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UNIQUE CONTINUATION FOR OPERATORS WITH PARTIALLY ANALYTIC COEFFICIENTS

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ABSTRACT. – A general unique continuation result for partial differential operators with partially analytic coefficients was obtained in Tataru (1995); however, certain technical assumptions were used there. A part of these assumptions were elliminated independently by Hörmander (1985), and Robbiano and Zuily (preprint). The aim of this note is to remove the remaining technical restriction and, at the same time, to provide a simple proof for the entire result. © Elsevier, Paris

1. Introduction

This article is devoted to the study of the unique continuation problem for partial differential operators whose coefficients are partially analytic. With an appropriate choice of coordinates this means that the coefficients are analytic with respect to some of the variables.

The first work in this direction is due to Robbiano [4], who obtained a partial result in the special case of the wave equation with time independent coefficients. His result was slightly improved by Hörmander [2] shortly afterwards. The first version of the general result below, for arbitrary operators with partially analytic coefficients, was proved by the author in [6]. However, the results in [6] required certain technical assumptions restricting the allowable class of analytic coefficients. Some of these assumptions were removed independently, using different methods, by Hörmander [3] and by Robbiano and Zuily [5]. Our aim here is to elliminate the remaining technical assumptions and, at the same time, to provide a simpler proof of the results.

The plan of the paper is as follows. In Section 2 we introduce the class of operators we work with and the appropriate notions of pseudoconvexity. In Section 3 we state our main unique continuation result and we relate it to the corresponding Carleman estimates. The rest of the paper is devoted to the proof of the Carleman estimates: Section 4 contains the discussion of the conjugation argument and the crucial conjugation result; in Section 5 we prove the Carleman estimates in the elliptic case, and in Section 6 we do the same for operators satisfying the principal normality condition.

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2. Definitions

Split the coordinates in R^n into $x = (x_a, x_b)$, so that the two components have dimension n_a, respectively n_b. Consider the foliation $\mathcal{F}$ of R^n with the surfaces $x_b = \text{const}$. The conormal bundle of the foliation is then:

$$N^*\mathcal{F} = \big\{(x,\xi) \in T^*R^n;\ \xi_a = 0\big\}.$$

Let $P(x, D)$ be a partial differential operator of order m with smooth coefficients in a region $A \times B \subset R^{n_a} \times R^{n_b}$. Assume that the coefficients of P are analytic in x_a. Denote by $p(x,\xi)$ the principal symbol of P.

DEFINITION 2.1. – *We say that the operator P is principally normal in the conormal bundle $N^*\mathcal{F}$ if*

$$\big|\{\bar{p}, p\}\big| \leqslant c|p||\xi|^{m-1}, \quad \text{in } N^*\mathcal{F} \tag{1}$$

and

$$|p_{x_a}| \leqslant c|p|, \quad \text{in } N^*\mathcal{F}. \tag{2}$$

We also need a stronger version of the principal normality condition which is tied to the analiticity of the coefficients. If the coefficients of P are analytic in x_a then the symbol $p(x,\xi)$ can be extended as a holomorphic function of x_a to a complex neighbourhood V of the set A.

DEFINITION 2.2. – *We say that the operator P is analytically principally normal in the conormal bundle $N^*\mathcal{F}$ if*

$$\big|\big\{p(z_a), p(\tilde{z}_a)\big\}\big| + \big|\big\{\overline{p(z_a)}, p(\tilde{z}_a)\big\}\big| \leqslant c\big|p(z_a)\big||\xi_b|^{m-1} \quad \text{in } N^*\mathcal{F} \text{ for } z_a, \tilde{z}_a \in V \tag{3}$$

and

$$|p_{z_a}| \leqslant c|p| \quad \text{in } N^*\mathcal{F},\ z_a \in V. \tag{4}$$

For most operators the principal normality seems to imply the analytic principal normality. This is certainly the case if the inequality

$$\big|q(x_b, \xi_b)\big| \leqslant c\big|p(x, 0, \xi_b)\big| \tag{5}$$

is equivalent to having finitely many derivatives of q vanish in certain directions; indeed, this latter property can be easily extended by analyticity. Simple examples where this happens are:

(a) If p vanishes simply on a codimension 1 surface.

(b) If $\operatorname{Re} p$, $\operatorname{Im} p$ vanish simply on two transversal surfaces.

In both these examples the inequality (5) follows from the condition $q = 0$ in $N^*\mathcal{F} \cap \operatorname{char} P$. In general, however, it seems conceivable that the principal normality does not always imply the analytic principal normality.

If the principal normality condition holds then the operators $P(x_a, x_b, 0, D_b)$ have the same strength in the sense that

$$\big|p(x_a, x_b, 0, \xi_b)\big| \leqslant c\big|p(\tilde{x}_a, x_b, 0, \xi_b)\big|,$$

for all $x_a, \tilde{x}_a \in A$. If the analytic principal normality condition is valid then the same holds for x_a, $\tilde{x}_a$ in a complex neighbourhood of A.

Two special cases where the analytic principal normality condition is trivial are:

(E) P is elliptic in the conormal bundle of the foliation $N^*\mathcal{F}$.

(H) P is of real principal type in $N^*\mathcal{F}$ and $N^*\mathcal{F}$ is invariant with respect to the null bicharacteristic flow.

Now we introduce the appropriate definitions of pseudoconvexity. Given a C^2 function ϕ, define the symbol

$$p_\phi(x,\xi,\tau) = p(x,\xi + \mathrm{i}\tau\nabla\phi).$$

Let Γ be a closed conic subset of the cotangent bundle $T^*\Omega$.

DEFINITION 2.3. – *Let S be a smooth oriented hypersurface which is a level surface of a smooth function ϕ, and $x_0 \in S$, $\nabla\phi(x_0) \neq 0$. We say that S is strongly pseudoconvex in Γ with respect to P at x_0 if*

$$\mathrm{Re}\left\{\bar{p},\{p,\phi\}\right\}(x_0,\xi) > 0 \quad \text{on } p(x_0,\xi) = \{p,\phi\}(x_0,\xi) = 0,\ \xi \neq 0,\ \xi \in \Gamma_{x_0}, \tag{6}$$

$$\{\bar{p}_\phi, p_\phi\}(x_0,\xi)/\tau\mathrm{i} > 0 \quad \text{on } p_\phi(x_0,\xi) = \{p_\phi,\phi\}(x_0,\xi) = 0,\ \xi \in \Gamma_{x_0},\ \tau > 0. \tag{7}$$

DEFINITION 2.4. – *A C^2 function ϕ is strongly pseudoconvex in Γ with respect to P at x_0 if*:

$$\mathrm{Re}\left\{\bar{p},\{p,\phi\}\right\}(x_0,\xi) > 0 \quad \text{on } \xi \in \Gamma_{x_0},\ p(x_0,\xi) = 0,\ \xi \neq 0, \tag{8}$$

$$\{\bar{p}_\phi, p_\phi\}(x_0,\xi)/\tau\mathrm{i} > 0 \quad \text{on } p_\phi(x_0,\xi) = 0,\ \xi \in \Gamma_{x_0},\ \tau > 0. \tag{9}$$

Following Proposition 28.3.3 in [1], it is easy to verify that:

