MATH 249 PROBLEM SET 1 (DUE SEPTEMBER 19)

(1) Let P be a finite poset, and $m \in \mathbb{N}$. Let $\underline{m}$ be the chain poset $1 < 2 < \cdots < m$. Show that the following numbers are equal:
   (a) The number of surjective order-preserving maps $\sigma : P \to \underline{m}$.
   (b) The number of chains $\hat{0} = I_0 < I_1 < \cdots < I_m = \hat{1}$ of length m in $J(P)$.

(2) Let P be a locally finite poset. Define $\eta \in I(P)$ (the incidence algebra) by $\eta(x, y) = 1$ if y covers x and $\eta(x, y) = 0$ otherwise. Show that $(1-\eta)^{-1}(x, y)$ is equal to the total number of maximal chains in $[x, y]$.

(3) Consider Young's lattice (the lattice of all partitions, ordered by containment). Calculate its Mobius function. That is, for each pair of partitions, $\lambda \subset \nu$, calculate $\mu(\lambda, \nu)$.

(4) Recall that for any positive integer n, the partition lattice Π_n is the poset of all partitions of $[n]$ (into blocks), where we define $\pi \leq \sigma$ in Π_n if and only if each block of π is contained in a block of σ. (In other words, π is a *refinement* of σ.) Find an EL-labeling of Π_n (and prove that it is one). Then identify the homotopy-type of the order complex $\Delta(\Pi_n - \hat{0} - \hat{1})$.

Note: Don't confuse Π_n with Young's lattice! Π_n is the poset of objects $(B_1, \ldots, B_k)$, where the disjoint union of the B_i's is $\{1, \ldots, n\}$. On the other hand, Young's lattice is the poset of all partitions $(\lambda_1, \ldots, \lambda_k)$, where $\lambda_1 \geq \cdots \geq \lambda_k$. We typically view this kind of partition as a Young diagram.