Homework due Thursday 3 March 2011

1) If (x_n) and (y_n) are two Cauchy sequences in a metric space (X, d), define $d((x_n), (y_n)) = \lim d(x_n, y_n)$. Is this a metric on the set of all Cauchy sequences? (Explain your answer.)

2) Define $\sim$ on the set of all sequences in a metric space (X, d) by

$$(x_n) \sim (y_n) \iff d(x_n, y_n) \to 0.$$

Show that exercise 1 defines a metric on the set $\hat{X}$ of equivalence classes of Cauchy sequences. For each $x \in X$ let $[(x)]$ denote the equivalence class of the constant sequence $x_n = x$. Observe that $x \mapsto [(x)]$ defines an isometric embedding of X into $\hat{X}$ so we may think of X as a sub metric space of $\hat{X}$. Show that X is dense in $\hat{X}$.

3) Two metrics d and D on a set X are said to be *equivalent* if there are constants k and K such that $d(x, y) \leq kD(x, y) \forall x, y \in X$ and $D(x, y) \leq Kd(x, y) \forall x, y \in X$.

Show that the metrics defined by the three norms $|| - ||_2, || - ||_1$ and $|| - ||_\infty$ on $\mathbb{R}^n$ are equivalent.

Show that equivalent metrics define the same topology.

4)Do exercises 17.4. and 17.5 using the $\epsilon - \delta$ definition of continuity.

5) Let (c_n) be an arbitrary sequence of 0's and 2's. Show that for each k, $\sum_{i=1}^{k} \frac{c_i}{3^i}$ is in the Cantor set, hence $\lim_{k\to\infty}(\sum_{i=1}^{k} \frac{c_i}{3^i})$ is in the Cantor set. If (c_n) and (d_n) are two such sequences, when is $\lim_{k\to\infty}(\sum_{i=1}^{k} \frac{c_i}{3^i}) = \lim_{k\to\infty}(\sum_{i=1}^{k} \frac{d_i}{3^i})$? Deduce that the Cantor set is uncountable.