LEMMA 2.5. – (a) *The strong pseudoconvexity condition for both functions and surfaces is stable with respect to small C^2 perturbations.*

(b) *If ϕ is as in Definition* 2.3 *then $\psi = \mathrm{e}^{\lambda\phi}$ satisfies the strong pseudoconvexity condition in* (2.4) *if λ is large enough.*

In the proof of the Carleman estimates we shall use the following equivalent formulation of the strong pseudoconvexity condition (see Hörmander [1], XXVIII):

LEMMA 2.6. – (a) *Assume that the operator P is principally normal. Then a C^2 function ϕ is strongly pseudoconvex in Γ with respect to P at x_0 iff for large enough c*

$$c^{-1}\tau\left(\xi^2+\tau^2\right)^{m-1} \leqslant c\tau^{-1}|p_\phi|^2 - \mathrm{i}\{\bar{p}_\phi, p_\phi\} \quad \text{in } \Gamma. \tag{10}$$

(b) *If in addition P is elliptic in Γ then* (10) *can be replaced by the stronger inequality*

$$c^{-1}\left(\xi^2+\tau^2\right)^{m-1} \leqslant c|p_\phi|^2 - \mathrm{i}\tau\{\bar{p}_\phi, p_\phi\} \quad \text{in } \Gamma. \tag{11}$$

3. Results

Our main result is

THEOREM 1. – *Let P be a partial differential operator whose coefficients are smooth overall and analytic in the leaves of the foliation $\mathcal{F}$. Assume that P is analytically principally normal in $N^*\mathcal{F}$.*

Let $\Sigma = \{\phi = 0\}$ be an oriented hypersurface and $x_0 \in \Sigma$. Suppose Σ is strongly pseudoconvex with respect to P in the conormal bundle of the foliation $N^*_{x_0}\mathcal{F}$.

If u is a solution to $P(x,D)u = 0$ near x_0 so that $\operatorname{supp} u \subset \{\phi \leqslant \phi(x_0)\}$ *then $x_0 \notin \operatorname{supp} u$.*

In other words, this says that we have unique continuation across surfaces that are strongly pseudoconvex in $N^*\mathcal{F}$. This result was first proved in Tataru [6] in case (H) under the assumption that the coefficients[2] of P are independent of x_a and in case (E) under the assumption that the coefficients[2] of P are entire functions of type 2 in x_a. Recently Hörmander [2] and Robbiano and Zuily [5] have independently proved the complete result in case (E), and for principally normal operators, under the assumption that the coefficients[2] of P do not depend on x_a on $N^*\mathcal{F}$. Here we allow the coefficients of P to depend (analytically) on x_a on $N^*\mathcal{F}$, subject, of course, to the adapted version of principal normality condition.

Remark 3.1. – A natural question to ask is whether the second part of the principal normality condition is truly necessary for the unique continuation. While the answer to this is not obvious, it is clear that this condition is necessary for the corresponding Carleman estimates.

Remark 3.2. – To keep the calculus simple we assume that the coefficients of P are smooth as functions of x_b. However, the arguments can be easily adapted to C^1 coefficients in case (E) and C^2 coefficients in case (H). In general the coefficients need to be as good as required in Fefferman–Phong's inequality.

The unique continuation result follows from some suitable Carleman estimates. Below we present these estimates and show how they imply Theorem 1.

THEOREM 2. – *Let $A \times B$ be a bounded subset of $R^{n_a} \times R^{n_b}$ and P be a partial differential operator in $A \times B$ whose coefficients are smooth overall and analytic in x_a. Assume that*:

(i) *P is analytically principally normal in $N^*_{A\times B}\mathcal{F}$.*

(ii) *ϕ is analytic in a neighbourhood of $A \times B$ and strongly pseudoconvex with respect to P in $N^*_{A\times B}\mathcal{F}$.*

Then there exist $c, d > 0$ such that for each small enough $\varepsilon > 0$ and large enough τ we have

$$\begin{aligned}\tau\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}u\right|^2_{m-1,\tau} \leqslant c\Big(&\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}P(x,D)u\right|^2_0 \\ &+\left|\mathrm{e}^{\tau(\phi-d\varepsilon)}P(x,D)u\right|^2_0+\left|\mathrm{e}^{\tau(\phi-d\varepsilon)}u\right|^2_{m-1,\tau}\Big)\end{aligned} \tag{12}$$

whenever u is a distribution supported in $A \times B$ for which the right hand side is finite.

The following stronger estimate holds in the elliptic case (E):

THEOREM 3. – *Let $A \times B$ be a bounded subset of $R^{n_a} \times R^{n_b}$ and P be a partial differential operator in $A \times B$ whose coefficients are smooth overall and analytic in x_a. Assume that*:

(i) *P is elliptic in $N^*_{A\times B}\mathcal{F}$.*

(ii) *ϕ is analytic in a neighbourhood of $A \times B$ and strongly pseudoconvex with respect to P in $N^*_{A\times B}\mathcal{F}$.*

Then there exist $c, d > 0$ such that for each small enough $\varepsilon > 0$ and large enough τ we have

$$\begin{aligned}\tau^{-1}\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}u\right|^2_{m,\tau} \leqslant c\Big(&\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}P(x,D)u\right|^2_0 \\ &+\left|\mathrm{e}^{\tau(\phi-d\varepsilon)}P(x,D)u\right|^2_0+\left|\mathrm{e}^{\tau(\phi-d\varepsilon)}u\right|^2_{m-1,\tau}\Big),\end{aligned} \tag{13}$$

whenever u is a distribution supported in $A \times B$ for which the right hand side is finite.

Note that in the classical Carleman estimates one assumes that the pseudoconvexity condition holds in the entire cotangent bundle; then the estimates have the form

$$\left|\mathrm{e}^{\tau\phi}u\right| \leqslant \left|\mathrm{e}^{\tau\phi}Pu\right|,$$

[2] This restriction applies only to the coefficients of the principal part.

(with appropriate norms). Here we only have the pseudoconvexity condition in the set $\{|\xi_a| = 0\}$. The Gaussian in the estimates cuts off exactly a neighbourhood of this region. The price to pay is the last two right hand side terms, which naively account for what happens away from this set. Theorems 2, 3 are proved in Sections 4, 5, 6. The other ingredient required for the proof of Theorem 1 is:

THEOREM 4. – *Suppose u satisfies*

$$\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}u\right| \leqslant c\mathrm{e}^{\gamma\tau} \tag{14}$$

for large enough τ. Then u is supported in $\{\phi \leqslant \gamma\}$.

This result was proved in Tataru [6]. For the reader's convenience we sketch the proof in Appendix A.

Now we can show how the Carleman estimates in Theorems 2, 3 combined with Theorem 4 yield the uniqueness result in Theorem 1.

Proof of Theorem 1. – After a standard perturbation and localization argument (see Hörmander [1], XXVIII) the uniqueness result reduces to the following statement:

Let $A \subset R^{n_a}$, convex and $B \subset R^{n_b}$. Let $r > 0$. Let ϕ be a strongly pseudoconvex function with respect to P on $N^*\mathcal{F}$ in $A \times B$.

Let u be a function satisfying the following conditions:

(a) *u is supported in $A \times B$.*

(b) *Pu is supported in $\phi \leqslant 0$.*

Then $u = 0$ in $\{\phi \leqslant 0\}$.

