)-space of length ~ 1 in x_1 and of length $\sim h^{\frac{2}{3}}$ in ξ_1. This amounts to a form of semi-classical second microlocalization. The presentation here is essentially self-contained and we refer to [23],[13],[17] and [3] for other approaches and references.

Let X be a compact $\mathcal{C}^\infty$ manifold and let $\Sigma \subset T^*X$ be a $\mathcal{C}^\infty$ compact hypersurface. We recall the standard class of semi-classical symbols on T^*X:

$$S^{m,k}(T^*X) = \{a \in \mathcal{C}^\infty(T^*X \times (0,1])\ :\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha\beta} h^{-m}\langle\xi\rangle^{k-|\beta|}\}\,.$$

We note that the symbols are tempered as $h \to 0$ (we make that assumption about any h-dependent functions appearing below).

The more general class, $S^{m,k}_\delta$, where the right hand side in the estimate is replaced by $C_{\alpha\beta}h^{-m-\delta(|\alpha|+|\beta|)}\langle\xi\rangle^{k-(1-\delta)|\beta|+\delta|\alpha|}$ has nice quantization properties as long as $0 \leq \delta < 1/2$. For any $0 \leq \delta < 1$ we now define a class of symbols associated to Σ:

$$a \in S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X) \Longleftrightarrow \begin{cases} \underline{\text{near } \Sigma}\ \ V_1\cdots V_{l_1} W_1 \cdots W_{l_2} a = \mathcal{O}(h^{-m-\delta l_2}\langle h^{-\delta} d(\Sigma, \bullet)\rangle^{k_1})\,, \\ \text{where } V_1, \cdots, V_{l_1} \text{ are vector fields tangent to } \Sigma \text{ and} \\ W_1, \cdots W_{l_2} \text{ are any vectorfields} \\ \\ \underline{\text{away from } \Sigma}\ \ \partial_x^\alpha\partial_\xi^\beta a(x,\xi;h) = \mathcal{O}(h^{-m-\delta k_1}\langle\xi\rangle^{k_2-|\beta|})\,. \end{cases} \tag{4.1}$$

By the distance, $d(\Sigma, \bullet)$ we mean the absolute value of any defining function of Σ, that is a modulus of a function which vanishes simply on Σ and which behaves like $\langle\xi\rangle$ away from Σ (near the infinity in T^*X) – we will use the second property below. We will later use a notation where the indices in $S^{m,k_1,k_2}_{\Sigma,\delta}$ are replaced by an order function and that will be particularly useful in the case of vector valued symbols. In the case above the order function is $h^{-m}\langle\xi\rangle^{k_2-k_1}\langle h^{-\delta} d(\Sigma, \bullet)\rangle^{k_1}$.

The basic properties of second microlocalization are described in the following proposition:

Proposition 4.1. *With the definitions introduced above and for $0 \leq \delta < 1$, there exist a class of operators, $\Psi^{m,k_1,k_2}_{\Sigma,\delta}(X)$, acting on $\mathcal{C}^\infty(X)$, and maps*

$$\begin{aligned} \mathrm{Op}_{\Sigma,h} &: S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X) \longrightarrow \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X) \\ \sigma_{\Sigma,h} &: \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X) \longrightarrow S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)/S^{m-1+\delta,k_1-1,k_2-1}_{\Sigma,\delta}(T^*X) \end{aligned}$$

such that

$$\sigma_{\Sigma,h}(A \circ B) = \sigma_{\Sigma,h}(A)\sigma_{\Sigma,h}(B) \tag{4.2}$$

$$0 \longrightarrow \Psi^{m-1+\delta,k_1-1,k_2-1}_{\Sigma,\delta}(X) \longrightarrow \Psi^{m,k_1,k_2}_{\Sigma,\delta}(X) \xrightarrow{\sigma_{\Sigma,h}} S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)/S^{m-1+\delta,k_1-1,k_2-1}_{\Sigma,\delta}(T^*X) \longrightarrow 0 \tag{4.3}$$

is a short exact sequence and

$$\sigma_{\Sigma,h} \circ \mathrm{Op}_{\Sigma,h} \ : \ S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X) \longrightarrow S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)/S^{m-1+\delta,k_1-1,k_2-1}_{\Sigma,\delta}(T^*X) \tag{4.4}$$

is the natural projection map. If $a \in S^{m,k_1,k_2}_{\Sigma,\delta}(T^*X)$ and $d(\operatorname{supp} a, \Sigma) \geq 1/C$ then $a \in S^{m+\delta k_1,k_2}(T^*X)$ and $\mathrm{Op}_{\Sigma,h}(a) = \mathrm{Op}_h(a) \in \Psi^{m+\delta k_1,k_2}(X)$ where $\Psi^{\bullet,\bullet}(X)$ is the standard class of semi-classical pseudodifferential operators on X.

To define $\Psi^{m,k_1,k_2}_{\Sigma,\delta}(X)$ we proceed locally and put Σ into a normal form $\Sigma_0 = \{\xi_1 = 0\}$ (locally). If we have a symbol, a, defined in a neighbourhood of some fixed point on $\{\xi_1 = 0\}$, say, $(0,0)$, then we can write $a = a(x, \xi, \lambda; h)$, $\lambda = h^{-\delta}\xi_1$, so that near $\{\xi_1 = 0\}$, (4.1) becomes

$$\partial_x^\alpha \partial_\xi^\beta \partial_\lambda^k a(x,\xi,\lambda;h) = \mathcal{O}(h^{-m})\langle\lambda\rangle^{k_1-k}\,. \tag{4.5}$$

We will use the notation

$$a = \widetilde{\mathcal{O}}\left(h^{-m}\langle\lambda\rangle^{k_1}\right) \iff \text{(4.5) holds.} \tag{4.6}$$

For such an a we can define an exact quantization

$$\widetilde{\mathrm{Op}}_h(a)u(x) = \frac{1}{(2\pi h)^n}\int a(x,\xi,h^{-\delta}\xi_1;h)e^{\frac{i}{h}\langle x-y,\xi\rangle}u(y)dyd\xi\,, \quad n = \dim X\,. \tag{4.7}$$

The main point is that the non-classical behaviour occurs entirely in the ξ variables and hence the composition formulæ hold (see for instance [21] for the standard semi-classical calculus): if $a = \widetilde{\mathcal{O}}(h^{-m_1}\langle\lambda\rangle^{k_1})$ and $b = \widetilde{\mathcal{O}}(h^{-m_2}\langle\lambda\rangle^{k_2})$ then

$$\begin{gathered} \widetilde{\mathrm{Op}}_h(a) \circ \widetilde{\mathrm{Op}}_h(b) = \widetilde{\mathrm{Op}}_h(a \,\sharp_h\, b) \bmod \Psi^{-\infty,-\infty}(X)\,, \\ a \,\sharp_h\, b(x,\xi,\lambda;h) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}(h\partial_\xi)^\alpha(a\restriction_{\lambda=h^{-\delta}\xi_1})D_x^\alpha b = \sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}(h\partial_{\xi'})^{\alpha'}(h\partial_{\xi_1} + h^{1-\delta}\partial_\lambda)^{\alpha_1} a\, D_x^\alpha b\,, \\ a \,\sharp_h\, b = \widetilde{\mathcal{O}}\left(h^{-m_1-m_2}\langle\lambda\rangle^{k_1+k_2}\right)\,. \end{gathered} \tag{4.8}$$

For simplicity we now put $m_1 = m_2 = 0$ and assume that ξ_1 is bounded so that in particular

$$h \leq \mathcal{O}(1) h^{1-\delta} \langle \lambda \rangle^{-1} . \tag{4.9}$$

We then see that (4.8) gives

$$\begin{aligned} a \sharp_h b = ab + h \partial_\xi a \, D_x b + h^{1-\delta} \partial_\lambda a \, D_{x_1} b + \frac{1}{2} h^{2(1-\delta)} \partial_\lambda^2 a \, D_{x_1}^2 b \ + \cdots + \frac{1}{p!} h^{p(1-\delta)} \partial_\lambda^p a \, D_{x_1}^p b \\ + \widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle \lambda \rangle} \langle \lambda \rangle^{k_1 + k_2} \right) , \quad p \geq (1-\delta)^{-1} . \end{aligned} \tag{4.10}$$

In preparation for the invariance result below let us assume that b is independent of λ that is, $\partial_x^\alpha \partial_\xi^\beta b = \mathcal{O}(1)$ for all $\alpha, \beta \in \mathbb{N}^n$, and that $\partial_{x_1} b = 0$ for $\xi_1 = 0$. Then $\partial_{x_1} b = c(x,\xi)\xi_1 = h^\delta c(x,\xi)\lambda = \widetilde{\mathcal{O}}(h^\delta \langle \lambda \rangle)$, and for $j \geq 1$

$$\frac{1}{j!} h^{j(1-\delta)} \partial_\lambda^j a \, D_{x_1}^j b = \widetilde{\mathcal{O}} \left(h \left(\frac{h^{1-\delta}}{\langle \lambda \rangle} \right)^{j-1} \langle \lambda \rangle^{k_1} \right) .$$

This gives

$$\begin{aligned} a \sharp_h b &= ab + h \sum_{i=1}^n \partial_{\xi_i} a \, D_{x_i} b + h^{1-\delta} \partial_\lambda a \, D_{x_1} b + \widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle \lambda \rangle} \langle \lambda \rangle^{k_1} \right) , \\ b \sharp_h a &= ab + h \sum_{i=1}^n \partial_{\xi_i} b \, D_{x_i} a + \widetilde{\mathcal{O}} \left(h^2 \langle \lambda \rangle^{k_1} \right) , \\ & \widetilde{\mathcal{O}} \left(h^2 \langle \lambda \rangle^{k_1} \right) = \widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle \lambda \rangle} \langle \lambda \rangle^{k_1} \right) , \end{aligned} \tag{4.11}$$

where we used the boundedness of ξ_1 – see (4.9). Hence under the above assumptions on b we have

$$\begin{aligned} b \sharp_h a - a \sharp_h b &= \frac{h}{i} H_b a - \frac{h^{1-\delta}}{i} \partial_{x_1} b \, \partial_\lambda a + \widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle \lambda \rangle} \langle \lambda \rangle^{k_1} \right) \\ &= \frac{h}{i} \left(H_b a - c(x,\xi) \lambda \partial_\lambda a \right) + \widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle \lambda \rangle} \langle \lambda \rangle^{k_1} \right) . \end{aligned} \tag{4.12}$$

Here $H_b = \sum_{j=1}^n \partial_{\xi_j} b \, \partial_{x_j} - \partial_{x_j} b \, \partial_{\xi_j}$ is the Hamilton vector field and $\partial_{x_1} b(x,\xi) = c(x,\xi)\xi_1$, as implied by the assumptions on b. The formula (4.12) is very natural as

$$(H_b a - c(x,\xi) \, \lambda \partial_\lambda a)(x,\xi,\lambda) \restriction_{\lambda = h^{1-\delta}\xi_1} = H_b \left(a(x,\xi,\lambda) \restriction_{\lambda = h^{1-\delta}\xi_1} \right) . \tag{4.13}$$

We now return to more general considerations. For the operator $\widetilde{\mathrm{Op}}_h(a)$ we define its principal symbol as the equivalence class of a in $\widetilde{\mathcal{O}}(h^{-m_1} \langle \lambda \rangle^{k_1}) / \widetilde{\mathcal{O}}(h^{-m_1+1-\delta} \langle \lambda \rangle^{k_1 - 1})$. This symbol map is clearly a homomorphism onto the quotient of symbol spaces:

$$\widetilde{\mathrm{Op}}_h(a) \mapsto [a] \in \widetilde{\mathcal{O}}(h^{-m_1} \langle \lambda \rangle^{k_1}) / \widetilde{\mathcal{O}}(h^{-m_1+1-\delta} \langle \lambda \rangle^{k_1 - 1}) . \tag{4.14}$$

We note that we are not concerned here with the behaviour at infinity in other directions than ξ_1. In the adaptation to a compact manifold the class $\widetilde{\mathcal{O}}(h^{-m}\langle\lambda\rangle^k)$ will only be used on compact subsets of the cotangent bundle.

For future reference we include here a version of Beals's characterization of pseudodifferential operators by stability under taking commutators. The proof, which is given in the appendix, follows from an adaptation of the proof in [4].

Lemma 4.1. *Let $A = A_h : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ and put $x' = (x_2, \cdots x_n)$. Then $A = \widetilde{\mathrm{Op}}_h(a)$ for $a = \widetilde{\mathcal{O}}(h^{-m}\langle\lambda\rangle^k)$ if and only if for all $N, p, q \geq 0$ and every sequence $\ell_1(x', \xi'), \cdots, \ell_N(x', \xi')$ of linear forms on $\mathbb{R}^{2(n-1)}$ there exist $C > 0$ for which*

$$\begin{aligned}\| \mathrm{ad}_{\ell_1(x', hD_{x'})} \circ \cdots \circ \mathrm{ad}_{\ell_N(x', hD_{x'})} \circ (\mathrm{ad}_{h^{1-\delta} D_{x_1}})^p \circ (\mathrm{ad}_{x_1})^q A u\|_{(q - \min(k,0))} \\ \leq C h^N (h^{1-\delta})^{p+q} \|u\|_{(\max(k,0))} ,\end{aligned}$$

where $\|u\|_{(p)} = \|u\|_{L^2} + \|(h^{1-\delta} D_{x_1})^p u\|_{L^2}$.

For the global definition we need the invariance of $\widetilde{\mathrm{Op}}_h\left(\widetilde{\mathcal{O}}(\langle\lambda\rangle^m)\right)$ under conjugation by h-Fourier Integral Operators which preserve $\Sigma_0 = \{\xi_1 = 0\}$ and that is the main technical result of this section. Before we state it we recall the following standard convention of semiclassical microlocal analysis: we say that $A = B$ microlocally near $((x, \xi), (x', \xi')) \in T^*\mathbb{R}^n$ if for any $a, a' \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ supported in sufficiently small neighbourhoods of $(x, \xi), (x', \xi')$ and equal to 1 in some neighbourhoods of $(x, \xi), (x', \xi')$ we have $\mathrm{Op}_h(a)(A - B)\,\mathrm{Op}_h(a') \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$. Similarly we say that $B = A^{-1}$ microlocally near $((x, \xi), (x', \xi'))$ if $\mathrm{Op}_h(a)(AB - I), (AB - I)\,\mathrm{Op}_h(a') \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$ and $\mathrm{Op}_h(a')(BA - I), (BA - I)\,\mathrm{Op}_h(a) \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$. When $(x, \xi) = (x', \xi')$ we say that the statement is microlocally true at (x, ξ). For basic information about h-Fourier Integral Operators we refer to [4].

Proposition 4.2. *Let U be a classical elliptic h-Fourier Integral Operator microlocally defined in a neighbourhood of $((0, 0), (0, 0))$. Assume that the canonical relation associated to U, κ, satisfies:*

$$\kappa(0, 0) = (0, 0) , \quad \kappa(\{\xi_1 = 0\} \cap V) \subset \{\xi_1 = 0\} , \quad (0, 0) \in V \subset T^*\mathbb{R}^n , \quad V \text{ open} . \tag{4.15}$$

Let $A = \widetilde{\mathrm{Op}}_h(a)$ where $a = \widetilde{\mathcal{O}}(\langle\lambda\rangle^m)$. Then, microlocally near $(0, 0)$,

$$U^{-1} A U = \widetilde{\mathrm{Op}}_h(b) , \quad b = a \circ K + \widetilde{\mathcal{O}}\left(\frac{h^{1-\delta}}{\langle\lambda\rangle}\langle\lambda\rangle^m\right) ,$$

where K is the natural lift of κ to the (x, ξ, λ) variables: $K(y, \eta, \mu) = (x, \xi, \lambda)$ if and only if $(x, \xi) = \kappa(y, \eta)$, $\lambda = (\xi_1/\eta_1)\mu$. Since the operators $\widetilde{\mathrm{Op}}_h(a)$ are microlocally 0 away from the diagonal this provides a complete microlocal description.

Proof. We start by observing that the proposition holds in the special cases $\kappa(x, \xi) = (x, \xi)$ and $\kappa(x, \xi) = (-x, -\xi)$. The first special case concerns conjugation with elliptic classical h-pseudodifferential operators and it follows from the discussion after (4.8). The second special case follows from the first one and the fact that the proposition is easily checked for $Uu(x) = u(-x)$. As a consequence we can asssume that κ preserves the sign of ξ_1:

$$\kappa(y,\eta) = (x,\xi) \implies \xi_1\eta_1 \geq 0\,. \tag{4.16}$$

We will prove the proposition by a deformation method inspired by the "Heisenberg picture of quantum mechanics" and for that we need the following geometric

Lemma 4.2. *Let κ be a smooth canonical transformation satisfying* (4.15) *and* (4.16). *Then we can find a piecewise smooth family of canonical transformations* $[0,1] \ni t \mapsto \kappa_t$ *satisfying* (4.15), (4.16) *and such that $\kappa_0 = id$ and $\kappa_1 = \kappa$.*

Proof. Let us denote by Σ the hypersurface given by $\xi_1 = 0$. We start by considering

$$\kappa\restriction_\Sigma \,:\, (y_1, 0; y', \eta') \longmapsto (x_1(y,\eta'), 0; \kappa'(y',\eta'))\,, \tag{4.17}$$

where for convenience we write $(y_1, y', \eta_1, \eta') \in T^*\mathbb{R}^n$ as $(y_1, \eta_1; y', \eta')$. We notice that κ' is independent of y_1 and that it is a canonical transformation mapping $(0,0)$ to $(0,0)$. In fact, the bicharacteristic leaves of Σ are given by $(y',\eta') = \text{const}$ and they are mapped into the bicharacteristic leaves. The property (4.16) is equivalent to

$$\frac{\partial x_1}{\partial y_1} > 0\,, \tag{4.18}$$

and we also have $x_1(0,0) = 0$.

We recall that every canonical transformation on a neighbourhood of $(0,0) \in T^*\mathbb{R}^{n-1}$ and mapping $(0,0)$ to $(0,0)$ can be smoothly deformed within the space of such objects to the identity. Applied to κ' this gives a family of canonical transformations κ'_t. Composing κ with the map $(y_1, \eta_1; y', \eta') \mapsto (y_1, \eta_1; (\kappa'_t)^{-1}(y', \eta'))$, we can make a deformation of κ to achieve $\kappa' = id$ in (4.17).

We have now reduced the problem to considering

$$\kappa(y_1, \eta_1; y', \eta') = (x_1(y,\eta') + \mathcal{O}(\eta_1), f(y,\eta)\eta_1; (y',\eta') + \mathcal{O}(\eta_1))\,, \tag{4.19}$$

where $f(y,\eta) > 0$. The projection

$$\text{graph}\,(\kappa) \ni (x,\xi; y,\eta) \longmapsto (x,\eta)$$

is a local diffeomorphism and hence κ has a generating function $\phi(x,\eta)$:

$$\kappa \,:\, (\phi'_\eta(x,\eta), \eta) \longmapsto (x, \phi'_x(x,\eta))\,, \tag{4.20}$$

with

$$\det \phi''_{x\eta}(0,0) \neq 0\,, \quad \frac{\partial \phi}{\partial x_1}(x; 0, \eta') = 0\,. \tag{4.21}$$

Since $\kappa' = id$ for $\eta_1 = 0$ we also get

$$\phi'_{\eta'}(x; 0, \eta') = x'\,, \quad \phi'_{x'}(x; 0, \eta') = \eta'\,. \tag{4.22}$$

It follows that $\phi(x; 0, \eta') = x' \cdot \eta' + C$ and we can choose $C = 0$. Consequently,

$$\phi(x,\eta) = x' \cdot \eta' + g(x,\eta)\eta_1\,, \quad g(0,0) = 0\,, \tag{4.23}$$

and the non-degeneracy assumption in (4.21) is equivalent to $\partial_{x_1} g \neq 0$. Actually (4.18) implies that

$$\frac{\partial g}{\partial x_1} > 0 \,. \tag{4.24}$$

Conversely, every ϕ satisfying (4.23) and (4.24) generates a canonical trasformation satisfying (4.19) (with $f > 0$ and $\partial_{y_1} x_1 > 0$). The second deformation is obtained by deforming g to x_1 through functions $\tilde{g}$ satisfying $\tilde{g}(0,0) = 0$ and $\partial_{x_1}\tilde{g} > 0$. We can for instance put $g_t(x,\eta) = tg(x,\eta) + (1-t)x_1$. □

We can now complete the proof of Proposition 4.2. Let κ_t be a piecewise smooth family with $\kappa_1 = \kappa$, $\kappa_0 = id$ and let U_t be a piecewise smooth family of classical elliptic h-Fourier Integral Operators defined microlocally near $(0,0)$ and associated to κ_t. We arrange also that $U_1 = U$ and $U_0 = Id$. For notational convenience we assume that our deformation is smooth in t – the piecewise smooth case follows from the same argument applied in several steps. With this in mind we have

$$hD_t(U_t) + U_t Q_t = 0 \,, \tag{4.25}$$

where Q_t is a smooth family of classical h-pseudodifferential operators of order 0 with the leading symbol q_t satisfying

$$\frac{d}{dt}\kappa_t(x,\xi) = (\kappa_t)_* \left(H_{q_t}(x,\xi) \right) \,. \tag{4.26}$$

For the reader's convenience we briefly recall the reason for this. The statement is equivalent to saying that for every $a \in \mathcal{C}^\infty_{\rm c}(T^*X)$ we have $H_{q_t}\kappa_t^* a = (d/dt)(\kappa_t^* a)$. On the other hand if $A = \Psi^{*,*}(\mathbb{R}^n)$ and $A_t = U_t^{-1} A U_t$ then (4.25) shows that $[Q_t, A_t] = (hD_t)A_t$. Taking A with the symbol given by a, Egorov's theorem gives $\kappa_t^* a$ as the symbol of A_t. Since the symbol of $[Q_t, A_t]$ is $(1/i)hH_{q_t}\kappa_t^* a$, (4.26) follows. We will proceed by a reversed argument for A's which are in the new class of operators: we use (4.26) to prove Egorov's theorem for the new class.

It follows from (4.26) that H_{q_t} is tangent to Σ and hence

$$\partial_{x_1} q_t(x_1, 0; x', \eta') = 0 \tag{4.27}$$

which was one of the assumptions on b in (4.12). From (4.25) we obtain

$$hD_t(U_t^{-1}) = -U_t^{-1} hD_t(U_t) U_t^{-1} = Q_t U_t^{-1} \,. \tag{4.28}$$

Let us now consider $A_t = U_t^{-1} A U_t$, so that A_1 is the operator we want to study and $A_0 = A$ is the given operator. From (4.25) and (4.28), we get

$$\begin{cases} hD_t(A_t) = [Q_t, A_t] \,, \\ A_0 = A \,. \end{cases} \tag{4.29}$$

We shall now construct C_t with the symbol $c_t = \widetilde{\mathcal{O}}(\langle\lambda\rangle^m)$, depending smoothly on t and such that microlocally near $(0,0)$

$$\begin{cases} hD_t(C_t) = [Q_t, C_t] + \widetilde{\mathrm{Op}}_h(\widetilde{\mathcal{O}}(h^\infty))\,, \\ C_0 = A\,. \end{cases} \tag{4.30}$$

Then it will follow that $C_t = A_t + \widetilde{\mathrm{Op}}_h(\widetilde{\mathcal{O}}(h^\infty))$ microlocally near $(0,0)$: $C_t - A_t$ solves an equation with null initial data and an $\widetilde{\mathrm{Op}}_h(\widetilde{\mathcal{O}}(h^\infty))$ inhomogeneous term. Since the equation has to be satisfied on the symbolic level it follows that $C_t - A_t = \widetilde{\mathrm{Op}}_h(\widetilde{\mathcal{O}}(h^\infty))$.

