Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III

Daniel Tataru *

Department of Mathematics

Northwestern University

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Abstract

In an earlier work [17] we have proved that the Strichartz estimate for second order hyperbolic operators hold in full if the coefficients are of class C^2. Here we improve this result and show that the same holds if the coefficients have two derivatives in $L^1(L^\infty)$. Then we use this result to improve the local theory for second order nonlinear hyperbolic equations beyond the results in [16].

1 Introduction

Denote by $x = (x_0, x_1, \cdots x_n) = (x_0, x')$ the coordinates in $R \times R^n$. Sometimes we refer to the first coordinate x_0 as "time" and to x' as "space" coordinates. The Strichartz estimates for solutions to the homogeneous wave equation in $R \times R^n$

$$\Box u = 0 \qquad u(0) = u_0 \quad u_t(0) = u_1$$

have the form

$$\||D|^{1-\rho} u\|_{L^p(L^q)} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2} \tag{1.1}$$

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Such an estimate holds for all pairs (ρ, p, q) satisfying the relations $2 \leq p \leq \infty$, $2 \leq q \leq \infty$ and

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \rho, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{1.2}$$

with the exception of the forbidden endpoint $(1, 2, \infty)$ in dimension $n = 3$. All (ρ, p, q) satisfying these relations are called in the sequel Strichartz pairs. If the equality holds in the second part of (1.2) then the corresponding pair is called a sharp Strichartz pair.

A straightforward consequence of (1.1) is an estimate for solutions to the inhomogeneous problem

$$\Box u = f \qquad u(0) = 0 \quad u_t(0) = 0$$

namely

$$\| |D|^{1-\rho} u \|_{L^p(L^q)} \lesssim \|f\|_{L^1(L^2)} \tag{1.3}$$

The simplest case of (1.3) is the well-known energy estimate

$$\|\nabla u\|_{L^\infty(L^2)} \lesssim \|f\|_{L^1(L^2)} \tag{1.4}$$

However, there is a larger family of estimates for solutions to the inhomogeneous wave equation where we also vary the norms in the right hand side,

$$\| |D|^{1-\rho} u \|_{L^p(0,T;L^q)} \lesssim \| |D|^{\rho_1} f \|_{L^{p_1'}(L^{q_1'})} \tag{1.5}$$

This holds for all Strichartz pairs (ρ, p, q), (ρ_1, p_1, q_1).

Estimates of the above type were first proved in the constant coefficient case in [3], [15]. Further references can be found in a more recent expository article [5]. The endpoint estimate $(p, q) = (2, \frac{2(n-1)}{n-3})$ was only recently obtained in [8] $(n \geq 4)$.

In this article we are interested in the variable coefficient case of these estimates, where we replace $\Box$ by a second order operator of the form

$$P(x, D) = \partial_i g^{ij}(x) \partial_j$$

which is strongly hyperbolic with respect to time.

If the coefficients g^{ij} are smooth then the estimates hold locally, see [12] (except for the endpoint). For C^2 coefficients, in dimension $n = 2, 3$, the estimates are proved in [13]. On the other hand in [14] they are shown to fail for C^s coefficients, $s < 2$.

In a previous article [17] we show that the full estimates hold in all dimensions for operators with C^2 coefficients, and we also obtain appropriate weaker estimates for operators with C^s coefficients for $0 < s < 2$. The main goal of this article is to prove that the estimates are still true if the coefficients have two derivatives in $L^1(L^\infty)$, and then to explore some consequences of this result.

In what follows we assume that the matrices $(g^{ij}(x))$, $(g^{ij}(x))^{-1}$ are uniformly bounded and of signature $(1,n)$. Furthermore, we also assume that the surfaces $x_0 = const$ are space-like uniformly in x, i.e. that $g^{00} > c > 0$.

Before we state our results observe that in the variable coefficient case there are at least two parameters that we need to take into account, namely the time T for which the estimates hold and the size of the second derivatives of the coefficients. To these we add, as in [17], a third parameter, μ, related to the constants in the estimates. This third parameter plays an essential role in the applications. Our first result is

Theorem 1. *Assume that $\partial_x^2 g \in L^1(L^\infty)$. Let (ρ,p,q) be a Strichartz pair. Then*

$$\||D'|^{1-\rho}u\|_{L^p(0,T;L^q)} \lesssim \mu^{\frac{1}{p}}\|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}}\|P(x,D)u\|_{L^1(L^2)}, \tag{1.6}$$

whenever the right hand side is finite provided that $\mu \geq 1$ and

$$T\|\partial_x^2 g\|_{L^1(L^\infty)} \leq \mu^2 \tag{1.7}$$

Observe that the relation (1.7) is nontrivial in terms of scaling since both sides are dimensionless. If we assume in addition that u is compactly supported then (1.6) holds for all $\mu > 0$ since one can use energy estimates to bound the $L^\infty(L^2)$ norm of ∇u by the $L^1(L^2)$ norm of $P(x,D)u$. Then it is more convenient to state the estimate in the form

$$\||D|^{-\rho}\nabla u\|_{L^p(0,T;L^q)} \leq \mu^{\frac{1}{p}}\|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}}\|P(x,D)u\|_{L^1(L^2)}, \tag{1.8}$$

This differs from (1.6) by a microlocal elliptic estimate away from the characteristic set of P.

The estimates in Theorem 1 also lead to some weaker Strichartz estimates in the case when the coefficients have less regularity. Following the terminology in [18] define the microlocalizable scale of spaces $\mathcal{X}^s$ by

$$\|u\|_{\mathcal{X}^s} = \sup_\lambda \lambda^s \|S_\lambda u\|_{L^1(L^\infty)}$$

where

$$1 = \sum_{\lambda=2^j} S_\lambda$$

is a standard Paley Littlewood decomposition. Then we consider operators with coefficients in the $\mathcal{X}^s$ spaces for $0 < s \leq 2$. To measure the regularity of the solutions we introduce the homogeneous Besov type spaces $\dot{B}_s^{pqr}$ with norms

$$\|u\|^r_{\dot{B}_s^{pqr}} = \sum_{\lambda=2^j} \lambda^s \|S_\lambda u\|^r_{L^p(L^q)}$$

Theorem 2. *Assume that P is in divergence form and that $g \in \mathcal{X}^s$, $0 \leq s \leq 2$. Let (ρ, p, q) be a Strichartz pair and*

$$\sigma = \frac{2-s}{2+s}$$

Then

$$\||D|^{1-\rho-\frac{\sigma}{p}} u\|_{\dot{B}_0^{pq\infty}} \lesssim \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \||D|^{-\sigma} Pu\|_{L^1(L^2)} \tag{1.9}$$

for all u compactly supported in $[0,T]$ and μ, T satisfying

$$T^s \|g\|^2_{\mathcal{X}^s} \leq \mu^{2+s}$$

Compared to the full estimates, here we loose $\dfrac{\sigma}{p}$ derivatives, but, on the other hand, we assume less about the right hand side of the equation. The reason we use Besov spaces on the left is that in general we do not have square summability of the dyadic pieces on the right. The argument we use for $s = 2$ to avoid this problem can be extended to $s > \frac{3}{2}$. Another borderline case is when $\nabla g \in L^2(L^\infty)$. Here we do not explore such issues in detail as they are not the main topic of the article. Our main results are the dyadic estimates.

As in Theorem 1, if $\mu \geq 1$ then the assumption that u has compact support is no longer needed and, modulo an elliptic estimate, (1.9) is equivalent to

$$\||D'|^{-\rho-\frac{\sigma}{p}} \nabla u\|_{L^p(\dot{B}_0^{q\infty})} \lesssim \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \||D'|^{-\sigma} Pu\|_{L^1(L^2)} \tag{1.10}$$

The estimates for solutions to the homogeneous equation follow easily from the above theorems combined with the energy estimates. Uniform energy estimates for a time T hold for instance if $1 \leq s \leq 2$ and

$$\|\partial_t g\|_{L^1(L^\infty)} \leq 1$$

Now let us turn our attention to the estimates for the inhomogeneous problem. Our first result is a generalization of (1.6).

Theorem 3. *Assume that the coefficients satisfy $\partial_x^2 g \in L^1(L^\infty)$. Let (ρ, p, q) be a Strichartz pair. Then*

$$\| |D'|^{1-\rho} u\|_{L^p(L^q)} \lesssim \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \|f_1\|_{L^1(L^2)} + \| |D'|^\rho f_2\|_{L^{p'}(L^{q'})} \tag{1.11}$$

for all u satisfying

$$P(x, D)u = f_1 + f_2$$

and $\mu \geq 1$ with

$$T \|\partial_x^2 g\|_{L^1(L^\infty)} \leq \mu^2$$

The proof of this result is similar to the proof of the corresponding result in [17]. The estimate (1.11) can also be rephrased in a manner similar to (1.8). The analogue of this result in the case when the coefficients are in $\mathcal{X}^s$, with $0 \leq s \leq 2$, is

Theorem 4. *Assume that P is in divergence form and that $g \in \mathcal{X}^s$, $0 \leq s < 2$. Let (ρ, p, q) be a Strichartz pair. Then*

$$\| |D|^{1-\rho-\frac{\sigma}{p}} u\|_{\dot{B}_0^{pq\infty}} \lesssim \mu^{\frac{1}{p}} \|\nabla u\|_{L^\infty(L^2)} + \mu^{-\frac{1}{p'}} \|D^{-\sigma} f_1\|_{L^1(L^2)} + \| |D|^{\rho-\frac{\sigma}{p}} f_2\|_{B_0^{p'q'\infty}} \tag{1.12}$$

for all u compactly supported in $[0, T]$ satisfying

$$P(x, D)u = f_1 + f_2$$

and

$$T^s \|g\|_{\mathcal{X}^s}^2 \leq \mu^{2+s}$$

Applied to solutions for the initial value problem this shows that the full Strichartz estimates hold for operators with C^2 coefficients.