To prove this, let

$$\phi_0 = \max\bigl\{\phi(x);\ x \in \operatorname{supp} u\bigr\}.$$

Then using the Carleman estimate (12) we obtain

$$\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}\mathrm{e}^{\tau\phi}u\right| \leqslant c\mathrm{e}^{\gamma\tau}, \quad \gamma = \max\{0, \phi_0 - d\varepsilon\}.$$

Hence Theorem 4 implies that $\phi_0 \leqslant \gamma$, which shows that $\phi_0 \leqslant 0$.

4. Proof of the Carleman estimates: the conjugation

To prove the Carleman estimates a first step is to conjugate the operator P by the exponential weight $\mathrm{e}^{\tau\phi}$. This yields

$$\mathrm{e}^{\tau\phi}P = P_\phi \mathrm{e}^{\tau\phi}, \quad P_\phi = P(x, D + \mathrm{i}\tau\nabla\phi).$$

Then with the notation $w = \mathrm{e}^{\tau\phi}u$ the inequality (13), for instance, reduces to

$$\tau^{-1}\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|_{m,\tau}^2 \leqslant c\bigl(\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}P_\phi w\right|_0^2 + \left|\mathrm{e}^{-d\varepsilon\tau}P_\phi w\right|_0^2 + \left|\mathrm{e}^{-d\varepsilon\tau}w\right|_{m-1,\tau}^2\bigr). \tag{15}$$

Without the Gaussian this would essentially be a subelliptic estimate for P_ϕ. As it is, what we need to do first is to find an approximate conjugate of P_ϕ with respect to the Gaussian,

$$\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}P_\phi = P_{\phi,\varepsilon/\tau}\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2} + \mathrm{O}\bigl(\mathrm{e}^{-c\tau}\bigr).$$

A simple computation yields

$$e^{-\frac{\varepsilon}{2\tau}D_a^2}x_a = \left(x_a + i\frac{\varepsilon}{\tau}D_a\right)e^{-\frac{\varepsilon}{2\tau}D_a^2}$$

and further

$$e^{-\frac{\varepsilon}{2\tau}D_a^2}x_a^\alpha = \left(x_a + i\frac{\varepsilon}{\tau}D_a\right)^\alpha e^{-\frac{\varepsilon}{2\tau}D_a^2}.$$

The crucial observation is that we can use the Weyl calculus to rewrite $(x_a + i(\varepsilon/\tau)D_a)^\alpha$ as $Op^w((x_a + i(\varepsilon/\tau)\xi_a)^\alpha)$. This suggests the good candidate for $P_{\phi,\varepsilon/\tau}$; namely, if:

$$P_\phi(x, D, \tau) = \sum c_\alpha(x)(D, \tau)^\alpha,$$

then set

$$P_{\phi,\varepsilon/\tau} = Op^w\, c_\alpha\left(x_a + i\frac{\varepsilon}{\tau}\xi_a\right)(D_a, \tau)^\alpha.$$

To do this we need to extend c_α as functions of x_a to the complex plane. In order to get a nice symbol for the extension we cannot just take the holomorphic extension of c_α; we need to use some cutoff.

The plan of the proof of the Carleman estimates is as follows:

(i) We define $P_{\phi,\varepsilon/\tau}$ and prove that it is a good conjugate of P_ϕ.
(ii) We discuss the calculus for operators of the same type as $P_{\phi,\varepsilon/\tau}$.
(iii) We prove a subelliptic estimate for P_ϕ.
(iv) We show that $P_{\phi,\varepsilon/\tau}$ is a small perturbation of P_ϕ in the appropriate sense; this allows us to transfer the subelliptic estimate to $P_{\phi,\varepsilon/\tau}$.
(v) Finally, we use the conjugation result to show that this implies the Carleman estimate for P.

In order to make the ideas clearer we carry out steps (iii) and (iv) first for the simpler elliptic case (E) in Section 5, and then for the general case in Section 6.

4.1. The conjugation

For $r > 0$ denote by A_r the r neighbourhood of A. Of course, both ϕ and the coefficients of P can be extended to holomorphic functions in x_a in a complex neighbourhood of A, say $A_{4r} + iB(0, 4r)$, for some $r > 0$.

Denote by H the space of bounded holomorphic functions in $A_{4r} + iB(0, 4r)$. Given a function $f \in H$ we truncate it as follows. Let χ be a smooth cutoff function supported in A_{4r}, which is 1 in A_{3r}. Let η be a smooth cutoff function supported in $B(0, 3r)$, which is 1 in $B(0, 2r)$. Now set

$$f^r(z) = \chi(\operatorname{Re} z)\eta(\operatorname{Im} z)f(z).$$

Then our candidate for the conjugate of f with respect to the Gaussian $e^{-\frac{\delta}{2}D_a^2}$ is the operator:

$$F_\delta = Op^w\left(f^r(x_a + i\delta\xi_a)\right).$$

Let μ be a smooth cutoff function supported in $A_{4r} + iR^n$ which is 1 in $A_{3r} + iR^n$. Define the remainder:

$$R_{f,\delta} = \mu F_\delta e^{-\frac{\delta}{2}D_a^2} - e^{-\frac{\delta}{2}D_a^2}f.$$

The following result shows that F_δ is a good conjugate for f:

PROPOSITION 4.1. – *Let X be a Banach space. Let $f \in H(X)$ and $R_{f,\delta}$ be as above. Then $R_{f,\delta} \in \mathrm{e}^{-r^2/2\delta} OPS^{-\infty}(A; X)$ uniformly in $0 \leqslant \delta \leqslant 1$.*

Proof. – Look at the kernel $K(x, y)$ of $R_{f,\delta}$. We have

$$K(x, y) = \mu(x) \int \left(f^r\left(\frac{x+w}{2} + \mathrm{i}\delta\xi\right) - f(y)\right) \mathrm{e}^{\mathrm{i}(x-w)\xi} \mathrm{e}^{-\frac{1}{2\delta}(w-y)^2}\, \mathrm{d}w\, \mathrm{d}\xi + \big(1-\mu(x)\big) a(y) \mathrm{e}^{-\frac{1}{2\delta}(x-y)^2}.$$

Since $y \in A$ while $1-\mu$ is supported away from A_{3r}, the conclusion follows immediately for the second RHS term. It remains to look at the first one. With the change of variable:

$$z = \frac{x+w}{2} + \mathrm{i}\delta\xi,$$

the corresponding integral becomes:

$$I(x, y) = \mu(x) \int \big(f^r(z) - f(y)\big) \mathrm{e}^{-\frac{1}{2\delta}(x-y)^2} \mathrm{e}^{\delta^{-1}(z-y)(2x-z-\bar{z})}\, \mathrm{d}z\, \mathrm{d}\bar{z}. \tag{16}$$

Write

$$f^r(z) - f(y) = \big(1 - \eta(\mathrm{Im}\, z)\big) f(y) + \eta(\mathrm{Im}\, z)\big(\chi(\mathrm{Re}\, z) f(z) - f(y)\big). \tag{17}$$

Corresponding to the first right hand side term, in the (w, ξ) coordinates, we get the integral

$$I_1(x, y) = \mu(x) f(y) \int \big(1 - \eta(\delta\xi)\big) \mathrm{e}^{\mathrm{i}(x-w)\xi} \mathrm{e}^{-\frac{1}{2\delta}(w-y)^2}\, \mathrm{d}w\, \mathrm{d}\xi$$

which can be explicitely integrated in w,

$$I_1(x, y) = \mu(x) f(y) \int \big(1 - \eta(\delta\xi)\big) \mathrm{e}^{\mathrm{i}(x-y)\xi} \mathrm{e}^{-\frac{\delta}{2}\xi^2}\, \mathrm{d}\xi$$

and the correct bound follows since $1-\eta$ is supported in $\{|\xi| > 3r\}$.

