Two sets A and B are said to have the *same cardinality* or be *equipotent* if there is a bijection $f : A \to B$. We then write $|A| = |B|$ or $\#(A) = \#(B)$.

A set is finite if it has the same cardinality as $\{1, 2, \cdots, n\}$ for some n.

Two finite sets have the same cardinality iff they have the same number of elements.

A set is said to be *infinite* if it is not finite.

A set is said to be *countable* if it is finite or has the same cardinality as $\mathbb{N}$. A bijection from $\mathbb{N}$ to that set is then a sequence which enumerates the set. A set is *countably infinite* if it is countable and infinite.

Comparing sizes of sets:

We say the cardinality of A is $\leq$ that of B ($|A| \leq |B|$ or $A \precsim B$) if there is an injective map from A to B.

Proposition 0.1. *$A \precsim B \iff$ there is a surjection $f : B \to A$.*

Proof. ($\Rightarrow$)If $g : A \to B$ is an injection, just define f to be g^{-1} on $f(A)$ and do anything on the rest of B.
($\Leftarrow$)For each $a \in A$ choose an $x \in B$ with $f(x) = a$ and define $g(a)$ to be x. This is clearly an injection. $\square$

Proposition 0.2. *A subset of a countable set is countable.*

Proof. If the set is finite this is obvious. If it is infinite, up to a bijection it is $\mathbb{N}$ so it's enough to prove that a subset of $\mathbb{N}$ is countable. For this just make the elements into a sequence by choosing the smallest one in the set, then the next smallest and so on. $\square$

Proposition 0.3. *$\mathbb{Z}$ is countable, $\mathbb{Z} \times \mathbb{Z}$ is countable, $\mathbb{Q}$ is countable.*

Proof.
$\mathbb{Z}$: all we have to do is arrange the elements of $\mathbb{Z}$ in a sequence. How about $(0, 1, -1, 2, -2, 3, -3, ...)$ or explicitly $c_1 = 0$ and for $n \geq 1, n \in \mathbb{N}$, $c_{2n+1} = n, \quad c_{2n} = -n$.

$\mathbb{Z} \times \mathbb{Z}$: By the previous argument it suffices to show that $\mathbb{N} \times \mathbb{N}$ is countable. But arrange the ordered pairs (m, n) as a matrix and start counting along northeast to southwest diagonals. Thus the sequence will begin: $(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), ...$ etc. It is a good exercise to give an explicit formula for the sequence.
This is by no means the only way to do it. Another way would be to define an injection from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{N}$ by sending (m, n) to $2^m 3^n$. By the unique factorisation of integers into products of primes, this is an injection which shows $\mathbb{N} \times \mathbb{N} \precsim \mathbb{N}$.

$\mathbb{Q}$: We can define a surjection from $\mathbb{Z} \times \mathbb{N}$ to $\mathbb{Q}$ by sending (m, n) to $\frac{m}{n}$. $\square$

Definition 0.1. *Let $2^{\mathbb{N}}$ denote the set of all sequences of zeros and ones. This is the same as the* power set *of $\mathbb{N}$, i.e. the set of all subsets of $\mathbb{N}$.*

Theorem 0.1. *$2^{\mathbb{N}}$ is uncountable.*

Proof. (Cantor's diagonal argument.)
Suppose you could list all such sequences as a sequence: write them $(a_n)_1, (a_n)_2, \ldots$. Call the nth term of the mth sequence $a_{m,n}$. Then define the sequence

$$b_n = a_{n,n} + 1 \mod 2$$

. This is again a sequence of 0's and 1's which differs from (a_n) (in the nth term) for every n so it's not on the list!

□

Corollary 0.1. *$\mathbb{R}$ is uncountable.*

Proof. By a result above, it suffices to show that $\mathbb{R}$ contains an uncountable subset. For this define for every sequence (a_n) as in the theorem, the element

$$\sum_1^\infty a_n 10^{-n}$$

(Strictly speaking we have not considered infinite sums yet but from 1B we know how to define them - they are the limits of the partial sums, which form a Cauchy sequence so converge.)

It suffices to show that $\sum_1^\infty a_n 10^{-n} = \sum_1^\infty b_n 10^{-n}$ (when the b_n are also 0's and 1's) implies $a_n = b_n$ for all n. To see this let's first look at the case $n = 1$. If a_1 were different from b_1 then we would have $\frac{a_1 - b_1}{10} = \sum_2^\infty (b_n - a_n)10^{-n}$. But the left hand side has absolute value $\frac{1}{10}$ while the right hand side has absolute value $\leq \sum_2^\infty \frac{1}{10^n}$ which is clearly impossible by summing the geometric series. (Note that the manipulations we have done on series are trivial consequences on limit theorems applied to the partial sums, e.g the triangle inequality $|\sum_0^\infty c_n| \leq \sum_0^\infty |c_n|$.) In general we may replace 1 in this argument by the smallest n for which $a_n \neq b_n$. □

Up to worries about considering the set of all sets we can consider the equality of the cardinality of two sets as defining an equivalence relation on sets. We would then like to show that these equivalence classes are ordered by $\precsim$. The key thing here is that $A \precsim B$ and $B \precsim A$ must imply $|A| = |B|$. This result is known as the Schroder-Bernstein theorem.

Theorem 0.2. *If $A \precsim B$ and $B \precsim A$ then $|A| = |B|$.*

Proof. Guest lecture. $\square$

Finally we would like to show that $\mathbb{R}$ has the same cardinality as $2^{\mathbb{N}}$.

Proposition 0.4. $|\mathbb{R}| \precsim |(0,1)|$.

Proof. Consider the function

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

Clearly f is defined for all $x \in \mathbb{R}$ and takes its values in $(-1,1)$. To see that it is injective suppose $\frac{x}{\sqrt{1+x^2}} = \frac{y}{\sqrt{1+y^2}}$. Squaring and simplifying we get $x = \pm y$. But $f(x) \neq f(-x)$ for $x \neq 0$. Hence f is injective. Moreover if $y \in (-1,1)$ then $\frac{y}{\sqrt{1-y^2}}$ is a real number whose image under f is y. $\square$

Proposition 0.5. $(0,1) \precsim 2^{\mathbb{N}}$

Proof. Define the map which sends a sequence (a_n) to

$$\sum_{n=1}^{\infty} a_n 2^n$$

This is just the binary expansion of a real number in base 2 so necessarily onto $[0,1]$. $\square$

Combining the above results and the Schroder Bernstein we see that $\mathbb{R}$ has the same cardinality as $2^{\mathbb{N}}$. The *continuum hypothesis* asserts that there is no set whose cardinality is strictly in between that of $\mathbb{N}$ and that of $\mathbb{R}$.