MATH 249 PROBLEM SET 4 (DUE OCTOBER 31)

(1) Let $M = (E, \mathcal{B})$ be a matroid and $S \subset E$. Show that

$$(M/S)^* = M^*|_{E-S}.$$

(2) Let f be a function from matroids to $\mathbb{C}$ such that:
   (a) If $E = \emptyset$ then $f(M) = 1$
   (b) Let $A = f(\text{coloop})$ and let $B = f(\text{loop})$. Then

$$f(M) = Af(M - e) \text{ for } e \text{ a coloop, and}$$
$$f(M) = Bf(M - e) \text{ for } e \text{ a loop.}$$

   (c) There exist constants $\alpha, \beta \neq 0$ such that when e is not a loop or coloop, we have $f(M) = \alpha f(M - e) + \beta f(M/e)$.

   Show that for all matroids M,

$$f(M) = \alpha^{|E|-r(E)} \beta^{r(E)} T_M(A/\beta, B/\alpha),$$

   where T_M is the Tutte polynomial.

(3) Recall that the chromatic polynomial is the polynomial enumerating proper colorings of a graph. Show that the chromatic polynomial satisfies a deletion-contraction recurrence. Then deduce that $\chi_G(\lambda) = (-1)^{|V|-k(G)} \lambda^{k(G)} T_G(1 - \lambda, 0)$, where $k(G)$ equals the number of connected components of the graph.

(4) How many positroids of rank 2 on the ground set [4] are there? Hint: it might help to use the three-term Plücker relation.