For the second right hand side term in (17) we write further:

$$\eta(\mathrm{Im}\, z)\big(\chi(\mathrm{Re}\, z) f(z) - f(y)\big) = \eta(\mathrm{Im}\, z) b(z, y)(z - y)$$

where b is holomorphic in the same domain as $\chi(\mathrm{Re}\, z) f(z)$, that is, in $A_{3r} + \mathrm{i}B(0, 4r)$.

With the $(z, \bar{z})$ coordinates the corresponding integral becomes:

$$I_2(x, y) = \int \eta(\mathrm{Im}\, z) b(z, y)(z - y) \mathrm{e}^{-\frac{1}{2\delta}(x-y)^2} \mathrm{e}^{\delta^{-1}(z-y)(2x-z-\bar{z})}\, \mathrm{d}z\, \mathrm{d}\bar{z}.$$

Then we can integrate by parts with respect to $\bar{z}$ to obtain:

$$I_2(x, y) = \mu(x) \int \bar{\partial}_z\big(\eta(\mathrm{Im}\, z) b(z, y)\big) \mathrm{e}^{-\frac{\tau}{2}(x-y)^2} \mathrm{e}^{\tau(z-y)(2x-z-\bar{z})}\, \mathrm{d}z\, \mathrm{d}\bar{z}.$$

Returning to the original coordinates, we have:

$$I_2(x, y) = I_{21}(x, y) + I_{22}(x, y)$$

where

$$I_{21} = \chi(x) \int \eta(\delta\xi)\bar{\partial} b\left(\frac{x+w}{2} + \mathrm{i}\delta\xi, y\right) \mathrm{e}^{\mathrm{i}(x-w)\xi} \mathrm{e}^{-\frac{1}{2\delta}(w-y)^2}\, \mathrm{d}w\, \mathrm{d}\xi$$

and

$$I_{22} = \chi(x) \int \eta'(\delta\xi) b\left(\frac{x+w}{2} + \mathrm{i}\delta\xi, y\right) \mathrm{e}^{\mathrm{i}(x-w)\xi} \mathrm{e}^{-\frac{1}{2\delta}(w-y)^2}\, \mathrm{d}w\, \mathrm{d}\xi.$$

For I_{21} it suffices to look at the support of the integrand. Indeed, we start with $y \in A$, while $x \in A_{4r}$ (the support of μ). But $(x+w)/2 \notin A_{3r}$ (the support of $\bar{\partial} b$). Since A is convex this implies $w \notin A_{2r}$. Hence $|w-y| > 2r$ in the support of the integrand and the correct bound follows.

For I_{22}, if there were no b, we could take advantage of the fact that

$$\operatorname{supp} \eta' \subset B(0,3r) \setminus B(0,2r) \tag{18}$$

and argue as for I_1. We contend that the result still holds even with b. First observe that without any restriction in generality we can restrict the integral to the region $|w-y| \leqslant r$. Since $x \in A_{4r}$ this implies that $(x+w)/2 \in A_{3r}$. Then for ξ in the support of η' we get that the function

$$b\left(\frac{x+w}{2} + \mathrm{i}\delta\xi, y\right)$$

is in effect holomorphic as a function of w in the complex region $|w-z| \leqslant r$. Then the estimate for I_{22} follows from (18) and the following lemma:

LEMMA 4.2. – *Let f be a holomorphic function in $|z| \leqslant r$; then*

$$\left| \int_{|x| \leqslant r} f(x) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{e}^{-\frac{1}{2\delta}x^2}\, \mathrm{d}x \right| \leqslant c_n r^n \mathrm{e}^{-\min\{\frac{\delta}{2}\xi^2, \frac{1}{2\delta}r^2\}} |f|_\infty,$$

where c_n depends only on the dimension.

Proof. – By rescaling the problem reduces to the case $\delta = 1$. In dimension 1 we make an appropriate change of the contour of integration to get the estimate. We can rewrite the integral as

$$\int_{-r}^{r} f(z) \mathrm{e}^{-\frac{\xi^2}{2}} \mathrm{e}^{-\frac{(z-\mathrm{i}\xi)^2}{2}}\, \mathrm{d}z.$$

If $|\xi| \leqslant r$ then we take as the new contour the broken line from $-r$ to $\mathrm{i}\xi$ to r; then the second exponent above has negative real part and the estimate follows. If $|\xi| > r$ then we take as the new contour the broken line from $-r$ to $ir \operatorname{sgn}\xi$ to r.

To prove the result in R^n assume without any restriction in generality that $\xi = (\xi_1, 0, \ldots, 0)$. Then use the one-dimensional result for the integration in x_1:

$$\left| \int_{|x| \leqslant r} f(x) \mathrm{e}^{\mathrm{i}x_1\xi_1} \mathrm{e}^{-\frac{1}{2\delta}x^2}\, \mathrm{d}x \right| \leqslant \int_{B^{n-1}(0,r)} \left(r^2 - |x'|^2\right)^{\frac{1}{2}} \mathrm{e}^{-\min\{\frac{\xi^2}{2}, \frac{r^2-x'^2}{2}\}} \mathrm{e}^{-\frac{x'^2}{2}}\, \mathrm{d}x'$$

$$\leqslant c_n r^n \mathrm{e}^{-\min\{\frac{\xi^2}{2}, \frac{r^2}{2}\}}. \qquad \square$$

To get similar estimates for the derivatives of $I(x, y)$ with respect to x, y observe first that the commutator of $R_{f,\delta}$ with D is an operator of the same type. Hence, it suffices to look at the x derivatives of the kernel. It is easiest to do this in (16). Differentiating there with respect to x yields a factor of $(x-y)/\delta$ and a factor of $(z-y)/\delta$. One can easily see that neither of them causes any trouble in the estimates.

Remark 4.3. – The above proof simplifies considerably if we assume that f is holomorphic in $A_{4r} + \mathrm{i}R^n$, decaying rapidly at infinity. Then the function η is no longer necessary, therefore the estimates for I_1 and I_{22} are not needed. For the coefficients of P this can be achieved by taking suitable coordinates and by multiplying P by $\mathrm{e}^{x_a^2}$, say.

If we use the above result with $\delta = \varepsilon/\tau$ and take X to be succesively the space of operators of order $0, 1, \ldots, m$ in x_b then we get:

COROLLARY 4.4. – *The following estimate holds*:

$$\left|\left(\chi(x) P_{\phi,\varepsilon/\tau} \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} - \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} P_\phi\right) w\right| \leqslant \mathrm{e}^{-\frac{\tau r^2}{2\varepsilon}} \left|\left(|D_a| + \tau\right)^{-n} w\right|_{m,\tau},$$

whenever $w \in H^m$ is supported in $A \times B$.

