Homework due Thursday 10 March 2011

1) Let A be an infinite set and F be a finite set. Show that A has the same cardinality as $A \cup F$.

2) Let (X, d) and (Y, D) be metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if

$$(\forall \epsilon > 0)(\exists \delta > 0) \text{ such that } d(x, y) < \delta \implies D(f(x), f(y)) < \epsilon$$

Show that a uniformly continuous function maps Caucy sequences to Cauchy sequences. Show that $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[1, \infty)$ but not on $(0, 1)$.

3) Let A be a dense subset of the metric space (X, d) and $f : A \to Y$ be a uniformly continuous function to the *complete* metric space (Y, D). Show that f admits a unique continuous extension to X.

4) You showed that any open subset U of $\mathbb{R}$ is a disjoint union of open intervals. Deduce that U is a countable union of disjoint open intervals.

5)Exhibit two open covers of $[0, 1)$, one of which has a finite subcover and one of which does not. Exhibit an open cover of $\mathbb{Q} \cap [0, 1]$ which has no finite subcover, and prove that it has no finite subcover.