THE WAVE MAPS EQUATION

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Abstract. The wave maps equation has become a very popular topic in recent years. The aim of these expository notes is to present a non-technical survey of the ideas and methods which have proved useful in the study of wave maps, leading up to the latest results. The remaining open problems are also stated and explained.

1. The equations

Let us begin with the Laplace equation,

$$-\Delta\phi = 0, \qquad \phi : \mathbb{R}^n \to \mathbb{R}$$

Its solutions are called harmonic functions and can be thought of as critical points for the Lagrangian

$$L^e(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi(x)|^2 dx.$$

The same equation is obtained if we look at vector valued functions, i.e. if we replace the target space $\mathbb{R}$ by $\mathbb{R}^m$.

However, the problem becomes considerably more interesting if instead of $\mathbb{R}^m$ we consider a Riemannian manifold (M, g). If

$$\phi : \mathbb{R}^n \to M$$

then its first order derivatives take values in the tangent space of M,

$$\partial_\alpha\phi : \mathbb{R}^n \to T_\phi M, \qquad \alpha = \overline{1, n}$$

Thus they are sections of the pull-back bundle

$$\phi^*(TM) = \cup_{x\in\mathbb{R}^n}\{x\} \times T_{\phi(x)}M$$

To measure the size of $\nabla\phi$ it is natural to use the metric g, therefore the modified Lagrangian has the form

$$L^e_M(\phi) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi(x)|^2_g dx$$

In order to describe the corresponding Euler-Lagrange equations we need to use covariant differentiation. To the metric g we associate the natural covariant differentiation on TM defined by its Levi-Civita connection. This induces a connection $\mathbf{D}$ on the pull-back bundle $\phi^*(TM)$. If V is a section of $\phi^*(TM)$ then we set

$$\mathbf{D}_X V = (\nabla_{\phi_* X} V)$$

With this notation, the Euler-Lagrange equations have the form

$$-\mathbf{D}_\alpha \partial_\alpha \phi = 0 \tag{1}$$

with the usual summation convention. To understand the type of this equation it is useful to write it in local coordinates on M. Then it has the form

$$-\Delta \phi^i = \Gamma^i_{jk}(\phi)\partial_\alpha \phi^j \partial_\alpha \phi^k$$

where Γ is the Riemann-Christoffel symbol on M. This equation is semilinear elliptic, and it is called the *harmonic maps equation.* Its solutions are called *harmonic maps.* The harmonic maps have their own story (see [8] and references therein) and at some point also had an impact on the study of wave maps.

We now switch to the wave equation,

$$\Box \phi = 0, \qquad \phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

where $\Box$ is the d'Allembertian, $\Box = \partial_t^2 - \Delta_x$. On the Minkowski space $\mathbb{R} \times \mathbb{R}^n$ we have the pseudo-Riemannian metric

$$(ds)^2 = -(dt)^2 + (dx)^2$$

Lifting indices with respect to this metric we can rewrite the wave equation in the form

$$-\partial^\alpha \partial_\alpha \phi = 0$$

where $\partial^t = -\partial_t$ and $\partial^x = \partial_x$.

We can also interpret the wave equation as the Euler-Lagrange equation for the Lagrangian

$$L^h(u) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}^n} -|\partial_t \phi|^2 + |\nabla_x \phi|^2 dtdx$$

but this is far less useful than in the elliptic case because it lacks coercivity. If instead we consider functions

$$\phi : \mathbb{R} \times \mathbb{R}^n \to M$$

then the corresponding Lagrangian has the form

$$L^h_M(\phi) = \int_{\mathbb{R}\times\mathbb{R}^n} -|\partial_t \phi|_g^2 + |\nabla_x \phi|_g^2 \; dtdx$$

Its Euler-Lagrange equation has the form

$$\mathbf{D}^\alpha \partial_\alpha \phi = 0 \tag{2}$$

and is called the *wave-maps equation*. In local coordinates it has the form

$$\Box\phi^i + \Gamma^i_{jk}(\phi)\partial^\alpha\phi^j\,\partial_\alpha\phi^k = 0$$

where Γ^i_{jk} are the usual Christoffel symbols. This can only be used as long as ϕ is continuous. Still, it shows that the wave-maps equation is essentially a semilinear wave equation.

In the analysis of nonlinear problems one often considers the linearized equations, which in our case can be easily computed:

$$\mathbf{D}^\alpha\mathbf{D}_\alpha\psi + R(\psi, \partial^\alpha\phi)\partial_\alpha\phi = 0 \tag{3}$$

where R is the Riemann curvature tensor. In local coordinates this is similar to the above equation for ϕ, only somewhat longer.

A special case of the wave-maps equation is when M is a submanifold of $\mathbb{R}^m$ with the euclidean metric. Then the second fundamental form plays a role. In our case it is convenient to interpret it as a symmetric quadratic form on the tangent space with values into the normal space,

$$S_p : TM \times TM \to NM, \qquad p \in M$$

$$\langle S(X,Y), N\rangle = \langle \partial_X Y, N\rangle$$

Then the wave-maps equation has the form

$$\Box\phi = -S_\phi(\partial^\alpha\phi, \partial_\alpha\phi)$$

In particular if M is a sphere, $M = S^{n-1}$, then the second fundamental form is

$$S_\phi(X,Y) = -\phi\langle X, Y\rangle$$

and the wave maps equation becomes

$$\Box\phi = \phi\langle\partial^\alpha\phi, \partial_\alpha\phi\rangle$$

Another interesting special case is when M is the hyperbolic space $\mathbb{H}^m$. We can think of $\mathbb{H}^m$ as the space-like hyperboloid

$$\phi_0^2 = 1 + \phi_1^2 + \cdots + \phi_m^2$$

in the Minkowski space $\mathbb{R} \times \mathbb{R}^m$ with the Lorentzian metric

$$(ds)^2 = -(dt)^2 + (dx_1)^2 + \cdots + (dx_n)^2$$

On the tangent space $T\mathbb{H}^m$ we use the induced inner product

$$\langle X, Y\rangle_L = -X_0Y_0 + X_1Y_1 + \cdots + X_mY_m$$

The second fundamental form is

$$S_\phi(X,Y) = \phi\langle X, Y\rangle_L$$

and the wave maps equation becomes

$$\Box\phi = -\phi\langle\partial^\alpha\phi, \partial_\alpha\phi\rangle_L$$

Finally, let us turn our attention to a special class of solutions. Let M_1 be a submanifold of M and ask the following question. Are the wave maps into M_1 also wave maps into M ? It is not difficult to see that the answer is affirmative iff M_1 is a totally geodesic submanifold of M, i.e. the M_1 geodesics are also M geodesics.