4.2. The calculus for the $P_{\phi,\varepsilon/\tau}$ operators

To keep things clear at the stage where we prove the Carleman estimates, we stop for a moment and set up a calculus for operators which are similar to $P_{\phi,\varepsilon/\tau}$. Let Y, Z be Banach spaces of differential (pseudodifferential) operators on R^{n_b}.

By $S^q(Y)$ we denote the Y valued S^q symbols in R^{n_a}. To a symbol $a \in S^q(Y)$ we associate the operator

$$A_\delta = OP^w\, a(x + \mathrm{i}\delta\xi), \quad 0 \leqslant \delta \leqslant 1.$$

We call $OP_\delta\, S^q(Y)$ the corresponding class of operators.

PROPOSITION 4.5. – *Let E, F be two Hilbert spaces of functions so that $Y \subset L(E, F)$. Let $a \in S^0(Y)$. Then we have*:

(a) $A_\delta : L^2(E) \to L^2(F)$, *uniformly in* $0 \leqslant \delta \leqslant 1$.

(b) $\delta^{-1}(A_0 - A_\delta) \in OP_\delta\, S^{-1}(Y) D_a$, *uniformly in* $0 \leqslant \delta \leqslant 1$.

Proof. – The symbols $a(x + \mathrm{i}\delta\xi)$ of A_δ are uniformly bounded in $S^0(L(E, F))$, therefore (a) is part of a classical result (see Hörmander [1], 18.1).

For (b) observe that $\delta^{-1}(A_\delta - A_0)$ has the symbol $\xi b_a(x + \mathrm{i}\delta\xi)$ where the symbols b_a are chosen so that

$$\xi b(x + \mathrm{i}\xi) = a(x + \mathrm{i}\xi) - a(x).$$

Next we take a look at composition of such operators. Let YZ be another Banach space of operators which contains all compositions of an operator in Y with an operator in Z.

PROPOSITION 4.6. – *Let $a \in S^k(Y)$ and $b \in S^l(Z)$. Then $A_\delta B_\delta \in OP_\delta\, S^{k+l}(YZ)$ uniformly in* $0 \leqslant \delta \leqslant 1$.

Proof. – Look for the Weil symbol of $A_\delta B_\delta$ of the form $c(x + \mathrm{i}\delta\xi)$. Then

$$\int c\left(\frac{x+y}{2} + \mathrm{i}\delta\xi\right) \mathrm{e}^{\mathrm{i}\xi(x-y)}\, \mathrm{d}\xi = \int a\left(\frac{x+z}{2} + \mathrm{i}\delta\eta\right) \mathrm{e}^{\mathrm{i}\xi(x-z)} c\left(\frac{z+y}{2} + \mathrm{i}\delta\mu\right) \mathrm{e}^{\mathrm{i}\mu(z-y)}\, \mathrm{d}\eta\, \mathrm{d}\mu\, \mathrm{d}z,$$

which after inverting the Fourier transform in ξ and making the change of variable

$$\frac{x+y}{2} := x, \quad x+z := 2y, \quad y+z := 2z, \quad \eta - \xi := \eta, \quad \mu - \xi := \mu$$

yields

$$c(x+\mathrm{i}\xi) = \int a\big(y+\mathrm{i}(\xi-\delta\eta)\big) b\big(z+\mathrm{i}(\xi-\delta\mu)\big) \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu}\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}\eta\, \mathrm{d}\mu. \tag{19}$$

Integrating in y, z in the RHS gives

$$c(x+\mathrm{i}\xi) = \int d(\eta,\mu,\xi) \mathrm{e}^{2\mathrm{i}x(\eta+\mu)}\, \mathrm{d}\eta\, \mathrm{d}\mu, \quad d \in \mathcal{S}_{\eta,\mu}\big(S_\xi^{k+l}(YZ)\big).$$

Since the Fourier transform of a Schwartz function is a Schwartz function it follows that:

$$c(x+\mathrm{i}\xi) \in S^{k+l}(YZ). \qquad \square$$

Next we get the commutator estimates.

PROPOSITION 4.7. – *Let $a \in S^k(Y)$ and $b \in \mathcal{S}^l(Z)$. Let W be another Banach space so that*

$$\big[a(z), b(\tilde{z})\big] \in S_z^k\big(S_{\tilde{z}}^l\big)(W). \tag{20}$$

Then $A_\delta B_\delta \in OP_\delta\, S^{k+l}(W) + \delta\, OP_\delta\, S^{k+l-1}(YZ)$ uniformly in $0 \leqslant \delta \leqslant 1$.

Proof. – As before, we denote by C_δ the commutator. Then the analogue of (19) is

$$\begin{aligned} c(x+\mathrm{i}\xi) = \int & a\big(y+\mathrm{i}(\xi-\delta\eta)\big) b\big(z+\mathrm{i}(\xi-\delta\mu)\big) \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu} \\ & - b\big(y+\mathrm{i}(\xi-\delta\eta)\big) a\big(z+\mathrm{i}(\xi-\delta\mu)\big) \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu}\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}\eta\, \mathrm{d}\mu \end{aligned}$$

which further gives

$$\begin{aligned} c(x+\mathrm{i}\xi) = \int & \big[b\big(y+\mathrm{i}(\xi-\delta\eta)\big), a\big(z+\mathrm{i}(\xi-\delta\mu)\big)\big] \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu} \\ & + a\big(y+\mathrm{i}(\xi-\delta\eta)\big) b\big(z+\mathrm{i}(\xi-\delta\mu)\big) \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu} \\ & - a\big(y+\mathrm{i}(\xi+\delta\eta)\big) b\big(z+\mathrm{i}(\xi+\delta\mu)\big) \mathrm{e}^{2\mathrm{i}(x-y)\eta} \mathrm{e}^{2\mathrm{i}(x-z)\mu}\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}\eta\, \mathrm{d}\mu. \end{aligned}$$

Arguing as in the previous lemma, by (20) the commutator is in $S^{k+l}(W)$. The difference of the last two terms, on the other hand, is in $\delta S^{k+l-1}(YZ)$, uniformly in δ. Hence the conclussion follows.

5. Proof of the Carleman estimates: The elliptic case

5.1. The subelliptic estimate for P_ϕ

The pseudoconvexity condition for ϕ in (11) implies that for large enough c,

$$c^{-1}\big(\xi^2+\tau^2\big)^m \leqslant c\big(\big|p_\phi(x,\xi,\tau)\big|^2 + \xi_a^2\big(\xi^2+\tau^2\big)^{m-1}\big) + 2\tau\{\mathrm{Re}\, p_\phi, \mathrm{Im}\, p_\phi\} \tag{21}$$

for x in a neighbourhood of $A \times B$, say $A_{2r} \times B$. Then Garding's inequality yields, for a larger c and sufficiently large τ,

$$c^{-1}|v|^2_{m,\tau} \leqslant c\big(|\tilde{P}_\phi v|^2 + |D_a v|^2_{m-1,\tau}\big) + 2\,\mathrm{Re}\langle P^r_\phi v, \mathrm{Im}\, P^i_\phi v\rangle, \tag{22}$$

for v supported in $A_{2r} \times B$. Here P^r_ϕ, P^i_ϕ are the selfadjoint, respectively the skew-adjoint parts of P_ϕ, whose principal symbols are $\mathrm{Re}\, p_\phi$, respectively $\mathrm{i}\,\mathrm{Im}\, p_\phi$.

