# The Schroeder-Bernstein Theorem

Suppose

$$H : Z \to Z$$

is a 1-to-1 function. For each $a \in Z$, the orbit of $a$ is the smallest subset of $Z$ which is closed under $H$ and which contains the point $a$. The next three easy lemmas refer to $H$ and $Z$. For each $a \in Z$, we let $O_a$ denote the orbit given by $a$.

**Lemma 1.** *Suppose $a, b \in Z$ and let $O_a$ and $O_b$ be the orbits given by $a$ and $b$ respectively. Then either $O_a = O_b$ or $O_a \cap O_b = \emptyset$.*

*Proof.* Exercise. □

**Definition 2.** Let $O_a$ be the orbit given by $a$. A point $b \in O_a$ is an Eve point if $b$ is not in the range of $H$.

An orbit cannot have two different Eve points.

**Lemma 3.** *Suppose $a \in Z$ and $O_a$ is the orbit given by $a$. If $O_a$ has an Eve point then that point is unique.*

*Proof.* Exercise. □

**Lemma 4.** *Suppose $a \in Z$ and $O_a$ is the orbit of $a$ given by $H$. Let $H_a$ be the restriction of $H_a$ to $O_a$ so that*

$$H_a : O_a \to O_a$$

*Then $O_a$ has no Eve point if and only if $H_a$ is a bijection from $O_a$ to $O_a$.*

*Proof.* Exercise. □

## Example

(1) Let $Z = \mathbb{R}$ and let $H$ be the function $H(x) = x^3$. Find all the $a \in \mathbb{R}$ such that $O_a$ is finite. Does every orbit have an Eve point?

(2) Let $Z = [0, \infty)$ and let $H$ be the function $H(x) = x^2$. Does every orbit have an Eve point? Find all the finite orbits.

(3) Let $Z = [0, \infty)$ and let $H$ be the function $H(x) = e^x$. Is any orbit finite? Does any orbit have an Eve point? □

**Theorem 5** (Schroeder-Bernstein). *Suppose that* $|X| \leq |Y|$ *and* $|Y| \leq |X|$. *Then* $|X| = |Y|$.

*Proof.* Let

$$f : X \to Y$$

and

$$g : Y \to X$$

be 1-to-1 functions. We must produce a bijection,

$$h : X \to Y.$$

Let $F : X \to X$ be the function $g^{-1} \circ f$ and let $G : Y \to Y$ be the function $f^{-1} \circ g$. Thus $F$ is a 1-to-1 function from $X$ to $X$ and $G$ is a 1-to-1 function from $Y$ to $Y$.

For each $a \in X$, let $O_a^X$ be the orbit of $a$ given by $F$ and for each $b \in Y$, let $O_b^Y$ be the orbit of $b$ given by $G$,

Note:

(1.1) For each $a \in X$, for each $c \in O_a^X$, $f(c) \in O_{f(a)}^Y$.

(1.2) For each $b \in Y$, for each $d \in O_b^Y$, $g(c) \in O_{g(b)}^X$.

Note:

- Suppose $a \in X$. Then $O_a^X$ has no Eve point if and only if $O_{f(a)}^Y$ has no Eve point.

Similarly,

- Suppose $b \in Y$. Then $O_b^Y$ has no Eve point if and only if $O_{g(b)}^X$ has no Eve point.

Fix $a \in X$. There are essentially three key observations (verify them!).

(I) Suppose $O_a^X$ has no Eve point. Then the restriction of $f$ to the orbit $O_a^X$ is bijection with the orbit $O_{f(a)}^Y$.

The next two key observations concern the situation where $O_a^X$ has an Eve point. Let $c$ be the (unique) Eve point of $O_a^X$.

(II) If $c$ is in the range of $g$ then $g^{-1}$ gives a bijection of $O_a^X$ with $O_{f(a)}^Y$.

(III) If $c$ is not in the range of $g$ then $f$ gives a bijection of $O_a^X$ with $O_{f(a)}^Y$.

Now we can easily define the desired bijection

$$h : X \to Y$$

by defining $h$ on each orbit $O_a^X$ where $a \in X$.

(4.1) If $O_a^X$ has no Eve point then for each $d \in O_a^X$, $h(d) = f(d)$.

(4.2) If $O_a^X$ has an Eve point and that Eve point is in the range of $g$ then for each $d \in O_a^X$, $h(d) = g^{-1}(d)$.

(4.3) If $O_a^X$ has an Eve point and that Eve point is not in the range of $g$ then for each $d \in O_a^X$, $h(d) = f(d)$.

It follows that $h$ is a bijection from $X$ to $Y$ (show this!). $\square$