SHARP COUNTEREXAMPLES FOR STRICHARTZ ESTIMATES FOR LOW REGULARITY METRICS

HART F. SMITH AND DANIEL TATARU

1. **Introduction.** In this paper we produce examples of time independent C^s metrics, for $0 \le s \le 2$, and solutions u to the wave equation for such metrics, which establish sharp lower bounds on the index of the Sobolev norm of the initial data of u required to bound mixed L^pL^q norms of u.

Consider a second order hyperbolic operator on $[0,1] \times \mathbb{R}^n$,

$$P(t,x,\partial_t,\partial_x) = \partial_t^2 - \partial_{x_i}\, g^{ij}(t,x)\, \partial_{x_j}$$

and the following estimates of Strichartz type

$$\|u\|_{L^p_tL^q_x([0,1]\times\mathbb{R}^n)} \le C\Big(\|u\|_{L^\infty_t([0,1];H^\gamma_x(\mathbb{R}^n))} + \|\partial_t u\|_{L^\infty_t([0,1];H^{\gamma-1}_x(\mathbb{R}^n))} + \|Pu\|_{L^1_t([0,1];H^{\gamma-1}_x(\mathbb{R}^n))}\Big)\,. \tag{1}$$

If the coefficients of P are smooth, then it is known that these estimates hold for (p,q) satisfying

$$\frac{1}{p} = \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2}-\frac{1}{q}\Big)\,, \qquad 2 \le q \le \frac{2(n-1)}{n-3}\,, \tag{2}$$

provided $(n,p,q) \ne (3,2,\infty)$, where the Sobolev index γ is given by

$$\gamma = \Big(\frac{n+1}{2}\Big)\Big(\frac{1}{2}-\frac{1}{q}\Big)\,.$$

On the other hand, in [3] there were constructed for each $s < 2$ examples of P with time independent coefficients of regularity C^s for which the same estimates fail to hold. The first author then showed in [1] that, in space dimensions 2 and 3, the estimates do hold if the coefficients of P are $C^{1,1}$. The second author subsequently showed in [4] that the estimates hold for C^2 metrics in all space dimensions, and that for operators with C^s coefficients, $0 < s < 2$, such estimates hold provided that γ is replaced by $\gamma + \sigma/p$, where $\sigma = \frac{2-s}{2+s}$. Indeed, [5] showed that such estimates hold under the condition that s derivatives

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of the coefficients belong to $L^1_t L^\infty_x$, which is important for applications to quasilinear wave equations.

The counterexamples of [3] do not coincide with the estimates established by [4], however. In this paper we remedy this gap, by producing examples of time independent C^s metrics, with $0 \le s \le 2$, which show that the results established in [4] are indeed best possible.

Theorem 1. *Let $0 \le s \le 2$, and suppose that (p,q) satisfy (2). Assume that the estimate (1) holds with a constant C depending only on the C^s norm of the coefficients, for all metrics $g^{ij}(x)$ sufficiently close in the uniform norm to the Euclidean metric δ^{ij}. Then*

$$\gamma \ge \Big(\frac{n+1}{2}\Big)\Big(\frac{1}{2}-\frac{1}{q}\Big)+\frac{\sigma}{p}\,, \qquad \sigma = \frac{2-s}{2+s}\,.$$

We remark that this construction also produces examples of C^s metrics, $1 \le s \le 2$, which show that the closely related spectral projection estimates for C^s metrics established by the first author [2] are best possible. For the spectral projection estimates, however, the counterexamples of [3] were already sharp.

2. **An explicit example.** Here we give a simple explicit construction, but which works only for $s \ge \frac{2}{3}$. In the next section we explain how to modify this construction in order to make it work for $0 \le s < \frac{2}{3}$. We work with variables $t \in \mathbb{R}$, $x \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$. For $\xi > 0$ a real number, we let

$$u_\xi(t,x,y) = e^{i\xi(t-x) - it(\frac{n-1}{2}) - \frac{1}{2}\xi\,|y|^2}\,,$$
$$P(y,\partial_t,\partial_x,\partial_y) = \big(\partial_t^2 - \big(1+|y|^2\big)\,\partial_x^2 - \Delta_y\,\big)\,,$$

and observe that

$$Pu_\xi(t,x,y) = -\left(\frac{n-1}{2}\right)^2 u_\xi(t,x,y)\,.$$

For an index $0 \le s \le 2$ we set

$$\delta = \frac{2}{2+s}\,, \qquad \sigma = \frac{2-s}{2+s}\,,$$

and note that $2\delta = 1+\sigma$.

