Theorem 0.1. *Let F be an ordered field. Then F is complete iff every bounded nondecreasing sequence converges.*

Proof. ($\implies$) This is very easy but let us prove it carefully. If (s_n) is a bounded non-decreasing sequence then $S = \{s_n | n \in \mathbb{N}\}$ is a subset of F which is bounded above. Let $s = \sup S$. We claim that $s_n \to s$. Let $\epsilon > 0$ be given. We must find an N for which $n \geq N \implies |s_n - s| < \epsilon$. By the epsilon condition for the supremum of a set, there is an N for which $s_N > s - \epsilon$.

Then for $n \geq N$ we have $s_n \geq s_N$ since (s_n) is nondecreasing so

$$s - \epsilon < s_N \leq s_n \leq s$$

which certainly implies

$$|s_n - s| < \epsilon.$$

($\impliedby$) First observe that by multiplying by -1 the condition implies that any bounded nonincreasing sequence converges as well. Let $S \subseteq F$ be a non-empty subset bounded above by K. We must show the existence of $\sup S$. Choose an element $s \in S$. We will construct inductively a pair $(x_n), (y_n)$ of sequences with the following three properties:

(i) $x_n \in S$ and y_n is an upper bound for S.

(ii) $x_{n+1} \geq x_n$ and $y_{n+1} \leq y_n$.

(iii) $|x_{n+1} - y_{n+1}| \leq \frac{1}{2}|x_n - y_n|$.

To do this first set $x_0 = s$ and $y_0 = K$. Then suppose $(x_1, x_2, x_3, ..., x_n)$ and $(y_1, y_2, y_3, ..., y_n)$ have been constructed satisfying (i),(ii) and(iii). Either $\dfrac{x_n + y_n}{2}$ is an upper bound for S or it isn't. If it is, put $x_{n+1} = x_n$ and $y_{n+1} = \dfrac{x_n + y_n}{2}$. Conditions (i),(ii) and(iii) are immediate. If $\dfrac{x_n + y_n}{2}$ is not an upper bound for S, there is a $t \in S$ with $t > \dfrac{x_n + y_n}{2}$. In this case set $x_{n+1} = t$ and $y_{n+1} = y_n$. Again (i),(ii) and(iii) are obvious. This ends the construction of the two sequences.

Note that (x_n) is a nondecreasing sequence bounded above by K and (y_n) is a nonincreasing sequence bounded below by s. Hence by our observation above there are x and y in F for which $x_n \to x$ and $y_n \to y$. We claim that $x = y$. This is surprisingly tricky. Clearly from (iii) $|x_n - y_n| \leq \dfrac{1}{2^n}|x_0 - y_0|$. Suppose for the moment we can show that $\frac{1}{2^n} \to 0$ (which is obvious enough if F is Archimedean). Then by choosing n sufficiently large we can suppose that $|x - x_n|, |y - y_n|$ and $|x_n - y_n|$ are all smaller than any given $\epsilon > 0$. Then

$$|x - y| = |x - x_n + x_n - y_n + y_n - y| \leq |x - x_n| + |x_n - y_n| + |y - y_n| < 3\epsilon$$

Since ϵ is arbitrary, $|x - y|$ must be zero.

Thus $x = y$ will follow from $\frac{1}{2^n} \to 0$. But $\frac{1}{2^n}$ is a nonincreasing bounded sequence so it has a limit $z \in F$. And by elementary properties of limits valid in any ordered field, the sequence $(2\frac{1}{2^n}) \to 2z$. But also the limit of the sequence $(\frac{1}{2^{n-1}})$ is the same as that of $(\frac{1}{2^n})$. Hence $2z = z$ which forces $z = 0$. (Thanks to Mr. ... for this trick.)

Finally we show that this common limit $x = y$ is the supremum of S, completing the proof. To see this first note that $t < y_n$ for any $t \in S$ and all n so $y \geq t$ for any $t \in S$ which means $x = y$ is an upper bound for S. If v were an upper bound with $v < x$ then there would be an x_n with $x_n > v$ since $(x_n) \to x$. Hence $x = y$ is the least upper bound.

$\square$