A special case when this happens is if M_1 is a geodesic γ of M. Then γ has one dimension and no curvature. Hence, with respect to the arclength parametrization, the wave maps equation into γ is nothing but the linear wave equation. Thus for any target manifold M we have at our disposal a large supply of wave maps associated to the geodesics of M.

2. The problems.

Consider the Cauchy problem for the wave-maps equation

$$\begin{cases} \mathbf{D}^\alpha \partial_\alpha \phi = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \\ \phi(0,x) = \phi_0(x), \ \partial_t(0,x) = \phi_1(x) & \text{in } \mathbb{R}^n \end{cases} \tag{4}$$

The initial data (ϕ_0, ϕ_1) must be chosen so that

$$\phi_0(x) \in M, \quad \phi_1(x) \in T_{\phi_0(x)} M, \quad x \in \mathbb{R}^n$$

We begin with two useful observations:

1. Energy. The wave maps equation has a conserved energy, namely the quadratic functional

$$E(\phi) = \frac{1}{2} \int_{\mathbb{R}^n} |\partial_t \phi|_g^2 + |\partial_x \phi|_g^2 dx$$

2. Scaling. The wave maps equation is invariant with respect to the dimensionless scaling

$$\phi(t,x) \to \phi(\lambda t, \lambda x) \qquad \lambda \in \mathbb{R}$$

Note however that the energy is scale invariant only in dimension $n = 2$.

2.1. Local well-posedness.

The natural problem is to consider initial data in Sobolev spaces

$$(\phi_0, \phi_1) \in H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$$

and to seek a solution

$$\phi \in C([-T,T]; H^s(\mathbb{R}^n)), \qquad \partial_t \phi \in C([-T,T]; H^{s-1}(\mathbb{R}^n))$$

with a lifespan T which depends on the initial data, or perhaps only on the size of the initial data. This is fairly easy to do if s is large enough, but it becomes increasingly difficult as s decreases.

For low s one has difficulties even with the definition of the Sobolev spaces. What is the meaning of $H^s(\mathbb{R}^n)$ for functions which take values into a manifold ? It is easier to answer this question for $s > \frac{n}{2}$. Then the H^s functions are continuous, therefore locally the image of ϕ is contained in the domain of a local map for M. Thus one can measure the regularity of ϕ using the corresponding local coordinates on M.

If $s \leq \frac{n}{2}$ then the problem becomes nonlocal, and the answer may depend on global properties of the manifold M. If for instance M embeds isometrically into $\mathbb{R}^m$ then one might use this to define the space of H^s functions with values in M for all $s \geq 0$. However, in this case one needs to ask whether these spaces depend on the isometric embedding or not.

Indeed, topological information is lost whenever $s < \frac{n}{2}$. Consider for instance the case when $M = S^n$ and ask how many times does an H^s function wrap around the sphere. As it was shown by Brezis-Nirenberg [2], this rotation number is well defined for $s = \frac{n}{2}$, but not for $s < \frac{n}{2}$.

Another piece of information comes from scaling. By rescaling the equation one can balance the size of the initial data and the lifespan of the solution. The initial data space[1] is scale invariant if $s = \frac{n}{2}$. Then one has the following relations concerning local well-posedness:

$$
\begin{array}{cccc}
s > \dfrac{n}{2} & \begin{array}{l}\text{small data}\\ \text{large time}\end{array} & \Leftrightarrow & \begin{array}{l}\text{large data}\\ \text{small time}\end{array} \\[2ex]
s = \dfrac{n}{2} & \begin{array}{l}\text{small data}\\ \text{small time}\end{array} & \Leftrightarrow & \begin{array}{l}\text{large data}\\ \text{small time}\end{array} \\[2ex]
s < \dfrac{n}{2} & \begin{array}{l}\text{small data}\\ \text{small time}\end{array} & \Leftrightarrow & \begin{array}{l}\text{large data}\\ \text{large time}\end{array}
\end{array}
$$

Given the above considerations, the first problem one is led to is

Problem 1. Prove local well-posedness for $s > \frac{n}{2}$.

As s decreases toward $\frac{n}{2}$ one gains better information concerning the lifespan of solutions. For $s = \frac{n}{2}$ a local result yields a global result, but one needs to distinguish between small and large data. Hence it makes sense to consider

Problem 2. Under reasonable assumptions on M prove global well-posedness for small data and $s = \frac{n}{2}$.

[1]In order to take advantage of scaling one should really use homogeneous Sobolev spaces, but this issue turns out not to be important unless $s = n/2$.

The need for assumptions on M stems from the fact that for $s = \frac{n}{2}$ the problem becomes nonlocal. For instance, using the special solutions contained on geodesics one easily sees that M needs to be geodesically complete. Likely this is not the only assumption which is needed.

An alternate venue which was pursued was to obtain weak global energy solutions using various penalization techniques, see for instance [20] [17], [6]. It would be interesting to understand whether in two dimensions these must coincide with the unique solutions obtained later in better classes of functions.

Finally, for $s < \frac{n}{2}$ a local result seems unreasonable except perhaps for very special cases. Thus one is led to consider the following:

Problem 3. Prove ill-posedness for $s < \frac{n}{2}$.

This was solved in [5] using the special solutions which are contained on geodesics, and will not be discussed further.

2.2. **Global solutions vs. blowup.** A local well-posedness result at scaling yields a global result for small data. However, the question of what happens for large data is of a different nature. Experience indicates that in general large data global well-posedness results can only be obtained by exploiting conserved or decreasing "energy" functionals.

For the wave maps equation the only conserved quantity that we know is the energy (and variations of it, e.g. momentum). Unfortunately in dimension $n \geq 3$ the energy is below scaling, and this essentially renders it useless. We call such problems *supercritical*:

Problem 4. Prove that large data solutions for the wave maps equation can blow up in finite time if $n \geq 3$.