We now fix s, and make a change of variables by scaling (t,x,y) by λ^σ, and replacing ξ by $r\lambda^{1-\sigma}$, to obtain

$$u_r^\lambda(t,x,y) = e^{ir\lambda(t-x) - i\lambda^\sigma t(\frac{n-1}{2}) - \frac{1}{2}r\lambda^{2\delta}|y|^2}\,,$$

which satisfies

$$P_\lambda u_r^\lambda(t,x,y) = -\left(\frac{n-1}{2}\right)^2 \lambda^{2\sigma} u_r^\lambda(t,x,y)\,,$$

where

$$P_\lambda(y,\partial_t,\partial_x,\partial_y)=P(\lambda^\sigma y,\partial_t,\partial_x,\partial_y)=\left(\partial_t^2-\left(1+\lambda^{2\sigma}|y|^2\right)\partial_x^2-\Delta_y\right).$$

We next fix a smooth, nonnegative bump function β supported in the interval $[1,2]$, and set

$$u^\lambda(t,x,y)=\frac{1}{(\log\lambda)^2}\int\beta\left((\log\lambda)^{-2}r\right)u_r^\lambda(t,x,y)\,dr\,.$$

The function u^λ is essentially a smooth bump function of size 1 localized to the set $|x-t|\le\lambda^{-1}\left(\log\lambda\right)^{-2},\ \ |y|\le\lambda^{-\delta}\left(\log\lambda\right)^{-1}$. Rather than obtain pointwise estimates, though, it is easier to work with weighted L^2 estimates. Thus, we note the following two inequalities:

$$\int|u^\lambda(t,x,y)|^2\,dx\,dy\approx\lambda^{-1-(n-1)\delta}\left(\log\lambda\right)^{-n-1}, \tag{3}$$

$$\int\left(1+\lambda^2\left(\log\lambda\right)^4|t-x|^2+\lambda^{2\delta}\left(\log\lambda\right)^2|y|^2\right)^n|u^\lambda(t,x,y)|^2\,dx\,dy \le C\,\lambda^{-1-(n-1)\delta}\left(\log\lambda\right)^{-n-1}. \tag{4}$$

The first follows by the Plancherel theorem applied to the x variable. The second follows by noting that

$$\begin{aligned}&\lambda^j\left(\log\lambda\right)^{2j}(t-x)^j\,u^\lambda(t,x,y)\\&=e^{-i\lambda^\sigma t(\frac{n-1}{2})}\int e^{ir\lambda(t-x)}\left(\log\lambda\right)^{2j}\partial_r^j\left(e^{-\frac12 r\lambda^{2\delta}|y|^2}\,\beta\left(\left(\log\lambda\right)^{-2}r\right)\right)\,dr\,,\end{aligned}$$

and applying Plancherel as before.

Together, (3) and (4) and the Schwarz inequality imply

$$\int\left(1+\lambda^2\left(\log\lambda\right)^4|t-x|^2+\lambda^{2\delta}\left(\log\lambda\right)^2|y|^2\right)^{-n}|u^\lambda(t,x,y)|^2\,dx\,dy \approx\ \lambda^{-1-(n-1)\delta}\left(\log\lambda\right)^{-n-1},$$

which by Holder's inequality implies that, for $2\le q\le\infty$,

$$\|u^\lambda(t,\,\cdot\,)\|_{L^q(\mathbb{R}^n)}\ge c\,\lambda^{-(1+(n-1)\delta)/q}\left(\log\lambda\right)^{-(n+1)/q}, \tag{5}$$

exactly the bounds for a suitably localized bump function of size 1. On the other hand, it is easy to compute the Sobolev space bounds

$$\|u_t^\lambda(t,\,\cdot\,)\|_{H^{\gamma-1}(\mathbb{R}^n)}+\|u^\lambda(t,\,\cdot\,)\|_{H^\gamma(\mathbb{R}^n)} \le C\,\lambda^{\gamma-(1+(n-1)\delta)/2}\left(\log\lambda\right)^{2\gamma-(n+1)/2}, \tag{6}$$

$$\|P_\lambda u^\lambda(t,\,\cdot\,)\|_{H^{\gamma-1}(\mathbb{R}^n)}\le C\,\lambda^{\gamma-1+2\sigma-(1+(n-1)\delta)/2}\left(\log\lambda\right)^{2(\gamma-1)-(n+1)/2}. \tag{7}$$

If the Strichartz estimate (1) holds uniformly for P_λ and u_λ, then by (5), (6) and (7) we must have

$$\lambda^{-\frac{1+(n-1)\delta}{q}}\left(\log\lambda\right)^{-\frac{n+1}{q}} \leq C\lambda^{\gamma-\frac{1+(n-1)\delta}{2}}\left(\log\lambda\right)^{2\gamma-\frac{n+1}{2}}\left(1+\left(\log\lambda\right)^{-2}\lambda^{2\sigma-1}\right). \tag{8}$$

If $s\geq 2/3$, then $2\sigma\leq 1$, and therefore we must have

$$\gamma\geq\left(\frac{1}{2}-\frac{1}{q}\right)\left(1+(n-1)\,\delta\right),$$

which compared to the smooth case involves a loss of derivatives of the following degree

$$\left(1+(n-1)\,\delta-\frac{n+1}{2}\right)\left(\frac{1}{2}-\frac{1}{q}\right)=(2\delta-1)\,\frac{1}{p}=\frac{\sigma}{p}\,.$$