Corollary 1. *Assume that $D^2 g \in L^1(0,T;L^\infty)$. Let $(\rho,p,q),(\rho_1,p_1,q_1)$ be Strichartz pairs. Then the following estimate holds*

$$\||D'|^{1-\rho_1} u\|_{L^{p_1}(0,T;L^{q_1})} \lesssim \||D'|^{\rho} P(x,D)u\|_{L^{p'}(0,T;L^{q'})} + \|u_0\|_{\dot H^1} + T^{-1}\|u_0\|_{L^2} + \|u_1\|_{L^2} \quad (1.13)$$

for

$$T\|\partial_x^2 g\|_{L^1(L^\infty)} \leq 1.$$

The corresponding result for $1 \leq s < 2$ is

Corollary 2. *Assume that the operator P has $\mathcal{X}^s$ coefficients*[1] *with $1 \leq s \leq 2$. Let $(\rho,p,q),(\rho_1,p_1,q_1)$ be Strichartz pairs. Then*

$$\||D'|^{1-\rho_1-\frac{\sigma}{p_1}} u\|_{L^{p_1}(0,1;\dot B_0^{q_1\infty})} \lesssim \||D'|^{\rho+\frac{\sigma}{p}} Pu\|_{L^{p'}(0,1;\dot B_0^{q'\infty})} + \|u_0\|_{H^1} + \|u_1\|_{L^2} \quad (1.14)$$

The weaker form of the special case of this result which we use in the study of the nonlinear wave equation corresponds to $s = 1$.

Corollary 3. *Assume that the coefficients of the operator P satisfy $\nabla g \in L^1(L^\infty)$. Let (ρ,p,q) be a Strichartz pair. Then*

$$\||D'|^{1-\rho_1-\frac{1}{3p}} u\|_{L^p(0,1;\dot B_0^{q\infty})} \lesssim \|Pu\|_{L^1(L^\infty)} + \|u_0\|_{H^1} + \|u_1\|_{L^2} \quad (1.15)$$

2 The FBI transform and error estimates

The essential tool in our proof of the Strichartz estimates is the FBI transform. The technique we use was first developed in [16]. The idea is quite simple, namely to obtain estimates for the FBI transform of the solution. This requires conjugating the operator P with respect to the FBI transform. The first part of [16] is devoted to proving L^2 error estimates for this

[1]For $s = 1$ this has to be replaced by $\nabla g \in L^1(L^\infty)$ so that we still have the energy estimates

conjugation. Here we cannot use directly the error estimates in [16], but the new estimates we prove are similar in spirit.

The FBI transform of a temperate distribution f is a holomorphic function in C^n defined as

$$(T_\lambda f)(z) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int e^{-\frac{\lambda}{2}(z-y)^2} f(y)\ dy \tag{2.16}$$

Set

$$z = x - i\xi, \qquad \Phi(z) = e^{-\lambda\xi^2}$$

Then the operator T_λ is an isometry from $L^2(R^n)$ onto the closed subspace of holomorphic functions in $L^2_\Phi(C^n)$. One inversion formula is provided by the adjoint operator:

$$f(y) = \lambda^{\frac{3n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} \int \Phi(z) e^{-\frac{\lambda}{2}(\bar{z}-y)^2} (T_\lambda f)(z)\ dx d\xi$$

The FBI transform of a function f can also be expressed in terms of the Fourier transform of f,

$$T_\lambda f(z) = \lambda^{\frac{5n}{4}} 2^{-\frac{n}{2}} \pi^{-\frac{3n}{4}} e^{\frac{\lambda}{2}\xi^2} \int e^{-\frac{\lambda}{2}(\eta-\xi)^2} e^{i\lambda x\eta} \hat{f}(\lambda\eta) d\eta \tag{2.17}$$

For additional information on the FBI transform we refer the reader to Delort [4].

Given a compactly supported symbol $a(x,\xi)$ we define the rescaled operators

$$A_\lambda = A(x, \frac{D}{\lambda})$$

Then our main result in [16] deals with the problem of conjugating the operator A_λ with respect to T_λ. All we need here is the result for C^2 symbols. In this case we set

$$\tilde{a}_\lambda = a + \frac{1}{-i\lambda} a_x (\partial_\xi - \lambda\xi) + \frac{1}{\lambda} a_\xi (\frac{1}{i}\partial_x - \lambda\xi)$$

Since we only consider this operator on holomorphic functions, we can also rewrite it in a complex fashion as

$$\tilde{a}_\lambda = a + \frac{2}{\lambda}(\bar{\partial} a)(\partial - i\lambda\xi), \qquad 1 < s \leq 2 \tag{2.18}$$

Define the remainder

$$R_{\lambda,a} = T_\lambda A_\lambda - \tilde{a}_\lambda T_\lambda$$

Then the result in [16] is (see Theorem 1 and Remark 1.2 there):

Theorem 5. *Assume that* $a \in C_x^2(C_0^\infty)$. *Then*

$$\|R_{\lambda,a}\|_{L^2 \to L^2_\phi} \lesssim \lambda^{-1}$$

and

$$\|(\partial_\xi - \lambda\xi)R_{\lambda,a}\|_{L^2 \to L^2_\phi} \lesssim \lambda^{-\frac{1}{2}}$$

The motivation for the second inequality is that it allows us to compute the regularity of the traces of the remainder on the characteristic cone

$$K = \{(x,\xi);\ p(x,\xi) = 0\}$$

by using the trace theorem:

Corollary 4. *Assume that* $a \in C_x^2(C_0^\infty)$. *Then*

$$\|R_{\lambda,a}\|_{L^2 \to L^2_\phi(K)} \lesssim \lambda^{-\frac{3}{4}}$$

In our case we need a modification of Theorem 5:

Theorem 6. *Assume that the symbol* a *satisfies* $\partial_x^2 a \in L^1(L^\infty)(C_0^\infty)$ *and that its Fourier transform in* x *is supported in* $B(0,\sqrt{\lambda})$. *Then*

$$\|\Phi^{\frac{1}{2}} R_{\lambda,a}\|_{L^\infty(L^2) \to L^2} \lesssim \lambda^{-\frac{3}{4}} \tag{2.19}$$

$$\|\Phi^{\frac{1}{2}} R_{\lambda,a}\|_{L^\infty(L^2) \to L^1_{x_0} L^2_{x'\xi'}(K)} \lesssim \lambda^{-\frac{3}{4}} \tag{2.20}$$

One can think of this result in terms of the previous one. More precisely, Theorem 5 implies Theorem 6 when the argument is supported in a time interval of length $\lambda^{-\frac{1}{2}}$. Then the summability with respect to such time intervals comes from the corresponding summability for the second derivatives of a.

Proof. Without any restriction in generality we can assume that the symbol a has the form

$$a(x,\xi) = b(x)c(\xi)$$

with $\partial_x^2 b \in L^1(L^\infty)$ and $c \in C_0^\infty$. This reduction can be achieved by writing a Fourier series for a as a function of ξ with respect to a larger region and then by truncating the terms in the expansion near the support of a.

First we prove the result for b, then we use Theorem 5 for c to obtain the desired estimate for $b(x)c(\xi)$.

The remainder $\Phi^{\frac{1}{2}} R_{\lambda,b}$ is given by

$$\Phi^{\frac{1}{2}} R_{\lambda,b} u(x,\xi) = \lambda^{\frac{3(n+1)}{4}} \int e^{i\lambda\xi(x-y)} e^{-\frac{\lambda}{2}(x-y)^2} (b(x) + b_x(x)(y-x) - b(y)) u(y) dy$$

Using the TT^* argument the estimate (2.19) reduces to

$$\|R^*_{\lambda,b} \Phi R_{\lambda,b}\|_{L^\infty(L^2) \to L^1(L^2)} \lesssim \lambda^{-\frac{3}{2}}$$

But a simple computation shows that $R^*_{\lambda,b} \Phi R_{\lambda,b}$ is a multiplication operator by

$$f(y) = \lambda^{\frac{n+1}{2}} \int e^{-\lambda(x-y)^2} (b(x) + b_x(x)(y-x) - b(y))^2 dx$$

Then to conclude we need to show that

$$\|f\|_{L^1(L^\infty)} \lesssim \lambda^{-\frac{3}{2}}$$

This can be easily done using the Taylor series expansion for b together with the Sobolev type estimate

$$\|\partial_x^2 b\|_{L^2(L^\infty)} \lesssim \lambda^{\frac{1}{4}} \|\partial_x^2 b\|_{L^1(L^\infty)}$$

which is true when the Fourier transform of b is supported in $|\xi| \leq \lambda^{\frac{1}{2}}$.

To prove the second estimate look first at the restriction of the operator $\Phi^{\frac{1}{2}} R_{\lambda,b}$ acting from a fixed time slice into a fixed time slice. This corresponds to the decomposition

$$\Phi^{\frac{1}{2}} R_{\lambda,b} u(x_0)_{|K} = \lambda^{\frac{3}{4}} \int e^{i\lambda\xi_0(x_0-y_0)} R_{x_0,y_0} u(y_0) dy_0$$

where

$$R_{x_0,y_0} u(x', \xi') = \lambda^{\frac{3n}{4}} \int e^{i\lambda\xi'(x'-y')} e^{-\frac{\lambda}{2}(x-y)^2} (b(x) + b_x(x)(y-x) - b(y)) u(y') dy'$$

We first estimate the operator R_{x_0,y_0} in L^2.