We try C_t with principal symbol c_t^0 (considered as an element of $\widetilde{\mathcal{O}}(\langle\lambda\rangle^m) \bmod \widetilde{\mathcal{O}}(h^{1-\delta}\langle\lambda\rangle^{m-1})$). From the discussion following (4.11) we see that c_t^0 should satisfy

$$\frac{\partial}{\partial t} c_t^0 = \left(H_{q_t} - r_t(x,\xi)\lambda \frac{\partial}{\partial \lambda} \right) c_t^0\,, \quad c_0^0 = a\,, \tag{4.31}$$

where

$$\frac{\partial}{\partial x_1} q_t = r_t(x,\xi)\xi_1\,.$$

If K_t is the transformation in (x,ξ,λ)-space corresponding to κ_t as in the statement of the proposition, it follows, in view of (4.13) and (4.26), that

$$c_t^0 = a \circ K_t\,.$$

Let C_t^0 be the h-pseudodifferential operator with symbol c_t^0. Then from (4.12) we get

$$hD_t(C_t^0) = [Q_t, C_t^0] + \widetilde{\mathrm{Op}}_h \left(\widetilde{\mathcal{O}} \left(h \frac{h^{1-\delta}}{\langle\lambda\rangle} \langle\lambda\rangle^m \right) \right)\,, \quad C_0^0 = A\,. \tag{4.32}$$

Iterating this argument, solving inhomogeneous transport equations similar to (4.31), we get c_t as an asymptotic sum. The leading term of $C_t = A_t + \widetilde{\mathrm{Op}}_h(\widetilde{\mathcal{O}}(h^\infty))$ is $c_1^0 = a \circ K$ and the proposition follows. $\square$

Once we have Proposition 4.2 the definition of the class $\Psi_{\Sigma,\delta}^{m,k_1,k_2}(X)$ mimics the standard procedure: $A \in \Psi_{\Sigma,\delta}^{m,k_1,k_2}(X)$ if and only if for any $m_0 \in \Sigma$ and for any h-Fourier Integral Operator, $U : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(\mathbb{R}^n)$, elliptic near $((0,0), m_0)$ and such that the corresponding canonical transformation, κ, satisfies

$$\kappa(m_0) = (0,0)\,, \quad \kappa(\Sigma \cap V) \subset \{\xi_1 = 0\}\,,$$

for some neighbourhood, V, of m_0, we have

$$UAU^{-1} = \widetilde{\mathrm{Op}}_h \left(\widetilde{\mathcal{O}}(h^{-m}\langle\lambda\rangle^{k_1}) \right)\,,$$

microlocally near $(0,0)$. Using the canonical relation of U, Proposition 4.2 and the local symbol map (4.14) we define a symbol of A near Σ. For m_0 outside any fixed neighbourhood of Σ we demand that $A \in \Psi^{m+\delta k_1, k_2}(X)$ microlocally near m_0 in both classical and semi-classical senses. Proposition 4.2 and the standard facts about the class $\Psi^{m,k}$ show that we obtain a well defined global symbol map $\sigma_{\Sigma,h}$ with the properties described in Proposition 4.1.

From the local definition we can define a global map $\mathrm{Op}_{\Sigma,h}$. To do that we introduce $\psi \in S^{0,-\infty}(T^*X)$ satisfying

$$\mathrm{supp}\psi \subset \{p : d(p,\Sigma) \leq 2\epsilon\}\,, \ \psi \equiv 1 \ \text{ on } \ \{p : d(p,\Sigma) \leq \epsilon\}\,.$$

For ϵ small enough we can find a finite open cover

$$\{p \ : \ d(p,\Sigma) \leq 2\epsilon\} \subset \bigcup V_j$$

such that for each j there exists a canonical transformation

$$\kappa_j : V \longrightarrow V_j\,, (0,0) \in V \subset T^*\mathbb{R}^n$$

with the inverse taking Σ to the model:

$$\kappa_j(\{\xi_1 = 0\} \cap V) = \Sigma \cap V_j\,.$$

We then *choose* elliptic h-Fourier Integral Operators, U_j, microlocally defined in neighbourhoods of $V \times V_j$ and associated to κ_j's.

Let ϕ_j be a partition of unity on $\{p : d(p,\Sigma) \leq 2\epsilon\}$ subordinate to the cover by V_j's. Let a_j be the unique symbol of the form $a_j = a_j(x, \xi_2, \cdots, \xi_n, \lambda; h)$ such that

$$(a_j)_{\lambda = h^{-\delta}\xi_1} = (\psi\phi_j a) \circ \kappa_j$$

and define

$$\mathrm{Op}_{\Sigma,h}(a) \stackrel{\mathrm{def}}{=} \mathrm{Op}_h\left((1-\psi)a\right) + \sum_j U_j \widetilde{\mathrm{Op}}_h\left(a_j\right) U_j^{-1}\,. \tag{4.33}$$

Clearly, $\mathrm{Op}_{\Sigma,h}(a) \in \Psi_{\Sigma,\delta}^{m,k_1,k_2}(X)$. By Proposition 4.2 every operator in $\Psi_{\Sigma,\delta}^{m,k_1,k_2}(X)$ is of this form and

$$\sigma_{\Sigma,h}(\mathrm{Op}_{\Sigma,h}(a)) \equiv a \bmod \Psi_{\Sigma,\delta}^{m-1+\delta,k_1-1,k_2-1}(X)\,.$$

This completes the proof of Proposition 4.1 and provides an explicit quantization $\mathrm{Op}_{\Sigma,h}$.

For $a \in S_\Sigma^{m,k_1,-\infty}(T^*X)$ we want to introduce a notion of essential support. Since it now has to depend on h, rather than introduce an equivalence class of families of sets, we will say that for h-dependent family of sets $V_h \subset T^*X$

$$\mathrm{ess\,supp}\, a \cap V_h = \emptyset \iff \exists\, 0 \leq \chi \in S^{0,0,-\infty}(T^*X)\,,\ \chi\restriction_{V_h} \geq 1\,,\ \chi a \in S^{-\infty,-\infty}(T^*X)\,.$$

We notice that

$$\mathrm{ess\,supp}\, a \cap V_h^j = \emptyset\,,\ j = 1, \cdots, N\,,\ \Longrightarrow\ \mathrm{ess\,supp}\, a \cap \left(V_h^1 \cup \cdots \cup V_h^N\right) = \emptyset\,,$$

and that the essential support behaves correctly under finite products and sums. We assume rapid vanishing at the ξ infinity to avoid the discussion of the "corner" $h = 0$, $|\xi| = \infty$. We note that essential support can be "supported" in small neighbourhoods depending on h due to fine localization properties of $\chi \in S_{\Sigma,\delta}^{0,0,-\infty}$.

The construction of $\mathrm{Op}_{\Sigma,h}$ shows, just as in the standard case, that if $\mathrm{Op}_{\Sigma,h}(a) = \mathrm{Op}_{\Sigma,h}(b)$, $b \in S_{\Sigma,\delta}^{m,k,-\infty}(T^*X)$, then $\mathrm{ess\,supp}\, a = \mathrm{ess\,supp}\, b$. Consequently for $A \in \Psi_{\Sigma,\delta}^{m,k,-\infty}(X)$ we can define

$$\mathrm{W}F_h(A) = \mathrm{ess\,supp}\, a\,, \quad A = \mathrm{Op}_{\Sigma,h}(a)\,. \tag{4.34}$$

We now introduce a more general notation $S_{\Sigma,\delta}(X,m)$ where $m = m(x,\xi,\lambda;h)$ is g-continuous with respect to the metric $g = dx^2 + d\xi^2/\langle\xi\rangle^2 + d\lambda^2/\langle\lambda\rangle^2$, uniformly with respect to h, that is m is an order function with respect to g – see [9], Sect.18.4. Here we put λ to be $h^{-\delta}d(\Sigma,\bullet)$ and require the estimates in the definition of $S^{m,k_1,k_2}_{\Sigma,\delta}$ be satisfied with m replacing $h^{-m}\langle\xi\rangle^{k_2-k_1}\langle h^{-\delta}d(\Sigma,\bullet)\rangle^{k_1}$. We then obtain a class of corresponding operators $\Psi_{\Sigma,\delta}(X,m)$.

A further generalization which we will require in this paper is to the vector valued case. Let $\mathcal{B}$ and $\mathcal{H}$ be two Banach spaces. We assume that they are equipped with $(x,\xi,\lambda;h)$ dependent norms $\|\bullet\|_{m_\mathcal{H}}$, $\|\bullet\|_{m_\mathcal{B}}$ which, for every fixed $(x,\xi,\lambda;h)$ are equivalent to some fixed norms (but without any uniformity). We assume that the norms are g-continuous with respect to the metric $g = dx^2 + d\xi^2/\langle\xi\rangle^2 + d\lambda^2/\langle\lambda\rangle^2$, uniformly with respect to h. We let m be a g-continuous order function as before. We then say that

$$\begin{gathered} a \in S_{\Sigma,\delta}(X,m,\mathcal{L}(\mathcal{B},\mathcal{H})) \Longleftrightarrow \\ \|a(x,\xi;h)u\|_{m_\mathcal{H}(x,\xi,\lambda;h)} \leq Cm(x,\xi,\lambda;h)\|u\|_{m_\mathcal{B}(x,\xi,\lambda;h)}\,, \\ \lambda = h^{-\delta}d(\Sigma,\bullet)\,, \quad \text{for all} \quad u\in\mathcal{B} \end{gathered} \tag{4.35}$$

and if this statement is stable under an application of the vector fields appearing in (4.1).

Applying the same procedure as in the scalar case we obtain a class of operators

$$\Psi_{\Sigma,\delta}(X;m,\mathcal{L}(\mathcal{B},\mathcal{H})) \ni A \,:\, \mathcal{C}^\infty(X)\otimes\mathcal{B} \longrightarrow \mathcal{C}^\infty(X)\otimes\mathcal{H}\,. \tag{4.36}$$

More precise mapping results can be established but as they will be clear in our context we refrain from too much general development. The symbol map and the invariance are the same as in the scalar case.

We conclude this section with two general results which will be useful later.

Lemma 4.3. *Let $A\in\Psi^{m,-\infty,-\infty}_{\Sigma,\delta}(X)$ and let us parametrize a neighbourhood of Σ in T^*X by $\Sigma\times[-\epsilon,\epsilon]$:*

$$\Sigma\times[-\epsilon,\epsilon]\ni(w,r)\longmapsto p\in T^*X\,, \quad r\restriction_\Sigma = 0\,,$$

so that for a measure L_Σ on Σ we have $L_\Sigma(dw)dr$ as the canonical measure on T^*X.

If in these coordinates we write $a(w,\lambda;h) = \sigma_{\Sigma,h}(A)(w,h^\delta\lambda;h)$ then

$$\operatorname{tr} A = \frac{h^{-n+\delta}}{(2\pi)^n}\int_\mathbb{R}\int_\Sigma a(w,\lambda;h)L_\Sigma(dw)d\lambda + \mathcal{O}(h^{-n+1})\,, \quad n = \dim X\,. \tag{4.37}$$

Proof. Because of the invariance properties given in Proposition 4.2 and the cyclicity of the trace we can work locally assuming that $\Sigma = \{\xi_1 = 0\}$. It is clear that an operator in $\Psi^{m,-\infty,-\infty}_{\Sigma,\delta}(X)$ is of trace class (it is a smoothing operator) and that the contribution from the symbol outside a fixed neighbourhood of Σ is $\mathcal{O}(h^\infty)$ (the operator is in $\Psi^{-\infty,-\infty}$ there, microlocally). Taking the trace of (4.7) gives

$$\operatorname{tr}\widetilde{\operatorname{Op}}_h(a) = \frac{h^{-n}}{(2\pi)^n}\int_{T^*\mathbb{R}^n} a(x,\xi,h^{-\delta}\xi_1;h)dxd\xi = \frac{h^{-n+\delta}}{(2\pi)^n}\int_{T^*\mathbb{R}^n} a(x,h^\delta\lambda,\xi',\lambda;h)(dxd\xi')d\lambda\,.$$

This expression is invariant up to terms in $\widetilde{\mathcal{O}}(h^{1-\delta}\langle\lambda\rangle^{-\infty})$ and they contribute $\mathcal{O}(h^{1-n})$ to the trace. This gives (4.37). $\square$

The next lemma deals with approximation by finite rank operators

Lemma 4.4. *Let $a \in S_\Sigma^{m,-\infty,-\infty}(T^*X)$ be supported in $W_h = \{p \in T^*X \;:\; d(p,\Sigma) \leq Mh^\delta\}$. Then there exists a finite rank operator R such that for $0 < h < h_0(M)$*

$$\mathrm{Op}_{\Sigma,h}(a) - R \in \Psi^{-\infty,-\infty}(X)\,, \quad \mathrm{rank}\, R = \mathcal{O}(Mh^{-n+\delta})\,, \quad n = \dim X\,.$$

Proof. Let $A = \mathrm{Op}_{\Sigma,h}(a)$. Then $A = \sum_{j=1}^J U_j A_j V_j$ where U_j, V_j are h-semiclassical Fourier Integral Operators of the form described in the definition of the class $\Psi_{\Sigma,\delta}$ and $A_j = \widetilde{\mathrm{Op}}_h(a_j)$ with $a_j = \widetilde{\mathcal{O}}(h^{-m}\langle\lambda\rangle^{-\infty})$. The construction of $\mathrm{Op}_{\Sigma,h}$ (see the discussion before (4.34)) shows that we can take $\mathrm{supp}\, a_j \subset \{|\lambda| \leq CM\}$, and we can also assume that (x,ξ) are bounded on the support of a_j's. If we construct finite rank operators, R_j, such that $A_j - R_j \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$ then $R = \sum_{j=1}^J U_j R_j V_j$, is a finite rank operator, rank $R \leq \sum_{j=1}^J \mathrm{rank}\, R_j$ and $A - R \in \Psi^{-\infty,-\infty}(X)$.

Consider the operator

$$Q = \left(\frac{h^{-\delta}}{M} hD_{x_1}\right)^2 + x_1^2 + (hD_{x_2})^2 + x_2^2 + \cdots (hD_{x_n})^2 + x_n^2\,,$$
$$Q = \widetilde{\mathrm{Op}}_h(q)\,, \quad q = \left(\frac{\lambda}{M}\right)^2 + x_1^2 + \cdots \xi_n^2 + x_n^2\,.$$

The standard analysis of harmonic oscillators shows that if $\lambda_j(Q)$ is the jth eigenvalue of Q then, for a bounded r, $\sharp\{\lambda_j(Q) \leq r\} = \mathcal{O}(Mh^{-n+\delta})$. If $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi(t) = 1$ for $t \leq \tilde{C}$, $\chi(t) = 0$ for $t > \tilde{2}C$, then $\chi(Q)$ is a finite rank operator and its rank is bounded by the counting function of eigenvalues of Q: rank $\chi(Q) = \mathcal{O}(Mh^{-n+\delta})$. The calculus discussed after (4.7) now shows that $\chi(Q)A_j - A_j \in \Psi^{-\infty,-\infty}(\mathbb{R}^n)$ and we can put $R_j = \chi(Q)A_j$. $\square$

5. **Two model problems**

Because of the natural homogeneity of our operator discussed in Sect.2 (see also [17] for a discussion of homogeneity in the language of blow-ups) the first model problem to consider is

$$P_\lambda - z = e^{-2\pi i/3}(D_t^2 + \mu t) + \lambda - z\,, \quad \lambda \in \mathbb{R}\,, \quad 1/C \leq \mu \leq C\,, \quad |\mathrm{Im}\, z| < C_1\,, \tag{5.1}$$

where C_1 will remain large but fixed. We recall from (2.4) that in relation to the original equation we should think of t as $h^{-2/3}x_n$, λ as $h^{-2/3}(R(x', hD_{x'}) - w)$, μ as $Q(0, x', hD_{x'})$ and z as $h^{2/3}(\zeta - w)$ where ζ is the original spectral parameter. For simplicity we shall put $\mu = 1$ in this section. All the estimates will clearly be uniform with respect to μ with all derivatives.

Let $0 > -\zeta_1 > -\zeta_2 > \cdots > -\zeta_k > \cdots$ be the zeros of the Airy function and let $e_j(t) = c_j Ai(t - \zeta_j)$ be the normalized eigenfunctions of

$$\begin{cases} (D_t^2 + t)e_j(t) = \zeta_j e_j(t)\,, & t \geq 0 \\ e_j(0) = 0\,. \end{cases}$$

We recall that the eigenfuctions e_j decay rapidly since for $t \to +\infty$ we have $Ai(t) \sim (2\sqrt{\pi})^{-1} t^{-\frac{1}{4}} \exp(-2t^{\frac{3}{2}}/3)$. We now take $N = N(C_1)$ the largest number such that

$$|\operatorname{Im} e^{-i2\pi/3}\zeta_N| \le C_1 .$$

To set up the model Grushin problem we define

$$\begin{gathered} R_+^0 : L^2([0,\infty)) \longrightarrow \mathbb{C}^N , \quad R_+^0 u(j) = \langle u, e_j\rangle , \quad 1 \le j \le N , \\ R_-^0 : \mathbb{C}^N \to L^2([0,\infty)) , \quad R_-^0 = (R_+^0)^* . \end{gathered} \tag{5.2}$$

Using this we put

$$\begin{gathered} \mathcal{P}_\lambda^0(z) = \begin{pmatrix} P_\lambda - z & R_-^0 \\ R_+^0 & 0 \end{pmatrix} \; : \; B_{\lambda,r} \times \mathbb{C}^N \longrightarrow L_r^2 \times \mathbb{C}^N , \\ L_r^2 = L^2([0,\infty), e^{rt} dt) , \quad B_{\lambda,r} = \{ u \in L_r^2 \; : \; D_t^2 u , \; tu \in L_r^2 , \; u(0) = 0 \} , \\ \|u\|_{B_{z,\lambda,r}} = \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L_r^2} + \|D_t^2 u\|_{L_r^2} + \|tu\|_{L_r^2} . \end{gathered} \tag{5.3}$$

Since the eigenvalues of P_λ are given by $\lambda + e^{-2\pi i/3}\zeta_j$ and e_j are the corresponding eigenfunctions, we see that for $r = 0$, $\mathcal{P}_\lambda^0(z)$ is bijective with a bounded inverse. Moreover we have

Lemma 5.1. *If*

$$\mathcal{P}_\lambda^0(z) \begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$$

then for $0 \le r \le r_0$ with $r_0 > 0$ small enough

$$\|D_t^2 u\|_{L_r^2} + \|tu\|_{L_r^2} + \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L_r^2} + |u_-| \le C(\|v\|_{L_r^2} + \langle \lambda - \operatorname{Re} z\rangle |v_+|) , \tag{5.4}$$

where C is independent of L and $|\bullet|$ a fixed norm on $\mathbb{C}^N$.

Proof. We first consider the case when $r = 0$. Since $R_+^0 u = v_+$ we can write $u = \sum_{j=1}^N v_+(j) e_j + \sum_{j=N+1}^\infty u_j e_j$. Writing $v = \sum_{j=1}^\infty v_j e_j$ we obtain from the equation $(P_\lambda - z)u + R_-^0 u_- = v$ the following relations

$$\begin{gathered} (\lambda - z + e^{-2\pi i/3}\zeta_j) v_+(j) + u_-(j) = v_j , \quad 1 \le j \le N , \\ (\lambda - z + e^{-2\pi i/3}\zeta_j) u_j = v_j , \quad j \ge N+1 . \end{gathered}$$

Consequently $u_j = v_j/(\lambda - z + e^{-2\pi i/3}\zeta_j)$ for $j \ge N+1$ and $u_-(j) = v_j - (\lambda - z + e^{-2\pi i/3}\zeta_j) v_+(j)$ for $1 \le j \le N$. It follows that

$$\langle \lambda - \operatorname{Re} z\rangle \|u\| + |u_-| \le C(\|v\| + \langle \lambda - \operatorname{Re} z\rangle |v_+|) .$$

Using this in the equation $(P_\lambda - z)u + R_-^0 u_- = v$ we see that

$$e^{-2\pi i/3}(D_t^2 + t)u = v - R_-^0 u_- - (\lambda - z)u .$$

If we call the right hand side w, we see that $\|w\| \le C(\|v\| + \langle \lambda - \operatorname{Re} z\rangle |v_+|)$. As in Lemma 4.3 of [32] we see that

$$\|w\|^2 = \|(D_t^2 + t)u\|^2 = \|D_t^2 u\|^2 + \|tu\|^2 + 2\|\sqrt{t} D_t u\|^2 ,$$

and that gives us control of the remaining components of the norm on $B_{z,\lambda,0}$. To consider the case of $r>0$ we observe that $L^2_r = e^{-rt/2}L^2$, $B_{z,\lambda,r} = e^{-rt/2}B_{z,\lambda,0}$ and hence we introduce the reduced operator

$$\begin{aligned}\mathcal{P}^r_\lambda(z) &= \begin{pmatrix} e^{rt/2} & 0 \\ 0 & 1 \end{pmatrix} \mathcal{P}^\lambda_0 \begin{pmatrix} e^{rt/2} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \mathcal{P}^0_\lambda(z) + \begin{pmatrix} -rD_t - r^2/4 & (e^{rt/2}-1)R_- \\ R_+(e^{-rt/2}-1) & 0 \end{pmatrix} .\end{aligned}$$

Standard interpolation shows that $D_t = \mathcal{O}(\langle\lambda\rangle^{-1/2}) : B_{z,\lambda,0} \to L^2$ and the super-exponential $(e^{-t^{3/2}/C})$ decay of e_j's shows that $R_+ t[(e^x-1)/x]\restriction_{x=rt/2} = \mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle^{-1}) : B_{z,\lambda,0} \to \mathbb{C}^N$. Similarly $t[(e^x-1)/x]\restriction_{x=-rt/2} R_- = \mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle^{-1}) : \mathbb{C}^N \to L^2$, if we equip $\mathbb{C}^N$ with the norm $\langle\lambda - \operatorname{Re} z\rangle | \bullet |$. Hence using the norms given in (5.4) we see that $\mathcal{P}^r_\lambda$ is a small perturbation of $\mathcal{P}^0_\lambda$ if r is small enough and that proves the lemma. $\square$

If we denote the inverse of $\mathcal{P}^0_\lambda(z)$ by

$$\mathcal{E}^0_\lambda(z) = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} ,$$

then

$$E_{-+} \in \operatorname{Hom}(\mathbb{C}^N, \mathbb{C}^N) , \quad (E_{-+})_{1\le i,j\le N} = -(\lambda - z + e^{-2\pi i/3}\zeta_j)\delta_{ij} . \tag{5.5}$$

We will now modify the Grushin problem so that we can have good global symbolic properties. For that we put, for $0<\delta\ll 1$,

$$e^{\lambda,\delta}_j(t) = \langle\delta\lambda\rangle^{\frac14} e_j(\langle\delta\lambda\rangle^{\frac12} t) , \tag{5.6}$$

which forms an orthonormal family. If $\Lambda = \langle\delta\lambda\rangle^{\frac12}$ we further notice that

$$\partial^k_\lambda \Lambda = \mathcal{O}_k(1)\left(\frac{\delta}{\Lambda^2}\right)^k \Lambda , \quad \|\partial^k_\lambda e^{\lambda,\delta}_j\|_{L^2_r} = \mathcal{O}_k(1)\left(\frac{\delta}{\Lambda^2}\right)^k , \tag{5.7}$$

where the last estimate follows from the superexponential decay of e_j's and their derivatives.

In particular we have that

$$\|e^{\lambda,\delta}_j - e_j\|_{L^2_r} \le C\delta|\lambda| . \tag{5.8}$$

Hence if we define $R^{\lambda,\delta}_+$, $R^{\lambda,\delta}_-$ by replacing e_j by $e^{\lambda,\delta}_j$ in the definitions of R^0_+, R^0_- we obtain

$$\begin{gathered}\mathcal{P}^\delta_\lambda(z) = \begin{pmatrix} P_\lambda - z & R^{\lambda,\delta}_- \\ R^{\lambda,\delta}_+ & 0 \end{pmatrix} : B_{z,\lambda,r}\times\mathbb{C}^N \longrightarrow L^2_r \times \mathbb{C}^N_{\langle\lambda-\operatorname{Re} z\rangle} , \\ \mathcal{P}^\delta_\lambda(z) - \mathcal{P}_\lambda(z) = \begin{pmatrix} 0 & \mathcal{O}(|\lambda|\delta) \\ \mathcal{O}(|\lambda|\delta) & 0 \end{pmatrix} : B_{z,\lambda,r}\times\mathbb{C}^N \longrightarrow L^2_r \times \mathbb{C}^N_{\langle\lambda-\operatorname{Re} z\rangle} .\end{gathered} \tag{5.9}$$

Thus for $|\lambda|\delta \ll 1$ we obtain the uniform invertibility of $\mathcal{P}^\delta_\lambda(z)$.

