FINAL EXAM, MATH 104. 2006

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Time allowed: 3 hours. Total points 400

1. a) We have seen that for every positive $x \in \mathbb{R}$ there is a unique positive number $\sqrt{x}$ satisfying $(\sqrt{x})^2 = x$. Prove that the function defined on the positive reals by $f(x) = \sqrt{x}$ is continuous. (Assume that $\sqrt{x}$ exists, do not prove it.)

b) Is the funciton $\sqrt{x}$ in part a) uniformly continuous?

2. Define the terms "metric space" and "Cauchy sequence". Give an example of a metric space, and a Cauchy sequence in it which does not converge. Prove that your sequence is Cauchy and that it does not converge.

3. Let (M, d) be a metric space, $f : M :\to \mathbb{R}$ be a function and y be a point in M. Give careful definitions of the following.

a) $\lim_{x\to y} f(x) = L$ for some real number L.

b) $\lim_{x\to y} f(x) = -\infty$.

Now let $g : \mathbb{R} \to M$ be given. Carefully define

a) g is continuous at $x = 0$.

b) $\lim_{x\to+\infty} g(x) = y$

NOTE: "Careful" does not mean long-winded. In each case your definition should be a brief mathematical sentence.

4. Suppose $f : [0, \infty) \to \mathbb{R}^+$ is uniformly continuous and suppose that

$$\lim_{b\to\infty} \int_0^b f(x)dx$$

exists and is finite. Prove that $\lim_{x\to\infty} f(x) = 0$.

5. Prove using the open cover definition of compactness that a compact subset of a metric space is closed and bounded.

6. Give an example of a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ that is zero for $|x| \geq 1$, $\int_{-\infty}^{\infty} f(x)dx = 0$ but which is not identically zero.

7. Give an example of an uncountable nowhere dense subset of $[0, 1]$ and prove that it is nowhere dense.

8. Let $f : M \to N$ be a continuous one-to-one surjective map. Prove that f is a homeomorphism if M is compact and give a couneterexamle to the assertion that f is a homeomorphism if N is compact.

9. Prove that $\mathbb{R}$ is uncountable.

10. Let X be a subset of $\mathbb{R}$. Suppose y is an upper bound for X. Show that y is the least upper bound for X iff $\forall \epsilon > 0$ there is an $x \in X$ for which $x > y - \epsilon$.

11. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the upper and lower Darboux sums for f and give Riemann's criterion for integrability in terms of them (without proof). Using this or otherwise show that if f is continous then it is Riemann integrable.

12. Give an example of a compact countable infinite subset of $\mathbb{R}$.

13. A metric space is called "countably compact" if every open cover has a countable subcover. Show that $\mathbb{R}$ is countably compact.