Lemma 5. *Suppose that* $\partial_x^2 b \in L^1(L^\infty)$. *Then*

$$\|R_{x_0,y_0}\|_{L^2 \to L^2} \lesssim \lambda^{-1} e^{-c\lambda(x_0-y_0)^2} \int_{x_0}^{y_0} \|(\partial_x^2 b)(s)\|_{L^\infty} ds \tag{2.21}$$

Proof. The operator $R^*_{x_0,y_0}R_{x_0,y_0}$ is a multiplication operator by

$$f(y') = \lambda^{\frac{n}{2}} \int e^{-\lambda(x-y)^2}(b(x)+b_x(x)(y-x)-b(y))^2 dx'$$

Since

$$(b(x)+b_x(x)(y-x)-b(y)) = \int_0^1 t(\partial_x^2 b)(tx+(1-t)y)(y-x,y-x)dt$$

we obtain

$$|b(x)+b_x(x)(y-x)-b(y)| \lesssim |x-y|^2 \int_{x_0}^{y_0} \|(\partial_x^2 b)(s)\|_{L^\infty} ds$$

Then

$$\begin{aligned} |f(y')| &\lesssim \lambda^{\frac{n}{2}}(\int_{x_0}^{y_0} \|(\partial_x^2 b)(s)\|_{L^\infty} ds)^2 \int |x-y|^4 e^{-\lambda(x-y)^2} dx' \\ &\lesssim \lambda^{-2} e^{-c(x_0-y_0)^2}(\int_{x_0}^{y_0} \|(\partial_x^2 b)(s)\|_{L^\infty} ds)^2 \end{aligned}$$

This implies the estimate in the Lemma.

Now we use (2.21) to conclude the proof of Theorem 6. We have

$$\begin{aligned} \|\Phi^{\frac{1}{2}} R_{\lambda,b} u_{|K}\|_{L^1_{x_0}(L^2_{x',\xi'})} &\lesssim \lambda^{\frac{3}{4}} \int \|R_{x_0y_0}\|_{L^2\to L^2} \|u(y_0)\|_{L^2} dx_0 dy_0 \\ &\lesssim \|u\|_{L^\infty(L^2)} \lambda^{\frac{3}{4}} \int \|R_{x_0y_0}\|_{L^2\to L^2} dx_0 dy_0 \end{aligned}$$

Then (2.20) for b follows if we verify that

$$\int \|R_{x_0y_0}\|_{L^2\to L^2} dx_0 dy_0 \lesssim c\lambda^{-\frac{3}{2}} \tag{2.22}$$

But by (2.21) we get

$$\int \|R_{x_0y_0}\|_{L^2\to L^2} \lesssim \lambda^{-1} \int \frac{1}{|x_0-y_0|} e^{-c\lambda(x_0-y_0)^2} \int_{x_0}^{y_0} \|(\partial_x^2 b)(s)\|_{L^\infty} ds dx_0 dy_0$$

We change the order of integration and use scaling to integrate with respect to x_0, y_0. This yields

$$\int \|R_{x_0y_0}\|_{L^2\to L^2} \lesssim \lambda^{-\frac{3}{2}} \int \|(\partial_x^2 b)(s)\|_{L^\infty} ds$$

which gives (2.22).

Finally, we need to put together the error estimates for b and c. We have

$$R_{\lambda,a} = R_{\lambda,b} C(\frac{D}{\lambda}) + (b + \lambda^{-1}(\partial_x b)(\partial_\xi - \lambda\xi))R_{\lambda,c}$$

The bounds for the first right hand side term follow immediately from the bounds for $R_{\lambda,b}$. For the second right hand side term observe that $b, \partial_x b$ are bounded therefore it suffices to prove the same bounds for the operators

$$R_{\lambda,c}, \qquad \lambda^{-1}(\partial_\xi - \lambda\xi))R_{\lambda,c}$$

In both cases these follow from the L^2 bounds in Theorem 5 since x, ξ are in a compact set.

3 Proof of Theorem 1.

The first part of the proof involves several localization type arguments. First we reduce the estimate to the case when $\mu = 1$ and u is supported in a cube of size 1. Then we use a Paley-Littlewood decomposition to reduce the problem to the corresponding dyadic estimates at fixed frequency λ. Finally we truncate the coefficients of P at frequency $\sqrt{\lambda}$.

A second stage of the proof is to obtain good L^2 estimates for the FBI transform of u. Conjugating the operator P with respect to the FBI transform T_λ we get two ode's in the "FBI" space. One of these ode's is along the gradient flow of $p(x,\xi)$ and provides an elliptic estimate away from the characteristic cone. The other one is along the Hamilton flow of p and corresponds to propagation of singularities. Exploiting the L^2 information coming from the two ode's we can reduce the dyadic inequalities to certain oscillatory integral estimates.

The last step is to prove the oscillatory integral estimates, using a complex interpolation argument. This requires sharp estimates on the regularity of the Hamilton flow. This argument is almost identical to the one in [17] and we do not repeat it here. Instead we just study the regularity of the Hamilton flow and set up the oscillatory integrals as in [17].

Reduction to the case $\mu = 1$. Because of the scale invariance we can assume without any restriction in generality that $\|\partial_x^2 g\|_{L^1(L^\infty)} \leq 1$ and that $T = \mu^2$. The interesting case is when $\rho < 1$ since all other estimates follow by Sobolev embeddings.

To achieve the reduction of (1.6) to the case $\mu = 1$ decompose the interval $[0,T]$ into maximal subintervals

$$0 = t_0 < t_1 < ..... < t_k = T$$

so that

$$\|P(x,D)u\|_{L^1(t_j,t_{j+1};L^2)} \leq \mu^{-1}\|P(x,D)u\|_{L^1(0,T;L^2)} \tag{3.23}$$

and

$$(t_{j+1}-t_j)\|\partial_x^2 g\|_{L^1(t_j,t_{j+1};L^\infty)} \leq 1 \tag{3.24}$$

for all j. We claim that the number k of intervals satisfies

$$\mu < k < 2\mu$$

The bound from below follows directly from (3.23). For the bound from above observe that the equality must hold either in (3.23) or in (3.24). The number k_1 of intervals for which the equality holds in (3.23) is at most μ. On an interval in which equality holds in the (3.24), on the other hand, we get

$$T^{-\frac{1}{2}}(t_{j+1}-t_j) + T^{\frac{1}{2}}\|\partial_x^2 g\|_{L^1(t_j,t_{j+1};L^\infty)} \geq 2$$

Summing up over all intervals with this property we get

$$2T^{\frac{1}{2}} \geq 2k_2$$

where k_2 is the number of such intervals. This gives

$$k_2 \leq T^{\frac{1}{2}} = \mu$$

therefore

$$k \leq k_1 + k_2 = 2\mu$$

If we apply the estimate (1.6) for $\mu = 1$ on each such interval we obtain

$$\||D'|^{-\rho}\nabla u\|_{L^p(t_j,t_{j+1};L^q)} \lesssim \|\nabla u\|_{L^\infty(0,T;L^2)} + \mu^{-1}\|P(x,D)u\|_{L^1(0,T;L^2)}$$

Raise this to the power p and sum up over j to obtain

$$\||D'|^{-\rho}\nabla u\|^p_{L^p(0,T;L^q)} \lesssim \mu\|\nabla u\|^p_{L^\infty(0,T;L^2)} + \mu^{1-p}\|P(x,D)u\|^p_{L^1(0,T;L^2)}$$

This implies (1.6) for u in $[0,T]$.

Reduction to a cube of size 1. If $\mu = 1$ then by rescaling we can take $T = 1$, $\|\partial_x^2 g\|_{L^1(L^\infty)} \leq 1$. First we observe that it suffices to prove the estimate with homogeneous spaces replaced by inhomogeneous spaces,

$$\|\langle D'\rangle^{-\rho}\nabla u\|_{L^p(0,1;L^q)} \lesssim \|\nabla u\|_{L^\infty(L^2)} + \|u\|_{L^\infty(L^2)} + \|P(x,D)u\|_{L^1(L^2)}, \tag{3.25}$$

where $\langle D'\rangle = (1+|D'|^2)^{\frac{1}{2}}$. Indeed, in order to obtain (1.6) from (3.25) we can use a Sobolev estimate

$$\||D'|^{-\rho}S_0'\nabla u\|_{L^p(L^q)} \lesssim \|S_0'\nabla u\|_{L^\infty(L^2)}$$

for low frequencies and (3.25) for $(1-S_0')u$.

A second observation is that, due to the variation of parameters formula, it suffices to prove the estimate for H^1 solutions to the homogeneous equation. These we can extend, still in H^1, to a larger time interval. Then truncating in time we reduce the problem to the case when u is supported in a fixed time interval, say of length 2, and $P(x,D)u \in L^2$.

Since the H^1 norm of u can be obtained from the energy estimate

$$\|\nabla u\|_{L^\infty(0,2;L^2)} \lesssim \|P(x,D)u\|_{L^2}$$

(3.25) reduces to

$$\||D'|^{-\rho}\nabla u\|_{L^p(0,2;L^q)} \lesssim \|P(x,D)u\|_{L^2}, \tag{3.26}$$

Now we decompose $P(x,D)u$ in components supported in cubes of size 1. Due to the the finite speed of propagation this yields a decomposition of u in components supported in cubes of fixed size $R > 1$. If the estimate holds in each such cube then we can sum up and obtain the global estimate since $p, q \geq 2$.

An elliptic estimate. We still need to prove (3.26) for u supported in a cube of size 1. We claim that it suffices to prove the weaker estimate

$$\||D|^{1-\rho}u\|_{L^p(L^q)} \lesssim \|P(x,D)u\|_{L^2}, \tag{3.27}$$

Clearly (3.27) implies (3.26) microlocally away from a conic neighborhood of $\xi' = 0$. But near $\xi' = 0$ the operator P is elliptic. Therefore, from the elliptic theory[2] we obtain that u is H^2 near $\xi' = 0$. This is more than sufficient for our estimates[3].

[2] note that P has C^1 coefficients

[3] $H^{\frac{3}{2}}$ would suffice

Reduction to dyadic estimates. Let $s(\xi)$ be a smooth symbol supported in $\frac{1}{2} \leq |\xi| \leq 2$ so that

$$\sum_j s(2^{-j}\xi) = 1$$

Given $\lambda = 2^j \geq 1$ let

$$S_\lambda = S(\frac{D}{\lambda})$$

be the standard multiplier which truncates at frequency λ, and

$$S_0 = \sum_{j<0} s(2^{-j}\xi)$$

the multiplier which cuts off the frequencies less than 1.

