MATH104 spring 2011 Jones. Makeup Midterm. Total points 150. Use your own scratch paper but give your final answers in the space provided. Don't hand in your scratch work.

YOUR NAME:

1 worth 30 points

What is a Caucy sequence?

A Cauchy sequence in a metric space (X, d) is a sequence (s_n) in X such that $(\forall \epsilon > 0)(\exists M \in \mathbb{N})$ such that $(m, n \geq M) \implies d(s_n, s_m) < \epsilon$.

What is a complete metric space?

One in which every Cauchy sequence converges.

Give an example of a complete metric space and an example of one that is not complete.

$\mathbb{R}$ is complete, $\mathbb{Q}$ isn't.

2 worth 30 points

Prove that a compact subset of a metric space is closed and bounded.

Let (X, d) be a metric space and $A \subseteq X$ be a compact subset.

(1) A is bounded: Fix any point $x \in X$ and consider the open sets $B(x, n)$ as n varies over $\mathbb{N}$. Given any other point $y \in X$, y is in $B(x, n)$ as soon as $n > d(x, y)$. So the $B(x, n)$ are an open cover of all of X, hence of A. Since A is compact there is a finite subcover $\{B(x, n_i) | i = 1, 2, .., k\}$ of A. But then $A \subseteq B(x, r)$ for any r which is greater than all the n_i. So by the definition of boundedness, A is bounded.

(2)A is closed. Choose a point x in the complement of A. We must show that there is an $r > 0$ such that $B(x, r) \cap A = \emptyset$. For each $n \in \mathbb{N}$alet $U(n) = \{y \in X | d(x, y) > \frac{1}{n}\}$. Then for any $a \in A$, $a \in U(n)$ as soon as $n > \frac{1}{d(x,y)}$. Hence the $U(n)$ form an open cover of A so by compactness there is a finite subcover $U(n_1), U(n_2), ..., U(n_k)$ and for any $r < \frac{1}{n_i}$ for all $i = 1, 2, ..., k$ we have $B(x, r) \cap U(n_i) = \emptyset$ for $i = 1, 2, .., k$ so that $B(x, r) \cap A = \emptyset$ as required.

3 True/False. Write T or F next to question. A correct answer is worth 4.

3.1. *Any finite metric space is complete. True, any Cauchy sequence must eventually stay on one point.*

3.2. *A compact subset of $\mathbb{Z}$ (with the metric inherited from $\mathbb{R}$) is finite. True, if the subset were infinite, there is no finite subcover of the open cover formed by the singletons of the set.*

3.3. *A continuous function from a connected metric space to $\mathbb{R}$ is uniformly continous. False. See example below.*

3.4. *Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 + y^2 + z^2$. Then $f^{-1}([1, 2])$ is compact. True, it's closed since f is continous. And bounded since it is contained in the closed ball of radius $\sqrt{2}$ centered at the origin.*

3.5. *Suppose the sequence* (s_n) *has a subsequence* (s_{n_k}) *with* $\lim_{k\to\infty} s_{n_k} = 1$. *Then* $\liminf_{n\to\infty} s_n = 1$. *False. The sequence can do anything it likes off the subsequence.*

3.6. *The intersection of two compact subsets of a metric space is compact. True, the intersection is a closed subset of a compact set, hence compact.*

3.7. *There is a sequence of elements of the Cantor set whose limit is not in the Cantor set. No, the Cantor set is closed.*

3.8. *The product of three irrational numbers is irrational. False-no way.*

3.9. *The set of all sequences with values in* $\mathbb{Q}$ *is countable. False, just the sequences with values in* $\{0, 1\}$ *are uncountable.*

3.10. *Let* f *and* g *be continuous functions from one metric space to another. If* $f(x) \neq g(x)$ *then there is a neighborhood* V *of* x *with* $f(a) \neq g(a)$ *for* $a \in V$. *True. By restricting* a *to be close to* x *we can make* $f(a)$ *close to* $f(x)$ *and* $g(a)$ *close to* $g(x)$.

4 50 points

Give examples of the following (just the example, no need to prove anything).
(1) An uncountbale metric space with countably many connected components.
$\mathbb{R} \setminus \mathbb{Z}$
(2) A function that is continuous but not uniformly continuous.
$\frac{1}{x}$ from $\mathbb{R}^+$ to $\mathbb{R}^+$.
(3) A sequence of real numbers with $\liminf = 1$ and $\limsup = 1.23456789$.
$s_n = 1$ when n is odd and 1.23456789 when n is even.
(4) A closed bounded subset of $\mathbb{Q}$ that is not compact.
$[0, 1] \cap \mathbb{Q}$.
(5) A compact uncountable subset of $\mathbb{R}^5$.
$\{(x_1, x_2, x_3, x_4, x_5) | x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 1\}$.