To see the boundedness of the inverse for all λ we need to make an assumption on z:

$$|\operatorname{Re} z| \ll \frac{1}{\delta}\,. \tag{5.10}$$

We will use the following abstract

Lemma 5.2. *Let $\mathcal{H}\,, V\,, \mathcal{D}\subset\mathcal{H}$ be complex Hilbert spaces and assume that*

$$\mathcal{P} = \left(\begin{array}{cc} P & R_- \\ R_+ & 0 \end{array}\right) \,:\, \mathcal{D}\oplus V \longrightarrow \mathcal{H}\oplus V\,, \quad R_- = R_+^*\,, \ \operatorname{Image} R_- \subset \mathcal{D}\,,$$

and that R_+ has a uniformly bounded right inverse. If $|\langle Pu, u\rangle| \geq \kappa \|u\|^2_{\mathcal{H}}$ and for $Q = P\,, P^*$, $\|Q\restriction_{\operatorname{Image} R_-}\|_{\mathcal{H}\to\mathcal{H}} = \mathcal{O}(\kappa)$ then

$$\mathcal{P}\left(\begin{array}{c} u \\ u_- \end{array}\right) = \left(\begin{array}{c} v \\ v_+ \end{array}\right) \quad\Longrightarrow\quad \kappa\|u\|_{\mathcal{H}} + \|u_-\|_V \leq C(\|v\|_{\mathcal{H}} + \kappa\|v_+\|_V)\,.$$

Proof. Considering

$$\left(\begin{array}{cc} \kappa^{-1} & 0 \\ 0 & 1 \end{array}\right)\mathcal{P}\left(\begin{array}{cc} 1 & 0 \\ 0 & \kappa \end{array}\right) = \left(\begin{array}{cc} \kappa^{-1}P & R_- \\ R_+ & 0 \end{array}\right)$$

we get a reduction to the case $\kappa = 1$. Let $\Pi : \mathcal{H} \to (\operatorname{kernel} R_+)^\perp = \operatorname{Image} R_-$ be the orthogonal projection. Then

$$\begin{aligned} \|(I-\Pi)u\|^2_{\mathcal{H}} &\leq |\langle P(I-\Pi)u, (I-\Pi)u\rangle| \\ &= |\langle (I-\Pi)v - (I-\Pi)P\Pi u, (I-\Pi)u\rangle| \\ &\leq \|v\|_{\mathcal{H}}\|(I-\Pi)u\|_{\mathcal{H}} + \|P\Pi u\|_{\mathcal{H}}\|(I-\Pi)u\|_{\mathcal{H}}\,, \end{aligned}$$

where $\mathcal{P}^t(u, u_-) = {}^t(v, v_+)$. By assumption there exists a uniformly bounded operator $P_+ : V \to (\ker R_+)^\perp \subset \mathcal{H}$ such that $R_+P_+v_+ = v_+$ and consequently $\Pi u = P_+v_+$. Thus

$$\|P\Pi u\|_{\mathcal{H}} = \|P\restriction_{\operatorname{Image} R_-} P_+v_+\| = \mathcal{O}(1)\|v_+\|_V\,,$$

and hence

$$\|(I-\Pi)u\|_{\mathcal{H}} \leq (\|v\| + \mathcal{O}(1)\|v_+\|_V)\,.$$

With $P_- = P_+^*$, we also have $P_-R_-u_- = u_-$, so that

$$\begin{aligned} u_- &= P_-(v - Pu) \\ &= P_-v - P_-\Pi P(I-\Pi)u - P_-P\Pi P_+v_+\,, \end{aligned}$$

and

$$\begin{aligned} \|u\|_V &\leq C(\|v\| + \|P^*\restriction_{\operatorname{Image} R_-}\|_{\mathcal{H}\to\mathcal{H}}\|(I-\Pi)u\|_{\mathcal{H}} + \|P\restriction_{\operatorname{Image} R_-}\|_{\mathcal{H}\to\mathcal{H}}\|v_+\|_V) \\ &\leq C(\|v\|_{\mathcal{H}} + \|(I-\Pi)u\|_{\mathcal{H}} + \|v_+\|_V)\,. \end{aligned}$$

Clearly $\|\Pi u\|_{\mathcal{H}} = \|P_+v_+\| \leq C\|v_+\|_V$ and

$$\|u\|_{\mathcal{H}} + \|u_-\|_V \leq C(\|v\|_{\mathcal{H}} + \|v_+\|_V)\,.$$

$\square$

Of course, if in addition we know that $\mathcal{P}$ is a Fredholm operator of index 0 then the lemma implies that $\mathcal{P}$ has a two sided inverse.

To check the hypotheses with $\mathcal{H} = L^2_r$, $V = \mathcal{C}^N$ (parameter independent norms) and $\kappa = \langle \lambda - \operatorname{Re} z\rangle$ for $|\lambda| \geq 1/(C\delta)$ and $|\operatorname{Re} z| \ll 1/\delta$ we first consider the case of λ positive:

$$\operatorname{Re}\left(e^{i\frac{\pi}{3}}(P_\lambda - z)\right) \geq \frac{\lambda - \operatorname{Re} z}{C}\,. \tag{5.11}$$

In fact,

$$\inf \left\{ \operatorname{Re}\left(e^{-i\pi/3}\sigma - e^{i\pi/3}z + e^{i\pi/3}\lambda\right) \; : \; \sigma > 0\,, |\operatorname{Im} z| \leq C_1 \right\} \geq \frac{\lambda - \operatorname{Re} z}{\widetilde{C}}\,,$$

which gives (5.11). For λ negative we have the same conclusion without multiplication by $e^{i\pi/3}$: $-\operatorname{Re}(P_\lambda - z) \geq -\lambda/C - |\operatorname{Re} z|$ so again the hypothesis holds.

To check that $\|(P_\lambda - z)\restriction_{\operatorname{Image} R_-}\| \leq C\langle \lambda - \operatorname{Re} z\rangle$ and that $\|(P_\lambda - z)^*\restriction_{\operatorname{Image} R_-}\| \leq C\langle \lambda - \operatorname{Re} z\rangle$, we note that in L^2_r

$$D_t^2 e_j^{\lambda,\delta} = \mathcal{O}(\langle \delta\lambda\rangle)\,, \quad t e_j^{\lambda,\delta} = \mathcal{O}(\langle\delta\lambda\rangle^{-\frac{1}{2}})\,.$$

We conclude that for $|\operatorname{Re} z| \ll 1/\delta$

$$\mathcal{P}^\delta_\lambda(z)\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix} \Longrightarrow \begin{cases} \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2_r} \leq C(\|v\|_{L^2_r} + \langle \lambda - \operatorname{Re} z\rangle |v_+|) \\ |u_-| \leq C(\|v\|_{L^2_r} + \langle\lambda - \operatorname{Re} z\rangle|v_+|) \end{cases}$$

As before the equation gives also the control of $D_t^2 u$ and tu and thus we obtain

$$\langle \lambda - \operatorname{Re} z\rangle\|u\|_{L^2_r} + \|D_t^2 u\|_{L^2_r} + \|tu\|_{L^2_r} + |u_-| \leq C(\|v\|_{L^2_r} + \langle\lambda - \operatorname{Re} z\rangle|v_+|)\,. \tag{5.12}$$

To simplify notation we will now write

$$\begin{aligned} \mathcal{B}_{z,\lambda,r} = B \otimes \mathbb{C}^N\,, \quad & \left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}_{z,\lambda,r}} = \langle \lambda - \operatorname{Re} z\rangle \|u\|_{L^2_r} + \|D_t^2 u\|_{L^2_r} \\ & \qquad + \|tu\|_{L^2_r} + |u_-|\,, \quad u(0) = 0\,, \\ \mathcal{H}_{z,\lambda,r} = H \otimes \mathbb{C}^N\,, \quad & \left\| \begin{pmatrix} v \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}_{z,\lambda,r}} = \|v\|_{L^2_r} + \langle\lambda - \operatorname{Re} z\rangle|v_+|\,. \end{aligned} \tag{5.13}$$

The mapping properties of $\mathcal{P}^\delta_\lambda(z)$ and of its inverse $\mathcal{E}^\delta_\lambda(z)$ given by (5.12) generalize to

$$\begin{aligned} \|\partial_\lambda^k \mathcal{P}^\delta_\lambda(z)\|_{\mathcal{L}(\mathcal{B}_{z,\lambda,r},\mathcal{H}_{z,\lambda,r})} \leq C_k\langle\lambda - \operatorname{Re} z\rangle^{-k}\,, \\ \|\partial_\lambda^k \mathcal{E}^\delta_\lambda(z)\|_{\mathcal{L}(\mathcal{B}_{z,\lambda,r},\mathcal{H}_{z,\lambda,r})} \leq C_k\langle\lambda - \operatorname{Re} z\rangle^{-k}\,, \end{aligned} \tag{5.14}$$

for $|\operatorname{Re} z| \ll 1/\delta$ and where to obtain the first estimate we used (5.7), noting that

$$\frac{\delta}{\langle\delta\lambda\rangle} = \frac{\delta\langle\lambda - \operatorname{Re} z\rangle}{\langle\delta\lambda\rangle}\frac{1}{\langle\lambda - \operatorname{Re} z\rangle} = \mathcal{O}(1)\langle\lambda - \operatorname{Re} z\rangle^{-1}\,.$$

The $-+$ component of $\mathcal{E}^\delta_\lambda$ is not quite as simple as (5.5) anymore. However we have

$$\begin{gathered}\|E^\delta_{-+}(z;p,\lambda) - \operatorname{diag}(z-\lambda-e^{-\frac{2\pi i}{3}}\zeta_j)\| \leq \mathcal{O}(|\lambda|\delta|)\langle\lambda - \operatorname{Re} z\rangle \leq \epsilon \ll 1 \\ \text{for } \ |\lambda| \leq \frac{1}{C\sqrt{\delta}}\,, \ |\operatorname{Re} z| \ll \frac{1}{\sqrt{\delta}}\,,\end{gathered} \tag{5.15}$$

and

$$\det E^\delta_{-+}(z;p,\lambda) = 0\,, \ |\lambda| \leq \frac{1}{C\sqrt{\delta}}\,, \ |\operatorname{Re} z| \ll \frac{1}{\sqrt{\delta}} \implies z = \lambda + e^{-\frac{2\pi i}{3}}\zeta_j\,, \tag{5.16}$$

which follows from Rouché's theorem. When $\mu \neq 1$ we need to replace ζ_j with $\zeta_j\mu^{\frac{2}{3}}$.

For $|\lambda| \gg 1 + |\operatorname{Re} z|$ we see that

$$E^\delta_{-+}(z;p,\lambda)^{-1} = \mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle^{-1}) \in \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)\,. \tag{5.17}$$

In fact, in the notation of Lemma 5.2 we recall the general facts that $E^{-1}_{-+} = -R_+P^{-1}R_-$ and that $|\langle Pu, u\rangle| \geq \kappa\|u\|^2$ implies $P^{-1} = \mathcal{O}(\kappa^{-1})$. Since for $|\lambda| \gg 1 + |\operatorname{Re} z|$ we checked this with $\kappa = \langle\lambda - \operatorname{Re} z\rangle$ and since R_+, R_- are bounded, the estimate follows.

This discussion is most relevant however for $\operatorname{Re}(\lambda - z)$ close to 0. When $|\lambda| \gg 1 + |\operatorname{Re} z|$ we can consider a simpler model problem and treat the t term entirely as a perturbation. Thus we define

$$\mathcal{P}^\sharp_\lambda(z) = \begin{pmatrix} P^\sharp_\lambda - z & R^{\lambda,\delta}_- \\ R^{\lambda,\delta}_+ & 0 \end{pmatrix}\,, \quad P^\sharp_\lambda = e^{-2\pi i/3}D_t^2 + \lambda\,.$$

If we define $\mathcal{B}^\sharp_{\lambda,r} \subset \mathcal{H}^\sharp_{\lambda,r} = L^2([0,\infty)) \otimes \mathbb{C}^N$, by

$$\begin{gathered}\begin{pmatrix} u \\ u_- \end{pmatrix} \in \mathcal{B}^\sharp_{\lambda,r} \iff \left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}^\sharp_{\lambda,r}} = \langle\lambda\rangle\|u\|_{L^2_r} + \|D_t^2 u\|_{L^2_r} + |u_-| < \infty\,, \quad u(0) = 0\,, \\ \begin{pmatrix} v \\ v_+ \end{pmatrix} \in \mathcal{H}^\sharp_{\lambda,r} \iff \left\| \begin{pmatrix} v \\ v_+ \end{pmatrix} \right\|_{\mathcal{H}^\sharp_{\lambda,r}} = \|v\|_{L^2_r} + \langle\lambda\rangle|v_+| < \infty\,.\end{gathered} \tag{5.18}$$

For $|\lambda| \gg 1 + |\operatorname{Re} z|$ we can again use Lemma 5.2 to obtain the inverse $\mathcal{E}^\sharp_\lambda(z)$ and we have the same estimates

$$\begin{gathered}\|\partial^k_\lambda \mathcal{P}^\sharp_\lambda(z)\|_{\mathcal{L}(\mathcal{B}^\sharp_{\lambda,r},\mathcal{H}^\sharp_{\lambda,r})} \leq C_k\langle\lambda\rangle^{-k}\,, \\ \|\partial^k_\lambda \mathcal{E}^\sharp_\lambda(z)\|_{\mathcal{L}(\mathcal{B}^\sharp_{\lambda,r},\mathcal{H}^\sharp_{\lambda,r})} \leq C_k\langle\lambda\rangle^{-k}\,.\end{gathered} \tag{5.19}$$

The same argument as the one used for (5.17) gives

$$E_{-+}(z;p,\lambda)^{-1} = \mathcal{O}(\langle\lambda\rangle^{-1})\,. \tag{5.20}$$

6. Symbol classes for the microlocal Grushin problem

The model problems of the previous section will allow us to consider the reduction to the boundary, via a Grushin problem, on the symbolic level.

If $P(h)$ is the scaled operator given by (3.7) then for $w \in W \Subset (0,\infty)$ and $|\operatorname{Im} z| \leq C$, $|\operatorname{Re} z| \ll 1/\delta$ (here $\delta > 0$ is as in the previous section)

$$\begin{aligned} P - z &\stackrel{\mathrm{def}}{=} h^{-\frac{2}{3}}(P(h) - w) - z \\ &= e^{-2\pi i/3}(D_t^2 + 2tQ(h^{\frac{2}{3}}t, x', hD_{x'}; h)) + h^{-\frac{2}{3}}(R(x', hD_{x'}; h) - w) \\ &\quad + F(h^{\frac{2}{3}}t, x')h^{\frac{2}{3}}D_t \end{aligned} \tag{6.1}$$

where $x = (x', x_n)$ are the variables given by (3.1) and $t = h^{-\frac{2}{3}}x_n$. We recall that $Q(0, x', \xi')$ is the second fundamental form on the boundary – see (3.5).

In the notation of Sect.4 we put

$$\Sigma_w = \{(x', \xi') \in T^*\partial\mathcal{O} \ : \ R(x', \xi') = w\}\,, \quad \lambda = h^{-\frac{2}{3}}(R(x', \xi') - w)\,.$$

Then with $B_{z,\lambda,r}$ and its norm given by (5.3) we can consider P as an element of

$$\Psi_{\Sigma_w, \frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(B_{z,\lambda,r}, L_r^2))\,.$$

The operator valued principal symbol of P becomes

$$p = e^{-2\pi i/3}(D_t^2 + 2tQ(h^{\frac{2}{3}}t, x', \xi')) + \lambda \in S_{\Sigma_w, \frac{2}{3}}(T^*\partial\mathcal{O}; 1, \mathcal{L}(B_{z,\lambda,r}, L_r^2))\,,$$

where the notation as in (4.35) and above. To introduce a Grushin problem we put $R_+ = R_+^{\lambda,\delta}$, $R_- = R_-^{\lambda,\delta}$ where the $R_\pm^{\lambda,\delta}$ notation is as in (5.9), with μ now equal to $2Q(0, x', \xi')$,

$$R_+ \in S_{\Sigma_w, \frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(L_r^2, \mathbb{C}^N))\,, \quad R_- \in S_{\Sigma_w, \frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathbb{C}^N, L_r^2))\,. \tag{6.2}$$

Analysis near the glancing hypersurface. Restricting to a small fixed neighbourhood of Σ_w amounts to assuming

$$\lambda = \mathcal{O}(h^{-\frac{2}{3}})\,,$$

and that

$$1/C \leq Q(0, x', \xi') \leq C\,.$$

We will consider P as a perturbation of the operator P_0:

$$P_0 = e^{-2\pi i/3}(D_t^2 + 2tQ(0, x', \xi')) + \lambda\,.$$

The first symbolic Grushin problem is exactly the same as in Sect.5 with $\mu = 2Q(0, x', \xi')$ but with the invariant meaning given by (6.2). Thus we put

$$\mathcal{P}_0(z) = \begin{pmatrix} P_0 - z & R_- \\ R_+ & 0 \end{pmatrix} \in S_{\Sigma_w, \frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{B}_{z,\lambda,r}, \mathcal{H}_{z,\lambda,r}))\,, \tag{6.3}$$

where the spaces and the norms are as in (5.13). As in (5.14) we obtain the same symbolic properties for the inverse $\mathcal{E}_0$ defined in a fixed h-independent neighbourhood of Σ_w:

$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^k \mathcal{E}_0(z) = \mathcal{O}_{\alpha,\beta,k}(\langle \lambda - \operatorname{Re} z\rangle^{-k}) \ : \ \mathcal{H}_{z,\lambda,r} \longrightarrow \mathcal{B}_{z,\lambda,r}\,.$$

In order to treat powers of t we define

$$T \stackrel{\text{def}}{=} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix},$$

observing that $s^m e^{-s} \leq m^m e^{-m}$, $s, m \geq 0$, implies

$$T^k = \frac{C^k k^k}{(r_2 - r_1)^k} \langle \lambda - \operatorname{Re} z \rangle^{-1} : \mathcal{B}_{z,\lambda,r_2} \longrightarrow \mathcal{H}_{z,\lambda,r_1}, \quad r_2 > r_1 .$$

Hence the powers of T can be considered provided that we simultaneously relax the exponentially weighted spaces. For $k = 1$ we have of course boundedness with $r_1 = r_2$ but with no decay in λ.

When we study the symbolic properties, the stability under commuting with T will be important.

Lemma 6.1. *If* ad_T *denotes the commutator with* T, $\operatorname{ad}_T A = [T, A]$, *then for* $|\operatorname{Re} z| \ll 1/\delta$

$$\operatorname{ad}_T^k \mathcal{P}_0 = \frac{\mathcal{O}_k(\delta^{-k/2})}{\langle \lambda - \operatorname{Re} z \rangle^{k/2}} \; : \; \mathcal{B}_{z,\lambda,r} \longrightarrow \mathcal{H}_{z,\lambda,r} .$$

Proof. A simple computation shows that

$$\operatorname{ad}_T \mathcal{P}_0 = \begin{pmatrix} 2ie^{2\pi i/3} D_t & tR_- \\ -R_+ t & 0 \end{pmatrix} .$$

Since $Id = \mathcal{O}(\langle \lambda - \operatorname{Re} z \rangle^{-1})$, $D_t^2 = \mathcal{O}(1) : B_{z,\lambda,r} \to L^2_r$, interpolation shows that $D_t = \mathcal{O}(\langle \lambda - \operatorname{Re} z \rangle^{-\frac{1}{2}}) : B_{z,\lambda,r} \to L^2_r$ (this is well known when there is no weight and here we can consider the conjugated operator $e^{rt/2} D_t e^{-rt/2}$ on the spaces with no weight). To see the bound on the norms of $R_+ t$, $tR_- = (R_+ t)^*$ we recall that $se_j(s) \in \mathcal{S}([0, \infty))$ and hence

$$u \longmapsto \langle u, t \langle \delta\lambda \rangle^{\frac{1}{4}} e_j(\langle \delta\lambda \rangle^{\frac{1}{2}} t) \rangle = \langle \delta\lambda \rangle^{-\frac{1}{2}} \langle u, \langle \delta\lambda \rangle^{\frac{1}{4}} (se_j(s)) \restriction_{s = \langle \delta\lambda \rangle^{\frac{1}{2}}} \rangle .$$

Since for $|\operatorname{Re} z| \leq 1/\delta$

$$\frac{\delta \langle \lambda - \operatorname{Re} z \rangle}{\langle \delta\lambda \rangle} = \mathcal{O}(1),$$

we get $R_+ t = \mathcal{O}(L \langle \lambda \rangle^{-\frac{1}{2}}) : L^2_r \to \mathbb{C}^N$. This proves the claim for $k = 1$. More generally,

$$\operatorname{ad}_T^k \mathcal{P}_0 = \begin{pmatrix} -2e^{2\pi i/3} \delta_{k2} & t^k R_- \\ (-1)^k R_+ t^k & 0 \end{pmatrix}, \quad k > 1 .$$

As before $Id = \mathcal{O}(\langle \lambda - \operatorname{Re} z \rangle^{-1}) : B_{z,\lambda,r} \to L^2_r$, and $(t^k R_-)^* = R_+ t^k = \mathcal{O}(1)(\delta \langle \lambda - \operatorname{Re} z \rangle)^{-\frac{k}{2}}) : L^2_r \to \mathbb{C}^N$, so the lemma follows. $\square$

We can generalize this further by combining the estimate of the lemma with the estimate (5.14):

$$\partial^\alpha_{x'} \partial^\beta_{\xi'} \partial^l_\lambda \operatorname{ad}_T^k \mathcal{P}_0(z) = \mathcal{O}(\delta^{-\frac{k}{2}} \langle \lambda - \operatorname{Re} z \rangle^{-l-\frac{k}{2}}) \; : \; \mathcal{B}_{z,\lambda,r} \longrightarrow \mathcal{H}_{z,\lambda,r}, \tag{6.4}$$

where we now dropped indicating the dependence of constants on k, l, α and β.

Since ad_T is a derivation we get the same estimates for the inverse:

$$\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \mathrm{ad}_T^k \mathcal{E}_0(z) = \mathcal{O}(\delta^{-\frac{k}{2}} \langle \lambda - \mathrm{Re}\, z\rangle^{-l-\frac{k}{2}}) \ : \ \mathcal{H}_{z,\lambda,r} \longrightarrow \mathcal{B}_{z,\lambda,r} \, . \tag{6.5}$$

We will consider the operator P as a perturbation of P_0 and in the estimates of the higher order terms we will not be concerned with the dependence on δ. That means that we can replace $\langle \lambda - \mathrm{Re}\, z\rangle$ by $\langle \lambda \rangle$ at the expense of δ dependent constants. Eventually, the constant h will be taken small enough depending on δ. Hence we write the formal expansion of the Grushin problem for P (for clarity of exposition we forget here about the h dependence in Q – it will certainly be included in the more general expansion (6.9)):

$$\begin{aligned}
\mathcal{P}(z) &= \begin{pmatrix} P-z & R_- \\ R_+ & 0 \end{pmatrix} \\
&\equiv \mathcal{P}_0(z) + \sum_{k=1}^\infty h^{\frac{2k}{3}} \begin{pmatrix} 2e^{-2\pi i/3} \partial_t^k Q(0,x',\xi') t^{k+1}/k! & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad + \sum_{k=1}^\infty h^{\frac{2k}{3}} \begin{pmatrix} \partial_t^{k-1} F(0,x') t^{k-1}/(k-1)! D_t & 0 \\ 0 & 0 \end{pmatrix} \\
&\equiv \mathcal{P}_0(z) + \sum_{k=1}^\infty h^{\frac{2k}{3}} T^k \mathcal{P}_k + \sum_{k=1}^\infty h^{\frac{2k}{3}} T^{k-1} \mathcal{D}_k \, ,
\end{aligned}$$

where

$$\mathcal{P}_k = \begin{pmatrix} 2e^{-2\pi i/3} t \partial_t^k Q(0,x',\xi')/k! & 0 \\ 0 & 0 \end{pmatrix} , \quad \mathcal{D}_k = \begin{pmatrix} \partial_t^{k-1} F(0,x')/(k-1)! D_t & 0 \\ 0 & 0 \end{pmatrix} , \quad k \geq 1 \, .$$

The symbolic properties of $\mathcal{P}_k$ and $\mathcal{D}_k$ are

$$\begin{aligned}
&\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \mathrm{ad}_T^{\tilde{k}} \mathcal{P}_k(z) = \mathcal{O}(\langle \lambda \rangle^{-l-\tilde{k}/2}) : \mathcal{B}_{z,\lambda,r} \longrightarrow \mathcal{H}_{z,\lambda,r} \, , \\
&\partial_{x'}^\alpha \partial_{\xi'}^\beta \partial_\lambda^l \mathrm{ad}_T^{\tilde{k}} \mathcal{D}_k(z) = \mathcal{O}(\langle \lambda \rangle^{-1/2-l-\tilde{k}/2}) : \mathcal{B}_{z,\lambda,r} \longrightarrow \mathcal{H}_{z,\lambda,r} \, ,
\end{aligned} \tag{6.6}$$

where we neglected a number of simplifying features and we shall in fact proceed as if they were absent (for instance, that in the first estimate the left hand side vanishes when $l \neq 0$ or $\tilde{k} \neq 0$). The power of $-1/2$ in the estimate for $\mathcal{D}$ came from the mapping property $D_t = \mathcal{O}(\langle \lambda - \mathrm{Re}\, z\rangle^{-1/2}) : B_{z,\lambda,r} \to L^2_r$, which was used in Lemma 6.1.