We replace (3.27) with the stronger estimate

$$\| |D|^{1-\rho} u\|_{L^p(L^q)} \lesssim \|u\|_{\dot{H}^1} + \|P(x,D)\|_{L^r(L^2)} \tag{3.28}$$

for some $1 < r < 2$. The motivation for this choice of r is that for $r = 1$ it is more difficult to use the Paley-Littlewood theory while for $r = 2$ the commutator estimates become harder due to the low regularity of the coefficients.

We claim that (3.28) reduces to the corresponding dyadic estimates,

$$\lambda^{-\rho}\|S_\lambda u\|_{L^p(L^q)} \lesssim \|S_\lambda u\|_{L^2} + \lambda^{-1}\|P(x,D)S_\lambda u\|_{L^r(L^2)} \tag{3.29}$$

To obtain (3.28) we square (3.29) and use the Paley-Littlewood theory to sum up the dyadic pieces. Then (3.28) follows provided that we can handle the commutators and show that

$$\sum_{\lambda=2^j} \|[P,S_\lambda]u\|^2_{L^r(L^2)} \lesssim \|u\|^2_{H^1}$$

or equivalently

$$\sum_{\lambda=2^j} \|[g,S_\lambda]\nabla v\|^2_{L^r(L^2)} \lesssim \|v\|^2_{L^2} \tag{3.30}$$

for v compactly supported. If $\tilde{S}_\lambda$ is another multiplier whose support is slightly larger then we can rewrite the operator on the left as

$$\begin{aligned}
[g,S_\lambda]\nabla &= S_\lambda(\nabla g) + [g,\nabla S_\lambda] \\
&= S_\lambda(\nabla g) + [g,\nabla S_\lambda]\tilde{S}_\lambda - \nabla S_\lambda g(1-\tilde{S}_\lambda)
\end{aligned}$$

The estimate for the first term $S_\lambda(\nabla g)$ is trivial. For the second term we get the orthogonality from $\tilde{S}_\lambda$ so it suffices to know the commutator estimate

$$[g, \nabla S_\lambda] : L^2 \to L^2$$

This holds since g is Lipschitz (see Taylor [18]). Finally, for the third term it suffices to show that

$$\|\nabla S_\lambda g(1-\tilde{S}_\lambda)\|_{L^2 \to L^r(L^2)} \lesssim \lambda^{\frac{r-2}{2r}}$$

Since the supports of the symbols S_λ and $(1-\tilde{S}_\lambda)$ are $O(\lambda)$ away, it follows that only the frequencies of size λ and higher in g are relevant. In other words, if U_λ is a multiplier selecting the frequencies of size λ and higher, then

$$\nabla S_\lambda g(1-\tilde{S}_\lambda) = \nabla S_\lambda(U_{\frac{\lambda}{10}} g)(1-\tilde{S}_\lambda)$$

therefore

$$\begin{aligned} \|\nabla S_\lambda g(1-\tilde{S}_\lambda)\|_{L^2 \to L^r(L^2)} &\lesssim \|U_\lambda \nabla g\|_{L^{\frac{2r}{2-r}}(L^\infty)} \\ &\lesssim \|U_\lambda \nabla g\|_{L^\infty}^{1-\frac{2-r}{2r}} \|U_\lambda \nabla g\|_{L^1(L^\infty)}^{\frac{2-r}{2r}} \\ &\lesssim \lambda^{\frac{r-2}{2r}} \end{aligned}$$

At the last step we have used the fact that $\nabla u \in L^\infty$ and $\nabla^2 u \in L^1(L^\infty)$.

This concludes the reduction to the dyadic estimates. As the reader may have noticed, up to this point we had a large leeway as far as the function spaces are concerned. Naively, the reason for this is that in the FBI calculus we need two derivatives on the coefficients for each derivative in the error estimates, while in the usual commutator estimates one derivative on the coefficients suffices. Now we return to the original set-up and strengthen (3.28) to

$$\lambda^{-\rho}\|S_\lambda u\|_{L^p(L^q)} \lesssim \|S_\lambda u\|_{L^\infty(L^2)} + \lambda^{-1}\|PS_\lambda u\|_{L^1(L^2)} \tag{3.31}$$

Truncating the coefficients at frequency $\sqrt{\lambda}$. We claim that it suffices to prove the dyadic estimate (3.31) when the coefficients have Fourier transform supported in $|\xi| \le \sqrt{\lambda}$.

Given $\lambda > 0$ define the multiplier

$$U_\lambda = \sum_{j \ge 0} S_{2^j\lambda}$$

which selects the frequencies λ and higher. Then we replace g^{ij} by $(1-U_{\epsilon\sqrt{\lambda}})g^{ij}$ where ϵ is a small parameter. Then we modify the factor $\|PS_\lambda u\|_{L^1(L^2)}$ in (3.31) at most by

$$\|(U_{\epsilon\sqrt{\lambda}}g^{ij})\partial_i\partial_j S_\lambda u\|_{L^1(L^2)} \lesssim \lambda\|D^2 g\|_{L^1(L^\infty)}\|u\|_{L^\infty(L^2)}$$

But this is controlled by the first right hand side term. Hence it suffices to prove the estimates with g^{ij} replaced by $(1-U_{\epsilon\sqrt{\lambda}})g^{ij}$, which have Fourier transform supported in $|\xi| < \lambda^{\frac{1}{2}}$. In particular, since the coefficients of p are Lipschitz, it follows that

$$|\Delta p(x,\xi)| \ll \lambda^{\frac{1}{2}} \qquad \frac{1}{4} \leq |\xi| \leq 4 \tag{3.32}$$

Once we have truncated the coefficients we can use again the L^2 well-posedness of the hyperbolic problem to reduce the estimate to

$$\lambda^{-\rho}\|S_\lambda u\|_{L^p(L^q)} \lesssim \|S_\lambda u\|_{L^\infty(L^2)} + \lambda^{-1}\|PS_\lambda u\|_{L^2} \tag{3.33}$$

This is a minor reduction which simplifies a bit the analysis of the function g below.

Using the FBI transform. Set

$$w = \Phi^{\frac{1}{2}} T_\lambda S_\lambda u$$

Then we try to get good L^2 estimates for w. The function $S_\lambda u$ can then be recovered from

$$S_\lambda u = T_\lambda^* \Phi^{\frac{1}{2}} w$$

Observe first that w is concentrated in the region

$$U = \{|x| \leq 2, \ \ \frac{1}{4} \leq |\xi| \leq 4\};$$

Outside this region we have

$$\|w\|_{L^2(U^c)} \leq e^{-c\lambda}\|S_\lambda u\|_{L^2}$$

which is a straightforward consequence of the representation formula for the FBI transform in terms of the Fourier transform of the argument.

Hence it suffices to get good estimates for w in the region U. Set

$$(\lambda p + 2(\bar{\partial} p)(\partial - i\lambda\xi))\Phi^{-\frac{1}{2}} w = \Phi^{-\frac{1}{2}} g \tag{3.34}$$

where

$$g = \Phi^{\frac{1}{2}}(\lambda R_{\lambda,p} S_\lambda u + \lambda^{-1} T_\lambda P(x,D) S_\lambda u)$$

We can use the error estimates to get good bounds for g in terms of the right hand side of (3.33).

Lemma 6. *The following estimates hold:*

$$\|g\|_{L^2} \leq \lambda^{\frac{1}{4}}\|S_\lambda u\|_{L^\infty(L^2)} + \lambda^{-1}\|P(x,D)S_\lambda u\|_{L^2} \tag{3.35}$$

$$\|g_{|K}\|_{L^1_{x_0}(L^2_{x'\xi'})} \lesssim \lambda^{\frac{1}{4}}\|S_\lambda u\|_{L^\infty(L^2)} + \lambda^{-1}\|P(x,D)S_\lambda u\|_{L^2} \tag{3.36}$$

In (3.36) the $L^2_{\xi'}$ norm of $g_{|K}$ is taken with respect to the surface measure on K. The information about g in this lemma is all we need in the sequel.

The first order equation (3.34) Since $\Phi^{-\frac{1}{2}}w$ is holomorphic as a function of $z = x - i\xi$, we have

$$(\partial - i\lambda\xi)\Phi^{-\frac{1}{2}}w = \Phi^{-\frac{1}{2}}(\partial_x - i\lambda\xi)w = i(\partial_\xi - \lambda\xi)\Phi^{-\frac{1}{2}}w = i\Phi^{-\frac{1}{2}}\partial_\xi w$$

(Recall that $\Phi = e^{-\lambda\xi^2}$). Then from (3.34) we obtain in effect two pieces of information, namely

$$[(p_x\partial_\xi - p_\xi\partial_x) - i\lambda(p - p_\xi \cdot \xi)]\, w = -ig \tag{3.37}$$

respectively

$$[(p_x\partial_x + p_\xi\partial_\xi) + \lambda(p - ip_x \cdot \xi)]\, w = g \tag{3.38}$$

The first equation is an ode along the Hamilton flow of p, while the second equation is an ode along the gradient curves of p. Our strategy is now to use (3.37) to obtain good estimates for w on the characteristic cone K, and then to use (3.38) to obtain good decay rates away from the cone. For all the analysis away from the cone we only use the fact that the coefficients are Lipschitz. The $L^1(L^\infty)$ bound on the second derivatives of g is only needed for the analysis of the Hamilton flow on the cone.

If we take the inner product of (3.38) with pw and integrate by parts then we obtain

$$\lambda\|pw\|^2_{L^2} = \frac{1}{2} < (|\nabla p|^2 + p\Delta p)w, w > + \Re < pw, g >$$

Then by (3.32) we get

$$\lambda^{\frac{1}{2}}\|pw\|_{L^2} \lesssim \|w\|_{L^2} + \lambda^{-\frac{1}{2}}\|g\|_{L^2} \tag{3.39}$$

Away from the characteristic cone this estimate improves the L^2 bound on w by a $\lambda^{-\frac{1}{2}}$ factor. In the physical space this corresponds to a gain of 1/2 derivative, which suffices for the Strichartz estimates simply by the Sobolev embeddings. Then we can cut away this region and assume without any restriction of generality that w, g are supported in a neighborhood of the cone.