Of course this further gives

$$c_1\tau|v|^2_{m-1,\tau} \leqslant |P_\phi v|^2 + \tau^{-1}|D_a v|^2_{m-1,\tau}.$$

However, (22) is of interest to us because this is the estimate we shall transfer to $P_{\phi,\varepsilon/\tau}$.

5.2. The subelliptic estimates for $P_{\phi,\varepsilon/\tau}$

We would like to show that 22 is still true with P_ϕ replaced by $P_{\phi,\varepsilon/\tau}$. To achieve this we shall prove that $P_{\phi,\varepsilon/\tau}$ is a small perturbation of P in the appropriate sense. From Proposition 4.5 with $\delta = \varepsilon/\tau$ we immediately obtain:

LEMMA 5.1. – *If $P_{\phi,\varepsilon/\tau}$ is as above then*

$$\big|(P_{\phi,\varepsilon/\tau} - P_\phi)v\big| \leqslant c\left|\frac{\varepsilon D_a}{\tau}v\right|_{m,\tau}$$

for all v supported in A_{2r}.

Now we need to look at the inner products arising in the proof of the estimates. By $P^r_{\phi,\varepsilon/\tau}$, $P^i_{\phi,\varepsilon/\tau}$ we denote the selfadjoint, respectively the skew-adjoint parts of P_ϕ, whose principal symbols[3] are $\mathrm{Re}\, p_{\phi,\varepsilon/\tau}$, respectively $\mathrm{i}\,\mathrm{Im}\, p_{\phi,\varepsilon/\tau}$.

LEMMA 5.2. – *Let P_ϕ, $P_{\phi,\varepsilon/\tau}$ be as above. Then*

$$\big|\,\mathrm{Re}\big\langle P^r_\phi v, P^i_\phi v\big\rangle - \mathrm{Re}\big\langle P^r_{\phi,\varepsilon/\tau} v, P^i_{\phi,\varepsilon/\tau} v\big\rangle\big| \leqslant c\left(\frac{\varepsilon}{\tau}|v|^2_{m,\tau} + \left|\frac{\varepsilon D_a}{\tau}v\right|^2_{m,\tau}\right),$$

whenever v is supported in A_{2r}.

Proof. – The left hand side in the inequality can be written as $|\langle Dv, v\rangle|$ with

$$D = \big[P^r_\phi, P^i_\phi\big] - \big[P^r_{\phi,\varepsilon/\tau}, P^i_{\phi,\varepsilon/\tau}\big].$$

With $\delta = \varepsilon\tau^{-1}$ we get

$$P_{\phi,\varepsilon/\tau} \in OPS_\delta\, S^{-\infty}\big(C^\infty_{x_b}\big)(D,\tau)^m.$$

Then, by Proposition 4.5,

$$P_{\phi,\varepsilon/\tau} - P_\phi \in \delta\, OPS_\delta\, S^{-1}\big(C^\infty_{x_b}\big)(D,\tau)^m D_a.$$

[3] modulo operators of the same type but of order $m-1$.

Hence from Proposition 4.7 we obtain

$$\left[P_\phi^r, P_\phi^i\right] - \left[P_{\phi,\varepsilon/\tau}^r, P_{\phi,\varepsilon/\tau}^i\right] \in \delta\, OPS_\delta\, S^{-2}\big(C^\infty\big)(D,\tau)^{2m} + \delta^2\, OPS_\delta\, S^{-1}\big(C^\infty\big)(D,\tau)^{2m} D_a$$

which implies the desired conclusion.

Using Lemmas 5.1, 5.2 in (22) we get, for small ε:

$$c^{-1}|v|_{m,\tau}^2 \leqslant c\left(\left|\tilde{P}_{\phi,\varepsilon/\tau} v\right|^2 + |D_a v|_{m-1,\tau}^2 + \left|\frac{\varepsilon D_a}{\tau} v\right|_{m,\tau}^2\right) + 2\,\mathrm{Re}\left\langle P_{\phi,\varepsilon/\tau}^r v, \mathrm{Im}\, P_{\phi,\varepsilon/\tau}^i v\right\rangle.$$

This implies that

$$|v|_{m,\tau}^2 \leqslant c\left(\tau\left|P_{\phi,\varepsilon/\tau} v\right|^2 + |D_a v|_{m-1,\tau}^2 + \left|\frac{\varepsilon D_a}{\tau} v\right|_{m,\tau}^2\right). \tag{23}$$

5.3. Conclusion

For w supported in A set

$$v = \mu_1 \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w,$$

where μ_1 is a cutoff function supported in $A_2 r$ which is 1 in A_r. If μ is as in Section 4.1 then

$$\left|(1-\mu) P_{\phi,\varepsilon/\tau} v\right| \leqslant \left(\frac{\varepsilon}{\tau}\right)^N |v|_{m,\tau}$$

since operators in $OP_\delta\, S^0$ have kernel decaying like $\delta^{-n}(1+\delta^{-1}x)^{-N}$ away from the diagonal.

Hence from (23) we get

$$|v|_{m,\tau}^2 \leqslant c\left(\tau\left|\mu P_{\phi,\varepsilon/\tau} v\right|^2 + |D_a v|_{m-1,\tau}^2 + \left|\frac{\varepsilon D_a}{\tau} v\right|_{m,\tau}^2\right).$$

We claim we can substitute v by $\mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w$ in the estimate above. Indeed, this follows from the decay of the kernel of $\mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2}$ off the diagonal,

$$\left|(1-\mu_1)\mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|_{m,\tau} \leqslant \mathrm{e}^{-\frac{\tau}{2\varepsilon} r^2}\left|\big(|D_a| + \tau\big)^{-N} w\right|_{m,\tau}.$$

Hence we obtain

$$\begin{aligned}\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|_{m,\tau}^2 \leqslant c\Bigg(&\tau\left|\chi P_{\phi,\varepsilon/\tau} \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|^2 + \left|D_a \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|_{m-1,\tau}^2 + \left|\frac{\varepsilon D_a}{\tau} \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|_{m,\tau}^2 \\ &+ \mathrm{e}^{-\frac{\tau}{2\varepsilon} r^2}\left|\big(|D_a| + \tau\big)^{-N} w\right|_{m,\tau}^2\Bigg).\end{aligned}$$

For the first RHS term we use the conjugation result in Corrolary 4.4. The next two RHS terms are controlled by the LHS in a region $|\xi_a| \leqslant c\tau$. Outside this region, the Gaussian provides exponential decay. Thus, we get

$$\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|_{m,\tau}^2 \leqslant c\big(\tau\left|P_\phi \mathrm{e}^{-\frac{\varepsilon}{2\tau} D_a^2} w\right|^2 + \mathrm{e}^{-c\varepsilon\tau}\left|\big(|D_a| + \tau\big)^{-N} w\right|_{m,\tau}^2\big)$$

which implies (15). □

6. Proof of the Carleman estimates: Principally normal operators

The proof follows the same line as the proof of Theorem 3, but we need to be more careful with the choice of the function spaces we use. We start by defining the correct function spaces and we relate them to the operator P_ϕ. Then we give the appropriate conjugation lemma. Finally, we prove the subelliptic estimate for P_ϕ and then we transfer it to $P_{\phi,\varepsilon/\tau}$.