Indeed, for a reasonably large class of manifolds M self-similar blowup solutions were constructed in [20],[23], [3]. It is however interesting to note that for positively curved target manifolds M (e.g. the sphere) such counterexamples have been obtained for all $n \geq 3$, while for negatively curved manifolds the current counterexamples only apply to dimensions $n \geq 7$.

The other easier case is in dimension $n = 1$, where the energy is above scaling. This problem is called *subcritical*:

Problem 5. Prove that large data solutions for the wave maps equation are global if $n = 1$.

This is true and was proved in [10].

It remains to discuss the most difficult case, $n = 2$. Here the energy is precisely at scaling, which makes it very difficult to exploit. We call this problem *critical*.

Suppose that M is a manifold for which the wave-maps equation is known to be well-posed for small data in the energy space $H^1 \times L^2$. Combining this with the finite speed of propagation, the large data problem reduces to a nonconcentration argument, which asserts that energy cannot focus at the tip of a light cone. Such nonconcentration arguments are known for other critical semilinear wave equations (see [21] and references therein), but none are available yet for wave maps.

Furthermore, numerical evidence in [9] and [1] indicates that the outcome seems to depend on the geometry of M. For the sphere blow-up appears to occur above a certain energy threshold, while for the hyperbolic space no such phenomena occurs.

Open Problem 6. Consider the wave maps equation in $2+1$ dimensions. Prove that

a) blow-up of large energy solutions occurs for certain target manifolds (e.g. the sphere).

b) large energy solutions are global for other target manifolds (e.g. the hyperbolic space).

Results of this type have only been previously obtained for special classes of solutions, see e.g. [4].

2.3. **The main result.** The rest of these notes is devoted to the following result:

Theorem 7. *Let $n \geq 2$. For all "reasonable" target manifolds M the wave maps equation is globally well-posed for initial data which is small in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$.*

This result was proved in a recent article [27] of the author, but this proof would not have been possible without earlier work in [12], [14], [29] and [26].

Here a "reasonable" target manifold is a manifold which admits an uniform isometric embedding into an Euclidean space $\mathbb{R}^m$. By Nash's theorem in [19] such an embedding exists for all smooth compact manifolds, and also for uniformly smooth unbounded manifolds with a positive injectivity radius.

In the next two sections we outline the evolution of ideas which led to the above result, and attempt to provide the reader with a road map for the proof. The actual proof is quite technical, and it would be hopeless to try to present it here. Hopefully, significant simplifications can be achieved given sufficient time.

3. Wave maps as a semilinear equation

Since in local coordinates the wave maps equation looks like a semilinear wave equation, it is natural to try to treat it as such. The usual approach is to consider the nonlinear term as a small perturbation of the linear operator. Suppose more generally that we are trying to solve a semilinear wave equation

$$\Box\phi = N(\phi), \qquad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1 \tag{5}$$

with initial data (ϕ_0, ϕ_1) in some Sobolev space X_0. Then the idea is to consider the nonlinear term as a small perturbation with respect to the linear equation and to absorb the nonlinearity in a fixed point argument.

We introduce the operators H and $\Box^{-1}$ which give the solution ϕ for the linear equation

$$\Box\phi = f, \qquad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1 \tag{6}$$

in the form

$$\phi = H(\phi_0, \phi_1) + \Box^{-1} f$$

There are two cases to consider:

(a) X_0 is above scaling. Then we seek some Banach spaces X for the solution ϕ and Y for the nonlinearity with the following properties:

(a1) (linear mapping property) The solution ϕ to (6) satisfies

$$\|\chi(t)\phi\|_X \lesssim \|(\phi_0, \phi_1)\|_{X_0} + \|f\|_Y$$

for any smooth compactly supported χ.

(a2) The truncated nonlinear term $\chi(t)N$ has the mapping property

$$\chi(t)N : X \to Y, \qquad \text{Lipschitz continuous}$$

If this is done we can use the contraction principle in X to solve the nonlinear problem

$$\phi = H(\phi_0, \phi_1) + \Box^{-1}(\chi(t)N(\phi))$$

If $\chi = 1$ in some time interval $[-T, T]$ then the function ϕ must also solve (5) in $[-T, T]$. Thus we obtain

(a3) For each initial data $(\phi_0, \phi_1) \in X_0$ there is some time T which depends only on the size of the initial data and a function ϕ in X which solves the equation in $[-T, T]$ and has a Lipschitz dependence on the initial data.

(b) X_0 is scale invariant. Then we seek some Banach spaces X for the solution ϕ and Y for the nonlinearity which are compatible with the scaling and have the following properties:

(b1) (linear mapping property) The solution ϕ to (6) satisfies

$$\|\phi\|_X \lesssim \|(\phi_0, \phi_1)\|_{X_0} + \|f\|_Y$$

(b2) The nonlinear term N maps

$$N : X \to Y \qquad \text{Lipschitz continuous}$$

If this is done then we recast the equation (5) in the form

$$\phi = H(\phi_0, \phi_1) + \Box^{-1} N(\phi)$$

which we solve in X using the contraction principle. This yields

(b3) For each small initial data $(\phi_0, \phi_1) \in X_0$ there is an unique global solution ϕ in X which solves the equation and depends smoothly on the initial data.

The small initial data is needed in order to gain the small Lipschitz constant in the fixed point argument. This is different than in problems above scaling, where the small Lipschitz constant is usually obtained by shortening the time interval $[-T, T]$.

Next we consider possible strategies for implementing this in the case of the wave maps equation.

3.1. **Energy estimates.** In this case the choice of spaces is made based on the energy estimates for the linear equation (6), namely

$$\|\nabla\phi\|_{L^\infty(H^{s-1})} \lesssim \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} + \|f\|_{L^1 H^{s-1}}$$

This corresponds to choosing

$$X_0 = H^s \times H^{s-1}, \quad X = C(H^s) \cap C^1 H^{s-1}, \quad Y = L^1 H^{s-1}$$

The nonlinear mapping property *(a2)* yields the constraint

$$s > \frac{n}{2} + 1$$

which is quite far from scaling.

