Midterm.

1 worth 20 points

What is a real number?

A real number is an element of a complete ordered field. Any two complete ordered fields are isomorphic (by a unique isomorphism) which justifies the notation $\mathbb{R}$ for the real numbers. There are various constructions of $\mathbb{R}$. One is by "Dedekind Cuts" where the elements of $\mathbb{R}$ are subsets of the rationals (called "cuts") with certain properties. Another construction is to consider the set of equivalence classes of Cauchy sequences on which one defines field operations and order.

2 True/False. Write T or F next to question. A correct answer is worth 4, a wrong one is worth -2.

2.1. *Let $f : X \to Y$ be a map between metric spaces. A subset A of Y is connected iff $f^{-1}(Y)$ is connected.*

False, a map that sends any disconnected metric space to a single point is a counterexample.

2.2. $\{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ *is a subfield of* $\mathbb{R}$.

True-it is clearly closed under multiplication and addition and has additive inverses. If either a or b is non-zero one may divide by the rational $a^2 - 2b^2$ (which is non-zero since $\sqrt{2}$ is irrational) and find that $(a+b\sqrt{2})(\frac{a-b\sqrt{2}}{a^2-2b^2}) = 1$ so multiplicative inverses are ok.

2.3. $\{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ *is uncountable.*

False- it is in bijection with $\mathbb{Q} \times \mathbb{Q}$ which is countable.

2.4. *The Cantor set has a countable dense subset.*

True, we showed that any subset of a separable metric space is separable.

2.5. *On $\mathbb{R}$ define $\sim$ by $x \sim y$ iff $x - y - \sqrt{2}$ is irrational. $\sim$ is an equivalence relation on $\mathbb{R}$.*

False-The relation is not even symmetric.(Try multiples of $\sqrt{2}$.)

2.6. *Given any metric space there is another metric on it which is bounded and defines the same topology (i.e. the same open sets).*

True-we showed that $\frac{d(x,y)}{1+d(x,y)}$ is a metric which is bounded and defines the same open sets.

2.7. *There is a metric on* $(0,1)$ *which defines the same topology as the usual metric but for which* $(0,1)$ *is complete.*

True, there is a homeomorphism from $(0,1)$ to $\mathbb{R}$.

2.8. *No nonempty open subset of* $\mathbb{Q}$ *is closed.*

False. Singletons are always closed.

2.9. *Every nonempty open subset of* $\mathbb{Q}$ *is closed.*

False. $\{q \in \mathbb{Q} | q^2 < 2\}$ is both open and closed.

2.10. *Let* $f : X \to Y$ *be a map between metric spaces with metrics* d *and* D. f *is continuous iff* $(\forall \epsilon > 0) \quad (\exists \delta > 0)$ *such that* $d(a,b) < \delta \Rightarrow D(f(a), f(b)) < \epsilon$.

False. This is the definition of *uniform* continuity.

2.11. *A closed bounded subset of a complete metric space is compact.*

False. (IMPORTANT) This is true in $\mathbb{R}^n$ but not in general. For a counterexample consider an infinite discrete metric space.

2.12. $\{(x,y) | x^2 - y^2 = 1\}$ *is a compact subset of* $\mathbb{R}^2$.

False. It's closed but certainly not bounded.

2.13. $[0,1) \cap \mathbb{Q}$ *is countable so let* (s_n) *be a a sequence taking all values in* $[0,1) \cap \mathbb{Q}$. *Then* $\liminf s_n = 1$.

False. Any sequence enumerating all these rationals must have infinitely many terms less than any $\epsilon > 0$ so $\liminf = 0$.

2.14. *The only connected subsets of the Cantor set are singletons (i.e. subsets with one element).*

True and False (ugh! mea culpa) I intended to say that the subsets under consideration are non-empty in which case it is true (one of the discarded middle thirds will eventually lie between two distinct points). But the empty set is by definition connected.....

2.15. *Suppose* (X,d) *is a metric space and* A, B C *are three subsets of* X *with* $A \subseteq B$ *and* $A \subseteq C$. *Then* A *is compact in* B *iff* A *is compact in* C.

True. Compactness of a subset is an intrinsic notion.

3 40 points

Let $f : X \to Y$ and $g : Y \to Z$ be functions between metric spaces.The composition $g \circ f$ is the function from X to Z defined by $g \circ f(x) = g(f(x))$. Prove using the $\epsilon - \delta$ definition of continuity that if f is continuous at x and g is continous at $f(x)$ then $g \circ f$ is continuous at x.

Let us use the same letter d for both metrics to simplify life. If x is a point in X and $\epsilon > 0$ is given we must find a δ so that $(d(x,t) < \delta) \Rightarrow d(g(f(x)), g(f(t))) < \epsilon$. But since g is continous at $f(x)$ there is an $\eta > 0$ with $d(f(x), w) < \eta \Rightarrow d(g(f(x)), g(w)) < \epsilon$. Now by the continuity of f at x, there is a $\delta > 0$ so that $(d(x,t) < \delta) \Rightarrow d(f(x)), f(t)) < \eta$. But then $d(g(f(x)), g(f(t))) < \epsilon$ as required.

4 40 points

The Bolzano Weierstrass theorem asserts that closed bounded intervals in $\mathbb{R}$ are sequentially compact. Assuming only this theorem, prove that closed balls in $\mathbb{R}^3$ are sequentially compact.

Let C be a closed ball in $\mathbb{R}^3$ of radius r, centred at (a, b, c) and (x_n, y_n, z_n) be a sequence in C. Since $(x_n - a)^2 + (y_n - b)^2 + (z_n - c)^2 \leq r^2$ we have that x_n, y_n and z_n lie in closed intervals of width $2r$. By BW (Bolzano-Weierstrass), x_n has a convergent subsequence x_{n_k} so define the sequence $(a_k, b_k, c_k) = (x_{n_k}, y_{n_k}, z_{n_k})$. It is a subsequence of the original one with the property that the first coordinate of any subsequence of it converges. Now by BW applied to (b_k),take a subsequence of (a_k, b_k, c_k) so that the second coordinate converges. We obtain a sequence $((d_\ell, e_\ell, f_\ell))$ which is a subsequence of the original sequence and for which both the first and second coordinates converge. Finally apply BW to $((f_\ell))$ to obtain a subsequence for which all three coordinates converge, i.e. a convergent subsequence in $\mathbb{R}^3$. The limit of this convergent subsequence must be in C since it is closed.