We shall invert $\mathcal{P}$ as an operator valued h-pseudo-differential operator on $\partial\mathcal{O}$ and we shall adopt an aproach used in Sect.1 of [23]. We shall work in symplectic local coordinates introduced in Sect.4 and in which $\lambda = h^{-\frac{2}{3}}\xi_1$ and $\xi' = (\xi_1, \xi'')$. The passage from the local to global constructions is justified by the calculus developed in Sect.4. We recall first that the composition formula (4.8)

$$a \sharp_h b = \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} (h\partial_\xi)^\alpha a D_x^\alpha b$$

can be rewritten as follows. Let us introduce *formal* operators

$$Q(x,\xi,D_x,h) = q(z,\xi + hD_x, h) \equiv \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \partial_\xi^\alpha q \, (hD_x)^\alpha \, , \quad q = Q(1) \, ,$$

where $q = a\,, b$ and $Q = A\,, B$ respectively. Then

$$a \sharp_h b = A \circ B(1)\,,$$

where by $A \circ B$ we mean the operator obtained from a formal composition of A and B and $A \circ B(1)$ is that operator applied to the constant function 1. To invert a asymptotically we want to construct b such $a \sharp_h b \equiv 1$ and that can be obtained by inverting the infinite order formal differential operator A. If B is its formal inverse then $A \circ B(1) = \mathrm{Id}(1) = 1$ and $b = B(1)$.

To apply the same procedure in our situation we write

$$\mathcal{P} = \sum_0^\infty h^{\frac{2k}{3}} T^k \left(\mathcal{P}_k(x', \xi', \lambda) + h^{\frac{2}{3}} \mathcal{D}_{k+1}(x', \xi') \right)\,, \tag{6.7}$$

where $\mathcal{P}_k$ and $\mathcal{D}$ satisfy (6.6). Then we introduce a formal operator of infinite order ($\alpha = (\alpha_1, \alpha''))$

$$\begin{aligned}
\mathfrak{P} &= \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \partial_{\xi'}^\alpha \left(\mathcal{P}(x', \xi', \lambda, z; h) \right) (h D_{x'})^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \left[\partial_{\xi''}^{\alpha''} \left(\partial_{\xi_1} + h^{-\frac{2}{3}} \partial_\lambda \right)^{\alpha_1} \mathcal{P} \right] (x', \xi', \lambda, z; h) (h D_{x'})^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \left[(h\partial_{\xi''})^{\alpha''} \left(h\partial_{\xi_1} + h^{\frac{1}{3}} \partial_\lambda \right)^{\alpha_1} \mathcal{P} \right] (x', \xi', \lambda, z; h) D_{x'}^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{n-1}} \frac{1}{\alpha!} \left(\sum_{k \in \mathbb{N}} h^{\frac{2k}{3}} T^k \left(\left[(h\partial_{\xi''})^{\alpha''} \left(h\partial_{\xi_1} + h^{\frac{1}{3}} \partial_\lambda \right)^{\alpha_1} \mathcal{P}_k \right] (x', \xi', \lambda, z; h) + h^{\frac{2}{3}} (h\partial_{\xi'})^\alpha \mathcal{D}_{k+1} \right) \right) D_{x'}^\alpha\,,
\end{aligned} \tag{6.8}$$

where the leading term is given by $\mathcal{P}_0(x', \xi', \lambda, z) = \mathcal{P}_0(z)$ defined by (6.3).

To invert the infinite order formal differential operator $\mathfrak{P}$ we introduce a general class of such operators:

$$\mathfrak{A} = \sum_{\substack{\alpha \in \mathbb{N}^{n-1} \\ k, l, a, b, \in \mathbb{N}}} \left(h^{\frac{2}{3}} T \right)^k \left(h^{\frac{2}{3}} \langle \lambda \rangle^{-\frac{1}{2}} \right)^l h^a \left(\frac{h^{\frac{1}{3}}}{\langle \lambda \rangle} \right)^b \mathcal{A}_{\alpha, a, b, k, l}(x', \xi', \lambda, z) D_{x'}^\alpha\,, \tag{6.9}$$

with

$$\partial_{x'}^{\tilde\alpha} \partial_{\xi'}^\beta \partial_\lambda^{\tilde l} \mathrm{ad}_T^{\tilde k} \mathcal{A}_{\alpha, a, b, k, l} = \mathcal{O}(\langle \lambda \rangle^{-\tilde l - \tilde k/2}) : \mathcal{B}_{z, \lambda, r} \longrightarrow \mathcal{H}_{z, \lambda, r}\,.$$

We note that in the expansion of $\mathfrak{P}$ we have $l = 0$ except for the $l = b = 1$, $a = 0, \alpha = 0$ term coming from $\mathcal{D}$.

Let $\mathcal{C}_{z,\lambda,r}$ denote either $\mathcal{B}_{z,\lambda,r}$ or $\mathcal{H}_{z,\lambda,r}$ or more generally some other family of Banach spaces. If $\mathfrak{B}$ is of the same form as $\mathfrak{A}$ but with $\mathcal{A}_{\alpha,a,b,k,l}$ replaced by by $B_{\alpha,a,b,k,l}$ satisfying

$$\partial_{x'}^{\tilde\alpha} \partial_{\xi'}^\beta \partial_\lambda^{\tilde l} \mathrm{ad}_T^{\tilde k} \mathcal{B}_{\alpha, a, b, k, l} = \mathcal{O}(\langle \lambda \rangle^{-\tilde l - \tilde k/2}) : \mathcal{C}_{z, \lambda, r} \longrightarrow \mathcal{B}_{z, \lambda, r}\,. \tag{6.10}$$

then

$$\mathfrak{C} \stackrel{\text{def}}{=} \mathfrak{A} \circ \mathfrak{B}$$

is of the same form with

$$\partial_{x'}^{\tilde{\alpha}}\partial_{\xi'}^{\beta}\partial_{\lambda}^{\tilde{l}}\mathrm{ad}_T^{\tilde{k}}\mathcal{C}_{\alpha,a,b,k,l} = \mathcal{O}(\langle\lambda\rangle^{-\tilde{l}-\tilde{k}/2}) : \mathcal{C}_{z,\lambda,r} \longrightarrow \mathcal{H}_{z,\lambda,r}\,.$$

This leads to the construction of formal inverses in the sense of expansions of the form (6.9):

Lemma 6.2. *If $\mathfrak{A}$ is of the form* (6.9) *and $\mathcal{A}_0$ is invertible with $\mathcal{A}_0^{-1} = \mathcal{B}_0$ satisfying*

$$\mathcal{B}_0 = \mathcal{O}(1) : \mathcal{H}_{z,\lambda,r} \longrightarrow \mathcal{B}_{z,\lambda,r}\,,$$

then there exists $\mathfrak{B}$ of the form (6.9) *(with $\mathcal{A}_\bullet$ replaced by $\mathcal{B}_\bullet$) satisfying* (6.10) *with $\mathcal{C}_{z,\lambda,r} = \mathcal{H}_{z,\lambda,r}$, and such that*

$$\mathfrak{A}\circ\mathfrak{B} = \mathrm{Id}\,, \quad \mathfrak{B}\circ\mathfrak{A} = \mathrm{Id}\,.$$

Proof. By multiplying $\mathfrak{A}$ by $\mathfrak{B}_0 = \mathcal{B}_0$ from the left we obtain $\mathfrak{C}$ of the form (6.9) and satisfying (6.10) with $\mathcal{C}_{z,\lambda,r} = \mathcal{B}_{z,\lambda,r}$ and $\mathcal{C}_0 = \mathrm{Id}$. Hence the formal series $\mathfrak{D} = I + (I-\mathfrak{C}) + (I-\mathfrak{C})\circ(I-\mathfrak{C}) + \cdots$ is of the form (6.9) (we note that the at first mysterious terms $l \neq 0$ in (6.9) arise from commuting $h^{2/3}T$ through). The left inverse is then obtained by taking $\mathfrak{B} = \mathfrak{D}\circ\mathfrak{B}_0$ and since we can also construct a right inverse the standard argument shows that they are the same as formal expansions. □

When we apply Lemma 6.1 to $\mathfrak{P}$ we obtain an inverse $\mathfrak{E}$ of the form (6.9). From this we get a parametrix of $\mathcal{P}(z)$ valid in the region $|\lambda| = \mathcal{O}(h^{-\frac{2}{3}})$:

$$\begin{aligned}\mathcal{E}(x',\xi',\lambda,z;h) &= \mathfrak{E}(x',\xi',D_{x'},\lambda,z;h)(1)\\ &= \sum_{k,l,a,b\in\mathbb{N}} \left(h^{\frac{2}{3}}T\right)^k \left(h^{\frac{2}{3}}\langle\lambda\rangle^{-\frac{1}{2}}\right)^l h^a \left(\frac{h^{\frac{1}{3}}}{\langle\lambda\rangle}\right)^b \mathcal{E}_{k,l,a,b}(x',\xi',\lambda;z)\,, \end{aligned} \tag{6.11}$$

with

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\partial_{\lambda}^{\tilde{l}}\mathrm{ad}_T^{\tilde{k}}\mathcal{E}_{k,l,a,b}(x',\xi',\lambda;z) = \mathcal{O}(1)\langle\lambda\rangle^{-\tilde{l}-\frac{\tilde{k}}{2}} : \mathcal{H}_{z,\lambda,r} \longrightarrow \mathcal{B}_{z,\lambda,r}\,. \tag{6.12}$$

At the moment this construction is formal. Roughly speaking, when $x_n \leq h^\epsilon$, $\epsilon < 2/3$, so that $t < h^{-2/3+\epsilon}$, we will be able to consider these asymptotic expansions modulo $\mathcal{O}(h^\infty)$ and they will then give good microlocal parametrices when we use calculus of Sect.4. This will be carried out in Sect.8.

We remark here that in the region where $1/C \leq |R(x',\xi') - 1| \leq 1/2$ (we put $w = 1$ for simplicity and note that this condition still implies that $Q(0,x',\xi')$ is bounded from below) we should obtain an analogous expansion with integral powers of h only. In coordinates used above, $R(x',\xi') - 1 = \xi_1$ so that

$$\frac{h^{\frac{1}{3}}}{\langle\lambda\rangle} = \frac{h^{\frac{1}{3}}}{(1+(h^{-\frac{2}{3}}\xi_1)^2)^{\frac{1}{2}}} = \frac{h}{|\xi_1|}(1+(h^{\frac{2}{3}}\ \xi_1^{-1})^2)^{-\frac{1}{2}}\,.$$

Replacing $\langle\lambda\rangle$ by $\langle\lambda^3\rangle^{\frac{1}{3}}$ does not change anything in the discussion above and we still get a similar expansion for $\mathcal{E}$. When $1/C \leq |R(x',\xi') - 1| \leq 1/2$ it now reduces to

$$\mathcal{E}(x',\xi',\lambda,z;h) = \sum_{k,l\in\mathbb{N}} \left(h^{\frac{2}{3}}T\right)^k h^l \widetilde{\mathcal{E}}_{k,l}(x',\xi',\lambda,z;h) \tag{6.13}$$

where $\tilde{\mathcal{E}}_{k,l}$ satisfy (6.12) which here becomes

$$\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\partial_{\lambda}^{\tilde{l}}\mathrm{ad}_T^{\tilde{k}}\tilde{\mathcal{E}}_{k,l,a,b}(x',\xi',\lambda;z)=\mathcal{O}(h^{\frac{\tilde{k}}{3}}):\mathcal{H}_{z,\lambda,r}\longrightarrow\mathcal{B}_{z,\lambda,r}\,. \tag{6.14}$$

Analysis away from the glancing hypersurface. This is the region where $|R(x',\xi')-w|>1/C$. The expansion (6.13) treated the case when this quantity is also bounded by $|w|/2$ but we need to consider the cases where $R(x',\xi')$ and $Q(0,x',\xi')$ can get small or large. However we now have $Q\ll|\lambda|=h^{-\frac{2}{3}}|R-w|$ and thus we would like to treat $tQ(h^{\frac{2}{3}}t,x',\xi')$ entirely as a perturbation.

We now put (changing the notation of Sect.4 from $P_\lambda^\sharp$ to $P_0^\sharp$):

$$P_0^\sharp=e^{-2\pi i/3}D_t^2+\lambda\,,\quad\lambda=h^{-\frac{2}{3}}(R(x',\xi')-w)\,.$$

We use the same $R_\pm$ as in the definition (6.3) of $\mathcal{P}_0(z)$ and we define (again changing the notation of Sect.4, now from $\mathcal{P}_\lambda^\sharp$ to $\mathcal{P}_0^\sharp$):

$$\mathcal{P}_0^\sharp=\begin{pmatrix}P_0^\sharp-z & R_-\\ R_+ & 0\end{pmatrix}\;:\;\mathcal{B}_{\lambda,r}^\sharp\longrightarrow\mathcal{H}_{\lambda,r}^\sharp\,, \tag{6.15}$$

where the spaces are as in (5.18). Again we consider $\mathcal{P}_0^\sharp$ as an element of

$$S_{\Sigma_w,\frac{2}{3}}(\partial\mathcal{O};1,\mathcal{L}(\mathcal{B}_{\lambda,r}^\sharp,\mathcal{H}_{\lambda,r}))\,,$$

where the norms where modified in the obvious way. For $|\lambda|\gg|\operatorname{Re}z|$ we see from (5.19) that $\mathcal{P}_0^\sharp$ is invertible:

$$\mathcal{E}_0^\sharp\circ\mathcal{P}_0^\sharp=Id_{\mathcal{B}_{\lambda,r}^\sharp}\,,\quad\mathcal{P}_0^\sharp\circ\mathcal{E}_0^\sharp=Id_{\mathcal{H}_{\lambda,r}^\sharp}\,,\quad\mathcal{E}_0^\sharp\in S_{\Sigma_w,\frac{2}{3}}(\partial\mathcal{O};\mathcal{L}_{m_\lambda^\sharp}(\mathcal{H}_{\lambda,r}^\sharp,\mathcal{B}_{\lambda,r}^\sharp))\,.$$

When we assume, as we do here, that $|\lambda|\geq h^{-\frac{2}{3}}/C$ we do not have to treat λ with special care and the symbol properties are

$$\begin{aligned}\partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\mathrm{ad}_T^k\mathcal{P}_0^\sharp(z)&=\mathcal{O}\left(\langle\xi'\rangle^{-|\beta|}\langle\lambda\rangle^{-\frac{k}{2}}\right)\;:\;\mathcal{B}_{\lambda,r}^\sharp\longrightarrow\mathcal{H}_{\lambda,r}^\sharp\,,\\ \partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\mathrm{ad}_T^k\mathcal{E}_0^\sharp(z)&=\mathcal{O}\left(\langle\xi'\rangle^{-|\beta|}\langle\lambda\rangle^{-\frac{k}{2}}\right)\;:\;\mathcal{H}_{\lambda,r}^\sharp\longrightarrow\mathcal{B}_{\lambda,r}^\sharp\,,\end{aligned} \tag{6.16}$$

where we note that $|\lambda|^{-k/2}\sim(h^{-1/3}\langle\xi'\rangle)^{-k}$. We also note that $Q(0,x',\xi')=\mathcal{O}(h^{2/3})|\lambda|$.

Writing the expansion

$$tQ(h^{\frac{2}{3}}t,x',\xi')\equiv\sum_{k=1}^{\infty}\frac{t^kh^{\frac{2(k-1)}{3}}}{(k-1)!}\partial_t^{k-1}Q(0,x',\xi')\,, \tag{6.17}$$

we see that if we consider $\mathcal{P}(z)$ defined as before but now for $|\lambda|\gg h^{-\frac{2}{3}}/C$ then

$$\begin{aligned}\mathcal{P}(z)&\equiv\sum_{l=0}^{\infty}\left(h^{\frac{2}{3}}T\right)^l\mathcal{P}_l^\sharp(x',\xi',x;h)\,,\\ \partial_{x'}^{\alpha}\partial_{\xi'}^{\beta}\mathrm{ad}_T^k\mathcal{P}_l^\sharp&=\mathcal{O}(1)\langle\xi'\rangle^{-|\beta|}\left(\frac{h^{\frac{1}{3}}}{\langle\xi'\rangle}\right)^k\;:\;\mathcal{B}_{\lambda,r}^\sharp\longrightarrow\mathcal{H}_{\lambda,r}^\sharp\,.\end{aligned} \tag{6.18}$$

To invert $\mathcal{P}(z)$ microlocally in the region where $1/C \geq |R(x',\xi')-1|$ and to infinite order at the boundary we can proceed as outlined before (6.7). Thus we put

$$\begin{aligned}\mathfrak{P}^\sharp &= \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}\left((h\partial_{\xi'})^\alpha \mathcal{P}\right) D_{x'}^\alpha \\ &\equiv \sum_{l=0}^{\infty}\sum_{\alpha\in\mathbb{N}^{n-1}} \frac{1}{\alpha!}\left(h^{\frac{2}{3}}T\right)^l \left((h\partial_{\xi'})^\alpha \mathcal{P}_l^\sharp\right) D_{x'}^\alpha\,.\end{aligned} \tag{6.19}$$

The more general class to consider in this case is given by formal operators of the following form:

$$\mathfrak{A} = \sum_{\substack{\alpha\in\mathbb{N}^{n-1}\\ l,a\in\mathbb{N}, a\geq|\alpha|}} \left(h^{\frac{2}{3}}T\right)^l \left(\frac{h}{\langle\xi'\rangle}\right)^a \mathcal{A}_{\alpha,l,a}(x',\xi',z;h) D_{x'}^\alpha\,, \tag{6.20}$$

with

$$\partial_{x'}^\alpha\partial_{\xi'}^\beta \mathrm{ad}_T^k \mathcal{A}_{\alpha,l,a} = \mathcal{O}(1)\langle\xi'\rangle^{-|\beta|}\left(\frac{h^{\frac{1}{3}}}{\langle\xi'\rangle}\right)^k \;:\; \mathcal{B}_{\lambda,r}^\sharp \longrightarrow \mathcal{H}_{\lambda,r}^\sharp\,. \tag{6.21}$$

A modification of Lemma 6.2 shows that if the leading term of $\mathfrak{A}$, $\mathcal{A}_0$, is invertible with the inverse satisfying the same estimates then $\mathfrak{A}$ has an inverse of the form (6.20). Aplying this to $\mathfrak{P}^\sharp$ we obtain a formal inverse, $\mathfrak{E}^\sharp$, of the form (6.20) satisfying the estimates (6.21) with $\mathcal{H}_{\lambda,r}^\sharp$ and $\mathcal{B}_{\lambda,r}^\sharp$ exchanged.

That gives an inverse of $\mathcal{P}(z)$ in the region $|R(x',\xi')-w|\geq 1/C$ and to infinite order at the boundary:

$$\begin{aligned}\mathcal{E}^\sharp(x',\xi',z;h) &= \mathfrak{E}^\sharp(x',\xi',D_{x'},z;h)(1) \\ &= \sum_{l,a\in\mathbb{N}} \left(h^{\frac{2}{3}}T\right)^l \left(\frac{h}{\langle\xi'\rangle}\right)^a \mathcal{E}_{a,l}^\sharp(x',\xi',z;h)\,,\end{aligned} \tag{6.22}$$

with

$$\partial_{x'}^\alpha\partial_{\xi'}^\beta \mathrm{ad}_T^k \mathcal{E}_{l,a}^\sharp = \mathcal{O}(1)\langle\xi'\rangle^{-|\beta|}\left(\frac{h^{\frac{1}{3}}}{\langle\xi'\rangle}\right)^a \;:\; \mathcal{H}_{\lambda,r}^\sharp \longrightarrow \mathcal{B}_{\lambda,r}^\sharp\,. \tag{6.23}$$

We will use this construction in Sect.7 to obtain a parametrix for the global Grushin problem for the operator $P(h)$.

7. **Estimates away from the boundary**

As in Sect.6 we put

$$P - z \stackrel{\mathrm{def}}{=} h^{-\frac{2}{3}}(P(h)-w) - z\,,\quad w\in W\Subset(0,\infty)\,.$$

We now assume that $|\operatorname{Re} z|\leq L$, $|\operatorname{Im} z|\leq C$, with C fixed but L allowed to grow. We will also use the following notation

$$D(\alpha) = \{x\in\mathbb{R}^n\setminus\mathcal{O} \;:\; d(x,\partial\mathcal{O})>\alpha\}\,,$$

where as before we identified the deformed exterior Γ with $\mathbb{R}^n \setminus \mathcal{O}$.

The purpose of this section is to establish the following

Proposition 7.1. *Let $0 < \epsilon < \frac{2}{3}$. There exists $h_0 = h_0(L)$ such that for $0 < h < h_0(L)$ and every $v \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ satisfying*

$$\operatorname{supp} v \subset \{x : d(x, \partial\mathcal{O}) > h^\epsilon\}$$

there exist $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \mathcal{O})$ and $v_0 \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ such that

$$\begin{gathered} (P - z)u = v + v_0 \,, \quad u\restriction_{\partial\mathcal{O}} = 0 \,, \\ \|u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{h^{-1+\frac{3}{2}\epsilon}/C_\gamma}\|u\|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)h^\epsilon))} + e^{h^{-1+\frac{3}{2}\epsilon}/C}\|v_0\|_{H^k(\mathbb{R}^n\setminus\mathcal{O})} \\ +h^{-\epsilon}\left(\|\chi(hD_{x_n})^2u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + \|\chi\left(-h^2\Delta_{\partial\mathcal{O}} - w\right)u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + \|(1-\chi)u\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})}\right) \\ \leq C_{\gamma,k,L}h^{\frac{2}{3}-\epsilon}\|v\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \,, \end{gathered} \tag{7.1}$$

for any $\gamma > 0$,$k \in \mathbb{R}$ and where $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O}; [0,1])$ is equal to one near $\partial\mathcal{O}$.

The proposition could be stated as giving maps $E_\epsilon : v \mapsto u$, $K_\epsilon : v \mapsto v_0$ satisfying $(P - z)E_\epsilon = I + K_\epsilon$ and with boundedness properties

$$\begin{aligned} E_\epsilon &= \mathcal{O}(h^{\frac{2}{3}-\epsilon}) \ : \ L^2(D(h^\epsilon)) \to H^2(\mathbb{R}^n \setminus \mathcal{O}) \\ K_\epsilon &= \mathcal{O}\left(e^{-h^{-1+\frac{3}{2}\epsilon}/C}\right) \ : \ L^2(D(h^\epsilon)) \to H^k(\mathbb{R}^n \setminus \mathcal{O}) \,, \end{aligned} \tag{7.2}$$

or in fact with somewhat better mapping properties given by (7.1). Since K_ϵ does *not* preserve the support condition an iteration producing an exact inverse is not allowed. The existence of an inverse is in any case unlikely once as we allow complex values of z.