3.2. **Strichartz estimates.** The nonlinear estimates can be improved if one takes advantage of the dispersive effect of the wave equation. A quantitative way of measuring that is provided by the Strichartz estimates. For solutions to (6) they yield

$$\|\nabla\phi\|_{L^\infty(H^{s-1})} + \|\nabla\phi\|_{L^4 L^\infty} \lesssim \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} + \|f\|_{L^1 H^{s-1}}$$

for $n = 2$, $s = \frac{5}{4}$, respectively

$$\|\nabla\phi\|_{L^\infty(H^{s-1})} + \|\nabla\phi\|_{L^2 L^\infty} \lesssim \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} + \|f\|_{L^1 H^{s-1}}$$

for[2] $n \geq 3$, $s = \frac{n+1}{2}$.

[2]This is false but almost true for $n = 3$.

Then we modify the choice of X to

$$X = \{\phi \in C(H^s), \nabla\phi \in C(H^{s-1}) \cap L^2L^\infty\}$$

and we gain the improved range

$$s \geq \frac{5}{4} \quad (n = 2), \qquad s > 2 \quad (n = 3), \qquad s \geq \frac{n+1}{2} \quad (n > 3).$$

3.3. **The null condition.** So far we have treated the wave-maps equation as a generic semilinear wave equation of the form

$$\Box\phi = (\nabla\phi)^2$$

Unfortunately, for such equations a counterexample of Lindblad [16] shows that the Strichartz estimates actually give the sharp result at least in low dimension ($n = 2, 3$). In this counterexample, blow-up occurs due to concentration along a light ray, or, in other words, due to nonlinear interaction of waves which travel in the same direction.

The first key insight into wave maps came from Klainerman who observed that the quadratic form which occurs in the wave maps equation is not generic, but instead has a special structure which he called the null condition. Exploiting this observation he was able to prove in [11] that for $n \geq 3$ solutions with small, smooth and compactly supported data are global. In two dimensions this was more difficult and the same result was proved later by Sideris [24].

To explain the meaning of the null condition it is useful to switch to the Fourier space. We denote by τ the time Fourier variable and by ξ the space Fourier variable. The symbol of the wave operator is

$$p(\tau, \xi) = \tau^2 - \xi^2$$

therefore solutions to the wave equation are concentrated in frequency near the characteristic cone

$$K = \{\tau^2 - \xi^2 = 0\}$$

A solution to the wave equation which is concentrated in frequency near some $(\tau, \xi) \in K$ travels roughly in the normal direction to the cone,

$$\nabla_{\tau,\xi}p(\tau, \xi) = (2\tau, -2\xi)$$

Consider now two waves $\phi^{(1)}$ and $\phi^{(2)}$ which are concentrated in frequency near $(\tau^{(1)}, \xi^{(1)}) \in K$, respectively $(\tau^{(2)}, \xi^{(2)}) \in K$.

Their product $\phi^{(1)}\phi^{(2)}$ has Fourier transform

$$\widehat{\phi^{(1)}\phi^{(2)}} = \int_{\mathbb{R}^{n+1}} \widehat{\phi^{(1)}}(s, \eta)\widehat{\phi^{(2)}}(\tau - s, \xi - \eta)ds\, d\eta$$

which is concentrated in frequency near

$$(\tau,\xi) = (\tau^{(1)} + \tau^{(2)}, \xi^{(1)} + \xi^{(2)})$$

If we consider the wave equation

$$\Box \phi = \phi^{(1)}\phi^{(2)}$$

the largest input comes from (τ,ξ) near the cone K. But the only way we can have both

$$(\tau,\xi) \in K, \quad (\tau^{(1)},\xi^{(1)}) \in K, \quad (\tau^{(2)},\xi^{(2)}) \in K$$

is if all three vectors are parallel. This computation indicates that for the wave equation with generic quadratic nonlinearities the worst nonlinear interaction is that between waves which travel in the same direction.

A partial physical space interpretation of the same phenomena is that waves which travel in different direction intersect on a smaller set therefore have less interaction.

Replace now the product $\phi^{(1)}\phi^{(2)}$ by the quadratic form arising in the wave maps equation,

$$Q_0(\phi^{(1)},\phi^{(2)}) = \partial^\alpha \phi^{(1)} \partial_\alpha \phi^{(2)}$$

Its Fourier transform is given by

$$\mathcal{F}Q_0(\phi^{(1)}\phi^{(2)}) = \int_{\mathbb{R}^{n+1}} q_0(s,\eta,\tau-s,\xi-\eta)\widehat{\phi^{(1)}}(s,\eta)\widehat{\phi^{(2)}}(\tau-s,\xi-\eta)ds\, d\eta$$

where the symbol q_0 of the quadratic form Q_0 is given by

$$q_0(\tau^{(1)},\xi^{(1)},\tau^{(2)},\xi^{(2)}) = \tau^{(1)}\tau^{(2)} - \xi^{(1)}\xi^{(2)}$$

The interesting observation is that q_0 vanishes if both $(\tau^{(1)},\xi^{(1)})$ and $(\tau^{(2)},\xi^{(2)}$ are on the cone and collinear. This is a consequence of the relation

$$2q_0(\tau^{(1)},\xi^{(1)},\tau^{(2)},\xi^{(2)}) = p(\tau^{(1)}+\tau^{(2)},\xi^{(1)}+\xi^{(2)}) - p(\tau^{(1)},\xi^{(1)}) - p(\tau^{(1)},\xi^{(1)})$$

Thus the worst kind of bilinear interaction cannot occur in the wave maps equation. Then we say that this nonlinearity satisfies the null condition.

3.4. **The $X^{s,b}$ spaces.** How can one choose the spaces X and Y in order to best take advantage of the null condition ? One answer to this is provided by the $X^{s,b}$ spaces introduced by Klainerman and Machedon. These spaces are associated to the wave equation essentially in the same way the H^s spaces are associated to the Laplacian. More precisely we define

$$\|u\|_{X^{s,b}} = \|(1+|\tau|+|\xi|)^s(1+||\tau|-|\xi||)^b \hat{u}(\tau,\xi)\|_{L^2}$$

The index s accounts for regular derivatives, while b corresponds to "wave" derivatives, i.e. powers of the wave operator.