We use (3.38) to decompose w into two parts,

$$w = w_1 + w_2$$

where w_1 solves the inhomogeneous equation

$$[(p_x\partial_x + p_\xi\partial_\xi) + \lambda(p - ip_x\cdot\xi)]\, w_1 = -ig, \qquad w_{1|K} = 0 \tag{3.40}$$

and w_2 solves the homogeneous equation

$$[(p_x\partial_x + p_\xi\partial_\xi) + \lambda(p - ip_x\cdot\xi)]\, w_2 = 0, \qquad w_{2|K} = w \tag{3.41}$$

Similarly we split the corresponding part of $S_\lambda u$ into $u_1 + u_2$ with

$$u_i = T^*_\lambda \Phi^{\frac{1}{2}} a(x,\xi) w_i$$

where a is a smooth cutoff selecting a small neighborhood of $K \cap U$.

The estimate for w_1. If we multiply (3.40) by $\frac{w_1}{p}$ and integrate by parts in a neighborhood $K_\epsilon = \{|p| \leq \epsilon\}$ of K we obtain

$$2\Re < \frac{w_1}{p}, g >_{K_\epsilon} = 2\lambda \|w_1\|^2_{L^2(K_\epsilon)} + \|\frac{|\nabla p|}{p} w_1\|^2_{L^2(K_\epsilon)} - < \frac{\Delta p}{p} w_1, w_1 >_{K_\epsilon} + \|\frac{|\nabla p|}{p} w_1\|^2_{L^2(\{|p|=\epsilon\})}$$

By (3.32) this implies

$$\lambda \|w_1\|^2_{L^2(K_\epsilon)} \lesssim \|g\|^2_{L^2(K_\epsilon)} \tag{3.42}$$

On the other hand, if we multiply (3.40) by pw_1 and integrate by parts we obtain the analogue of (3.39) in K_ϵ. Summing this with (3.32) we obtain

$$\lambda^2 \|pw_1\|^2_{L^2(K_\epsilon)} + \lambda \|w_1\|^2_{L^2(K_\epsilon)} \lesssim \|g\|^2_{L^2(K_\epsilon)} \tag{3.43}$$

Then, if we use the bound (3.35) for g, the bound for u_1 corresponding to (3.31) follows from

$$\|T_\lambda \Phi^{\frac{1}{2}} a(x,\xi) w_1\|_{L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}} (\lambda^{\frac{1}{2}} \|pw_1\|^2 + \|w_1\|^2)$$

which is equivalent to

$$\left\| T_\lambda^* \Phi^{\frac{1}{2}} \frac{\lambda^{\frac{1}{4}} a(x,\xi)}{1+\lambda^{\frac{1}{2}}|p|} \right\|_{L^2 \to L^p(L^q)} \leq \lambda^{\rho+\frac{1}{2}}$$

By the "TT^*" argument this reduces to

$$\left\| T_\lambda^* \Phi \frac{\lambda^{\frac{1}{2}} a^2(x,\xi)}{(1+\lambda^{\frac{1}{2}}|p|)^2} T_\lambda \right\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho+1}$$

The weight inside is integrable across the level sets of p therefore we can foliate with respect to the level sets of p and reduce this to

$$\|T_\lambda^* \Phi a^2(x,\xi) \delta_{p=0} T_\lambda\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho+1}$$

which was proved in [16].

The gradient flow The next step in our analysis is to estimate w_2 using (3.41). To achieve that we need to compute the regularity of the gradient flow. Suppose we start with initial data (x,ξ) on the cone $K = \{p=0\}$. Denote by q the natural parameter along the flow, chosen so that $q=0$ on K. Set (x_q,ξ_q) the image of (x,ξ) along the flow. Then (x_q,ξ_q) solve the equations

$$\begin{cases} \partial_q x_q = p_x(x_q,\xi_q) & x(0) = x \\ \partial_q \xi_q = p_\xi(x_q,\xi_q) & \xi(0) = \xi \end{cases}$$

Theorem 7. *Assume that P has C^1 coefficients with Fourier transform supported in $B(0,\sqrt{\lambda})$. Then*

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta x_q| &\leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|-1}{2}} e^{c_{\alpha\beta}\sqrt{\lambda}|q|} & |\alpha|+|\beta|>0 \\ |\partial_x^\alpha \partial_\xi^\beta (\xi_q - \xi)| &\leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|-1}{2}} e^{c_{\alpha\beta}\sqrt{\lambda}|q|} & |\alpha|+|\beta|>0 \end{aligned}$$

Thus the gradient flow blows up exponentially on the $\lambda^{\frac{1}{2}}$ scale. This is, however, compensated by the fact that the fundamental solution to (3.41) exhibits Gaussian decay on precisely the same scale.

Proof. The linearization of the gradient flow is given by the system

$$\begin{cases} \partial_q y = p_{xx} y + p_{x\xi}\eta \\ \partial_q \eta = p_{\xi x} y + p_{\xi\xi}\eta \end{cases} \tag{3.44}$$

This implies the inequalities

$$\begin{cases} |\partial_q y| \leq c(\sqrt{\lambda}|y| + |\eta|) \\ |\partial_q \eta| \leq c(|y| + |\eta|) \end{cases}$$

If we take $(y,\eta) = (\partial_x x_q, \partial_x \xi_q)$ then the Cauchy data in (3.44) is $(I,0)$. Thus we obtain the bounds

$$|\partial_x x_q| \leq e^{c\sqrt{\lambda}|q|}, \qquad |\partial_x \xi_q| \leq \lambda^{-\frac{1}{2}} e^{c\sqrt{\lambda}|q|}$$

If we take $(y,\eta) = (\partial_\xi x_q, \partial_\xi \xi_q)$ then the Cauchy data in (3.44) is $(0,I)$. Then we get the bounds

$$|\partial_\xi x_q| \leq \lambda^{-\frac{1}{2}} e^{c\sqrt{\lambda}|q|}, \qquad |\partial_\xi (\xi_q - \xi)| \leq \lambda^{-\frac{1}{2}} e^{c\sqrt{\lambda}|q|}$$

The rest follows by induction.

The oscillatory integral for u_2. Clearly u_2 can be expressed in terms of the trace of w on the cone. This computation is carried out in the following theorem:

Theorem 8. *Assume that P has C^1 coefficients frequency localized in $|\xi| \leq \sqrt{\lambda}$. Then we have*

$$u_2 = \lambda^{-\frac{1}{2}} V_\lambda w_{|K}$$

where V_λ is an integral operator,

$$V_\lambda w = \lambda^{\frac{3(n+1)}{4}} \int_K e^{i\lambda\xi(x-y)} G(x,y,\xi) w dx d\xi$$

with a kernel G satisfying

$$|\partial_x^\alpha d_\xi^\beta G(x,y,\xi)| \leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|}{2}} e^{-c\lambda(x-y)^2} \tag{3.45}$$

Proof. We have

$$w_2(y) = \lambda^{\frac{3(n+1)}{4}} \int e^{i\lambda\xi(y-x)} e^{-\frac{\lambda}{2}(x-y)^2} w_2(x,\xi) dx d\xi$$

Now we choose $q = 0$ on the cone and $(x,\xi) \in K$. Then (q,x,ξ) can be interpreted as a new set of coordinates by

$$(q,x,\xi) \to (x_q,\xi_q)$$

In these new coordinates we write

$$w_2(y) = \lambda^{\frac{3(n+1)}{4}} \int e^{i\lambda\xi_q(y-x_q)} e^{-\frac{\lambda}{2}(x_q-y)^2} w_2(x_q,\xi_q) dx_q d\xi_q$$

Since w_2 solves the ode (3.41) we can represent it as

$$w_2(x_q,\xi_q) = F(x,\xi,q) w(x,\xi)$$

where F is the solution to the homogeneous equation

$$[\partial_q + \lambda(p(x_q,\xi_q) - ip_\xi(x_q,\xi_q)\cdot\xi_q)]F = 0 \qquad F(x,\xi,0) = 1$$

After the change of variable, w_2 has the form

$$w_2(y) = \lambda^{\frac{3(n+1)}{4}} \int_R \int_K e^{i\lambda\xi_q(y-x_q)} e^{-\frac{\lambda}{2}(x_q-y)^2} F(x,\xi,q) \frac{\partial(x_q,\xi_q)}{\partial(x,\xi,q)} w(x,\xi) dx d\xi dq$$

or, recombining further the exponents,

$$w_2(y) = \lambda^{\frac{3(n+1)}{4}} \int_R \int_K e^{i\lambda\xi(y-x)} e^{i\lambda(\xi_q-\xi)(y-x_q)} e^{-\frac{\lambda}{2}(x_q-y)^2} e^{i\lambda\xi(x-x_q)} F(x,\xi,q) \frac{\partial(x_q,\xi_q)}{\partial(x,\xi,q)} w(x,\xi) dx d\xi dq$$

Then

$$G(x,\xi,y) = \lambda^{\frac{1}{2}} \int_R e(x,\xi,y,q) dq$$

where

$$e(x,\xi,y,q) = e^{i\lambda(\xi_q-\xi)(y-x_q)} e^{-\frac{\lambda}{2}(x_q-y)^2} e^{i\lambda\xi(x-x_q)} F(x,\xi,q) \frac{\partial(x_q,\xi_q)}{\partial(x,\xi,q)}$$

Hence in order to obtain (3.45) it suffices to show that

$$|\partial_x^\alpha \partial_\xi^\beta e(x,\xi,y,q)| \leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|}{2}} e^{-c\lambda(x-y)^2} e^{-c\lambda q^2}$$