We start with a lemma which is essentially contained in Sect.5 of [32] (see (6.16) there).

Lemma 7.1. *Assume that $0 < h < h_0(L)$. If $v_1 \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ and*

$$\operatorname{supp} v_1 \subset D(\delta) \,, \quad \delta > 0 \,,$$

there exist $u_1 \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \mathcal{O})$ and $v_0^1 \in \mathcal{C}_c^\infty(\mathbb{R}^n \setminus \mathcal{O})$ such that

$$\begin{gathered} (P - z)u_1 = v_1 + v_0^1 \,, \quad u_1\restriction_{\partial\mathcal{O}} = 0 \,, \\ \|u_1\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{C_\gamma h^{-1}}\|u_1\|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)\delta))} + e^{C_\gamma h^{-1}}\|v_0^1\|_{H^k(\mathbb{R}^n\setminus\mathcal{O})} \leq C_\gamma h^{\frac{2}{3}}\|v\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \end{gathered} \tag{7.3}$$

for any k, $\gamma > 0$ and where $H^2(\mathbb{R}^n \setminus \mathcal{O})$ is equipped with the natural semiclassical norm.

Proof. We first solve

$$\left(h^{-\frac{2}{3}}(-h^2\Delta\restriction_{\widetilde{\Gamma}} - w) - z\right)\tilde{u} = v_1 \,, \quad \|\tilde{u}\|_{H^2(\widetilde{\Gamma})} \leq h^{\frac{2}{3}}\|v_1\|_{L^2(\widetilde{\Gamma})} \,, \tag{7.4}$$

where, as in [32], $\widetilde{\Gamma}$ extends the totally real submanifold $\Gamma \subset \mathbb{C}^n \setminus \mathcal{O}$ (given in (3.6) above) to a smooth totally real submanifold $\widetilde{\Gamma}$ in $\mathbb{C}^n$ chosen so that the symbol of $-\Delta\restriction_{\widetilde{\Gamma}}$ takes its values in $\{\zeta : \arg(1+i\theta) < -\arg\zeta < 2\pi/3+\epsilon\}$. That shows that we can invert $h^{-\frac{2}{3}}(-h^2\Delta_{\widetilde{\Gamma}} - w) - z$ as long as h is small enough depending on L. We now claim that

$$\|\tilde{u}\|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)\delta))} \leq C_k e^{-C_\gamma/h}\|v_1\|_{L^2(D(\delta))} \,. \tag{7.5}$$

This follows from the weighted estimates for the resolvent of $-h^2\Delta\restriction_{\widetilde{\Gamma}}$ with the spectral parameter away from the values of the symbol:

$$\left|\partial_x^\alpha\partial_y^\beta\left[(-h^2\Delta\restriction_{\widetilde{\Gamma}}-\zeta)^{-1}\right](x,y)\right|\leq C_{\alpha,\beta}e^{-d_{\widetilde{\Gamma}}(x,y)/C}\,,$$
$$d_{\widetilde{\Gamma}}(x,y)>\epsilon\,,\quad -\arg\zeta<\arg(1+i\theta)-\tilde{\epsilon}\,,\quad \epsilon\,,\tilde{\epsilon}>0\,,\quad \alpha,\beta\in\mathbb{N}^n\,,$$

which in turn follow from the standard conjugation by a smoothed out distance function – see for instance [22]. We can now truncate $\tilde{u}$ by a cut-off function equal to 1 on $D((1-2\gamma)\delta)$ and 0 near ∂O. This gives u_1 satisfying the boundary condition and an error term v_0^1 with an estimate following from (7.5). □

The more interesting analysis takes place near the boundary. Here too we will use an auxiliary operator

$$\begin{aligned}\widetilde{P}_\delta(h)=&e^{-2\pi i/3}\left((hD_{x_n})^2+2x_nQ(x',hD_{x'})\right)+R(x',hD_{x'})+hF(x_n,x')hD_{x_n}\\&+x_n\chi\left(\frac{x_n}{4\delta}\right)2e^{-2\pi i/3}\left(Q(x_n,x',hD_{x'})-Q(x',hD_{x'})\right)\,,\end{aligned}\tag{7.6}$$

where $\chi\in\mathcal{C}^\infty(\mathbb{R};[0,1])$, $\chi(t)\equiv1$ for $t\leq1/2$ and $\chi(t)\equiv0$ for $t>3/4$. We put $E(x_n,x',hD_{x'})=\chi(x_n/4\delta)e^{2\pi i/3}(Q(x_n,x',hD_{x'})-Q(x',hD_{x'}))$. Choosing $\delta>0$ small enough we can arrange that, in any coordinates, the full symbol of E satisfies

$$|E(x_n,x',hD_{x'})|\ll\langle\xi'\rangle^2\,.\tag{7.7}$$

We now make a change of variable $x_n=h^\epsilon s$ so that the operator becomes

$$\begin{aligned}\widetilde{P}_\delta(h)=&h^\epsilon\left[e^{-2\pi i/3}\left((\tilde{h}D_s)^2+2sQ(x',hD_{x'})\right)+h^{-\epsilon}R(x',hD_{x'})+h^{1-\frac{\epsilon}{2}}F(x')\tilde{h}D_s\right.\\&\left.+sE(h^\epsilon s,x',hD_{x'})\right]\,,\quad \tilde{h}=h^{1-\frac{3}{2}\epsilon}\,.\end{aligned}$$

Since we are interested in solving approximately a non-homogeneous equation with data supported in $1<s<h^{-\epsilon}\delta$ we modify $P(h)$ once more:

$$\begin{aligned}P_\delta^\sharp(h)=&h^\epsilon\left[e^{-2\pi i/3}\left((\tilde{h}D_s)^2+(2s+\chi(s))(Q(x',hD_{x'})+1-\chi(h^\epsilon s/4\delta)\right)+h^{-\epsilon}R(x',hD_{x'})\right.\\&\left.+h^{1-\frac{\epsilon}{2}}F(x_n,x')\tilde{h}D_s+sE(h^\epsilon s,x',hD_{x'})\right]\,,\end{aligned}\tag{7.8}$$

where χ is as in (7.6). The term $1-\chi(h^\epsilon s/4\delta))$ which is supported $\{s>2\delta h^{-\epsilon}\}$ is added to avoid domain problems.

We now have

Lemma 7.2. *Let $P^\sharp-z=h^{-2/3}(P_\delta^\sharp(h)-w)-z$ with w and z as above and let $\delta>0$ be small enough, $0<h<h_0(L)$. For any $\tilde{v}\in\mathcal{C}_{\rm c}^\infty([0,\infty)\times\partial\mathcal{O})$ there exists $\tilde{u}\in\mathcal{C}^\infty([0,\infty)\times\partial\mathcal{O})$ such that*

$$(P^\sharp-z)\tilde{u}=\tilde{v}\,,\quad \tilde{u}\restriction_{s=0}=0\,,$$

and

$$\begin{aligned}\|(\tilde{h}D_s)^2\tilde{u}\|_{L^2([0,\infty)\times\partial\mathcal{O})}+\|\tilde{u}\|_{L^2([0,\infty))\otimes H^2(\partial\mathcal{O})}+h^{-\epsilon}\|(-h^2\Delta_{\partial\mathcal{O}}-w)\tilde{u}\|_{L^2([0,\infty)\times\partial\mathcal{O})}\\\leq Ch^{\frac{2}{3}-\epsilon}\|\tilde{v}\|_{L^2([0,\infty)\times\partial\mathcal{O})}\,.\end{aligned}\tag{7.9}$$

If $\tilde v$ satisfies

$$\mathrm{supp}\tilde v \subset \{1 < s < \delta h^{-\epsilon}\}$$

then in addition

$$e^{C_\gamma h^{-1+\frac{3}{2}\epsilon}} \|\tilde u\restriction_{\{s<(1-\gamma)\}}\|_{H^k([0,\infty)\times\partial\mathcal{O})} + e^{C_\gamma h^{-1}} \|\tilde u\restriction_{\{s>(1+\gamma)\delta h^{-\epsilon}\}}\|_{H^k([0,\infty)\times\partial\mathcal{O})} \\ \leq \|v\|_{L^2([0,\infty)\times\partial\mathcal{O})}\,, \tag{7.10}$$

for any $\gamma > 0$. Here $H^2(\partial\mathcal{O})$ denotes the h-Sobolev space on $\partial\mathcal{O}$.

Proof. We will prove an a priori estimate: if $\tilde u \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty)\times\partial\mathcal{O})$ and $\tilde u\restriction_{s=0}= 0$ then

$$\begin{aligned} C\|(P^\sharp - z)\tilde u\|_{L^2([0,\infty)\times\partial\mathcal{O})} \geq h^{\epsilon-\frac{2}{3}} \Big(& \|(1+(\tilde h D_s)^2)\tilde u\|_{L^2([0,\infty)\times\partial\mathcal{O})} \\ & + h^{-\epsilon}\|(-h^2\Delta_{\partial\mathcal{O}} - w)\tilde u\|_{L^2([0,\infty)\times\partial\mathcal{O})} + \|(1-h^2\Delta_{\partial\mathcal{O}})\tilde u\|_{L^2([0,\infty)\times\partial\mathcal{O})} \\ & + h^{\epsilon}\|s(1-h^2\Delta_{\partial\mathcal{O}})\tilde u\|_{L^2([0,\infty)\times\partial\mathcal{O})}\Big)\,. \end{aligned} \tag{7.11}$$

Since we can consider $P^\sharp - z$ as an operator on the space defined by the norm on the right hand side of (7.11) this will give solvablity of the non-homogeneous problem and (7.9): the operator is seen to have index 0 by a deformation to $\mathrm{Im}\, z > 0$ and to a self-adjoint operator.

The problem involves two Planck constants, h, $\tilde h = h^{1-\frac{3}{2}\epsilon}$, so it is convenient to obtain an ordinary differential estimate in s first and then use an FBI transformation on $\partial\mathcal{O}$. Thus we consider the following operator on $[0,\infty)$:

$$\begin{aligned} \widetilde P =& e^{-2\pi i/3}\left((\tilde h D_s)^2 + (2s+\chi(s))(\mu + (1-\chi(h^\epsilon s/4\delta))\right) \\ & + \mathcal{O}(h^{1-\frac{\epsilon}{2}})\tilde h D_s + sE(h^\epsilon s)\mu + h^{-\epsilon}\omega - \tilde h^{\frac{2}{3}} z\,, \end{aligned}$$

where $\omega > \mu - C$, $0 < \mu < C\langle\omega\rangle$, $|\omega| > 1/C - \mu$.

We see (compare Lemma 5.2 of [32]) that for $\tilde h$ small enough and $w \in \mathcal{C}^\infty_{\mathrm{c}}([0,\infty))$, $w(0) = 0$ we have

$$\begin{aligned} C\|\widetilde P w\|_{L^2([0,\infty))} \geq & \|(1+(\tilde h D_s)^2)w\|_{L^2([0,\infty))} + \|(1+\mu)w\|_{L^2([0,\infty))} \\ & + h^{-\epsilon}\|\omega w\|_{L^2([0,\infty))} + h^\epsilon\|s(1+\mu)w\|_{L^2([0,\infty))}\,. \end{aligned} \tag{7.12}$$

The estimate (7.11) follows from this and an application of the FBI transformation in the $\partial\mathcal{O}$-variables as in the proofs of Propositions 5.1 and 5.2 of [32].

To obtain the better estimate under the support assumption, $\mathrm{supp}\,\tilde v \subset \{1 < s < \delta h^{-\epsilon}\}$, we introduce a weight function

$$\psi_\epsilon(s) = \frac{1}{C}\chi^0_\gamma(s) + \frac{h^{-\frac{3}{2}\epsilon}}{C}\chi^1_\gamma(h^\epsilon s)\,,$$

where $\chi^0_\gamma(s) \equiv 1$ for $s < 1-\gamma$, $\chi^0_\gamma(s) \equiv 0$ for $s > 1-\gamma/2$ and where $\chi^1_\gamma(s) \equiv 1$ for $s > 1+\gamma$, $\chi^1_\gamma(s) \equiv 0$ for $s < 1+\gamma/2$. On the level of ordinary differential estimates we see that $\exp(\psi_\epsilon/\tilde h)\widetilde P\exp(-\psi_\epsilon/\tilde h)$ still satisfies the estimate (7.12): in the outer region

$s > (1+\gamma)\delta h^{-\epsilon}$ the square of the gradient of ψ_ϵ is bounded by $h^{-\epsilon}/C$ and for C large enough it is dominated by the s term. Again this leads to an estimate for $P^\sharp - z$:

$$h^{\epsilon-\frac{2}{3}} \| e^{\psi_\epsilon / h^{1-\frac{3}{2}\epsilon}} (P^\sharp - z)\tilde{u} \|_{L^2([0,\infty)\times\partial\mathcal{O})} \geq \| e^{\psi_\epsilon / h^{1-\frac{3}{2}\epsilon}} \tilde{u} \|_{L^2([0,\infty)\times\partial\mathcal{O})} + \| e^{\psi_\epsilon / h^{1-\frac{3}{2}\epsilon}} \tilde{u} \|_{L^2([0,\infty)\times\partial\mathcal{O})} + h^\epsilon \| s e^{\psi_\epsilon / h^{1-\frac{3}{2}\epsilon}} \tilde{u} \|_{L^2([0,\infty)\times\partial\mathcal{O})} .$$

Since $\psi_\epsilon = 0$ on the support of $\tilde{v} = (P^\sharp - z)\tilde{u}$ this provides the weighted estimate (7.10) with $k = 0$. Elliptic regularity and interpolation give the estimate for all k, at the expense of all the constants involved. $\square$

Before proving Proposition 7.1 we need to translate the statement of Lemma 7.2 to the original coordinates. If we put $u(x_n, x') = \tilde{u}(h^{-\epsilon} x_n, x')$ and $v(x_n, x') = \tilde{v}(h^{-\epsilon} x_n, x')$, then (7.9) becomes

$$h^{-\epsilon} \| (hD_{x_n})^2 u \|_{L^2([0,\infty)_{x_n}\times\partial\mathcal{O})} + \| u \|_{L^2([0,\infty)_{x_n})\otimes H^2(\partial\mathcal{O})} + h^{-\epsilon} \| (-h^2 \Delta_{\partial\mathcal{O}} - w) u \|_{L^2([0,\infty)\times\partial\mathcal{O})} \leq h^{\frac{2}{3}-\epsilon} \| v \|_{L^2([0,\infty)_{x_n}\times\partial\mathcal{O})} \tag{7.13}$$

and if $\operatorname{supp} v \subset \{ h^\epsilon < x_n < \delta \}$ then in addition

$$e^{C_\gamma h^{-1+\frac{3}{2}\epsilon}} \| u \|^2_{H^k(\{x_n \leq (1-\gamma) h^\epsilon\}\times\partial\mathcal{O})} + e^{C_\gamma h^{-1}} \| u \|^2_{H^k(\{x_n > (1+\gamma)\delta\}\times\partial\mathcal{O})} \leq \| v \|_{L^2([0,\infty)\times\partial\mathcal{O})} . \tag{7.14}$$

We now note that $P^\sharp - z = P - z$ for $3h^\epsilon/4 < x_n < 2\delta$ – see (7.8). Hence if $v_2 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \partial\mathcal{O})$, $\operatorname{supp} v_2 \subset \{ x : h^\epsilon < d(x, \partial\mathcal{O}) < \delta \}$ then by cutting-off $u(x_n, x') = \tilde{u}(h^{-\epsilon} x_n, x')$ constructed in Lemma 7.2 be obtain $u_2 \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \mathcal{O})$ such that

$$(P - z) u_2 = v_2 + v_0^2 , \quad u_2 \restriction_{\partial\mathcal{O}} = 0 ,$$

and

$$\begin{aligned} h^{\frac{2}{3}-\epsilon} \| v_2 \|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \ \geq\ & h^{-\epsilon} \Big(\| (hD_{x_n})^2 u \|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + \| (-h^2 \Delta_{\partial\mathcal{O}} - w) \tilde{u} \|_{L^2([0,\infty)\times\partial\mathcal{O})} \Big) \\ & \| u_2 \|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{C h^{-1+\frac{3}{2}\epsilon}} \| u_2 \|_{H^k(\mathbb{R}^n\setminus D((1-\gamma)h^\epsilon))} \\ & + e^{C h^{-1}} \| u_2 \|_{H^k(D((1+\gamma)\delta))} + e^{C h^{-1+\frac{3}{2}\epsilon}} \| v_0^2 \|_{H^k(\mathbb{R}^n\setminus\mathcal{O})} . \end{aligned} \tag{7.15}$$

The proof of Proposition 7.1 is now quite straightforward:

Proof of Proposition 7.1. If $v \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^N \setminus \mathcal{O})$ satisfies $\operatorname{supp} v \subset D(h^\epsilon)$ we choose δ small enough for the operator $P^\sharp_\delta(h)$ given by (7.8) to satisfy the assumptions of Lemma 7.2 and decompose v:

$$v = v_1 + v_2 , \quad v_i \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n \setminus \mathcal{O}) , \quad \| v_1 \|_{L^2} + \| v_2 \|_{L^2} \leq 2 \| v \|_{L^2} , \quad \operatorname{supp} v_1 \subset D(\delta/2) , \quad \operatorname{supp} v_2 \subset D(h^\epsilon) \setminus D(\delta) .$$

We can then take $u = u_1 + u_2$, $v_0 = v_0^1 + v_0^2$ where u_1, v_0^1 are given by Lemma 7.1 and u_2, v_0^2 in the discussion following the proof of Lemma 7.2:

$$\begin{aligned}
&\|u\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{Ch^{-1+\frac{3}{2}\epsilon}}\|u\|_{L^2(\mathbb{R}^n\setminus D((1-\gamma)h^\epsilon)} + e^{Ch^{-1+\frac{3}{2}\epsilon}}\|v_0\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \\
&\qquad \leq \|u_1\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + \|u_2\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + e^{Ch^{-1+\frac{3}{2}\epsilon}}\|u_2\|_{L^2(D((1-\gamma)h^\epsilon))} \\
&\qquad\qquad + e^{Ch^{-1+\frac{3}{2}\epsilon}}\left(\|v_0^1\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + \|v_0^2\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})}\right) \\
&\qquad \leq Ch^{\frac{2}{3}-\epsilon}\left(\|v_1\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + \|v_2\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})}\right) \\
&\qquad \leq Ch^{\frac{2}{3}-\epsilon}\|v\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})},
\end{aligned}$$

which is almost the estimate of Proposition 7.1. Using (7.3) away from the boundary and (7.15) near the boundary we also estimate the remaining terms where we gain $h^{-\epsilon}$. □

8. **The global Grushin problem**

Throughout this section we will take $w \in W \Subset (0,\infty)$ and $z \in \mathbb{C}$ satisfying $|\operatorname{Re} z| \ll 1/\delta$ and $|\operatorname{Im} z| \leq C_1$.

To consider the global Grushin problem we introduce new spaces

$$\begin{aligned}
\mathcal{B}_{w,r,\epsilon} &= \left(H^2 \cap H_0^1\right)(\mathbb{R}^n \setminus \mathcal{O}) \times L^2(\partial\mathcal{O};\mathbb{C}^N), \\
\mathcal{H}_{w,r} &= L^2(\mathbb{R}^n \setminus \mathcal{O}) \times L^2(\partial\mathcal{O};\mathbb{C}^N),
\end{aligned} \tag{8.1}$$

which are the global modification of the spaces considered in Sect.5 and 6. Thus we introduce a weight function $\psi \in \mathcal{C}^\infty([0,\infty);[0,1])$ which satisfies $\psi(t) = t$ for $t < 1/2$ and $\psi(t) = 1$ for $t \geq 1$ and then give the norms as follows (see (5.13)):

$$\begin{aligned}
\left\|\begin{pmatrix} u \\ u_- \end{pmatrix}\right\|_{\mathcal{B}_{w,r,\epsilon}} =& h^{-\frac{2}{3}}\|e^{r\psi(x_n)/2h^{\frac{2}{3}}}(hD_{x_n})^2 u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + h^{-\frac{2}{3}}\|e^{r\psi(x_n)/2h^{\frac{2}{3}}}\chi(x_n/\delta)x_n u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \\
&+ \|e^{r\psi(x_n)/2h^{\frac{2}{3}}}\langle x_n\rangle^{-2}\langle h^{-\frac{2}{3}}(-h^2\Delta_{\partial\mathcal{O}} - w)\rangle u\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} \\
&+ h^{-\frac{2}{3}}\|e^{r\psi(x_n)/2h^{\frac{2}{3}}}(1-\chi(x_n/\delta))u\|_{H^2(\mathbb{R}^n\setminus\mathcal{O})} + h^{\frac{1}{3}}\|u_-\|_{L^2(\partial\mathcal{O};\mathbb{C}^N)},
\end{aligned}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R};[0,1])$, $\chi(t) = 1$ for $t < 1$ and $\chi(t) = 0$ for $t > 2$. The other norm is given by

$$\left\|\begin{pmatrix} v \\ v_+ \end{pmatrix}\right\|_{\mathcal{H}_{w,r}} = \|e^{r\psi(x_n)/2h^{\frac{2}{3}}}v\|_{L^2(\mathbb{R}^n\setminus\mathcal{O})} + h^{-\frac{1}{3}}\|\langle h^{-\frac{2}{3}}(-h^2\Delta_{\partial\mathcal{O}} - w)\rangle v_+\|_{L^2(\partial\mathcal{O};\mathbb{C}^N)}.$$

We first note that

$$\begin{pmatrix} h^{-\frac{2}{3}}(P(h) - w) - z & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{B}_{w,r,\epsilon} \longrightarrow \mathcal{H}_{w,r}. \tag{8.2}$$

In fact, we can decompose u with ${}^t(u,0) \in \mathcal{B}_{w,r,\epsilon}$ as $u_1 + u_2$ where $\operatorname{supp} u_2 \subset \{x_n \geq 2\delta\}$ and $\operatorname{supp} u_1 \subset \{x_n \leq 3\delta\}$ and the norm of u is comparable to the sum of norms of u_1 and u_2.

Then

$$\left\| \begin{pmatrix} u_2 \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\epsilon}} \sim \; h^{-\frac{2}{3}} \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} u_2 \|_{H^2(\mathbb{R}^n \setminus \mathcal{O})} \,,$$

so that

$$\| e^{r\psi(x_n)/2h^{\frac{2}{3}}} (h^{-\frac{2}{3}}(P(h) - w) - z) u_2 \|_{L^2(\mathbb{R}^n \setminus \mathcal{O})} \leq C \left\| \begin{pmatrix} u \\ 0 \end{pmatrix} \right\|_{\mathcal{B}_{w,r,\epsilon}} \,.$$

For the terms supported in the region where $x_n < 3\delta$ we can use (3.7) to describe the operator and the boundedness follows as in Sect.5.