It is easy to see that

$$\Box : X^{s,b} \to X^{s-1,b-1}$$

and that the wave equation parametrix $\Box^{-1}$ has the inverse mapping properties,

$$\Box^{-1} : X^{s-1,b-1}_{comp} \to X^{s,b}_{loc}$$

Furthermore, H^s solutions to the homogeneous wave equation are in $X^{s,b}$ for all b. Hence we seek spaces X, Y of the form

$$X = X^{s,b}, \qquad Y = X^{s-1,b-1}$$

The choice of b is related to scaling, and the optimal value is $b = \frac{1}{2}$.

Given the way the $X^{s,b}$ spaces are defined, the estimates *(a2)* for the nonlinearity reduce to weighted convolution estimates in the Fourier space. This venue was pursued by Klainerman-Machedon [12] who proved local well-posedness for the wave maps equation in $H^s \times H^{s-1}$ for all $s > \frac{n}{2}$, $n \geq 3$. The similar result in the more difficult case $n = 2$ was obtained shortly afterward by Klainerman-Selberg [14].

3.5. **Scale invariant results.** Can the same method be used in the scale invariant case $s = \frac{n}{2}$? Then the spaces X, Y should be compatible with scaling, which would appear to imply that we should use the homogeneous version of the $X^{s,b}$ spaces, namely

$$X = \dot{X}^{\frac{n}{2},\frac{1}{2}} = \{u; (|\tau| + |\xi|)^{\frac{n}{2}} ||\tau| - |\xi||^{\frac{1}{2}} \hat{u} \in L^2\}$$

$$Y = \dot{X}^{\frac{n}{2}-1,-\frac{1}{2}} = \{u; (|\tau| + |\xi|)^{\frac{n}{2}-1} ||\tau| - |\xi||^{-\frac{1}{2}} \hat{u} \in L^2\}$$

However, X is not well defined as a space of distributions, while Y does not even contain all the smooth compactly supported functions. Any naive attempt to fix one problem worsens the other.

Consider the nonlinear estimate one needs to prove:

$$f(X)\partial^\alpha X\, \partial_\alpha X \to Y$$

We can split this into bilinear pieces. Clearly X should be an algebra, and in addition two more multiplicative properties are required:

$$\begin{aligned} X \cdot X &\to X, \\ X \cdot Y &\to Y, \\ \partial^\alpha X \ \cdot \partial_\alpha X &\to Y \end{aligned} \tag{7}$$

A standard technique in harmonic analysis is to use a Littlewood-Paley decomposition to break up functions into dyadic pieces, i.e. which are frequency localized in dyadic regions. For $j \in \mathbb{Z}$ let us loosely denote by

X_j, respectively Y_j the spaces of functions in X, respectively Y which have Fourier transform supported in the region

$$\{|\xi| + |\tau| \in [2^{j-1}, 2^{j+1}]\}$$

The structure of X_j, Y_j is determined by that of X, Y, but the converse is also true. Indeed, on one hand, the rescaling must map one X^j into another. On the other hand, the square summability with respect to j is inherited from the initial data. Thus, we have

$$\|\phi\|_X^2 \approx \sum_{j\in\mathbb{Z}} \|P_j\phi\|_{X_j}^2 \tag{8}$$

where $P_j(D)$ are multipliers which select the frequency $\approx 2^j$ part of ϕ. A preliminary step in obtaining (7) is to prove their dyadic counterparts. When we multiply two frequency localized functions there are two possible types of interaction.

a) High-high interactions. If we multiply two functions of comparable frequency, the product is at the same frequency or lower. Thus we have dyadic estimates of the form

$$\begin{array}{c} P_j(X_k \cdot X_k) \to X_j, \\ P_j(X_k \cdot Y_k) \to Y_j, \\ P_j(\partial^\alpha X_k \cdot \partial_\alpha X_k) \to Y_j \end{array} \qquad j \leq k \tag{9}$$

b) High-low interactions. If we multiply a high frequency function with a low frequency one, the product is at high frequency. Hence the dyadic estimates are

$$\begin{array}{c} X_j \cdot X_k \to X_k, \\ X_j \cdot Y_k \to Y_k, \quad X_k \cdot Y_j \to Y_k \\ \partial^\alpha X_j \cdot \partial_\alpha X_k \to Y_k \end{array} \qquad j < k \tag{10}$$

Aside from the bilinear dyadic estimates, we also want to have the linear mapping property for the parametrix of the wave equation,

$$\Box^{-1} : Y_j \to X_j \tag{11}$$

If one attempts to produce function spaces X, Y which have all the desired properties, it soon becomes clear that there are two major hurdles to overcome.

i) The division problem, which is to find dyadic spaces X_j satisfying all the dyadic estimates in (9), (10) and (11). The name comes from the fact that in the Fourier space the parametrix $\Box^{-1}$ for the wave equation is essentially the division by $\tau^2 - \xi^2$. The function $(\tau^2 - \xi^2)^{-1}$ is not locally integrable and without considerable care this generates logarithmic divergences in the estimates.

ii) The summation problem, which is to somehow be able to sum up the dyadic estimates (9), (10) in order to obtain (7). A-priori, in both high-high and high-low interactions there is a logarithmic divergence in the summation with respect to the index k.

The division problem was solved in two consecutive articles of the author. The first one, [28], applies to high dimension ($n \geq 4$) and uses function spaces X, Y which are essentially modified $X^{s,b}$ spaces combined with Strichartz type norms. The second one [29] applies in low dimension and is much more involved. This time an essential part of the X and Y spaces consists of functions tied to rotating characteristic frames which are frequency localized in certain neighborhoods of sectors on the characteristic cone.

In that setup there is an additional gain of $2^{-\epsilon|j-k|}$ which insures the k summation for the high-high interactions. However, the k summation in the high-low interactions remains a problem. In [28], [29] this is avoided by replacing the l^2 dyadic summation in (8) with an l^1 summation. The main result has the form:

Theorem 8. *The wave maps equation is well-posed for initial data which is small in the homogeneous Besov space* $\dot{B}^{\frac{n}{2}}_{2,1} \times \dot{B}^{\frac{n}{2}-1}_{2,1}$.