This, in turn, would follow from

$$|\partial_x^\alpha \partial_\xi^\beta e^{i\lambda(\xi_q-\xi)(y-x_q)} e^{-\frac{\lambda}{2}(x_q-y)^2}| \leq c_{\alpha,\beta} \lambda^{\frac{|\alpha|}{2}} e^{-c\lambda(x_q-y)^2} e^{c\sqrt{\lambda}q} \tag{3.46}$$

$$|\partial_x^\alpha \partial_\xi^\beta \frac{\partial(x_q,\xi_q)}{\partial(x,\xi,q)}| \leq c_{\alpha,\beta}\lambda^{\frac{|\alpha|}{2}} e^{c\sqrt{\lambda}q} \tag{3.47}$$

$$|\partial_x^\alpha \partial_\xi^\beta e^{i\lambda\xi(x-x_q)} F(x,\xi,q)| \leq c_{\alpha,\beta}\lambda^{\frac{|\alpha|}{2}} e^{-c\lambda q^2} \tag{3.48}$$

The bounds (3.46) and (3.47) follow easily from Theorem 7. For (3.48) observe that

$$F(x,\xi,q) = e^{\lambda \int_0^q i p_x(x_s,\xi_s)\xi_s - p(x_s,\xi_s) ds}$$

therefore

$$e^{i\lambda\xi(x-x_q)} F(x,\xi,q) = e^{\lambda \int_0^q i p_x(x_s,\xi_s)(\xi_s-\xi) - p(x_s,\xi_s) ds}$$

Since $p \approx cq$ we obtain the desired Gaussian decay, therefore it remains to show that

$$\lambda |\partial_x^\alpha d_\xi^\beta (i p_x(x_q,\xi_q)(\xi_q-\xi) - p(x_q,\xi_q))| \leq c_{\alpha,\beta}\lambda^{\frac{|\alpha|+1}{2}} e^{c\sqrt{\lambda}q}$$

This again follows from Theorem 7.

Taking into account the bound (3.36), we need to prove the estimate

$$\|V_\lambda w\|_{L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}} \|H_p w\|_{L^1_{x_0}(L^2_{x'\xi'})(K)} \tag{3.49}$$

for all w supported in $K \cap U$.

Given a pair (x,ξ) we denote by (x_t,ξ_t) its image along the Hamilton flow. This map is homogeneous of order 1 with respect to ξ. Then, (3.49) is equivalent to

Theorem 9. *Let $a(x,\xi)$ be a smooth compactly supported function, which is 0 near $\xi = 0$ and 1 in $1/4 \leq |\xi| \leq 4$. Then*

$$\|V_\lambda a(x,\xi) L\|_{L^2(K\cap\{x_0=0\}) \to L^p(L^q)} \leq \lambda^{\rho+\frac{1}{4}} \tag{3.50}$$

where L is the transport operator along the Hamilton flow given by

$$(Lw)(x,\xi) = \begin{cases} 0 & \text{if} \quad x_0 < 0 \\ w(x_t,\xi_t) & \text{if} \quad x_{t0} = 0, \quad x_0 \geq 0 \end{cases}$$

This is further equivalent to the corresponding bound for the operator

$$Z = (V_\lambda a L)(V_\lambda a L)^* = V_\lambda a L L^* \delta_{p(x,\xi)=0} a V_\lambda^* \ , \tag{3.51}$$

namely

$$\|Z\|_{L^{p'}(L^{q'}) \to L^p(L^q)} \leq \lambda^{2\rho + \frac{1}{2}} \tag{3.52}$$

The operator LL^* is an integral operator along bicharacteristics, with kernel

$$l(t,s) = 1_{\{t \geq 0, \ s \geq 0\}}$$

This embeds into the analytic family

$$L_\theta = (\theta - 1)(t-s)^{\theta - 1} l(t,s)$$

From this point forward, the proof of (3.52) is identical to the proof in [17] provided that we obtain the same regularity results for the Hamilton flow. This is done in the next section.

The regularity of the Hamilton flow Here we obtain precise bounds on the derivatives of the flow map H_t with respect to ξ. Observe first that if the coefficients g^{ij} have Fourier transform supported in $|\xi| \leq \sqrt{\lambda}$ then the following relations hold:

$$\|\partial_x^\alpha g\|_{L^1(L^\infty)} \leq c_\alpha \lambda^{\frac{\alpha-2}{2}}, |\alpha| \geq 2 \tag{3.53}$$

$$\|\partial_x^\alpha g\|_{L^\infty} \leq c_\alpha \lambda^{\frac{\alpha-1}{2}}, |\alpha| \geq 1 \tag{3.54}$$

Then

Lemma 7. *Assume that the coefficients of P satisfy $\partial_x^2 g \in L^1(L^\infty)$, with Fourier transform supported in $B(0,\sqrt{\lambda})$. Then the following bounds hold*

$$\begin{aligned} &|\partial_\xi^\alpha x_t| \leq t(1+t\sqrt{\lambda})^{|\alpha|-1}, \qquad |\alpha| \geq 1 \\ &|\partial_\xi^\alpha \xi_t| \leq (1+t\sqrt{\lambda})^{|\alpha|-1} \qquad |\alpha| \geq 1 \end{aligned} \tag{3.55}$$

Proof. We use induction with respect to α. For $\alpha = 1$ the functions $(\partial_\xi x_t, d_\xi \xi_t)$ solve the linearized system

$$\frac{d}{dt}\begin{pmatrix} y \\ \eta \end{pmatrix} = A(t)\begin{pmatrix} y \\ \eta \end{pmatrix}, \qquad A(t) = \begin{pmatrix} p_{\xi x}(x_t,\xi_t) & p_{\xi\xi}(x_t,\xi_t) \\ -p_{xx}(x_t,\xi_t) & -p_{x\xi}(x_t,\xi_t) \end{pmatrix} \tag{3.56}$$

with initial data

$$\begin{pmatrix} y(0) \\ \eta(0) \end{pmatrix} = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$$

Since $A(t)$ is integrable, it follows that the solutions remain bounded. Furthermore, the Cauchy data of the first component is 0, therefore the first component can be bounded by ct.

For the induction step, compute

$$\frac{d}{dt}\partial_\xi^\alpha x_t = \partial_\xi^\alpha \frac{d}{dt} x_t = \partial_\xi^\alpha p_\xi(x_t,\xi_t)$$

and

$$\frac{d}{dt}\partial_\xi^\alpha \xi_t = \partial_\xi^\alpha \frac{d}{dt} \xi_t = -\partial_\xi^\alpha p_x(x_t,\xi_t)$$

Then the functions $(\partial_\xi^\alpha x_t,\ \partial_\xi^\alpha \xi_t)$ solve a system of the form

$$\frac{d}{dt}\begin{pmatrix} y \\ \eta \end{pmatrix} = A(t)\begin{pmatrix} y \\ \eta \end{pmatrix} + \begin{pmatrix} \bar{y}(t) \\ \bar{\eta}(t) \end{pmatrix}$$

with zero initial data, where the inhomogeneous term $(\bar{y}(t), \bar{\eta}(t))$ can be expressed in terms of the lower order derivatives as a sum of products of the form

$$(\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} p_\xi)(x_t,\xi_t)\prod_{k=1}^{|\alpha_1|}(\partial_\xi^{\alpha_1^k} x_t)\prod_{k=1}^{|\alpha_2|}(\partial_\xi^{\alpha_2^k}\xi_t), \qquad (\partial_x^{\alpha_1}\partial_\xi^{\alpha_2} p_x)(x_t,\xi_t)\prod_{k=1}^{|\alpha_1|}(\partial_\xi^{\alpha_1^k} x_t)\prod_{k=1}^{|\alpha_2|}(\partial_\xi^{\alpha_2^k}\xi_t)$$

where

$$1 \le |\alpha_1| + |\alpha_2| \le |\alpha|, \qquad 1 \le |\alpha_1^k|, |\alpha_2^k| \le |\alpha| - 1 \qquad \sum_k |\alpha_1^k| + \sum_k |\alpha_2^k| = |\alpha|$$

Then, given the induction hypothesis and the bounds (3.54), (3.53) we can bound $\bar{y}(t)$ and $\bar{\eta}(t)$ by

$$\bar{y}(t) \leq c_\alpha(1+t\sqrt{\lambda})^{|\alpha|-1}, \quad \|\bar{\eta}\|_{L^1(0,t)} \leq c_\alpha(1+t\sqrt{\lambda})^{|\alpha|-1}$$

If we just use the boundedness of the linearized flow this gives a $(1+t\sqrt{\lambda})^{|\alpha|-1}$ bound for both y and η. The additional t factor for y is obtained if we use this bound in the equation for y, which has bounded coefficients. □

We can use the above Lemma to produce an expansion of x_t, ξ_t in terms of powers of t:

Lemma 8. *Assume that the coefficients of P satisfy $\partial_x^2 g \in L^1(L^\infty)$ with Fourier transform supported in $B(0,\sqrt{\lambda})$. Then the following estimates hold:*

$$x_t = x + tp_\xi + t^2 g(t,x,\xi) \tag{3.57}$$

$$\xi_t = \xi + th(t,x,\xi) \tag{3.58}$$

where g, h satisfy the following bounds:

$$|\partial_\xi^\alpha h(t,x,\xi)|, |\partial_\xi^\alpha g(t,x,\xi)| \leq (1+t\sqrt{\lambda})^{\alpha-1}, \tag{3.59}$$

Proof. We have

$$\begin{aligned} x_t &= x + tp_\xi + \int_0^t (t-s)\frac{d^2}{ds^2}x_s ds \\ &= x + tp_\xi + \int_0^t (t-s)f(x_s,\xi_s)ds \end{aligned}$$

where f can be computed from the flow equation,

$$f(x,\xi) = p_{\xi x}p_\xi - p_{\xi\xi}p_x$$

It suffices to estimate the derivatives of the integrand in L^1,

$$\|\partial_\xi^\alpha f(x_t,\xi_t)\|_{L^1(0,t)} \leq c_\alpha t(1+t\sqrt{\lambda})^{|\alpha|-1},$$

By (3.54), (3.53) we have

$$\|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)\|_{L^1(L^\infty)} \leq c_{\alpha,\beta}(\sqrt{\lambda})^{|\alpha|-1}$$

so this can be easily done using the previous Lemma. The estimate for ξ_t is similar.