This would conclude the proof of Theorem 1 if the coefficients of the operators P_λ were uniformly bounded in C^s. While this is not true, the bound

$$\int_{|y|\geq\lambda^{-\delta}}\left|\partial^\alpha_{t,x,y}u^\lambda(t,x,y)\right|^2dx\,dy\leq C_{N,\alpha}\,\lambda^{-N}$$

shows that we can freely modify the coefficients of P_λ outside the ball $\{|y|\leq\lambda^{-\delta}\}$. Thus, let $a(y)$ denote a positive smooth function, such that

$$a(y)=|y|^2\quad\text{if}\quad|y|\leq 1\,,\qquad a(y)=0\quad\text{if}\quad|y|\geq 2\,,$$

and set

$$P^1_\lambda=\partial_t^2-\left(1+\lambda^{2\sigma-2\delta}\,a(\lambda^\delta y)\right)\partial_x^2-\Delta_y\,.$$

Note that

$$\left\|\lambda^{2\sigma-2\delta}\,a(\lambda^\delta y)\right\|_{C^0}\leq C\,\lambda^{2\sigma-2\delta}\,,\qquad\left\|\lambda^{2\sigma-2\delta}\,a(\lambda^\delta y)\right\|_{C^2}\leq C\,\lambda^{2\sigma}\,.$$

Since $(2\sigma-2\delta)(2-s)+2\sigma s=0$, it follows that P^1_λ has C^s coefficients, uniformly over λ. Furthermore, the coefficients of P^1_λ converge in the L^∞ norm to those of the usual d'Alembertian $\partial_t^2-\partial_x^2-\Delta_y$ as $\lambda\to\infty$.

3. **A modified example.** The reason that the previous example fails for $s<2/3$ is that $P_\lambda u^\lambda_r$ is too large. To remedy this, we seek modified functions u_ξ and operators P of the form

$$u_\xi(t,x,y)=a(y)e^{i\xi(t-x)+i\alpha t-\xi\phi(y)}\,,$$

$$P(y,\partial_t,\partial_x,\partial_y)=\partial_t^2-g(y)\,\partial_x^2-\Delta_y\,,$$

with a,g,ϕ smooth on some ball about 0, spherically symmetric, and with $a(0)=g(0)=1$, $\Delta\phi(0)>0$, $\alpha>0$, and

$$Pu_\xi=0\,.$$

Given such a function u_ξ and an operator P then we can substitute them in the argument of the previous section, and so obtain the desired counterexamples in the full range $0 \le s \le 2$.

We compute

$$Pu_\xi = \left(-(\xi+\alpha)^2 + \xi^2 g(y) - \xi^2 |\nabla\phi|^2 + \xi\Delta\phi - \frac{\Delta a}{a} + 2\xi \frac{\nabla a \cdot \nabla\phi}{a} \right) u_\xi \,.$$

By requiring that this vanish for all ξ we obtain the following nonlinear system for a, g, ϕ:

$$\begin{cases} \Delta a + \alpha^2 a = 0 \\ \Delta\phi + 2\dfrac{\nabla a \cdot \nabla \phi}{a} - 2\alpha = 0 \\ g = 1 + |\nabla\phi|^2 \end{cases}$$

The first equation permits an analytic, spherically symmetric solution,

$$a(y) = c_n \int_{S^{n-2}} e^{i\alpha\langle y,\eta\rangle} d\sigma(\eta)\,.$$

The function g is uniquely determined by the third equation, so it remains to solve the second equation for ϕ. Since a has zeros, we only obtain a local solution ϕ for y near 0. If we express $\nabla a/a$ and ϕ as formal power series near 0,

$$\frac{\nabla a(y)}{a(y)} = y \sum_{k=1}^{\infty} a_{2k}\, |y|^{2k-2}\,, \qquad \phi(y) = \sum_{k=1}^{\infty} b_{2k}\, |y|^{2k}\,,$$

then we derive the recurrence relation

$$k(2k+n-3) b_{2k} = -\sum_{j=1}^{k-1} 2j\, a_{2(k-j)}\, b_{2j}\,, \qquad k \ge 2\,,$$

with the initial condition

$$b_2 = \frac{\alpha}{n-1} > 0\,.$$

This implies that

$$|b_{2k}| < \max_{1\le j\le k-1} |a_{2(k-j)} b_{2j}|\,.$$

Since a is analytic near 0 we have

$$|a_{2k}| \le M^k\,,$$

where M^{-1} is the distance to the first complex 0 of a. Combined with the previous inequality, this inductively leads to the bound

$$|b_{2k}| \le M^{k-1} b_2\,,$$

which guarantees that the formal series for b generates an analytic function near 0.

We remark that for dimension $n = 2$ one can explicitly solve the above system to obtain

$$a(y) = \cos \alpha y\,, \qquad \phi(y) = y \tan \alpha y\,.$$

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Department of Mathematics, University of Washington, Seattle, WA 98195

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail address: hart@math.washington.edu

E-mail address: tataru@math.berkeley.edu