Let (X, d) be a metric space.

1) A sequence (x_n) in X converges to x if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) \text{ so that } (n \geq N) \implies d(x_n, x) < \epsilon$$

.

2) A sequence (x_n) is Cauchy if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N}) \text{ so that } (n, m \geq N) \implies d(x_n, x_m) < \epsilon$$

.

3) A metric space is said to be *complete* if every Cauchy sequence converges.

4) Given $a \in X$ and $r \in \mathbb{R}, r > 0$, the *open ball* (centred at x, radius r) is

$$B(a, r) = \{x \in X | d(a, x) < r\}$$

5) A subset $S \subseteq X$ is *bounded* if there is an $a \in X$ and an $r > 0$ so that $S \subseteq B(a, r)$.

6)A subset $U \subseteq X$ is *open* if it contains an open ball around each of its points, i.e.

$$(\forall x \in X)(\exists r > 0) \text{ such that } (B(x, r) \subseteq U)$$

7) A subset $F \subseteq X$ is *closed* if $X \setminus F$ is open. A subset X is closed iff if it contains the limit of any convergent sequence whose terms lie in F.

8) A *neighbourhood* of $x \in X$ is a set N so that there is an $r > 0$ with

$$x \in B(x, r) \subseteq N.$$

9) The *closure* of $E \subset X$ is the intersection of all closed sets containing E:

$$\bar{E} = \bigcap_{F \subseteq X, F closed, E \subseteq F} F$$

If $A \subseteq B$ are subsets of X, A is *dense* in B if $B \subseteq \bar{A}$.

10) The *interior* S° of $S \subseteq X$ is the set of all points in S for which S is a neighbourhood. Or:

$$S^\circ = \{x \in S | \exists r > 0 \text{ with } B(x, r) \subseteq S\}$$

11) The *boundary* ∂S of $S \subseteq X$ is

$$\partial S = \bar{S} \setminus S^\circ$$

12) A point $x \in X$ is called an *accumulation point* of $S \subseteq X$ if $(\forall \epsilon > 0), B(x, r)$ contains points of S besides x itself (x may or may not be in S). A subset E of X is closed iff it contains all of its accumulation points.

13) The collection $\{U \subseteq X | U open\}$ is called the *topology* on X defined by the metric d. A property of metric spaces is said to be "topological" if it depends only on the topology. Note that convergence of sequences is topological, neighbourhoods are topological, closure, interior and boundary are topological. Boundedness isn't.

14) The set $\mathcal{U}$ of all open subsets of X satisfies the following three properties:
(i) $\phi \in \mathcal{U}$ and $X \in \mathcal{U}$.
(ii) If U_i are in $\mathcal{U}$ for i in some indexing set I then $\bigcup_{i \in I} U_i \in \mathcal{U}$
(iii) If $U, V \in \mathcal{U}$ then $U \bigcap V \in \mathcal{U}$.
In general given just a set X we say a collection of subsets $\mathcal{U}$ is a topology on X it satisfies the three properties above. Then X is called a topological space and we can speak of closed subsets of X, convergence of sequences in X, closures, interiors and boundaries of subsets and any other topological properties. The most important property a topological space can have is that it is Hausdorff: any two distinct points are in disjoint neighbourhoods. Limits of sequences are unique in Hausdorff spaces.

15) Given metric spaces (X, d) and (Y, D) we define 3 metrics on $X \times Y$:

a) $d_2((x_1, y_1), (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + D(y_1, y_2)^2}$

b) $d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), D(y_1, y_2)\}$

c) $d_1((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + D(y_1, y_2)$

All three metrics define the same topology. The process is extended in the obvious way to Cartesian products of finitely many metric spaces. A sequence (x_n, y_n) tends to (x, y) iff $x_n \to x$ and $y_n \to y$.

16) The concept of metric space is hereditary: any subset of a metric space becomes a metric space by restricting the metric. It is said to "inherit" the metric. A subset with the inherited metric is called a sub-metric space or metric sub-space.

17) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is said to be *continuous* at x if

$$(\forall \epsilon > 0)(\exists \delta > 0) \text{ such that } d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$$

Continuity of a function at x is a topological notion.
f is continuous at x iff $(x_n \to x) \implies (f(x_n) \to f(x))$.

18) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is said to be *continuous* (on X) if f is continuous at every point of X.

19) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is continuous iff $f^{-1}(U)$ is open for every open set $U \subseteq Y$. (Recall that $f^{-1}(U) = \{x \in X | f(x) \in U\}$.)

20) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is continuous iff $f^{-1}(E)$ is closed for every closed set $E \subseteq Y$.

What it means for two metric spaces to be "the same" is a complicated question.

21) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is called an *isometry* if it is a bijection and $D(f(x), f(y)) = d(x, y) \quad \forall x, y \in X$. (An isometry is obviously continuous and its inverse is also an isometry.)

22) Given metric spaces (X, d) and (Y, D), a function $f : X \to Y$ is called a *homeomorphism* if it is a continous bijection whose inverse is also continuous. This is the same as saying that, for a subset $U \subseteq X$,

$$U \text{ is open} \iff f(U) \text{ is open} .$$

23) If A is a subset of a metric space X, an *open cover* $\{U_\alpha | \alpha \in I\}$ is a family of open subsets of X (indexed by α in some indexing set I) such that

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha$$

An *open subcover* of a cover $\mathcal{C} = \{U_\alpha | \alpha \in I\}$ of A is a subset of $\mathcal{C}$ which is also an open cover of A.

24) A subset A of a metric space X is said to be *compact* if every open cover of A has a finite subcover. A is a compact subset of X iff A is a compact subset of A with the inherited metric.

25) A compact subset of a metric space is closed and bounded. A closed subset of a compact metric space is compact.

26) A metric space X is called *sequentially compact* if every sequence in X has a convergent subsequence. $[0, 1]$ is sequentially compact. So is $[a_1, b_1] \times [a_2, b_2] \times ... \times [a_n, b_n]$ as a metric subspace of $\mathbb{R}^n$.

27) A metric space is sequentially compact iff it is compact.

28) The Heine Borel theorem: A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

29) The continuous image of a compact set is compact.

30) A continuous real-valued function on a compact metric space attains its maximum and minimum values.

31) Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space. Then if $f(x) > 0 \quad \forall x$, there is an $\epsilon > 0$ such that $f(x) > \epsilon \quad \forall x$.

32)If $f : X \to Y$ is a continous bijection between metric spaces and X is compact then f is a homeomorphism.

33) If U_α is an open cover of a metric space, a “Lebesgue number" for the cover is a number $\lambda < 0$ such that any set of diameter $< \lambda$ is contained in one of the U_α. If X is compact, any open cover has a Lebesgue number. (The diameter of a subset A of a metric space is $\sup\{d(a,b)|a,b \in A\}$).

34) If $|-|$ and $||-||$ are two norms on $\mathbb{R}^n$ then there exist $A, B > 0$ with

$$A|x| \leq ||x|| \leq B|x| \quad \forall x \in \mathbb{R}^n$$

.

35) A metric space is called *separable* if it contains a countable dense subset. $\mathbb{R}^n$ is separable. An uncountable metric space with the discrete metric isnot separable.

36) Any metric subspace of a separable metric space is separable.

37) A subset C of a metric space X is said to be *connected* if whenever U and V are *disjoint* open subsets of X,

$$C \subseteq U \cup V \Rightarrow C \subseteq U \text{ or } C \subseteq V$$

(the empty set is connected).

38) A (nonempty) subset of $\mathbb{R}$ is connected iff it is an interval.

39) As with compactness, a subset C of a metric space X is connected as a subset of X iff it is connected as a subset of itself (with the metric it inherits from X).

40) Continuous images of connected sets are connected.

41) Let $f : A \to \mathbb{R}$ (where $A \subseteq \mathbb{R}$) be continuous. Then if $I \subseteq A$ is an interval, so is $f(I)$ so that f takes all values in between any two given values $f(x)$ and $f(y)$ for $x, y \in I$.

42) If X is a metric space define $\sim$ on X by $x \sim y$ iff there is a connected subset of X containing x and y. $\sim$ is an equivalence relation whose equivalence classes are called the *connected components* of X.

43) A subset C of a metric space X is called *path connected* if for any 2 points x and y in C there is a continuous function $c : [0,1] \to C$ with $c(0) = x$ and $c(1) = y$(in which case we say there is a path in C connecting x and y).

44) Path connected implies connected but not vice versa.

45) On a metric space X define $\sim$ byt $x \sim y$ iff there is a path connecting x and y. $\sim$ is an equivalence relation whose equivalence classes are called the *path components* of X.