Another straightforward consequence of Lemma 7 is the following bound for the exponents in the kernel G:

Lemma 9. *Let G be as in (3.45). Then*

$$|\partial_\xi^\alpha G(x_t,\xi_t,y)| \le c_\alpha e^{-c\lambda(\xi_t-y)^2}(1+t\sqrt{\lambda})^{|\alpha|},$$

In a similar manner we obtain the related result for the exponent in the oscillatory term in our kernel[4]:

Lemma 10. *For ξ in a compact set and away from 0 we have*

$$|\partial_\xi\left(\lambda(x_t-\tilde{y})\xi_t\right) - \big(\lambda(x-\tilde{y}) + t^2\lambda g(t,x,\xi)\big)| \le (1+\lambda(x_t-\tilde{y})^2)(1+t\sqrt{\lambda}) \tag{3.60}$$

and

$$|\partial_\xi^\alpha \lambda((x_t-\tilde{y})\xi_t)| \le c_\alpha(1+\lambda(x_t-\tilde{y})^2)(1+t\sqrt{\lambda})^{|\alpha|}, |\alpha| \ge 2 \tag{3.61}$$

4 Proof of Theorem 2.

By rescaling the problem reduces to the case when $T=1$ and

$$\|g\|_{\mathcal{X}^s}^2 \le \mu^{2+s} \tag{4.62}$$

Now we follow the proof of Theorem 1 and successively simplify the problem by

- reduction to a dyadic estimate
- truncating the coefficients at frequency $\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}}$.

However, when we reach this point we can use directly Theorem 1 to prove the result.

The Paley-Littlewood decomposition The analysis below is related to the paradifferential calculus. Given a frequency λ define the operator P_λ as

$$P_\lambda = \partial_i g_\lambda^{ij}\partial_j$$

where g_λ^{ij} are the coefficients of P truncated at frequency $\frac{\lambda}{10}$,

$$g_\lambda^{ij} = (1-U_{\frac{\lambda}{10}})g^{ij}$$

[4]see [16] for an explicit computation

Our first step is to reduce the problem to the dyadic estimates

$$\lambda^{-\rho-\frac{\sigma}{p}}\|S_\lambda v\|_{L^p(L^q)} \lesssim \mu^{\frac{1}{p}}\|S_\lambda v\|_{L^\infty(L^2)} + \lambda^{-1-\sigma}\mu^{-\frac{1}{p'}}\|P_\lambda S_\lambda v\|_{L^1(L^\infty)} \tag{4.63}$$

This implies (1.9) provided we show that

$$\||D|^{-\sigma}(S_\lambda P - P_\lambda S_\lambda)u\|^2_{L^1(L^2)} \lesssim \mu\|\nabla u\|^2_{L^\infty(L^2)}$$

i.e.

$$\||D|^{1-\sigma}(S_\lambda g - g_\lambda S_\lambda)\nabla u\|^2_{L^1(L^2)} \lesssim \mu\|\nabla u\|^2_{L^\infty(L^2)}$$

Since the range of $S_\lambda g - g_\lambda S_\lambda$ consists of functions frequency localized at frequency λ this is equivalent to

$$\|(S_\lambda g - g_\lambda S_\lambda)v\|_{L^1(L^2)} \lesssim \mu\lambda^{\sigma-1}\|v\|_{L^\infty(L^2)} \tag{4.64}$$

where $v = \nabla u$. Because v is compactly supported, without any restriction in generality we can assume that g is compactly supported. Then we know that

$$\|g\|_{L^1(L^\infty)} \leq 1, \qquad \|g\|_{\mathcal{X}^s} \lesssim \mu^{\frac{2+s}{2}}$$

This implies the bound

$$\|g\|_{\dot{B}^{1\infty1}_{1-\sigma}} \lesssim \mu$$

where $1-\sigma = \frac{2s}{2+s}$.

Hence (4.64) would follow from

$$\|(S_\lambda g - g_\lambda S_\lambda)v\|_{L^1(L^2)} \leq \lambda^{\sigma-1}\|g\|_{\dot{B}^{1\infty1}_{1-\sigma}}\|v\|_{L^\infty(L^2)} \tag{4.65}$$

It suffices to prove this for a single dyadic piece of g at frequency ν, which satisfies

$$\|S_\nu g\|_{L^1(L^\infty)} \lesssim \nu^{-1+\sigma}\|g\|_{\dot{B}^{1\infty1}_{1-\sigma}}$$

If $\nu \gg \frac{\lambda}{10}$ then we need to show that

$$\|S_\lambda(S_\nu g)v\|_{L^1(L^2)} \lesssim \lambda^{\sigma-1}\nu^{1-\sigma}\|S_\nu g\|_{L^1(L^\infty)}\|v\|_{L^\infty(L^2)}$$

which is straightforward.

If $\nu \lesssim \frac{\lambda}{10}$, on the other hand, then we need to show that

$$\|[S_\lambda, (S_\nu g)]v\|_{L^1(L^2)} \lesssim \lambda^{\sigma-1}\nu^{1-\sigma}\|S_\nu g\|_{L^1(L^\infty)}\|v\|_{L^\infty(L^2)}$$

But this is a scale invariant estimate. Without any restriction in generality we can take $\nu = 1$ and show that

$$\|[S_\lambda, (S_1 g)]v\|_{L^1(L^2)} \lesssim \lambda^{\sigma-1}\|S_1 g\|_{L^1(L^\infty)}\|v\|_{L^\infty(L^2)}$$

We replace this by the stronger estimate

$$\|[S_\lambda, g]v\|_{L^1(L^2)} \lesssim \lambda^{-1}\|\nabla g\|_{L^1(L^\infty)}\|v\|_{L^\infty(L^2)}$$

which we rescale to the case $\lambda = 1$

$$\|[S_1, g]v\|_{L^1(L^2)} \lesssim \|\nabla g\|_{L^1(L^\infty)}\|v\|_{L^\infty(L^2)}$$

The last bound follows since the kernel $K(y, \tilde{y})$ of $[S_1, g]$,

$$K(y, \tilde{y}) = \hat{s}(y - \tilde{y})(g(y) - g(\tilde{y}))$$

satisfies

$$K(y, \tilde{y}) \lesssim (1 + |y' - \tilde{y}'|)^{-N}(1 + |y_0 - \tilde{y}_0|)^{-N} \int_{y_0}^{\tilde{y}_0} \|\nabla g(s)\|_\infty ds$$

and

$$\int \left[(1 + |y_0 - \tilde{y}_0|)^{-N} \int_{y_0}^{\tilde{y}_0} \|\nabla g(s)\|_\infty \, ds\right] dy_0 d\tilde{y}_0 \lesssim \|\nabla g\|_{L^1(L^\infty)}$$

Truncating the coefficients. Now we show that it suffices to prove the dyadic estimate under the additional assumption that the coefficients of P_λ have Fourier transform supported in $B(0, \mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}})$. To achieve this we need to prove that if we truncate the coefficients at frequency $\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}}$ then the change in the second right hand side term in (4.63) is controlled by the first right hand side term, i.e. that

$$\|\partial(U_{\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}}} g_\lambda)\partial S_\lambda u\|_{L^1(L^2)} \leq \mu\lambda^{1+\sigma}\|S_\lambda u\|_{L^\infty(L^2)}$$

Indeed,

$$\begin{aligned}
\|\partial(U_{\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}}} g_\lambda)\partial S_\lambda u\|_{L^1(L^2)} &\leq \lambda^2\|U_{\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}}} g\|_{L^1(L^\infty)}\|S_\lambda u\|_{L^\infty(L^2)} \\
&\leq \lambda^2(\mu^{\frac{1}{2}}\lambda^{\frac{2}{2+s}})^{-s}\|g\|_{\mathcal{X}^s}\|S_\lambda u\|_{L^\infty(L^2)} \\
&\leq \mu\lambda^{1+\sigma}\|S_\lambda u\|_{L^\infty(L^2)}
\end{aligned}$$

At the last step we have used the bound (4.62) for g.

Conclusion. Observe that if the coefficients of P have Fourier transform supported in the region $\{|\xi| \leq \mu^{\frac{1}{2}}\lambda^{\frac{2}{2-s}}\}$ then

$$\|\partial_x^2 g\|_{L^1(L^\infty)} \leq \mu^{\frac{2-s}{2}}\lambda^{\frac{2(2-s)}{2+s}}\|g\|_{\mathcal{X}^s}$$

which gives

$$\|\partial_x^2 g\|_{L^1(L^\infty)} \leq (\mu\lambda^\sigma)^2$$

Then (4.63) is a special case of (1.8). □

5 Quasilinear hyperbolic equations

Consider a quasilinear second order hyperbolic equation in $R^n \times R$,

$$\partial_i g^{ij}(u)\partial_j u = N(u, \partial u) \tag{5.66}$$

with Cauchy data

$$u(0) = u_0, \quad u_t(0) = u_1 \tag{5.67}$$

Then the classical theory (see [6], and also [18] and references therein) says that this problem is locally well-posed in $H^s \times H^{s-1}$ for $s > \frac{n}{2} + 1$. This condition insures that the coefficients of the principal part are C^1 and that ∇u is bounded.

The Strichartz estimates in [16] allowed us to obtain the same result for a lower s provided that the nonlinearity is a most quadratic in ∇u,

$$N(u, u) = G(u)Q(\nabla u, \nabla u) \tag{5.68}$$

The stronger estimates in this paper enable us to push this result even further. Results in the same direction were independently proved by H. Bahouri and J-Y. Chemin [2], [1] using a different method.