On the positive side, the above Besov space is scale invariant, therefore the solutions which are obtained are global and depend smoothly on the initial data.

On the negative side, the Besov space is smaller than the corresponding Sobolev space. Due to the embedding $\dot{B}^{\frac{n}{2}}_{2,1} \subset L^\infty$ the solutions obtained are small in L^∞. Hence the problem becomes local with respect to the target manifold M, and the geometry of M does not come into the picture.

One might ask whether it is possible to also solve the summation problem by making a better choice of the function spaces X, Y. At the time the answer was not clear. However, we now know that this is not the case; indeed, the results in [5] show that there is no uniformly continuous dependence of the solutions on the initial data in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2}-1}$. Thus the semilinear approach to wave maps can go no further than this.

4. Wave maps as a nonlinear equation

The main idea of this section is that at scaling the wave maps equation behaves as a genuinely nonlinear wave equation. As in the semilinear case, we begin with a presentation of the common approach for nonlinear waves.

4.1. **Local solutions for the nonlinear wave equation.** Suppose that we have a nonlinear wave equation

$$P(\phi, \partial\phi, \partial^2\phi) = 0, \qquad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1 \tag{12}$$

Definition 9. The problem (12) is well-posed in $H^{s_0} \times H^{s_0-1}$ if:

For each $M > 0$ there is some $T > 0$ so that:

(i) *(a-priori bound for smooth solutions)* For each smooth initial data (ϕ_0, ϕ_1) with

$$\|(\phi_0, \phi_1)\|_{H^{s_0}\times H^{s_0-1}} \leq M \tag{13}$$

there is an unique smooth solution ϕ in $[-T, T]$ which satisfies uniform bounds

$$\|\phi\|_{C(-T,T;H^s)\cap C^1(-T,T;H^{s-1})} \leq C\|(\phi_0, \phi_1)\|_{H^s\times H^{s-1}}, \qquad s \geq s_0$$

with C depending only on M, s.

(ii) *(weak stability estimates)* There is *some* $s < s_0$ so that for any two smooth solutions ϕ, ψ whose initial data is subject to (13) we have

$$\|\phi - \psi\|_{C(-T,T;H^s)\cap C^1(-T,T;H^{s-1})} \leq C\|(\phi_0 - \psi_0, \phi_1 - \psi_1)\|_{H^s\times H^{s-1}}$$

for *some* $s < s_0$, with C depending only on M, s.

(iii) *(rough solutions as limits of smooth solutions)* For any initial data (ϕ_0, ϕ_1) which satisfies (13) there is a solution $\psi \in C(-T, T; H^{s_0})\cap C^1(-T, T; H^{s_0-1})$, depending continuously on the initial data, which can be obtained as the unique limit of smooth solutions.

Most notably one looses the Lipschitz dependence on the initial data, which implies that the solution can no longer be obtained via a fixed point argument. For the wave maps equation, because we are at scaling, there is also another twist: M must be small while the lifespan T must be infinite (arbitrarily large).

Note that if s_0 is above scaling then the smallness of M can be gained by rescaling. Consequently, in what follows we restrict ourselves to small data solutions.

To outline a standard proof of well-posedness we first introduce some notations. Begin with the initial data space, where we set

$$X_0 = H^{s_0} \times H^{s_0-1}, \qquad X_0^s = H^s \times H^{s-1}$$

Next we need some Banach space X which measures the regularity of $H^{s_0} \times H^{s_0-1}$ solutions. In effect it is better to have a family of spaces X_s which measure the regularity of $H^s \times H^{s-1}$ solutions. Usually these spaces are related by multipliers just like the Sobolev spaces,

$$X^s = |D|^{s_0-s}X$$

One might be tempted to set

$$X^s = C(-T,T;H^s) \cap C^1(-T,T;H^{s-1}),$$

but this is rarely enough. It is always convenient to take M small (which can be achieved by scaling in general, or by hypothesis for wave-maps).

A main difficulty in obtaining estimates for nonlinear equations is that on one hand in order to obtain bounds for the solution one needs bounds for the coefficients, while on the other hand the bounds for the coefficients follow from bounds for the solution.

A way out of this circular argument is provided by a bootstrap lemma of the form

Lemma 10. *a) For all smooth solution ϕ to (12) in $[-T,T]$ we have*

$$\left.\begin{array}{c} \|\phi[0]\|_{X_0} \ll 1 \\ \\ \|\phi\|_X \leq 2 \end{array}\right\} \implies \|\phi\|_X \leq 1$$

b) In addition for $s > s_0$ and $\alpha > 0$

$$\left.\begin{array}{c} \|\phi[0]\|_{X_0^s} \ll \alpha \\ \\ \|\phi\|_X \leq 2 \\ \\ \|\phi\|_{X^s} \leq 2\alpha \end{array}\right\} \implies \|\phi\|_{X^s} \leq \alpha$$

This is used in a continuity argument as follows. Let s be large and $\alpha > 0$ arbitrary. Given some smooth initial data (ϕ_0, ϕ_1) satisfying

$$\|\phi[0]\|_{X_0} \ll 1, \qquad \|\phi[0]\|_{X_0^s} \ll \alpha$$

we consider a smooth one parameter family of initial data sets

$$(\phi_0^h, \phi_1^h) \qquad h \in [0,1]$$

uniformly satisfying similar bounds with

$$(\phi_0^0, \phi_1^0) = (0,0), \qquad (\phi_0^1, \phi_1^1) = (\phi_0, \phi_1)$$

Here we assume for simplicity that 0 is a solution and that the linearization at 0 is the linear wave equation. Then for small h there is a smooth solution ϕ^h in $[-T,T]$ depending smoothly on h which satisfies

$$\|\phi^h\|_X \leq 2, \qquad \|\phi^h\|_{X_s} \leq 2\alpha \tag{14}$$

From the lemma we obtain

$$\|\phi^h\|_X \leq 1, \qquad \|\phi^h\|_{X_s} \leq \alpha$$

By continuity it follows that a smooth solution exists and satisfies (14) for a larger set of values of h. By the same token one argues that the set $A \subset [0,1]$ of those h for which a solution satisfying (14) is both open and closed in $[0,1]$. Hence $A = [0,1]$, therefore for $h = 1$ we obtain a solution with initial data (ϕ_0, ϕ_1).