We now want to correct (8.2) to an invertible operator by introducing

$$\begin{aligned} R_{-,w} \; &: \; L^2(\partial\mathcal{O}; \mathbb{C}^N) \longrightarrow L^2(\mathbb{R}^n \setminus \mathcal{O}) \,, \\ R_{+,w} \; &: \; (H^2 \cap H^1_0)(\mathbb{R}^n \setminus \mathcal{O}) \longrightarrow L^2(\partial\mathcal{O}; \mathbb{C}^N) \,, \end{aligned}$$

as the antidiagonal terms. They are obtained from symbols appearing in Sect.5 and 6 and the calculus of Sect.4: let $e_j^{\lambda,\delta}$ be given by (5.6) and let us put

$$\tilde{e}^{\delta}_{j,w}(m, x_n) = \chi(x_n)(e^{h^{-\frac{2}{3}}(R(m)-w),\delta}(h^{-\frac{2}{3}} x_n) \,, \quad m \in T^*\partial\mathcal{O} \,,$$

where as before χ is one near x_n equal to 0 and vanishes for large x_n. From $\tilde{e}^{\delta}_{j,w}$ we obtain a family of operator valued symbols:

$$\tilde{e}^{\delta}_{j,w} \in S_{\Sigma_w, \frac{2}{3}} \left(\partial\mathcal{O}; h^{\frac{1}{3}}, \mathcal{L}(L^2([0,\infty)), \mathbb{C}^N) \right) \,,$$

defined by

$$\tilde{e}^{\delta}_w(j) u(m) = \int_0^\infty \tilde{e}^{\delta}_{j,w}(m, x_n) u(x_n) dx_n \,.$$

Using coordinates (3.1) on $\mathbb{R}^n \setminus \mathcal{O}$ we then put

$$R_{+,w} = \mathrm{Op}_{\Sigma_w, h}(\tilde{e}^{\delta}_w) : L^2(\mathbb{R}^N \setminus \partial O) \longrightarrow L^2(\partial\mathcal{O}; \mathbb{C}^N) \,. \tag{8.3}$$

where $\mathrm{Op}_{\Sigma_w,h}$ is given by Proposition 4.1. The other operator is defined simply as the adjoint:

$$R_{-,w} = R^*_{+,w} \,.$$

We first observe that

$$\mathcal{P}_w(z) = \begin{pmatrix} h^{-\frac{2}{3}}(P(h) - w) - z & R_{-,w} \\ R_{+,w} & 0 \end{pmatrix} \; : \; \mathcal{B}_{w,r,\epsilon} \longrightarrow \mathcal{H}_{w,r} \,. \tag{8.4}$$

We will show that for $0 < h < h_0(\delta)$ the operator $\mathcal{P}_w(z)$ has an inverse

$$\mathcal{E}_w(z) = \begin{pmatrix} E_w(z) & E_{+,w}(z) \\ E_{-,w}(z) & E_{-+,w}(z) \end{pmatrix} \; : \; \mathcal{H}_{w,0} \longrightarrow \mathcal{B}_{w,0,\epsilon} \,, \tag{8.5}$$

with $E_{-+,w}(z) \in \Psi^{0,1,2}_{\Sigma_w, \frac{2}{3}} \left(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N) \right)$.

For convenience we now put $w = 1$ and drop the corresponding subscript. It will be clear that the analysis is uniform for w in a fixed compact subset of $(0, \infty)$. Using the results of Sect.5,6 and 7 we will construct approximate inverses which will lead to the global inverse of $\mathcal{P}(z)$. We start with the most interesting case. Let $\mathcal{B}_{\lambda,r}$ and $\mathcal{H}_{\lambda,r}$ be the spaces of functions on $[0,\infty)$ introduced in Sect.5 and translated from the t to the x_n coordinate but where we dropped the dependence on z (the weight fuction ψ is the same as in the previous norms):

$$\left\| \begin{pmatrix} w \\ w_+ \end{pmatrix} \right\|_{\mathcal{H}_{\lambda,r}} = \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} w \|_{L^2([0,\infty))} + h^{-\frac{1}{3}} \langle \lambda \rangle |w_+|_{\mathbb{C}^N} \,,$$

$$\left\| \begin{pmatrix} u \\ u_+ \end{pmatrix} \right\|_{\mathcal{B}_{\lambda,r}} = \langle \lambda \rangle \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} u \|_{L^2([0,\infty))} + h^{-\frac{2}{3}} \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} (hD_{x_n})^2 u \|_{L^2([0,\infty))}$$

$$+ h^{-\frac{2}{3}} \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} x_n u \|_{L^2([0,\infty))} + h^{\frac{1}{3}} |u_-|_{\mathbb{C}^N} \,.$$

With this notation we can state

Lemma 8.1. *Let $0 < \epsilon < \frac{2}{3}$ and let us put*

$$\chi_1 \stackrel{\text{def}}{=} \tilde{\chi}_1 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix} ,$$

where χ is as in (8.1), $\tilde{\chi}_1 \in \Psi^{0,0}(\partial\mathcal{O})$, $\mathrm{WF}_h(\tilde{\chi}_1 - Id) \subset \{m : d(m,\Sigma) \geq C\}$, $\mathrm{WF}_h(\tilde{\chi}_1) \subset \{m : d(m,\Sigma) \leq 2C\}$.

There exist $\mathcal{E}_1^R(z) \in \Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$, $\mathcal{E}_1^L(z) \in \Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$, such that

$$\mathcal{P}(z)\mathcal{E}_1^R(z) - \chi_1 = \mathcal{R}_1^R(z) \,, \quad \mathcal{E}_1^L(z)\mathcal{P}(z) - \chi_1 = \mathcal{R}_1^L(z) \,,$$

where

$$\mathcal{R}_1^R(z) \in \Psi_{\Sigma,\frac{2}{3}}\left(\partial\mathcal{O}; h^N \langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{H}_{\lambda,r})\right) \,,$$
$$\mathcal{R}_1^L(z) \in \Psi_{\Sigma,\frac{2}{3}}\left(\partial\mathcal{O}; h^N \langle\lambda\rangle^{-N}, \mathcal{L}(\mathcal{B}_{\lambda,r}, \mathcal{B}_{\lambda,r})\right) \,,$$

for any N. Here the constants are allowed to depend on z and δ.

Proof. Sect.4 and the first part of Sect.6 give us an operator $\widetilde{\mathcal{E}}_1 \in \Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{B}_{\lambda,r}))$ with $\mathrm{WF}_h(\widetilde{\mathcal{E}}_1) \subset \{m : d(m,\Sigma) \leq 2C\}$ such that

$$\mathcal{P}(z)\widetilde{\mathcal{E}}_1(z) - Id = \widetilde{\mathcal{R}}_1^R(z) \,, \quad \widetilde{\mathcal{E}}_1(z)\mathcal{P}(z) - Id = \widetilde{\mathcal{R}}_1^L(z) \,,$$

where for any $A \in \Psi^{0,0}(\partial\mathcal{O})$ with $\mathrm{WF}_h(A) \subset \{m : d(m,\Sigma) \leq C\}$ and any k we have

$$A\widetilde{\mathcal{R}}_1^\bullet = \begin{pmatrix} x_n^k & 0 \\ 0 & 0 \end{pmatrix} B_k^\bullet + h^k A_k^\bullet \,,$$

$$A_k^R, B_k^R \in \Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}_{\lambda,r}, \mathcal{H}_{\lambda,r})) \,, \quad A_k^L, B_k^L \in \Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{B}_{\lambda,r}, \mathcal{B}_{\lambda,r})) \,.$$

The lemma follows after we introduce the cut-off to the region where $x_n < h^\epsilon$, noting that for $0 < \epsilon < 2/3$ the operator

$$\begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix}$$

is bounded on $\mathcal{B}_{\lambda,r}$ and $\mathcal{H}_{\lambda,r}$. $\square$

Similarly we will now apply the second part of Sect.6. We again translate the spaces appearing there to the x_n setting: we now have $\mathcal{H}^\sharp_{\lambda,r}$ and $\mathcal{B}^\sharp_{\lambda,r}$ with norms

$$\left\| \begin{pmatrix} w \\ w_+ \end{pmatrix} \right\|_{\mathcal{H}^\sharp_{\lambda,r}} = \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} w \|_{L^2([0,\infty))} + h^{-\frac{1}{3}} \langle \lambda \rangle |w_+|_{\mathbb{C}^N} \,,$$

$$\left\| \begin{pmatrix} u \\ u_- \end{pmatrix} \right\|_{\mathcal{B}^\sharp_{\lambda,r}} = \langle \lambda \rangle \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} u \|_{L^2([0,\infty))} + h^{-\frac{2}{3}} \| e^{r\psi(x_n)/2h^{\frac{2}{3}}} (hD_{x_n})^2 u \|_{L^2([0,\infty))} + h^{\frac{1}{3}} |u_-|_{\mathbb{C}^N} \,.$$

Lemma 8.2. *Let $0 < \epsilon < \frac{2}{3}$ and let us put*

$$\chi_2 \stackrel{def}{=} \tilde{\chi}_2 \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix} \,,$$

where $\tilde{\chi}_2 \in \Psi^{0,0}(\partial\mathcal{O})$, $\mathrm{WF}_h(\tilde{\chi}_2 - Id) \subset \{m : d(m,\Sigma) \leq C\}$, $\mathrm{WF}_h(\tilde{\chi}_2) \subset \{m : d(m,\Sigma) \geq C/2\}$.

There exist $\mathcal{E}^R_2(z) \in \Psi(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}^\sharp_{\lambda,r}, \mathcal{B}^\sharp_{\lambda,r}))$, $\mathcal{E}^L_2(z) \in \Psi(\partial\mathcal{O}; 1, \mathcal{L}(\mathcal{H}^\sharp_{\lambda,r}, \mathcal{B}^\sharp_{\lambda,r}))$, such that

$$\mathcal{P}(z)\mathcal{E}^R_2(z) - \chi_2 = \mathcal{R}^R_2(z) \,, \quad \mathcal{E}^L_2(z)\mathcal{P}(z) - \chi_2 = \mathcal{R}^L_2(z) \,,$$

where

$$\mathcal{R}^R_2(z) \in \Psi_{\Sigma,\frac{2}{3}} \left(\partial\mathcal{O}; h^N \langle \lambda \rangle^{-N}, \mathcal{L}(\mathcal{H}^\sharp_{\lambda,r}, \mathcal{H}^\sharp_{\lambda,r}) \right) \,,$$
$$\mathcal{R}^L_2(z) \in \Psi_{\Sigma,\frac{2}{3}} \left(\partial\mathcal{O}; h^N \langle \lambda \rangle^{-N} \mathcal{L}(\mathcal{B}^\sharp_{\lambda,r}, \mathcal{B}^\sharp_{\lambda,r}) \right) \,,$$

for any N.

Proof. This follows from the same argument as that in the proof of Lemma 8.1 but with the simpler calculus: as in the second part of Sect.6 we can now use the standard semi-classical calculus as we are away from the glancing hypersurfcase, Σ. ☐

The two lemmas give

Proposition 8.1. *There exist $\mathcal{E}^\bullet_3(z) : \mathcal{H}_r \to \mathcal{B}_{r,\epsilon}$, $\bullet = R\,, L$, such that*

$$\mathcal{P}(z)\mathcal{E}^R(z) - \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix} = \mathcal{R}^R_3(z) \,, \quad \mathcal{E}^L(z)\mathcal{P}(z) - \begin{pmatrix} \chi(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix} = \mathcal{R}^L_3(z) \,,$$

where for any N

$$\langle h^2 \Delta_{\partial\mathcal{O}} \rangle^N \mathcal{R}^R_3(z) \langle h^2 \Delta_{\partial\mathcal{O}} \rangle^N = \mathcal{O}(h^N) : \mathcal{H}_r \to \mathcal{H}_r \,,$$
$$\langle h^2 \Delta_{\partial\mathcal{O}} \rangle^N \mathcal{R}^L_3(z) \langle h^2 \Delta_{\partial\mathcal{O}} \rangle^N = \mathcal{O}(h^N) : \mathcal{B}_{r,\epsilon} \to \mathcal{B}_{r,\epsilon} \,,$$

where $\langle h^2 \Delta_{\partial\mathcal{O}} \rangle^N$ is applied to both components, and

$$E^L_{-+}(z) = E^R_{-+}(z) \in \Psi^{0,1,2}_{\Sigma,\frac{2}{3}} \left(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N) \right) \,.$$

Proof. We simply put $\mathcal{E}^R(z) = \mathcal{E}_1^R(z) + \mathcal{E}_2^R(z)$ and choose $\tilde{\chi}_1$ and $\tilde{\chi}_2$ so that $\tilde{\chi}_1 + \tilde{\chi}_2 = 1$. The spaces $\mathcal{B}_{r,\epsilon}$ and $\mathcal{H}_r$ were defined so that after truncation to the h^ϵ neighbourhood of the boundary, the operators in $\Psi_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; \mathcal{L}(\bullet_{\lambda,r}, \bullet_{\lambda,r}))$ microlocalized to a neighbourhood of Σ and operators in $\Psi(\partial\mathcal{O}; \mathcal{L}(\bullet^\sharp_{\lambda,r}, \bullet^\sharp_{\lambda,r}))$ microlocalized away from Σ, $\bullet = \mathcal{H}, \mathcal{B}$, are bounded on $\mathcal{H}_r$ and $\mathcal{B}_{r,\epsilon}$ respectively. $\square$

Combining this with the estimates of Sect.7 we can obtain the full inverse:

Proposition 8.2. *For $0 < \epsilon < \frac{2}{3}$ and $0 < h < h_0(\delta)$ there exists $\mathcal{E}_w(z) : \mathcal{H}_{w,0} \longrightarrow B_{w,0,\epsilon}$ such that*

$$\mathcal{P}_w(z)\mathcal{E}_w(z) - Id = 0\,, \quad \mathcal{E}_w(z)\mathcal{P}_w(z) - Id = 0\,,$$

and $E_{w,-+} \in \Psi^{0,1,2}_{\Sigma_w,\frac{2}{3}}\left(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)\right)$.

Proof. We first construct an approximate right inverse and for that we put

$$\tilde{\mathcal{E}}^R(z) = \mathcal{E}^R(z)\begin{pmatrix} \tilde{\chi}(x_n/h^\epsilon) & 0 \\ 0 & Id \end{pmatrix} + \begin{pmatrix} E_\epsilon(1 - \tilde{\chi}(x_n/h^\epsilon)) & 0 \\ 0 & 0 \end{pmatrix},$$

where E_ϵ is given by (7.2), $\mathcal{E}^R(z)$ by Proposition 8.2, and $\tilde{\chi} \in \mathcal{C}^\infty(\mathbb{R}; [0,1])$ has its support close to 0 and $\chi = 1$ on the support of $\tilde{\chi}$. Then

$$\mathcal{P}(z)\tilde{\mathcal{E}}^R(z) = Id + \mathcal{K}_\epsilon(z)\,, \quad \mathcal{K}_\epsilon = \mathcal{R}_3^R(z) + \begin{pmatrix} K_\epsilon(1 - \tilde{\chi}(x_n/h^\epsilon)) & 0 \\ 0 & 0 \end{pmatrix}$$

where again K_ϵ and $\mathcal{R}_3^R$ are as in (7.2) and Proposition 8.2 respectively. We have $\mathcal{K}_\epsilon = \mathcal{O}(h^\infty) : \mathcal{H}_0 \to \mathcal{H}_0$ and hence for h small enough $(I + \mathcal{K}_\epsilon)^{-1} = I + \mathcal{A}_\epsilon$, $\mathcal{A}_\epsilon = \mathcal{O}(h^\infty) : \mathcal{H}_0 \to \mathcal{H}_0$. Thus we can put $\mathcal{E}(z) = \tilde{\mathcal{E}}^R(z)(I + \mathcal{A}_\epsilon(z))$.

The operator $\mathcal{A}_\epsilon(z)$ has of course much better mapping properties: $K_\epsilon(1 - \tilde{\chi}(x_n/h^\epsilon))$ maps L^2 to H^k for any k and $\mathcal{R}_3^R$ is smoothing in the tangential variables. If we write

$$\mathcal{A}_\epsilon(z) = \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{pmatrix}$$

then it follows that $A_{12}(z)$ and $A_{22}(z)$ are smoothing in the tangential variables. Hence

$$\begin{aligned} E_{-+}(z) &= E^R_{-+}(z) + E^R_{-+}(z)A_{22}(z) + E^R_{-}(z)A_{12}(z) \\ &= E^R_{-+}(z) + R(z)\,, \quad R(z) \in \Psi^{-\infty,-\infty}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))\,, \end{aligned}$$

that is, E_{-+} remains essentially the same. $\square$

The precise structure of $E_{w,-+}(z)$ comes directly from Sect.5 and 6. Using the standard formula

$$\left(h^{-\frac{2}{3}}(P(h) - w) - z\right)^{-1} = E_w(z) - E_{w,+}(z)E_{w,-+}(z)^{-1}E_{w,-}(z) \tag{8.6}$$

as in [8] we identify the resonances with the values of z for which $E_{w,-+}(z)$ is *not* invertible. This is summarized in the main consequence of this section:

Theorem 8.1. *Let $W \Subset (0,\infty)$ be a fixed set. For every $w \in W$ and $z \in \mathbb{C}$, $|\operatorname{Re} z| \ll 1/\sqrt{\delta}$, $|\operatorname{Im} z| \leq C_1$ there exists $E_{w,-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,\frac{2}{3}}(\partial\mathcal{O}, \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$, where $\Sigma_w = \{p \in T^*\partial\mathcal{O} : R(p) = w\}$, $N = N(C_1)$ such that for $0 < h < h_0(\delta)$:*

(i) If the multiplicity of the pole of the meromorphic continuation of $(\Delta_{\mathbb{R}^n\setminus\mathcal{O}} - \zeta)^{-1}$ is given by $m_{\mathcal{O}}(\zeta)$ then

$$m_{\mathcal{O}}(h^{-2}(w + h^{\frac{2}{3}}z)) = \frac{1}{2\pi i}\operatorname{tr}\oint_{|\tilde{z}-z|=\epsilon} E_{w,-+}(\tilde{z})^{-1}\frac{d}{d\tilde{z}}E_{w,-+}(\tilde{z})d\tilde{z}\,, \quad 0 < \epsilon \ll 1\,. \tag{8.7}$$

(ii) If $E^0_{w,-+}(z;p) = \sigma_{\Sigma_w,h}(E_{w,-+}(z))(p;h)$, $p \in T^*\partial\mathcal{O}$, $\sigma_{\Sigma_w,h}$ given in Proposition 4.1, then

$$E^0_{w,-+}(z;p) = \mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle)\,.$$

In addition for $|\lambda| \leq 1/(C\sqrt{\delta})$ we have

$$\|E^0_{w,-+}(z;p;h) - \operatorname{diag}(z - \lambda - e^{-2\pi i/3}\zeta_j(p))\|_{\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)} \leq \epsilon \ll 1\,, \tag{8.8}$$

and

$$\det E^0_{w,-+}(z;p;h) = 0 \iff z = \lambda + e^{-2\pi i/3}\zeta_j(p) \text{ for some } 1 \leq j \leq N \tag{8.9}$$

where the zero is simple. Here $\zeta_j(p) = \zeta_j(2Q(p))^{\frac{2}{3}}$.

(iii) For $|\lambda| \geq 1/(C\sqrt{\delta})$, $E^0_{w,-+}$ is invertible and

$$E^0_{w,-+}(z;p;h)^{-1} = \mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle^{-1})\,.$$

Proof. The first part follows (8.6) as in [8]: the multiplicity is given by the trace of the integral of the left hand side divided by $2\pi i$. We then use (8.6), the cyclicity of the trace and the identity $E_{w,-}(z)E_{w,+}(z) = -\partial_z E_{w,\pm}(z)$. For the second part we recall the symbolic constructions of $\mathcal{E}$ and $\mathcal{E}^\sharp$ given in Sect.6. They show that the principal symbol is $\mathcal{O}(\langle\lambda - \operatorname{Re} z\rangle)$. For $|\lambda| \leq 1/(C\sqrt{\delta})$ the only contribution comes from $\mathcal{E}$ and then the statement follows from (5.15) and (5.16).

For the last part we first note that in the region of overlap between the two symbolic constructions, the expansions (6.13) and (6.19) agree. Consequently we need to check the boundedness of the inverse of the principal symbol separately for each construction, assuming $|\lambda| \gg 1 + |\operatorname{Re} z|$. That follows from (5.17) and (5.20) in Sect.5. ∎

9. **Trace formula for the reduced problem**

In this section we will use Theorem 8.1 and the calculus developed in Sect.4 to establish a local trace formula for $E_{-+}(z)$ (for simplicity we now put $w = 1$ and drop that index). We introduce a new parameter $L \ll 1/\sqrt{\delta}$ which will give the size of $\operatorname{Re} z$ in the region we will work in. From now on the constant h_0 will depend on both L and δ. All other constants are assumed to be *independent* of L and δ unless it is indicated otherwise. The main point is that the leading symbol has nice uniform properties for large L but the lower order terms and the derivatives can grow as L gets large (that is, as δ gets small – see Sect.6).

We first need an auxiliary operator $\widetilde{E}_{-+}(z)$ which microlocally differs from $E_{-+}(z)$ for $|\lambda| \leq L+C$ only and is globally invertible. Thus let $r \in S^{0,0,-\infty}_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}, \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$, a multiple of $Id_{\mathbb{C}^N}$, be supported in $|\lambda| \leq L+C$ and satisfy

$$(E^0_{-+}(z;p;h) + r(w, h^{\frac{2}{3}}\lambda))^{-1} = \mathcal{O}(1) : \mathbb{C}^N \longrightarrow \mathbb{C}^N ,$$

which is possible because of *(ii)* of Theorem 8.1. If we use R to denote $\mathrm{Op}_{\Sigma,\frac{2}{3}}(r)$ then for $0 < h < h_0$ and

$$\Lambda = \langle h^{-\frac{2}{3}}(-h^2\Delta_{\partial\mathcal{O}} - 1)\rangle , \quad \Lambda \in \Psi^{0,1,2}_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}) , \tag{9.1}$$

the operator $\Lambda^{-1}(E_{-+}(z) + R) : L^2(\partial\mathcal{O}) \to L^2(\partial\mathcal{O})$ is uniformly invertible. We now use Lemma 4.4 to find a finite rank operator K such that $R - K \in \Psi^{-\infty,-\infty}(\partial\mathcal{O})$, rank $(K) = \mathcal{O}(Lh^{1-n+\frac{2}{3}})$. Putting $\widetilde{E}_{-+}(z) = E_{-+}(z) + K$ we obtain:

Lemma 9.1. *There exists an operator* $\widetilde{E}_{-+}(z) \in \Psi^{0,1,2}_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$ *such that for* $0 < h < h_0$

$$\widetilde{E}_{-+}(z)^{-1} , \ (\Lambda^{-1}\widetilde{E}_{-+}(z))^{-1} , \ \widetilde{E}_{-+}(z)^{-1}E_{-+}(z) = \mathcal{O}(1) \ : \ L^2(\partial\mathcal{O}; \mathbb{C}^N) \longrightarrow L^2(\partial\mathcal{O}; \mathbb{C}^N) ,$$

and

$$E_{-+}(z) - \widetilde{E}_{-+}(z) = \mathcal{O}(1) \in \mathcal{L}(L^2(\partial\mathcal{O}; \mathbb{C}^N), L^2(\partial\mathcal{O}; \mathbb{C}^N)) \text{ is independent of } z \text{ and}$$
$$\text{rank } (E_{-+}(z) - \widetilde{E}_{-+}(z)) = \mathcal{O}(Lh^{1-n+\frac{2}{3}}) .$$

We note that by Lemma 4.1 the inverse is a pseudodifferential operator in our exotic class: $\widetilde{E}_{-+}(z)^{-1} \in \Psi^{0,-1,-2}_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N))$. From (8.9) we also see that $0 < h < h_0$

$$\mathrm{dist}\left(z, \bigcup_j^N \left(\mathbb{R} + e^{-2\pi i/3}\zeta_j(\Sigma)\right)\right) \geq \frac{1}{\mathcal{O}(1)} \implies E_{-+}(z)^{-1} \in \Psi^{0,-1,-2}_{\Sigma,\frac{2}{3}}(\partial\mathcal{O}; \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)) . \tag{9.2}$$

To have this satisfied for a sufficiently large set of z's (in addition to the obvious set $\mathrm{Im}\, z > \mathrm{Im}(e^{-2\pi i/3}) \min_\Sigma \zeta_1$) we need to make geometric assumptions on $\mathcal{O}$. We will assume that

$$\min_\Sigma \zeta_{j_0+1}(p) > \max_\Sigma \zeta_{j_0}(p) , \tag{9.3}$$

which is (1.4). We also assume that N is much larger than j_0. Condition (9.3) implies that

$$\min_\Sigma \zeta_{j+1}(p) > \max_\Sigma \zeta_j(p) , \ 1 \leq j \leq j_0 , \tag{9.4}$$

since $j \mapsto \zeta_{j+1} - \zeta_j$ is decreasing. That is well known from Sturm's comparison theorem. We remark that asymptotically the zeros of the Airy function, or rather their negatives, satisfy $\zeta_j \sim (3j/2)^{2/3}$. Hence if we allow j_0 to grow then $\mathcal{O}$ needs to be closer to the ball.