To avoid distracting technicalities we assume that the functions G, g^{ij} are smooth, bounded and have bounded derivatives up to a sufficiently high order. Also we assume that the coefficients g^{ij} are uniformly hyperbolic in time. Then our main result is

Theorem 10. *The quasilinear problem (5.66)-(5.67) is locally well-posed in $H^s \times H^{s-1}$ for*

$$s > \frac{n}{2} + \frac{5}{6} \quad , \quad n = 2$$
$$s > \frac{n}{2} + \frac{2}{3} \quad , \quad n \geq 3$$

One should note, though, that at this point the counterexamples for the quasilinear equation are no better than those for the semilinear equation, see [10], [11].

The same method can be used for second order hyperbolic equations of the form

$$g^{ij}(u, \nabla u)\partial_i \partial_j u = N(u, \nabla u) \tag{5.69}$$

Differentiating once we obtain equations which are essentially of the form (5.66), therefore

Theorem 11. *The quasilinear problem (5.69)-(5.67) is locally well-posed in $H^s \times H^{s-1}$ for*

$$\begin{aligned} s &> \frac{n+2}{2} + \frac{5}{6} \quad , \quad n = 2 \\ s &> \frac{n+2}{2} + \frac{2}{3} \quad , \quad n \geq 3 \end{aligned}$$

Proof. of Theorem 10

The proof of this result is similar to the corresponding proof in [16] therefore here we only sketch the main arguments.

The energy estimates A first step in the proof is to obtain good bounds for the H^s norm of the solutions in terms of the initial data. The main estimate states that

$$\|\nabla u(t)\|_{H^{s-1}} \leq c\|\nabla u(0)\|_{H^{s-1}} e^{c\int_0^t \|\nabla u(s)\|_\infty ds} \tag{5.70}$$

This holds for any s and implies that the solutions do not blow up in any H^s norm for as long as

$$\int_0^t \|\nabla u(s)\|_\infty ds$$

remains finite, i.e. for as long as $\nabla u \in L^1(L^\infty)$. The proof of this result is also of interest to us. For $s = 1$ it suffices to multiply the equation by u_t, integrate by parts and use Gronwall's lemma. For other values of s we introduce the auxiliary function

$$v = |D'|^{s-1} u$$

and compute

$$P(x, u, D)v = [P(x, u, D)|D'|^{s-1}]u + |D|^{s-1}\Gamma(u)(\nabla u)^2)$$

A careful estimate of the right hand side at fixed time yields[5]

$$\| [P(x,u,D)|D'|^{s-1}]u + |D|^{s-1}\Gamma(u)(\nabla u)^2\|_{L^2} \leq \|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{s-1}} \tag{5.71}$$

Then (5.70) follows from the H^1 energy estimates for v combined with Gronwall's lemma.

This argument goes back to work of Klainerman [9] for integer s; for noninteger s the proof of the estimate (5.71) requires better commutator estimates as proved later in [7].

Thus, in order to obtain a local well-posedness result on has to determine for what s is it possible to close the argument and recover locally in time the $L^1(L^\infty)$ regularity for ∇u. This we do using the Strichartz estimates. In effect what we can get from the Strichartz estimates is $L^4(L^\infty)$ for $n=2$, $L^{2+\epsilon}(L^\infty)$ for $n=3$ respectively $L^2(L^\infty)$ for $n \geq 4$.

The Strichartz estimates. We apply the Strichartz estimates in Corollary 3 to the auxiliary function v defined above. To fix the indices suppose for instance that $n \geq 4$. If we apriori assume that the coefficients satisfy

$$\|\nabla g\|_{L^1(L^\infty)} \leq 1$$

then we obtain locally

$$\||D'|^{-\frac{n+1}{2}-\frac{1}{6}}\nabla v\|_{L^2(\dot{B}_0^{\infty\infty})} \leq \|\nabla v(0)\|_{L^2} + \|P(x,u,D)v\|_{L^1(L^2)}$$

Combine this with the bound (5.71) for $P(x,u,D)v$ to get

$$\||D'|^{-\frac{n+1}{2}-\frac{1}{6}}\nabla v\|_{L^2(\dot{B}_0^{\infty\infty})} \leq c(1+\|\nabla u\|_{L^1(L^\infty)})\|\nabla u\|_{L^\infty(H^{s-1})}$$

which for u gives

$$\|\nabla u\|_{L^2(L^\infty)} \leq c(1+\|\nabla u\|_{L^1(L^\infty)})\|\nabla u\|_{L^\infty(H^{s-1})} \qquad s > \frac{n}{2}+\frac{2}{3}$$

Closing the loop. Since

$$\|\nabla g\|_{L^1(0,T;L^\infty)} \leq \|\nabla u\|_{L^1(0,T;L^\infty)}$$

all we need to do is to estimate ∇u in terms of itself and show that for a small time it has to stay in $L^1(L^\infty)$. We have

$$\begin{aligned} \|\nabla u\|_{L^1(0,T;L^\infty)} &\leq \sqrt{T}\|\nabla u\|_{L^2(0,T;L^\infty)} \\ &\leq \sqrt{T}(1+\|\nabla u\|_{L^1(0,T;L^\infty)})\|\nabla u\|_{L^\infty(0,T;H^{s-1})} && \text{(Strichartz)} \\ &\leq \sqrt{T}(1+\|\nabla u\|_{L^1(0,T;L^\infty)})\|\nabla u(0)\|_{H^{s-1}} e^{c\int_0^T \|\nabla u(t)\|_{L^1(0,T;L^\infty)}dt} && \text{(Energy)} \end{aligned}$$

[5]here we assume an apriori bound on the L^∞ norm of u

Hence in order to recover our assumption that

$$\|\nabla g\|_{L^1(0,T;L^\infty)} \leq 1$$

it suffices to choose T so that

$$\sqrt{T}\|\nabla u(0)\|_{H^{s-1}} \leq c \ll 1$$

Thus for such a time T we obtain uniform H^s bounds on the solution if the initial data is in a bounded set in H^s for $s > \frac{n}{2} + \frac{2}{3}$.

Stability estimates To complete the proof of the well-posedness result we now show that the solutions depend continuously (even Lipschitz) on the initial data. Such a dependence cannot hold, though, in the H^s norm. Here we contend ourselves with the energy norm.

Lemma 11. *Assume that $n \geq 3$. Let u, v be smooth solutions to (5.66) in $[0,T]$. Then*

$$\|\nabla(u-v)\|_{L^\infty(0,T;L^2)} \leq c(\|u|_{H^s}, \|v\|_{H^s}, \|\nabla u\|_{L^2(L^\infty)}, \|\nabla v\|_{L^2(L^\infty)})\|\nabla(u-v)(0)\|_{L^2} \quad (5.72)$$

The proof is identical to the one in [16]. The difference

$$w = u - v$$

solves the linear equation

$$P(x,u,D)w = A_1\nabla w + A_0 w \tag{5.73}$$

where A_1 is linear in $\nabla u, \nabla v$,

$$A_1 = G_1(u,v)\nabla(u,v)$$

while A_0 contains a term which is linear in the second order derivative of v and a quadratic term in $\nabla u, \nabla v$,

$$A_0 = G_2(u,v)\partial'\nabla v + G_3(u,v)(\nabla(u,v))^2$$

Using the Strichartz estimates for u, v one can estimate the coefficients A_0, A_1 in mixed L^p spaces. Then, due to the Strichartz estimates for w, it is easy to establish the H^1 well-posedness for (5.73) and further, to show that the right hand side remains in L^2 locally.

This argument is tight in dimension $n = 3$. For $n \geq 4$ one has more freedom and the stability estimates are also valid in smaller Sobolev spaces. By contrary, for $n = 2$ the above argument has to be modified since then $D_x\nabla v$ can only be estimated in negative Sobolev spaces. Thus, the best result one can obtain in a similar manner is a stability estimate in $H^{\frac{5}{6}} \times H^{-\frac{1}{6}}$.

References

[1] Hajer Bahouri and Jean-Yves Chemin. Equations d'ondes quasilineaires et effet dispersif. preprint.

[2] Hajer Bahouri and Jean-Yves Chemin. Equations d'ondes quasilineaires et estimations de strichartz. preprint.

[3] Philip Brenner. On $L_p - L_{p'}$ estimates for the wave-equation. *Math. Z.*, 145(3):251–254, 1975.

[4] Jean-Marc Delort. *F.B.I. transformation.* Springer-Verlag, Berlin, 1992. Second microlocalization and semilinear caustics.

[5] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Anal.*, 133(1):50–68, 1995.

[6] Thomas J. R. Hughes, Tosio Kato, and Jerrold E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Arch. Rational Mech. Anal.*, 63(3):273–294 (1977), 1976.

[7] Tosio Kato and Gustavo Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.*, 41(7):891–907, 1988.

[8] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[9] Sergiu Klainerman. Long-time behavior of solutions to nonlinear evolution equations. *Arch. Rational Mech. Anal.*, 78(1):73–98, 1982.

[10] Hans Lindblad. Counterexamples to local existence for semi-linear wave equations. *Amer. J. Math.*, 118(1):1–16, 1996.

[11] Hans Lindblad. Counterexamples to local existence for quasilinear wave equations. *Math. Res. Lett.*, 5(5):605–622, 1998.

[12] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson-Sjölin estimates. *J. Amer. Math. Soc.*, 6(1):65–130, 1993.

[13] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[14] Hart F. Smith and Christopher D. Sogge. On Strichartz and eigenfunction estimates for low regularity metrics. *Math. Res. Lett.*, 1(6):729–737, 1994.

[15] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[16] Daniel Tataru. Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation. preprint, `/http://www.math.nwu/~tataru/nlw`.

[17] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients ii. preprint, `/http://www.math.nwu/~tataru/nlw`.

[18] Michael E. Taylor. *Pseudodifferential operators and nonlinear PDE.* Birkhäuser Boston Inc., Boston, MA, 1991.