For the weak stability part of the estimates we need to consider the linearized equations,

$$P^{lin}(\phi)\psi = 0, \qquad \psi(0) = \psi_0, \quad \partial_t \psi(0) = \psi_1 \tag{15}$$

around some solution ϕ. These equations have variable coefficients, and for them we need the next Lemma:

Lemma 11. *There is $s < s_0$ so that the linearized equations (15) are uniformly well-posed in $H^{s-1} \times H^s$ for all smooth solutions ϕ to (12) in $[-T, T]$ with $\|\phi[0]\|_{X_0} \ll 1$.*

Note that from the previous lemma we obtain that the solutions ϕ must be bounded in X, which provides us with some regularity for the coefficients of P^{lin}. A robust way to prove the lemma is to obtain an estimate of the form

$$\|\psi\|_{X^s} \lesssim \|(\psi_0, \psi_1)\|_{H^s \times H^{s-1}} \tag{16}$$

4.2. **The paradifferential calculus.** Since it is difficult to prove nonlinear estimates, one usually attempts to reduce them to linear estimates. A convenient tool which can be used for that is the paradifferential calculus. Given a Littlewood-Paley decomposition of a solution ϕ to (12), the basic principle is that we can transform the equation into an infinite system

$$P^{lin}(\phi_{<k})\phi_k = \text{error} \tag{17}$$

where $\phi_{<k}$, respectively ϕ_k loosely denote the part of ϕ which is at frequency less than 2^{k-10}, respectively $\approx 2^k$.

The term *"error"* means an acceptable error, i.e. which is small in an appropriate sense. To measure this we need Banach spaces Y, respectively Y_s so that the forward parametrix for $P^{lin}(\phi_{<k})$ has the linear mapping properties

$$[P^{lin}(\phi_{<k})]^{-1} : Y_k \to X_k, \qquad [P^{lin}(\phi_{<k})]^{-1} : Y_{s,k} \to X_{s,k}$$

One advantage in doing this is that the equations (17) are frequency localized. Thus the statements in Lemma 10(a), Lemma 10(b) and Lemma 11 become more or less equivalent after such a transformation.

Another advantage is that we peel off some less important parts of the nonlinearity, and retain the main contributions only. This amounts to saying that the low frequency contributions of the high-high frequency interactions is negligible.

4.3. **Wave maps.** Can such a strategy be implemented for the critical wave maps equation ? As it turns out, the answer is affirmative. This shows that although the wave maps equation has the form of a semilinear wave equation, its behavior at scaling is genuinely nonlinear.

The first step in this direction was carried out by Tao, initially in high dimension in [25] and then in low dimension [26]. He considered the case of a target which is a sphere, and proved only part (i) in Definition 9 of well-posedness (i.e. the wave maps counterpart of the bootstrap Lemma 10).

However, as it turns out, similar ideas can be used for more general target manifolds. Klainerman and Rodnianski [13] did this for $n \geq 5$, Shatah-Struwe [22] and Nahmod-Stefanov-Uhlenbeck [18] for $n \geq 4$ and Krieger [15] for the hyperbolic space and $n = 3$.

Finally, the result in low dimension ($n \geq 2$) for general target manifolds was proved in a very recent article of the author [27]. Parts (ii), (iii) in Definition 9 of nonlinear well-posedness are also obtained there.

To keep the exposition simple we begin the discussion below using Tao's set-up of the spherical target. Then we move on to general target manifolds and use the spherical case for comparison.

4.4. **The spaces.** The first step in the proof of Theorem 7 is to construct Banach spaces X, Y for the solutions, respectively the inhomogeneous terms in the wave equation. Just as in Section 3.5, we omit the description of these spaces because it is too technical.

The spaces used by Tao [26] and also later in [27] are variations of the spaces introduced in [29]. The Y space is essentially the same, but the X space is slightly enlarged in order to gain the key algebra property for $X \cap L^\infty$. This already removes the logarithmic divergence in the first relation in (7). However, the logarithmic divergence in the last two relations in(7) is genuinely nonlinear and cannot be removed. Instead, the best one can do is to use the paradifferential calculus.

We note that there seems to be considerable room for improvement in this functional setup. Hopefully, given enough time it is possible that a simpler framework will emerge.

4.5. **Paradifferential calculus and the trilinear estimate.** The paradifferential form of the wave maps equation into the sphere is

$$\Box \phi^i_k = 2\phi^i_{<k} \partial^\alpha \phi^j_{<k} \partial_\alpha \phi^j_k + \text{error}$$

However, this is not entirely satisfactory. Instead one uses the equation for the sphere to get $\phi^2 = 1$ which yields $\phi^j \partial_\alpha \phi^j = 0$. In paradifferential form this gives good control over the product $\phi^j_{<k} \partial_\alpha \phi^j_k$. Thus we replace the above equation with

$$\Box \phi^i_k = 2(\phi^i_{<k} \partial^\alpha \phi^j_{<k} - \phi^j_{<k} \partial^\alpha \phi^i_{<k}) \partial_\alpha \phi^j_k + \text{error} \tag{18}$$

In doing this, one obtains antisymmetric matrices

$$(A^\alpha_{<k})^{ij} = (\phi^i_{<k} \partial^\alpha \phi^j_{<k} - \phi^j_{<k} \partial^\alpha \phi^i_{<k})$$

as the coefficients of the first order term, which is crucial later on.

Passing from the original equation to its paradifferential form turns out to be quite nontrivial. In particular, it cannot be done using bilinear estimates; as it turns out, in addition one needs a trilinear estimate, which is central in Tao's work. To explain it let us begin with a consequence of the dyadic estimates (9), (10), namely

$$P_k \left(X_{k_1} \partial^\alpha X_{k_2} \partial_\alpha X_{k_3} \right) \to Y_k$$

The $2^{\epsilon|j-k|}$ gain in (9) already shows that there is a gain above as well unless we have a high-low-low type interaction, i.e. $k = \max\{k_1, k_2, k_3\}$. Tao goes one step further, and proves that there is an additional gain unless $k_1 < \min\{k_2, k_3\}$. This shows that the only nonremovable logarithmic divergence occurs when the undifferentiated term is the smallest in frequency.