6.1. Function spaces

Due to the analytic principal normality condition, it follows that

$$c^{-1}\big|p(z_a,x_b,0,\xi_b)\big| \leqslant \big|p(\tilde z_a,x_b,0,\xi_b)\big| \leqslant c\big|p(z_a,x_b,0,\xi_b)\big|,$$

for z_a, $\tilde z_a$ in a small complex neighbourhood of A. This leads us to introduce a reference symbol

$$q(x_b,\xi_b) = p\big(x_a^0,x_b,0,\xi_b\big),$$

for some fixed $x_a^0 \in A$. Correspondingly we define the classes of symbols in R^b:

$$S_Q^k = \big\{a(x_b,\xi_b) \in S^k;\ \big|a(x_b,\xi_b)\big| \leqslant c\big|q(x_b,\xi_b)\big||\xi_b|^{k-m}\big\}.$$

If we extend as before the coefficients of P to C^{n_a} then the analytic principal normality yields the following properties for the extended symbol $p(z_a,x_b,\xi)$:

$$p(z_a,x_b,0,\xi_b) \in \mathcal{S}\big(S_Q^m\big), \tag{24}$$

$$\big\{p(z_a,x_b,0,\xi_b),\, p(\tilde z_a,x_b,0,\xi_b)\big\},\ \big\{p(z_a,x_b,0,\xi_b),\, p(\tilde z_a,x_b,0,\xi_b)\big\} \in \mathcal{S}\big(S_Q^{2m-1}\big). \tag{25}$$

We first observe that a cQ bound for a symbol implies a similar estimate for the operator.

LEMMA 6.1. – *Suppose Q, R are operators of order m with smooth coefficients so that Q is principally normal and $|r(x_b,\xi_b)| \leqslant c|q(x_b,\xi_b)|$. Then*

$$|Rv| \leqslant c\big(|Qv| + |v|_{m-1}\big).$$

Proof. – Without any restriction in generality we can assume that the principal symbol of r is real. Then we use Fefferman–Phong's theorem.

The above lemma suggests the introduction the Hilbert spaces H_Q^k with norm defined by

$$|v|_{k,Q}^2 = |Qv|_{k-m}^2 + |v|_{k-1}^2.$$

Then the operators with symbols in S_Q^k map H_Q^j into H^{j-k}.

Define also the weighted function spaces $H_{Q,\tau}^m$ with norms:

$$|u|_{m,\tau,Q}^2 = |Qu|^2 + \big|(D_a,\tau)u\big|_{m-1,\tau}^2.$$

These spaces are translation invariant in x_a so they will interact in a simple way with the Gaussian. We can also use Lemma 6.1 to relate them to the operators P_ϕ:

COROLLARY 6.2. – *The following estimate holds*:

$$|P_\phi v| \leqslant c|v|_{m,\tau,Q} \quad \textit{and} \quad |v|_{m,\tau,Q} \leqslant c\big(|P_\phi v| + \big|(D_a,\tau)v\big|_{m-1,\tau}\big).$$

6.2. The conjugation

PROPOSITION 6.3. – *Let P_ϕ, $P_{\phi,\varepsilon/\tau}$ be as above. Then*

$$\left|\left(\chi P_{\phi,\varepsilon/\tau}\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2} - \mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}P_\phi\right)w\right| \leqslant \mathrm{e}^{-c\tau}\left|\left(|D_a|+\tau\right)^{-N}w\right|_{m,Q}.$$

Proof. – We use Proposition 4.1; the part of P_ϕ containing m D_b derivatives is in $H(S_Q^m)$ therefore the corresponding remainder is estimated by the RHS. For the rest of P_ϕ we just use the H^k spaces.

6.3. The subelliptic estimate for P_ϕ

The pseudoconvexity condition (10) for ϕ implies the estimate

$$c^{-1}\tau^2\left(\xi^2+\tau^2\right)^{m-1} \leqslant c\left(\left|p_\phi(x,\xi,\tau)\right|^2 + \xi_a^2\left(\xi^2+\tau^2\right)^{m-1}\right) + 2\{\mathrm{Re}\,p_\phi, \mathrm{Im}\,p_\phi\}. \tag{26}$$

Then the Fefferman–Phong inequality yields (with a larger c)

$$c^{-1}\tau^2|v|_{m-1,\tau}^2 \leqslant c\left(|P_\phi v|^2 + |D_a v|_{m-1,\tau}^2\right) + 2\,\mathrm{Re}\langle \mathrm{Re}\,P_\phi v, \mathrm{Im}\,P_\phi v\rangle.$$

Using Lemma 6.1 this further gives

$$c^{-1}|v|_{m,\tau,Q}^2 \leqslant c\left(|P_\phi v|^2 + |D_a v|_{m-1,\tau}^2\right) + 2\,\mathrm{Re}\langle \mathrm{Re}\,P_\phi v, \mathrm{Im}\,P_\phi v\rangle. \tag{27}$$

6.4. The subelliptic estimate for $P_{\phi,\varepsilon/\tau}$

Now we shall prove that $P_{\phi,\varepsilon/\tau}$ is a small perturbation of P_ϕ as far as the estimate (27) is concerned.

LEMMA 6.4. – *If $P_{\phi,\varepsilon/\tau}$ is as above then*

$$\left|P_{\phi,\varepsilon/\tau}v\right| \leqslant c|v|_{m,\tau,Q}$$

and

$$\left|(P_{\phi,\varepsilon/\tau} - P_\phi)v\right| \leqslant c\left|\frac{\varepsilon D_a}{\tau}v\right|_{m,\tau,Q},$$

for all V supported in A_{2r}.

Proof. – (a) All the terms in $P_{\phi,\varepsilon/\tau}v$ are controlled by $|(D_a,\tau)v|_{m-1,\tau}$ except for $P(x_a + \mathrm{i}(\varepsilon D_a/\tau), x_b, 0, D_b, 0)$. But according to (24) this operator is in the class $OP_\delta S^0(S_X)$, therefore it is bounded from $L^2(H_Q^m)$ into $L^2(L^2)$.

(b) This follows in a similar way from part (b) of Proposition 4.5.

Now we need to look at the inner products arising in the proof of the estimates:

LEMMA 6.5. – *Let P_ϕ, $P_{\phi,\varepsilon/\tau}$ be as above. Then*

$$\left|\mathrm{Re}\langle P_\phi^r v, \mathrm{Im}\,P_\phi^i v\rangle - \mathrm{Re}\langle P_{\phi,\varepsilon/\tau}^r v, P_{\phi,\varepsilon/\tau}^i v\rangle\right| \leqslant \frac{\varepsilon}{\tau}\left(|v|_{m,\tau,Q}^2 + \left|\frac{\varepsilon D_a}{\tau}v\right|_{m,\tau,Q}^2\right). \tag{28}$$

Proof. – By (24),

$$P_{\phi,\varepsilon/\tau} = P_\delta^m + P_\delta^{m-1}(D_a,\tau) + \cdots + P_\delta^0(D_a,\tau)^m,$$

where $p^m \in \mathcal{S}(S_Q^m)$ and $p^j \in \mathcal{S}(S^j)$ for $j = 0, m-1$. Then $P_{\phi,\varepsilon/\tau} - P_\phi$ has the form

$$\Delta P_{\phi,\varepsilon/\tau} = P_{\phi,\varepsilon/\tau} - P_\phi = \delta\big(Q_\delta^m + (D_a,\tau)Q_\delta^{m-1} + \cdots + (D_a,\tau)^m Q_\delta^0\big)D_a$$

where the Q^j's are in the same class as P^j's.

