Homework 3 (two was going to Wassermann's talk) and midterm.

1) Show that if ϕ and ψ are pure states on a C*-algebra with $||\phi - \psi|| < 2$ then the GNS representations π_ϕ and π_ψ are unitarily equivalent. (Hint: show that if they are inequivalent then $||\phi - \psi|| = 2$ by considering $\pi_\phi \oplus \pi_\psi$ and a 2x2 matrix argument. By the bicommuant theorem there things you might not expect to see in $(\pi_\phi \oplus \pi_\psi)(A)''$.)

2) If ω_ξ and ω_ν are vector states on a C*-algebra then

$$||\omega_\xi - \omega_\nu|| \leq 2\sqrt{1 - |\langle \xi, \nu \rangle|^2}$$

Let tr denote the usual trace on $B(\mathcal{H})$ defined on trace-class operators.

3) If p and q are projections in $B(\mathcal{H})$ and ϕ_p and ϕ_q are the two corresponding pure quasi-free states on $CAR(\mathcal{H})$ show that if $p - q$ is Hilbert-Schmidt,

$$||\phi_p - \phi_q||^2 \leq 2tr((p-q)^2)$$

Hints: Begin with the finite dimensional case and chose onb's $\{v_i\}$ and $\{w_i\}$ of the images of $1-p$ and $1-q$ respectively. Reduce to the case of general position and estimate $||\phi_p - \phi_q||$ in terms of the determinant of the matrix $(|\langle v_i, w_j \rangle|^2)$. Also calculate $tr((p-q)^2)$ in terms of the same matrix. Extend to infinite dimensions by diagonalising $(p-q)^2$.

4) The irreducible representations determined by ϕ_p and ϕ_q are unitarily equivalent if $p - q$ is Hilbert-Schmidt.

Hint: Most of the Hilbert-Schmidt norm sits on a finite dimensional space.

5) Show that if α is an automorphism of a C*-algebra A and ϕ and $\phi \circ \alpha$ are equivalent pure states then there is a unitary u on $\mathcal{H}_\phi$ with

$$u\pi_\phi(a)u^* = \pi_\phi(\alpha(a))$$

for all $a \in A$. Show that u is unique up to multiplication by a complex number.

6) Let $\mathcal{H}$ be $L^2(S^1)$ and p be the Hardy space projection onto functions with only positive Fourier modes (boundary values of holomorphic functions). Let $f : S^1 \to S^1$ be a smooth loop. Show that $p - fpf^*$ is Hilbert-Schmidt and calculate its Hilbert-Schmidt norm.

7) Deduce from the above the existence of a projective unitary representation of the group of smooth loops LS^1 on the GNS Hilbert space of the quasi-free state for p.

8) (Extra credit)Investigate the continuity of this projective representation.