The need for a trilinear estimate seems to be connected to the choice of the X and Y spaces. Hopefully, a more careful choice could lead to better bilinear estimates which would imply the trilinear one.

4.6. **The gauge transformation.** The right hand side of (18) cannot be treated as an error term. The idea used by Tao, inspired from similar work on harmonic maps (see Helein's book [7] and references therein), is to do a gauge transformation,

$$\phi_k \to U_{<k} \phi_k$$

for some $U_{<k}$ in $L^\infty \cap X$. Ideally $U_{<k}$ should be orthogonal and, in order to cancel the $A^\alpha_{<k}$ it should satisfy

$$U_{<k} A^\alpha_{<k} + \partial^\alpha U_{<k} = 0$$

This would imply that A^α's are antisymmetric and justifies in part the earlier choice of the paradifferential formulation of the wave maps equation.

However, the above system can be solved exactly only if the following compatibility conditions hold:

$$\partial^\alpha A^\beta_{<k} - \partial^\beta A^\alpha_{<k} = [A^\alpha_{<k}, A^\beta_{<k}]$$

which are not true in general. However we do have a good control over the curl of A so there is hope to find at least an approximate gauge transformation. This is obtained inductively using a paradifferential type setup,

$$U_k = -U_{<k}(\phi^i_{<k}\phi^j_k - \phi^j_{<k}u^i_k)$$

The main estimate in this context is

$$\Box(U_{<k}\phi_k) = \text{error} \tag{19}$$

This allows one to use linear estimates for $U_{<k}\phi_k$ and closes the loop.

4.7. **Embedded manifolds and Moser type estimates.** Assume now that the target manifold M is isometrically embedded in $\mathbb{R}^n$, which is the setup considered in [27]. Then the wave maps equation has the form

$$\Box\phi^i = -S^i_{jl}(\phi)(\partial^\alpha\phi^j, \partial_\alpha\phi^l) \tag{20}$$

The image of the second fundamental form is contained in the normal space of M, therefore we have the compatibility condition

$$S^i_{jl}(\phi)\partial^\alpha\phi^i = 0$$

Taking the two relations above into account, we arrive at a paradifferential formulation of (20) of the form

$$\Box\phi_k = -2A^\alpha_{<k}\partial_\alpha\phi_k + \text{error}$$

where $A^\alpha_{<k}$ are the antisymmetric matrices

$$(A^\alpha_{<k})_{ij} = ([S^i_{jl}(\phi)]_{<k} - [S^j_{il}(\phi)]_{<k})\partial_\alpha\phi^l_{<k} \tag{21}$$

The gauge transformation is defined inductively as

$$U_k = U_{<k}([S^i_{jl}(\phi)]_{<k} - [S^j_{il}(\phi)]_{<k})\phi^l_k$$

and the analysis is not very different from the case of the sphere. However, here are two new problems which we need to consider.

a) If the manifold is unbounded then the restriction $\phi \in L^\infty$ becomes unreasonable. The solution is to simply drop the boundedness assumption.

b) We need to bound the expression $S(\phi)$ in X in terms of ϕ in X. $X \cap L^\infty$ is an algebra so this is not a problem if S is analytic. However, this restriction is too severe. Instead, we need a Moser type estimate. The classical Moser estimates have the form

$$\|f(u)\|_{H^s} \leq c(\|u\|_{L^\infty})(1 + \|u\|_{H^s})$$

Its counterparts in our case, proved in [27], have the form

$$\|f(\phi)\|_X \leq c\|\phi\|_X(1 + \|\phi\|^N_X)$$

$$\|f(\phi)\|_{X^s} \leq c\|\phi\|_{X^s}(1+\|\phi\|_X^N)$$

Here f is a bounded function with sufficiently many bounded derivatives. In exchange, no boundedness condition is imposed on the functions in X. Also N is a large enough integer (probably 2 or 3 should do but this is not easy to verify).

To prove these Moser type estimates the usual paradifferential calculus is no longer good enough. Instead one needs to consider a family of multilinear paradifferential expansions for the nonlinear expression $f(\phi)$.

4.8. **The linearized equation.** Let ϕ be a wave map with values in M. Then a solution ψ to the linearized equations around ϕ takes values in the tangent bundle of M,

$$\psi(x) \in T_\phi(x)M \tag{22}$$

Since the wave maps equation is translation invariant, it follows that in particular the functions $\partial_\alpha\phi$ are solutions for the linearized equations. This is exploited in the higher dimensional approach in [13], [22] and [18].

One possibility at this point, followed for instance in [13], [22], is to express the linearized equations in a local basis in TM. However, making this global requires that the manifold M is parallelizable, an extra assumption which we want to avoid.

Instead, we simply linearize the equation (20),

$$\Box\psi^i = -2S^i_{jl}(\phi)(\partial^\alpha\phi^j, \partial_\alpha\psi^l) - (\partial_m S)^i_{jl}(\phi)(\partial^\alpha\phi^j, \partial_\alpha\phi^l)\psi^m \tag{23}$$

Next we write the appropriate paradifferential formulation. Recall that we measure ψ in a lower regularity space X^s with $s < n/2$. Consequently, every time a low frequency in ψ contributes to a higher frequency output in the nonlinearity, there is an additional gain compared to the case when $s = \frac{n}{2}$. Then in the error term we can include the second right hand side term entirely, and also the part of the first term where ψ does not have the highest frequency. Hence we arrive at

$$\Box\psi^i_k = -2S^i_{jl}(\phi)_{<k}(\partial^\alpha\phi^j_{<k}, \partial_\alpha\psi^l_k) + \text{ error}$$

On the other hand, the compatibility condition (22) yields

$$S^i_{jl}(\phi)\psi^i = 0$$

Differentiating this we conclude that the expression

$$S^i_{jl}(\phi)_{<k}\partial_\alpha\psi^i_k$$

should also be better behaved. This leads us to the paradifferential formulation of the linearized equations

$$\Box\psi = -2A^{\alpha}_{<k}\partial_\alpha\psi + \text{ error}$$

where, unsurprisingly, A^α are the same as in (21). But this can be dealt with as before by making a gauge transformation.

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