To state the trace formula we need to introduce two domains, $W \subset \Omega$, in $\mathbb{C}$ and a suitable contour, γ:

$$
\begin{gathered}
W = \{z : a_2 < \operatorname{Re} z < a_4 \, , a_1 < -\operatorname{Im} z < a_3\} \subset \Omega = \{z : b_2 < \operatorname{Re} z < b_4 \, , b_1 < -\operatorname{Im} z < b_3\} \\
\cos\frac{\pi}{6} \max_{\Sigma} \zeta_{j_0 - 1} < b_1 < a_1 < \cos\frac{\pi}{6} \min_{\Sigma} \zeta_{j_0} \, , \\
\cos\frac{\pi}{6} \max_{\Sigma} \zeta_{j_0} < a_3 < b_3 < \cos\frac{\pi}{6} \min_{\Sigma} \zeta_{j_0 + 1} \, , \\
\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 = \ \partial \widetilde{W} \\
W \subset \widetilde{W} = \{z : c_2 \leq \operatorname{Re} z \leq c_4 \, , c_1 \leq -\operatorname{Im} z \leq c_3\} \subset \Omega \, ,
\end{gathered}
\tag{9.5}
$$

where we took the positive orientation, γ_1, γ_3 are the horizontal pieces, $-\operatorname{Im} z = c_1$, $-\operatorname{Im} z = c_3$, respectively and γ_2, γ_4 are the vertical ones, $\operatorname{Re} z = c_2$, $\operatorname{Re} z = c_4$, respectively – see Fig.2. As a choice of the constants we take

$$
a_2 = -\frac{L}{2} \, , \ a_4 = \frac{L}{2} \, , \ a_1 - b_1 \, , b_3 - a_3 = \mathcal{O}(1) \, . \tag{9.6}
$$

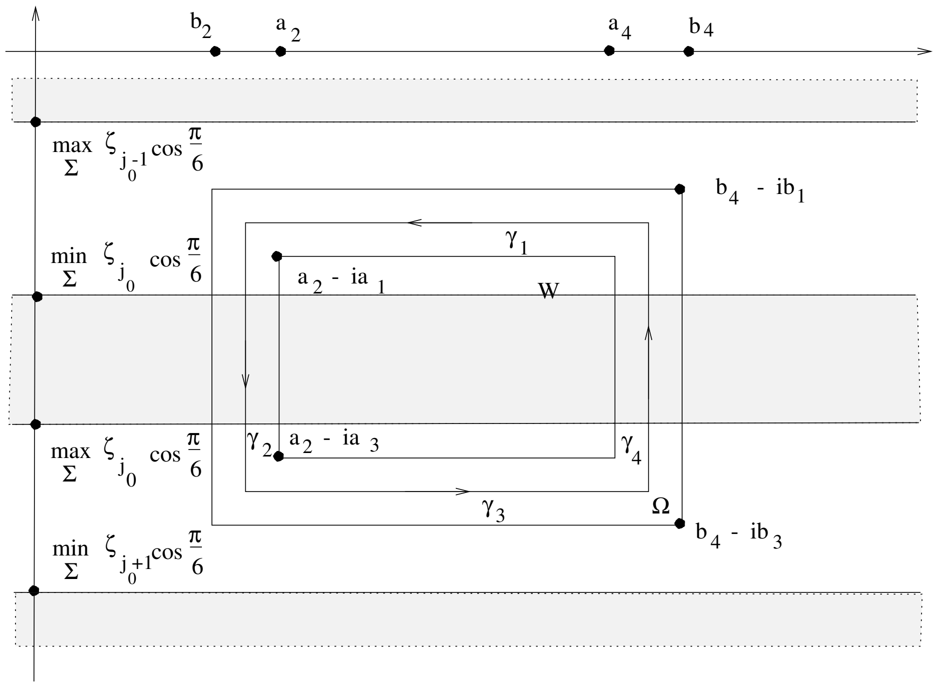

FIGURE 2. The regions $W \subset \Omega$ and the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. When $j_0 = 1$ there is *no* upper shaded region.

To state the trace formula we need one more definition:

$$
\operatorname{Res}(h) \stackrel{\text{def}}{=} \{z \ : \ m_{\mathcal{O}}(h^{-2}(1 + h^{\frac{2}{3}} z)) > 0\} \, ,
$$

where $m_{\mathcal{O}}$ is as in (8.7) and we count the elemets of $\operatorname{Res}(h)$ according to their multiplicities given by $m_{\mathcal{O}}$.

Theorem 9.1. *Let $\widetilde{E}_{-+}(z)$ be as in Lemma 9.1 and $W \subset \Omega \subset \mathbb{C}$, γ be as in* (9.5) *with assumption* (9.3) *satisfied. Let $f(z)$ be a holomorphic function in Ω such that*

$$|f(z)| \leq 1 \quad \text{if} \quad b_2 < \operatorname{Re} z < a_2 \quad \text{and} \quad a_4 < \operatorname{Re} z < b_4 \,, \quad \epsilon > 0 \,.$$

Then for $0 < h < h_0$

$$\begin{aligned} \sum_{j=1,3} \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_j} f(z) \left(E_{-+}(z)^{-1} \frac{d}{dz} E_{-+}(z) - \widetilde{E}_{-+}(z)^{-1} \frac{d}{dz} \widetilde{E}_{-+}(z) \right) dz \\ = \sum_{z \in \operatorname{Res}(h) \cap W} f(z) + \mathcal{O}(1) L h^{1-n+\frac{2}{3}} \,. \end{aligned} \tag{9.7}$$

The first step of the proof of Theorem 9.1 does not depend on the assumption (9.3): we reduce the problem to a finite dimensional one using a Grushin problem for $E_{-+}(z)$. This is given in the following

Lemma 9.2. *Let $\widetilde{E}_{-+}(z)$ be as in Lemma 9.1, Λ as in* (9.1)*, and let $M = \operatorname{rank}(E_{-+}(z) - \widetilde{E}_{-+}(z))) = \mathcal{O}(1) L h^{1-n+\frac{2}{3}}$. Then there exist operators $R_+ \in \mathcal{L}(L^2(\partial\mathcal{O}; \mathbb{C}^N), \ell^2(\mathbb{Z}_M))$, $R_-(z) \in \mathcal{L}(\ell^2(\mathbb{Z}_M), L^2(\partial\mathcal{O}; \mathbb{C}^N))$ such that*

$$\begin{aligned} \mathcal{Q}(z) &= \begin{pmatrix} \Lambda^{-1} E_{-+}(z) & R_-(z) \\ R_+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(L) & \mathcal{O}(L) \\ \mathcal{O}(1) & 0 \end{pmatrix} \; : \; L^2(\partial\mathcal{O}; \mathbb{C}^N) \oplus \ell^2(\mathbb{Z}_M) \longrightarrow L^2(\partial\mathcal{O}; \mathbb{C}^N) \oplus \ell^2(\mathbb{Z}_M) \end{aligned}$$

has a bounded inverse

$$\begin{aligned} \mathcal{F}(z) = \mathcal{Q}(z)^{-1} &= \begin{pmatrix} F(z)\Lambda & F_+(z) \\ F_-(z)\Lambda & F_{-+}(z) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(1/L) & \mathcal{O}(1) \\ \mathcal{O}(1/L) & \mathcal{O}(1) \end{pmatrix} \; : \; L^2(\partial\mathcal{O}; \mathbb{C}^N) \oplus \ell^2(\mathbb{Z}_M) \longrightarrow L^2(\partial\mathcal{O}; \mathbb{C}^N) \oplus \ell^2(\mathbb{Z}_M) \,. \end{aligned}$$

In particular

$$E_{-+}(z)^{-1} = F(z) - F_+(z) F_{-+}(z)^{-1} F_-(z) \,. \tag{9.8}$$

Proof. Let $e_1, \cdots, e_M, e_{M+1}, \cdots$ be an orthonormal basis of $L^2(\partial\mathcal{O}; \mathbb{C}^N)$ such that

$$\operatorname{Image}(\Lambda^{-1}(E_{-+}(z) - \widetilde{E}_{-+}(z)))^* = (\operatorname{Ker}(\Lambda^{-1} E_{-+}(z) - \widetilde{E}_{-+}(z)))^{\perp} = \operatorname{span}\{e_1, \cdots e_M\} \,,$$

where e_j's are independent of z as the difference $E_{-+} - \widetilde{E}_{-+}$ is. In particular,

$$e_{M+1}, e_{M+2}, \cdots \in \operatorname{Ker}(E_{-+}(z) - \widetilde{E}_{-+}(z)) \,. \tag{9.9}$$

We put

$$R_+ u(j) = \langle u, e_j \rangle \,, \; 1 \leq j \leq M \,, \quad R_-(z) u_- = \sum_{j=1}^{M} u_-(j) \Lambda^{-1} \widetilde{E}_{-+}(z) e_j = \Lambda^{-1} \widetilde{E}_{-+}(z) R_+^* u_- \,.$$

We want to solve

$$\Lambda^{-1}E_{-+}(z)u + R_-(z)u_- = v\,, \tag{9.10}$$

$$R_+u = v_+\,. \tag{9.11}$$

We will find formulæ for u, u_- in terms of v, v_+. For that let us assume that we have a solution of (9.10) and (9.11). Writing $u = \sum_{j=1}^{\infty} u_j e_j$ we see from (9.11) that $u_j = v_+(j)$ for $j \leq M$ and hence

$$u = R_+^* v_+ + u'\,, \quad u' \in \mathrm{Image}(I - \Pi)\,, \quad \Pi = R_+^* R_+\,.$$

Hence (9.10) becomes

$$\Lambda^{-1}(E_{-+}(z)u' + \widetilde{E}_{-+}(z)R_+^* u_-) = v - \Lambda^{-1}E_{-+}(z)R_+^* v_+\,.$$

From (9.9) we see that $\Lambda^{-1}E_{-+}(z)u' = \Lambda^{-1}\widetilde{E}_{-+}(z)u'$ and consequently we can rewrite the previous equation as

$$\Lambda^{-1}\widetilde{E}_{-+}(z)(u' + R_+^* u_-) = v - \Lambda^{-1}E_{-+}(z)R_+^* v_+\,.$$

Thus

$$u' + R_+^* u_- = \widetilde{E}_{-+}(z)^{-1}\Lambda v - \widetilde{E}_{-+}(z)^{-1}E_{-+}(z)R_+^* v_+\,,$$

from which it follows that

$$\begin{aligned} u' &= (I - \Pi)\widetilde{E}_{-+}(z)^{-1}\Lambda v - (I - \Pi)\widetilde{E}_{-+}(z)^{-1}E_{-+}(z)R_+^* v_+\,, \\ u_- &= R_+\widetilde{E}_{-+}(z)^{-1}\Lambda v - R_+\widetilde{E}_{-+}(z)^{-1}E_{-+}(z)R_+^* v_+\,, \end{aligned}$$

that is

$$\begin{aligned} &F(z) = (I - \Pi)\widetilde{E}_{-+}(z)^{-1}\,, \qquad F_+(z) = R_+^* - (I - \Pi)\widetilde{E}_{-+}(z)^{-1}E_{-+}(z)R_+^* \\ &F_-(z) = R_+\widetilde{E}_{-+}(z)^{-1}\,, \qquad F_{-+}(z) = -R_+\widetilde{E}_{-+}(z)^{-1}E_{-+}(z)R_+^*\,, \end{aligned} \tag{9.12}$$

which completes proof of the lemma □

Remark 9.1. We note here that following an argument from [26], Lemma 9.2 applied with $1/\delta, L = \mathcal{O}(1)$ provides a finer local upper bound on the number of resonances than that given in Theorem 4 of [32]: if $0 < h < h_0$ then

$$\sum_{\substack{|1-\mathrm{Re}\,\zeta| \leq Ch^{\frac{2}{3}} \\ -\mathrm{Im}\,\zeta < Ch^{\frac{2}{3}}}} m_{\mathcal{O}}(\zeta) = \mathcal{O}(1)h^{1-n+\frac{2}{3}}\,. \tag{9.13}$$

In fact, it follows from (9.8) that the poles of $E_{-+}(z)^{-1}$ are given, with multiplicities, by the poles of $F_{-+}(z)^{-1}$ and these are in turn the zeros of $\det F_{-+}(z)$. We now recall that if A is a $M \times M$ matrix and $\|A\|_{\mathcal{L}(\ell^2(\mathbb{Z}^M);\ell^2(\mathbb{Z}^M))} = \mathcal{O}(1)$ then $|\det A| \leq \exp \mathcal{O}(M)$. Hence $|\det F_{-+}(z)| = \exp(\mathcal{O}(h^{1-n+\frac{2}{3}}))$ and Jensen's inequality gives a bound for the number of its zeros in $|z| \leq C$ as $\mathcal{O}(h^{1-n+\frac{2}{3}})$. Rescaling to the original spectral coordinates gives (9.13).

For the next lemma we need to recall a *lower modulus theorem* which has a long tradition in function theory and is essentially due to H. Cartan – see Theorem 4 in Sect.11.3 of [14] and references given there. Suppose that g is holomorhic in $D(z_0, 2eR)$ and $g(z_0) = 1$. Then for any $\eta > 0$

$$\log|g(z)| \geq -\log\left(\frac{15e^3}{\eta}\right)\log\max_{|z-z_0|<2eR}|g(z)|\,,\quad z\in D(z_0,R)\setminus\mathcal{D}\,,\tag{9.14}$$

where $\mathcal{D}$ is a union of discs with the sum of radii less than ηR. For our purposes the values of the fixed constants in (9.14) will not be relevant.

Lemma 9.3. *Let us assume that* (9.3) *holds and let us take f satisfying assumptions of Theorem 9.1. If N is large enough compared to j_0 then with γ_j's as in* (9.5), *we have*

$$\left|\int_{\gamma_j} f(z)\frac{d}{dz}\log\det F_{-+}(z)dz\right| = \mathcal{O}(1)Lh^{1-n+\frac{2}{3}}\,,\quad j=2\,,4\,,\tag{9.15}$$

where $F_{-+}(z)$ is given by Lemma 9.1.

Proof. We will prove the claim for $j = 2$, the other case being analogous. Using Remark 9.1 we first note that a deformation of the contour within the region where $|f(z)| \leq 1$ gives an $\mathcal{O}(h^{1-n+\frac{2}{3}})$ error. We also observe, again as in Remark 9.1, that

$$\det F_{-+}(z) = \mathcal{O}(1)e^{CLh^{1-n+\frac{2}{3}}}\,, (\det F_{-+}(z_0))^{-1} = \det\left(F_{-+}(z_0)^{-1}\right) = \mathcal{O}(1)e^{CLh^{1-n+\frac{2}{3}}}\,,$$

where the last estimate follows from writing (see (9.12))

$$F_{-+}(z_0)^{-1} = -R_+E_{-+}(z_0)^{-1}\widetilde{E}_{-+}(z_0)R_+^*\,,$$

and from Theorem 8.1 and Lemma 9.1.

Using this and (9.14) with η small enough (see Fig.3) we can find a contour $\tilde{\gamma}_2$ close to γ_2 and with the same end points on which $F_{-+}(z)$ has no zeros and in addition $|\log\det F_{-+}(z)| = \mathcal{O}(Lh^{1-n+\frac{2}{3}})$ for z on the contour (we choose some determination of the logarithm in the neighbourhood of the zero free contour). We now write

$$\begin{aligned}
&\left|\int_{\tilde{\gamma}_2} f(z)\frac{d}{dz}\log\det F_{-+}(z)dz\right| = \\
&\quad\left|-\int_{\gamma_2}\frac{d}{dz}f(z)\log\det F_{-+}(z)dz + f(a_2-ia_j)\log\det F_{-+}(a_2-ia_j)|_{j=1}^{j=3}\right| \\
&\quad\leq \int_{\tilde{\gamma}_2}\left|\frac{d}{dz}f(z)\right||\log\det F_{-+}(z)|\,d|z| + \mathcal{O}(Lh^{1-n+\frac{2}{3}})\,.
\end{aligned}$$

Since by Cauchy inequality, $f'(z) = \mathcal{O}(1)$ on $\tilde{\gamma}_2$ this gives the estimates (9.15). □

Remark 9.2. A more elementary argument for (9.15) can be given by adapting the methods of Sect.8 of [26]. We would have to take a product over a larger set of zeros in defining G (in some rectangle containing γ_2 but of size independent of L). The bound would then come from Harnack's inequality as in [26]. Alternatively, we could use Caratheodory's inequality

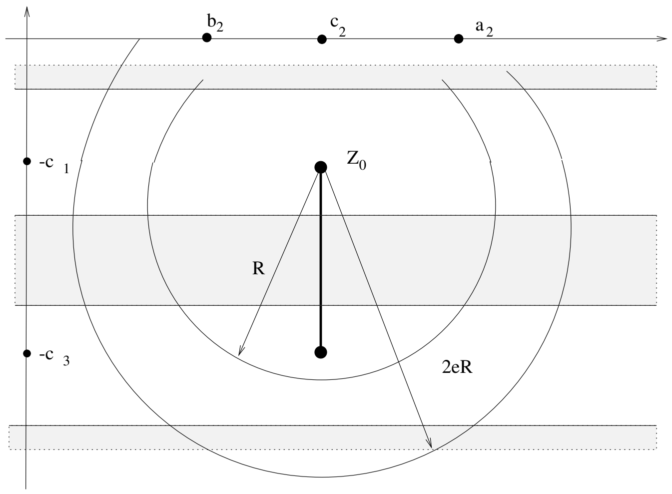

FIGURE 3. The discs in which (9.14) is applied.

directly as in the proof of (9.14). It seemed however worthwhile to recall that useful result even though we do not use its full power.

To conclude the proof of the main result of this section we adapt some of the arguments of [26]:

Proof of Theorem 9.1. Using (9.8) and (9.12) we write

$$\begin{aligned} & E_{-+}(z)^{-1}\frac{d}{dz}E_{-+}(z) - \widetilde{E}_{-+}(z)^{-1}\frac{d}{dz}\widetilde{E}_{-+}(z) = \\ & -\left(\Pi\widetilde{E}_{-+}(z)^{-1} + F_+(z)F_{-+}(z)^{-1}F_-(z)\right)\frac{d}{dz}E_{-+}(z)\,, \end{aligned} \tag{9.16}$$

where $\Pi = R_+^* R_+$ is as in the proof of Lemma 9.2. Both terms on the right hand side are of rank $\mathcal{O}(Lh^{1-n+\frac{2}{3}})$. The first term, $-\Pi\widetilde{E}_{-+}(z)^{-1}$, is holomorphic in Ω and hence

$$-\sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\ \Pi\widetilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)dz = \sum_{j=2,4}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\ \Pi\widetilde{E}_{-+}(z)^{-1}\partial_z E_{-+}(z)dz\,. \tag{9.17}$$

On $\gamma_2\cup\gamma_4$ we have $|f(z)|\leq 1$ and the trace, and hence the entire integral, can be estimated by $\mathcal{O}(Lh^{1-n+\frac{2}{3}})$. Consequently we can absorb it into the error term in (9.7).

It remains to study

$$\begin{aligned} & -\sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\ \left(F_+(z)F_{-+}(z)^{-1}F_-(z)\frac{d}{dz}E_{-+}(z)\right)dz = \\ & -\sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\ \left(F_{-+}(z)^{-1}F_-(z)\frac{d}{dz}E_{-+}(z)F_+(z)\right)dz\,. \end{aligned} \tag{9.18}$$

Using $\dot{\mathcal{F}}(z) = -\mathcal{F}(z)\dot{\mathcal{Q}}(z)\mathcal{F}(z)$, we get (since R_+ is independent of z)

$$-F_-(z)\frac{d}{dz}E_{-+}(z)F_+(z) = \frac{d}{dz}F_{-+}(z) + F_-(z)\Lambda\frac{d}{dz}R_-(z)F_{-+}(z)\,. \tag{9.19}$$

We first consider the contribution to (9.18) of the last term in (9.19):

$$\begin{aligned}\sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\,&\left(F_{-+}(z)^{-1}F_-(z)\Lambda\frac{d}{dz}R_-(z)F_{-+}(z)\right)dz\\ &= \sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\,\left(F_-(z)\Lambda\frac{d}{dz}R_-(z)\right)dz\\ &= -\sum_{j=2,4}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\,\left(F_-(z)\Lambda\frac{d}{dz}R_-(z)\right)dz\,,\end{aligned} \tag{9.20}$$

as in (9.17), and as there we conclude that the last expression is $\mathcal{O}(Lh^{1-n+\frac{2}{3}})$.

We are now left with

$$\sum_{j=1,3}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\,\left(F_{-+}(z)^{-1}\frac{d}{dz}F_{-+}(z)\right)dz =$$

$$\sum_{\mathrm{Res}(h)\cap W} f(z) + \mathcal{O}(h^{1-n+\frac{2}{3}}) - \sum_{j=2,4}\frac{1}{2\pi i}\int_{\gamma_j} f(z)\mathrm{tr}\,\left(F_{-+}(z)^{-1}\frac{d}{dz}F_{-+}(z)\right)dz\,, \tag{9.21}$$

where we noted, from the assumption on f and (9.13) with $L\sim 1$, that $\sum_{\mathrm{Res}(h)\cap\widetilde{W}\setminus W} f(z) = \max_{\mathrm{Res}(h)\cap\widetilde{W}\setminus W}|f(z)|\mathcal{O}(h^{1-n+\frac{2}{3}}) = \mathcal{O}(h^{1-n+\frac{2}{3}})$, $\gamma = \partial\widetilde{W}$. Since $\mathrm{tr}F_{-+}^{-1}\partial_z F_{-+} = \partial_z(\log\det F_{-+})$ the last term on the right hand side of (9.21) is $\mathcal{O}(Lh^{1-n+\frac{2}{3}})$ by Lemma 9.3. This gives (9.7). □

10. Proof of the main theorem

We will now use Theorem 9.1 to prove, under the assumption (9.3), the asymptotic formula for the number of resonances in a band corresponding to the jth zero of the Airy function. We formulate it here in the semiclassical setting:

Proposition 10.1. *For $0 < a < b$ let us write*

$$\begin{aligned} N_h([a,b];j) &= \sum_{\substack{a<\mathrm{Re}\,z<b\\ \kappa(j)h^{\frac{2}{3}}-Ch<-\mathrm{Im}\,z<K(j)h^{\frac{2}{3}}+Ch}} m_{\mathcal{O}}(h^{-2}z)\,,\\ \kappa(j) = 2^{\frac{2}{3}}\zeta_j\cos\frac{\pi}{6}\min_{S\partial\mathcal{O}} Q^{\frac{2}{3}}\,, &\quad K(j) = 2^{\frac{2}{3}}\zeta_j\cos\frac{\pi}{6}\max_{S\partial\mathcal{O}} Q^{\frac{2}{3}}\,,\end{aligned} \tag{10.1}$$

where $m_{\mathcal{O}}$ is the multiplicity of a pole of the meromorphic continuation of $(-\Delta_{\mathbb{R}^n\setminus\mathcal{O}} - \zeta)^{-1}$, $-\zeta_j$ is the jth zero of the Airy function and Q is the second fundamental form of $\partial\mathcal{O}$. Let us assume that

$$\frac{\max_{S\partial\mathcal{O}} Q}{\min_{S\partial\mathcal{O}} Q} < \left(\frac{\zeta_{j+1}}{\zeta_j}\right)^{\frac{3}{2}}\,. \tag{10.2}$$

Then for any $\epsilon > 0$

$$N_h([a,b];j) = (1+\mathcal{O}(\epsilon))\,\frac{h^{1-n}}{(2\pi)^{n-1}}\int_{a\leq|\xi'|^2_{x'}\leq b} dx'd\xi' + \mathcal{O}_\epsilon(h^{1-n+\frac{2}{3}})\,, \tag{10.3}$$

where $|\xi'|^2_{x'}$ is the induced metric on $T^*_{x'}\partial\mathcal{O}$.

In addition there are no resonances of $-h^2\Delta_{\mathcal{O}}$ in

$$\begin{aligned} K(j-1)h^{\frac{2}{3}} + Ch < -\operatorname{Im} z < \kappa(j)h^{\frac{2}{3}} - Ch\,, \\ K(j)h^{\frac{2}{3}} + Ch < -\operatorname{Im} z < \kappa(j+1)h^{\frac{2}{3}} - Ch\,, \end{aligned} \tag{10.4}$$

with the convention that $K(0) = -\infty$.

