Homework due Tuesday 8 Feb.

1) Let $\mathbb{R}(x)$ be the field of rational functions (with real coefficients) in the variable x. Let $\mathcal{P}$ be the set of elements of $\mathbb{R}(x)$ of the form $\frac{P(x)}{Q(x)}$ where P and Q are polynomials such that the coefficient of the highest power of x is positive. Show that $\mathcal{P}$ satisfies the properties required to make $\mathbb{R}(x)$ into an ordered field.

Show that $\mathbb{R}(x)$ is not Archimedean, hence not complete.

We take the definition of convergence of sequences in an ordered field F to be as for $\mathbb{R}$:

$$s_n \to s \iff (\forall \epsilon \in F, \epsilon > 0)(\exists N \in \mathbb{N}) \text{ such that } (n \geq N) \implies (|s_n - s| < \epsilon)$$

Find a sequence of non-zero elements of $\mathbb{R}(x)$ that tends to zero (and prove that it does).

Exhibit a bounded monotonic sequence in $\mathbb{R}(x)$ that does not converge (and prove that it doesn't).

2) Show that in $\mathbb{R}$, and for any $n \in \mathbb{N}$, any positive element has a unique positive nth. root.

3)Show that the sum of two Cauchy sequences is Cauchy. Same with the product.

4) Do 12.3, 12.4 and 12.10