Again, we need to get an estimate for $\langle Dv, v\rangle$ with

$$D = \big[P_\phi^r, P_\phi^i\big] - \big[P_{\phi,\varepsilon/\tau}^r, P_{\phi,\varepsilon/\tau}^i\big] = [\operatorname{Im} P_{\phi,\varepsilon/\tau}, \operatorname{Re}\Delta P_{\phi,\varepsilon/\tau}] - [\operatorname{Im}\Delta P_{\phi,\varepsilon/\tau}, \operatorname{Re} P_\phi].$$

As in Lemma 5.2, we have:

$$D \in \delta\, OP_\delta\, S^{-1}\big(S^0\big)(D,\tau)^{2m} + \delta^2\, OP_\delta\, S^{-1}\big(S^0\big)(D,\tau)^{2m} D_a.$$

All the terms in $\langle Dv, v\rangle$ can be estimated by the right hand side in (28) except for those terms containing at least $2m-1$ D_b derivatives. Hence we only need to do special estimates for two types of commutators:

(a) $[P_\delta^m, Q_\delta^{m-1}(D_a,\tau)D_a]$,

(b) $[P_\delta^m, Q_\delta^m D_a]$.

The commutators of both P_δ^m and Q_δ^{m-1} with D_a yield operators of the same type, i.e., in $OPS_\delta(S_Q^m)$, respectively $OPS_\delta(S^{m-1})$.

For part (a) we use Proposition 4.7 with $Y = S_Q^m$, $Z = S^{m-1}$, $YZ = S_Q^{2m-1}$ and $W = S^{2m-2}$ to get

$$\big[P_\delta^m, Q_\delta^{m-1}(D_a,\tau)D_a\big] \in OPS_\delta\big(S_Q^{2m-1}\big)(D_a,\tau) + \delta\, OPS_\delta\big(S^{2m-2}\big)(D_a,\tau)D_a^2.$$

The first component does contain $2m-1$ D_b derivatives, but is in the corresponding Q space, therefore it has the correct mapping properties.

Finally, for part (b), by (24), (25) we can use Proposition 4.7 with $Y = Z = S_Q^m$, $YZ = L(H_Q^m, (H_Q^m)')$ and $W = S_Q^{2m-1}$ to get

$$\big[P_\delta^m, Q_\delta^m D_a\big] \in OPS_\delta\big(S_Q^{2m-1}\big)D_a + OPS_\delta\big(L\big(H_Q^m, \big(H_Q^m\big)'\big)\big).$$

Using the above two Lemmas in (27) we get:

$$c^{-1}|v|_{m,\tau,Q}^2 \leqslant c\left(|P_{\phi,\varepsilon/\tau}v|^2 + |D_a v|_{m-1,\tau}^2 + \left|\frac{\varepsilon}{\tau}D_a v\right|_{m,\tau,Q}\right) + 2\operatorname{Re}\big\langle P_{\phi,\varepsilon/\tau}^r v, P_{\phi,\varepsilon/\tau}^i v\big\rangle \tag{29}$$

and further

$$|v|_{m,\tau,Q}^2 \leqslant c\left(\tau|P_{\phi,\varepsilon/\tau}v|^2 + |D_a v|_{m-1,\tau}^2 + \left|\frac{\varepsilon}{\tau}D_a v\right|_{m,\tau,Q}\right). \tag{30}$$

6.5. Conclusion

Now we insert the cutoff function χ and then elliminate μ, μ_1 just as we did in the elliptic case, but estimating the remainders using the H_Q^m instead of the H^m norm. This yields:

$$\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|_{m,\tau,Q}^2 \leqslant c\Bigg(\tau\left|\chi P_{\phi,\varepsilon/\tau}\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|^2 + \left|D_a\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|_{m-1,\tau}^2 + \left|\frac{\varepsilon}{\tau}D_a\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|_{m,\tau,Q} + \left|\left(|D_a|+\tau\right)^{-N}w\right|_{Q,\tau}\Bigg).$$

Use the conjugation lemma for the first RHS term. The other two are controlled by the LHS in the region $|\xi_a| \leqslant c\tau$; outside this region we get the exponential decay from the Gaussian as before, we get

$$\left|\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|_{m,\tau,Q}^2 \leqslant c\left(\tau\left|\chi P_{\phi,\varepsilon/\tau}\mathrm{e}^{-\frac{\varepsilon}{2\tau}D_a^2}w\right|^2 + \mathrm{e}^{-c\varepsilon\tau}\left|\left(|D_a|+\tau\right)^{-N}w\right|_{m,Q,\tau}^2\right). \tag{31}$$

To conclude we use Corollary 6.2 to bound the last term by

$$\left|\left(|D_a|+\tau\right)^{-N}w\right|_{m,Q,\tau}^2 \leqslant c\left|\left(|D_a|+\tau\right)^{-N}P_\phi w\right|^2 + \left|\left(|D_a|+\tau\right)^{-N}w\right|_{m-1,\tau}^2.$$

A. Proof of Theorem 4

Without any restriction in generality we can assume that $\gamma = 0$. Let v be a function whose Fourier transform has compact support. Consider the function $g : R \to R$,

$$g(t) = \int \delta\big(t - \phi(x)\big)u(x)v(x)\,\mathrm{d}x.$$

Its Fourier transform is the entire function

$$\hat{g}(z) = \langle v, \mathrm{e}^{\tau\phi}u\rangle.$$

Clearly

$$\left|\hat{g}(z)\right|_{m-1.\tau} \leqslant c\mathrm{e}^{c|z|}, \quad z \in C,$$

while

$$\left|\hat{g}(z)\right|_{m-1.\tau} \leqslant c\big(1+|z|^{m-1}\big), \quad z \in R.$$

On the other hand, (14) shows that $\hat{g}$ is bounded on the negative imaginary axis. Hence, we can use the Phragmen–Lindelof theorem to conclude that

$$\left|\hat{g}(z)\right|_{m-1.\tau} \leqslant c\big(1+|z|^{m-1}\big), \quad \operatorname{Im} z < 0.$$

This implies that $g(t) = 0$ when $t > 0$. Hence, if h is a smooth function compactly supported in R^+ then:

$$\int g(t)h(t) = 0$$

which is equivalent to

$$\int u(x)v(x)h\big(\phi(x)\big)\,\mathrm{d}x = 0.$$

This holds for any v whose Fourier transform has compact support and, by density for any v. Consequently we get $u = 0$ in $\operatorname{supp} h(\phi(x))$, i.e., $u = 0$ in $\phi > 0$.

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