Theorems 1.1 and 1.2 are easy consequences of this proposition: we can apply a dyadic decomposition and then use (10.3) in each dyadic piece (see for instance Sect.7 of [28]).

To obtain Proposition 10.1 we will apply the pseudodifferential calculus to the expression on the left in the trace formula of Theorem 9.1. The parameter L is now becoming large and since δ can be chosen depending on L we drop it altogether.

Proposition 10.2. *Let f, W, γ_j and j_0 be as in Theorem 9.1. Then*

$$\begin{aligned} \sum_{\mathrm{Res}(h)\cap W} f(\lambda) =& \frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma\times\mathbb{R}} f(\lambda + e^{-2\pi i/3}\zeta_{j_0}(w))1\!\!1_{I(w)}(\lambda)L_\Sigma(dw)d\lambda \\ &+ \mathcal{O}(Lh^{1-n+\frac{2}{3}}) + \mathcal{O}_{f,L}(h^{2-n})\,, \end{aligned} \tag{10.5}$$

where

$$I(w) = \{\lambda\in\mathbb{R}\;:\;\lambda + e^{-2\pi i/3}\zeta_{j_0}(w)\in W\}\,, \tag{10.6}$$

the coordinates $(w, h^{\frac{2}{3}}\lambda)$ are as in Lemma 4.3, and we consider ζ_{j_0} as a function on Σ.

Proof. If we apply Lemma 4.3 to the term appearing in (9.7),

$$\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_3} f(z)\mathrm{tr}\,\left((E_{-+}(z)^{-1} - \widetilde{E}_{-+}(z)^{-1})\frac{d}{dz}E_{-+}(z)\right)dz\,, \tag{10.7}$$

then in the notation of Theorem 8.1 we obtain

$$\begin{aligned} &\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma\times\mathbb{R}}\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_3} f(z)\mathrm{tr}\,\Big((E^0_{-+}(z;w,\lambda)^{-1} \\ &\qquad -\widetilde{E}^0_{-+}(z;w,\lambda)^{-1})\frac{d}{dz}E^0_{-+}(z;w,\lambda)\Big)\,dz L_\Sigma(dw)d\lambda + \mathcal{O}_{L,f}(h^{2-n})\,. \end{aligned}$$

By construction, $(E^0_{-+})^{-1} - (\widetilde{E}^0_{-+})^{-1}$ is $\mathcal{O}(h^\infty\langle\lambda\rangle^{-\infty})$ for $|\lambda|\geq C+L$ and thus we can insert $1\!\!1_{[-C-L,C+L]}(\lambda)$ into the integral at the expense of a $\mathcal{O}(h^\infty)$ error. That allows us to split the integrals into two terms corresponding to E^0_{-+} and $\widetilde{E}^0_{-+}$.

Since $\widetilde{E}^0_{-+}(z;w,\lambda)^{-1}$ is holomorphic in Ω we can change the contour of integration to $\gamma_2\cup\gamma_4$. The function f satisfies $|f(z)|\leq 1$ there and hence the contribution of the term involving $\widetilde{E}^0_{-+}$ is $\mathcal{O}(Lh^{1-n+\frac{2}{3}})$ (the factor L comes from length of the range of integration in the λ variable) and (10.7) becomes

$$\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma\times\mathbb{R}}\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_3} f(z)\mathrm{tr}\,\left(E^0_{-+}(z;w,\lambda)^{-1}\frac{d}{dz}E^0_{-+}(z;w,\lambda)\right)dzL_\Sigma(dw)1\!\!1_{[-C-L,C+L]}(\lambda)d\lambda$$
$$+\mathcal{O}(Lh^{1-n+\frac{2}{3}})+\mathcal{O}_{L,f}(h^{2-n}).$$

We now note that $\mathrm{tr}E^0_{-+}\partial_z E^0_{-+}=\partial_z\log\det E^0_{-+}$ has a simple pole with residue 1 at $z=\lambda+e^{-2\pi i/3}\zeta_{j_0}(w)$ – see (8.9). For $\lambda\in I(w)$, with $I(w)$ given by (10.6), that pole is in W (and it is then the only pole there) and $E^0_{-+}(z;w,\lambda)^{-1}$ is uniformly bounded on $\gamma_2\cup\gamma_4$. Hence at the expense of an $\mathcal{O}(h^{1-n+\frac{2}{3}})$ error we can close the contour which gives

$$\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma\times\mathbb{R}}\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_3} f(z)\mathrm{tr}\,\left(E^0_{-+}(z;w,\lambda)^{-1}\frac{d}{dz}E^0_{-+}(z;w,\lambda)\right)dz1\!\!1_{I(w)}(\lambda)L_\Sigma(dw)d\lambda$$
$$=\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma\times\mathbb{R}} f(\lambda+e^{-2\pi i/3}\zeta_{j_0}(w))1\!\!1_{I(w)}(\lambda)L_\Sigma(dw)d\lambda+\mathcal{O}(Lh^{1-n+\frac{2}{3}}).$$

The remaining part of the integral obtained by inserting $1-1\!\!1_{I(w)}(\lambda)$ is $\mathcal{O}(h^{1-n+\frac{2}{3}})$. In fact, we can deform $\gamma_2\cup\gamma_4$ away from its endpoints so that $E^0_{-+}(z)^{-1}$ remains bounded on the deformed contour and the deformation takes place in the region where $|f(z)|\leq 1$ – see Fig.4. Strictly speaking we have to replace W by a slightly larger $\widetilde{W}$ here, such that $|f(z)|\leq 1$ for $z\in\widetilde{W}\setminus W$. Since the number of resonances in $\widetilde{W}\setminus W$ is $\mathcal{O}(h^{1-n+\frac{2}{3}})$ by Remark 9.1 we commit a controllable error on the right hand side of (9.7)). The contributions from residues consequently give a term of order $\mathcal{O}(h^{1-n+\frac{2}{3}})$. □

We will apply this proposition with

$$f_\epsilon(z)=\sqrt{\frac{\epsilon}{2\pi}}e^{-\epsilon(z-z_0)^2/2}\,,\quad z_0=-i\frac{a_1+a_3}{2}\,,\quad \epsilon L\ll 1\ll\epsilon L^2\,, \tag{10.8}$$

where we recall from (9.6) that $a_2+a_4=0$. In Ω, we have

$$\begin{aligned}\mathrm{Re}\left(-\frac{\epsilon}{2}(z-z_0)^2\right)&=-\frac{\epsilon}{2}\left(\mathrm{Re}(z-z_0)\right)^2+\mathcal{O}(\epsilon)\,,\\ \mathrm{Im}\left(-\frac{\epsilon}{2}(z-z_0)^2\right)&=-\epsilon\,\mathrm{Re}(z-z_0)\,\mathrm{Im}(z-z_0)=\mathcal{O}(L\epsilon)\,,\end{aligned}$$

and hence

$$f_\epsilon(z)=(1+\mathcal{O}(\epsilon L))\sqrt{\frac{\epsilon}{2\pi}}e^{-\epsilon(\mathrm{Re}(z-z_0))^2/2}\,. \tag{10.9}$$

In particular, $\arg f_\epsilon=\mathcal{O}(\epsilon L)$ and

$$|f_\epsilon(z)|\leq(1+\mathcal{O}(\epsilon L))\sqrt{\frac{\epsilon}{2\pi}}e^{-\epsilon L^2/2}\,,\quad z\in\Omega\,,\ \mathrm{Re}\,z\in[b_2,a_2]\cup[a_4,b_4]\,. \tag{10.10}$$

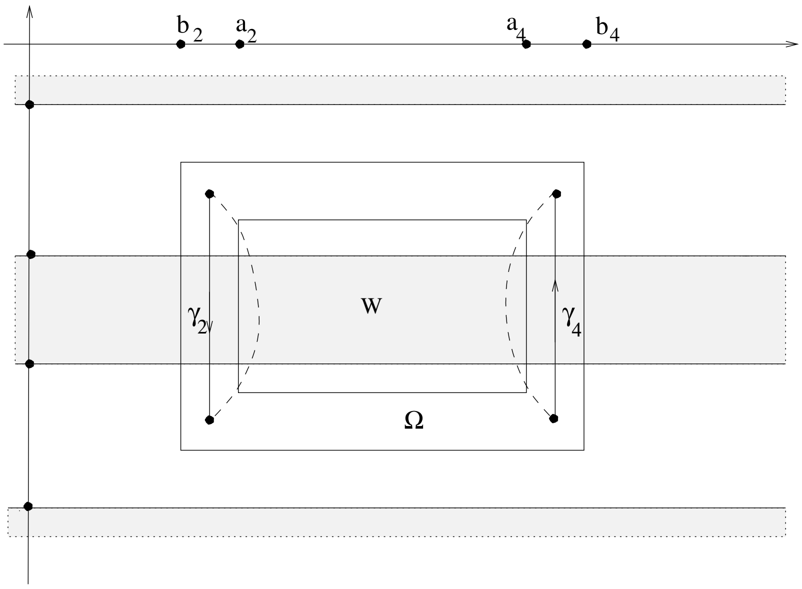

FIGURE 4. Contour deformation for $\lambda \notin I(w)$.

Applying Proposition 10.2 with $f(z) = \left((1+\mathcal{O}(\epsilon L))\sqrt{\epsilon/2\pi}e^{-\epsilon L^2/2}\right)^{-1} f_\epsilon(z)$ (which satisfies the required assumptions because of (10.10)) we obtain

$$\begin{aligned}\sum_{z\in \mathrm{Res}(h)\cap W} \sqrt{\frac{\epsilon}{2\pi}} e^{-\epsilon(\mathrm{Re}(z-z_0))^2/2} &= (1+\mathcal{O}(\epsilon L))\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_\Sigma L_\Sigma(dw) \\ &\quad + \mathcal{O}(1)L\sqrt{\epsilon}e^{-\epsilon L^2/2}h^{1-n+\frac{2}{3}} + \mathcal{O}_{\epsilon,L}(h^{2-n}) \\ &= (1+\mathcal{O}(\epsilon L))\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_\Sigma L_\Sigma(dw) + \mathcal{O}_{\epsilon,L}(h^{2-n}),\end{aligned} \tag{10.11}$$

where we used the assumptions on L and ϵ in (10.8).

Proof of Proposition 10.1. We now reintroduce the parameter $w \in W$ as appearing in Theorem 8.1 and we take $[a,b] \subset W \Subset (0,\infty)$. We generalize previous notation by writing

$$\mathrm{Res}(h;w) = \{z : m_{\mathcal{O}}(h^{-2}(w+h^{\frac{2}{3}}z)) > 0\},$$

with points included according to the multiplicities given by $m_{\mathcal{O}}$, $\mathrm{Res}(h) = \mathrm{Res}(h;1)$. Then

$$\begin{aligned}\int_{w-Lh^{\frac{2}{3}}/2}^{w+Lh^{\frac{2}{3}}/2} \sqrt{\frac{\epsilon}{2\pi}} e^{-\epsilon h^{-\frac{4}{3}}(v-w)^2/2} N_h(dv;j_0) &= \sum_{z\in\mathrm{Res}(h;w)\cap W} \sqrt{\frac{\epsilon}{2\pi}} e^{-\epsilon(\mathrm{Re}\, z)^2/2} \\ &= (1+\mathcal{O}(\epsilon L))\frac{h^{1-n+\frac{2}{3}}}{(2\pi)^{n-1}}\int_{\Sigma_w} L_{\Sigma_w}(dx'd\xi') \\ &\quad + \mathcal{O}_{\epsilon,L}(h^{2-n}).\end{aligned} \tag{10.12}$$

If we multiply the left hand side by $h^{-\frac{2}{3}}$ and integrate it in w over $[a,b]$ we obtain

$$\begin{aligned} &h^{-\frac{2}{3}}\int_a^b \int_{w-Lh^{\frac{2}{3}}/2}^{w+Lh^{\frac{2}{3}}/2} \sqrt{\frac{\epsilon}{2\pi}} e^{-\epsilon h^{-\frac{4}{3}}(v-w)^2/2} N_h(dv;j_0)dw \\ &\quad = h^{-\frac{2}{3}}\int_a^b \int_{v-Lh^{\frac{2}{3}}/2}^{v+Lh^{\frac{2}{3}}/2} \sqrt{\frac{\epsilon}{2\pi}} e^{-\epsilon h^{-\frac{4}{3}}(v-w)^2/2} dw N_h(dv;j_0) + \mathcal{O}(\sqrt{\epsilon}Lh^{1-n+\frac{2}{3}}) \\ &\quad = N_h([a,b],j_0) + \mathcal{O}(Lh^{1-n+\frac{2}{3}}) \,. \end{aligned}$$

The error term came from the change of order of integration: the symmetric difference of integration domains is contained in

$$\bigcup_{c\in\{a,b\}} \left\{ (v,w) \;:\; |v-c| \,,\, |w-c| \leq Lh^{\frac{2}{3}}/2 \right\} \,.$$

The integral in w is normalized and hence after the integration with respect to w first we are left with

$$\mathcal{O}(1) \sum_{c\in\{a,b\}} \int_{|v-c|\leq Lh^{\frac{2}{3}}} N_h(dv,j_0) = \mathcal{O}(Lh^{1-n+\frac{2}{3}})$$

where we used Remark 9.1.

To summarize, we have shown that

$$N_h([a,b],j_0) + \mathcal{O}(\sqrt{\epsilon}Lh^{1-n+\frac{2}{3}}) = (1+\mathcal{O}(\epsilon L))\frac{h^{1-n}}{(2\pi)^{n-1}} \int_{\Sigma_w}\int_a^b\int_{\Sigma_w} L_{\Sigma_w}(dx'd\xi')dw + \mathcal{O}_{\epsilon,L}(h^{2-n}) \,.$$

Since $L_{\Sigma_w}(dx'd\xi')dw = dx'd\xi'$, the integral on the right hand side is the integral on the right hand side of (10.1) with ϵ replaced by ϵL. By taking $L = \epsilon^{-\frac{2}{3}}$, as in view of (10.8) we may, we obtained (10.1) with ϵ replaced by $\epsilon^{\frac{1}{3}}$. As it is arbitrarily small this completes the proof of (10.3).

To see the pole free regions we apply Theorem 8.1. For the rescaled variables z, $\zeta = h^{-2}(w + h^{\frac{2}{3}}z)$, the condition (10.4) translates to

$$K(j-1) + Ch^{\frac{1}{3}} < -\operatorname{Im} z < \kappa(j) - Ch^{\frac{1}{3}} \,,\; K(j) + Ch^{\frac{1}{3}} < -\operatorname{Im} z < \kappa(j+1) - Ch^{\frac{1}{3}} \,.$$

The assumptions on the curvature and z show that all but one of the terms $z - \lambda - e^{-2\pi i/3}\zeta_j(p)$ have moduli bounded from below. Hence (8.8) and (8.9) show that

$$E_{-+}(z) = A(z)G_{-+}(z)B(z) \,,$$

where $A(z), B(z) \in \Psi^{0,0,0}_{\Sigma_w,\frac{2}{3}}(\partial\mathcal{O}, \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$ are invertible, $G_{-+}(z) \in \Psi^{0,1,2}_{\Sigma_w,\frac{2}{3}}(\partial\mathcal{O}, \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N))$ has the principal symbol satisfying

$$\operatorname{Im} G^0_{-+}(z) \geq Ch^{\frac{1}{3}} \; Id_{\mathbb{C}^N}$$

near Σ_w and $\operatorname{Im} G^0_{-+}(z) \geq \langle\xi\rangle^2 h^{-\frac{2}{3}}/\widetilde{C}$ away from Σ_w. If C is chosen large enough then by (6.11) the imaginary part of the full symbol of $G_{-+}(z)$ is bounded from below by a positive symbol in $S^{-\frac{1}{3},0,2}_{\Sigma_w,\frac{2}{3}}$. An adaptation of the sharp Gårding inequality (with h replaced by $h^{\frac{1}{3}}$; see for instance Sect.2 of [24] or Proposition 3.1 of [32]) gives

$$\|E_{-+}(z)w\|_{L^2} \geq Ch^{\frac{1}{3}}\|w\|_{L^2}\,, w \in \mathcal{C}^\infty(\partial\mathcal{O}, \mathbb{C}^N)\,.$$

Since $E_{-+}(z)$ is a Fredholm deformation of an invertible operator it is consequently invertible and hence $h^{-2}(w + h^{\frac{2}{3}}z)$ is not a resonance. $\square$

Appendix

We present here the proof of Lemma 4.1. For that we let

$$a_h(x,\xi) \stackrel{\text{def}}{=} a(x, h^{1-\delta}\xi_1, h\xi_2, \cdots, h\xi_n; h)\,,$$

so that we can write $A = a_h(x, D_x)$ and we will be concerned with that operator and some of its commutators. We have

$$\langle A\psi, \phi\rangle = \frac{1}{(2\pi)^n}\int\int e^{i\langle x,\xi\rangle}a_h(x,\xi)\widehat{\psi}(\psi)\overline{\phi(x)}dxd\xi\,, \tag{A.1}$$

with $\widehat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int e^{-i\langle x,\xi\rangle}\psi(x)dx$, and for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Let us put

$$\tau_{x_0}u(x) = u(x - x_0)\,, \quad \widehat{\tau}_{\xi_0}u(x) = e^{i\langle x,\xi_0\rangle}u(x)\,,$$

and recall that

$$\mathcal{F}\widehat{\tau}_{\xi_0} = \tau_{\xi_0}\mathcal{F}\,, \quad \mathcal{F}\tau_{x_0} = \widehat{\tau}_{-\xi_0}\mathcal{F}\,.$$

We have

$$B \stackrel{\text{def}}{=} \mathrm{ad}^{\alpha'}_{x'}\mathrm{ad}^{\beta'}_{hD_{x'}}\mathrm{ad}^{q}_{x_1}\mathrm{ad}^{p}_{h^{1-\delta}D_{x_1}}A = (-i)^{|\alpha'|+|\beta'|+q+p}\,h^{|\beta'|+(1-\delta)p}b_h(x,D)\,,$$

$$b_h(x,\xi) = (-\partial_{\xi'})^{\alpha'}\partial^{\beta'}_{x'}(-\partial_{\xi_1})^q\partial^p_{x_1}a_h(x,\xi)\,.$$

Let us now assume we have the commutator estimate in the lemma with $k \geq 1$. Then if $\|u\|_{(-q)} = \|\langle h^{1-\delta}D_{x_1}\rangle^{-q}u\|_{L^2}$ is the norm dual to $\|\bullet\|_{(q)}$, we get

$$|\langle B\tau_{x_0}\widehat{\tau}_{\xi_0}\psi, \tau_{y_0}\widehat{\tau}_{\eta_0}\phi\rangle| \leq Ch^{|\alpha'|+|\beta'|+(1-\delta)(p+q)}\|\tau_{x_0}\widehat{\tau}_{\xi_0}\psi\|_{(k)}\|\tau_{y_0}\widehat{\tau}_{\eta_0}\phi\|_{(-q)}\,. \tag{A.2}$$

For fixed $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$

$$\|\tau_{x_0}\widehat{\tau}_{\xi_0}\psi\|_{(k)} \sim \langle h^{1-\delta}\xi_{0,1}\rangle^k\,, \quad \|\tau_{y_0}\widehat{\tau}_{\eta_0}\psi\|_{(-q)} \sim \langle h^{1-\delta}\eta_{0,1}\rangle^{-q}\,,$$

and hence the right hand side of (A.2) is

$$\mathcal{O}(1)h^{|\alpha'|+|\beta'|+(1-\beta)(p+q)}\langle h^{1-\delta}\xi_{0,1}\rangle^k\langle h^{1-\delta}\eta_0\rangle^{-q}\,.$$

Using (A.1) we rewrite the left hand side of (A.2) as

$$\frac{1}{(2\pi)^n}\left|h^{|\beta'|+(1-\delta)p}\int\int e^{i\langle x,\xi\rangle}b_h(x,\xi)\widehat{\psi}(\xi-\xi_0)e^{-i\langle x_0,\xi-\xi_0\rangle}e^{-i\langle\eta_0,x-y_0\rangle}\bar{\phi}(x-y_0)dxd\xi\right|\,.$$

Decomposing the first exponent in the integral as

$$\langle x,\xi\rangle = \langle y_0,\xi_0\rangle + \langle x-y_0,\xi_0\rangle + \langle\xi-\xi_0,y_0\rangle + \langle x-y_0,\xi-\xi_0\rangle$$

we rewrite it further as

$$\frac{h^{|\beta'|+(1-\delta)p}}{(2\pi)^n}\left|\int\int b_h(x,\xi)e^{i\langle x-y_0,\xi-\xi_0\rangle}\widehat{\psi}(\xi-\xi_0)\bar{\phi}(x-y_0)e^{i(\langle\xi-\xi_0,y_0-x_0\rangle+\langle x-y_0,\xi_0-\eta_0\rangle)}dxd\xi\right|$$
$$= h^{|\beta'|+(1-\delta)p}\left|\mathcal{F}((\tau_{y_0,\xi_0}\chi)b_h)(\eta_0-\xi_0,x_0-y_0)\right|\,,\quad \chi(x,\xi)=e^{i\langle x,\xi\rangle}\widehat{\psi}(\xi)\bar{\phi}(x)\,.$$

Summing up, we get

$$\begin{gathered}\left|\mathcal{F}\Big((\tau_{y_0,\xi_0}\chi)\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a_h\Big)(\eta_0-\xi_0,x_0-y_0)\right|=\\ \mathcal{O}(1)h^{|\alpha'|+(1-\delta)q}\langle h^{1-\delta}\xi_{0,1}\rangle^k\langle h^{1-\delta}\eta_{0,1}\rangle^{-q}\,.\end{gathered}\tag{A.3}$$

Here (as in [4], Sect.8) χ can be replaced by any fixed C_0^∞ function. Writing $\zeta=\eta_0-\xi_0$, $z=x_0-y_0$, we therefore get

$$\begin{gathered}\mathcal{F}\left(\partial_{\xi'}^{\tilde{\alpha}'}\partial_{\xi_1}^{\tilde{q}}\partial_{x'}^{\tilde{\beta}'}\partial_{x_1}^{\tilde{p}}(\tau_{y_0,\xi_0}\chi)\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a_h\right)(\zeta,z)=\\ \mathcal{O}(1)h^{|\alpha'|+(1-\delta)q}\langle h^{1-\delta}\xi_{0,1}\rangle^k\langle h^{1-\delta}(\xi_{0,1}+\zeta_1)\rangle^{-q}\,,\end{gathered}$$

which putting $\tilde{\alpha}=(\tilde{q},\tilde{\alpha}')$, $\tilde{\beta}=(\tilde{p},\tilde{\beta}')$ we rewrite as

$$z^{\tilde{\alpha}}\zeta^{\tilde{\beta}}\mathcal{F}\left((\tau_{y_0,\xi_0}\chi)\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a_h\right)(\zeta,z)=\mathcal{O}(1)h^{|\alpha'|+(1-\delta)q}\langle h^{1-\delta}\xi_{0,1}\rangle^k\langle h^{1-\delta}(\xi_{0,1}+\zeta_1)\rangle^{-q}\,.$$

It follows that

$$\mathcal{F}\left((\tau_{y_0,\xi_0}\chi)\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a_h\right)(\zeta,z)=\mathcal{O}(1)h^{|\alpha'|+(1-\delta)q}\langle h^{1-\delta}\xi_{0,1}\rangle^{k-q}\langle\zeta\rangle^{-N}\langle z\rangle^{-N}\,,$$

for any N and consequently

$$\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a_h(x,\xi)=\mathcal{O}(1)h^{|\alpha'|+(1-\delta)q}\langle h^{1-\delta}\xi_1\rangle^{k-q}\,,$$

that is

$$(\partial_{\xi'}^{\alpha'}\partial_{\xi_1}^{q}\partial_{x'}^{\beta'}\partial_{x_1}^{p}a)(x,h^{1-\delta}\xi_1,h\xi';h)=\mathcal{O}(1)\langle h^{1-\delta}\xi_1\rangle^{k-q}\,,$$

which gives the lemma for $k\geq 0$. When $k<0$ we check that the assumption is satisfied for $\langle h^{1-\delta}D_{x_1}\rangle^{-k}A$ with k replaced by 0. The composition formula then gives the result.

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