On the equation $\Box u = |\nabla u|^2$ in $5+1$ dimensions

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Abstract

The aim of this article is to complete the local theory for one of the most resilient semilinear hyperbolic equations. The main difficulty in such problems is to construct the function spaces in which we seek the solutions. Such spaces must, on one hand, be large enough, so that they contain the solutions; but on the other hand, they must be small enough, so that they have suitable multiplicative properties.

1 Introduction

This article is devoted to the study of the local well-posedness problem for the semilinear hyperbolic equation

$$\Box u = G(u)(\nabla u)^2 \qquad \text{in} \quad R^n \times R \tag{1.1}$$

with Cauchy data at time 0

$$u(0) = u_0 \in H^s(R^n), \qquad u_t(0) = u_1 \in H^{s-1}(R^n)$$

The main question is to determine the the values of s for which this problem is locally well-posed.

The local theory for semilinear hyperbolic equations has received considerable attention in recent years. The first major breakthrough was the discovery of the Strichartz estimates in the 70's, which has eventually generated some progress for all such problems (see for instance [3], [13] and references therein). However, the Strichartz estimates do not provide sharp results for most equations where the nonlinearity involves ∇u. A second turning point was Klainerman and Machedon's idea of obtaining convolution estimates in the Fourier

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space (see e.g. [6], [7], [8]). This has led to the $X^{s,\theta}$ spaces[1] which are multiplier weighted L^2 spaces associated to the wave operator in the same way the H^s spaces are associated to the Laplacian. Using multiplicative estimates for $X^{s,\theta}$ spaces one again obtains an improvement for all problems, but sharp results only for some. Going beyond this point is very technical and requires various modifications of the $X^{s,\theta}$ spaces (see e.g. [11], [16] and [17]); at this point the difficulty shifts from proving the estimates to constructing the spaces. The equation we consider in this article falls within this category.

Now return to the problem at hand. A scaling analysis leads to the scaling exponent $s_c = \frac{n}{2}$. Since even smooth data can lead to blow-up for this equation (see Lindblad [12]), it follows that local well-posedness can only hold for $s \geq \frac{n}{2}$. Naively, one might conjecture that this should indeed be the case. However, even at fixed frequency there is an additional obstruction which further limits this range, arising from the regularity properties of $|\nabla u|^2$. Overall, the conjectured sharp range of s is

$$s > \max\{\frac{n}{2}, \frac{n+5}{4}\}$$

which is consistent with Lindblad's counterexamples. In dimensions $n = 2, 3$ this can be proved using the Strichartz estimates, while in dimension $n = 4$ it can be proved within the framework of the $X^{s,\theta}$ spaces. These approaches fail if $n \geq 5$; This failure is related to the interaction of high and low frequencies in the multiplicative estimates. Our main result considers exactly the most difficult case $n \geq 5$.

Theorem 1 *Assume that $n \geq 5$. Then the equation (1.1) is locally well-posed in H^s for all $s > \frac{n}{2}$. More precisely, given an initial data $(u_0, u_1) \in H^s \times H^{s-1}$ there exists a unique H^s local solution u within the space F^s defined below. Furthermore, the solution depends analytically on the initial data.*

Observe that we can always reduce the problem to the case when the initial data is small. Indeed, it suffices to prove that local well-posedness holds for initial data in the homogeneous space $\dot{H}^s \times \dot{H}^{s-1}$. Due to scaling it is enough to prove this for small data in $\dot{H}^s \times \dot{H}^{s-1}$. Then we use the finite speed of propagation and localize the initial data to a compact set; thus we achieve the reduction to small data in $H^s \times H^{s-1}$. Note that one has to be a bit cautious about the localization since small functions in $\dot{H}^s$ are small locally only after subtracting a local average.

The theorem is proved using a fixed point argument in an appropriate function space. If we introduce the parametrix V for the wave equation, i.e. $Vf = u$ if

$$\begin{cases} \Box u = f \\ u(0) = 0 \\ u_t(0) = 0 \end{cases} \tag{1.2}$$

[1]Klainerman and Machedon use the notation $H^{s,\theta}$ for these spaces. However, we prefer to use the letter X instead; on one hand the H notation has been used with many different meanings before and may sometimes lead to confusion, while, on the other hand, the X notation is consistent with the one used by Bourgain for KdV and by Kenig-Ponce-Vega for the nonlinear Schroedinger equation

then we can rewrite the equation as

$$u = v + VN(u), \qquad N(u) = G(u)(\nabla u)^2$$

where v solves the corresponding homogeneous equation

$$\begin{cases} \Box v = 0 \\ v(0) = u_0 \\ v_t(0) = u_1 \end{cases} \tag{1.3}$$

However, a successful fixed point argument in this context would yield a global result for data in a scale invariant space[2] such as $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$. Obtaining local results for $s > \frac{n}{2}$ requires a modified fixed point argument where we use an appropriate truncation,

$$u = \chi(t)(v + VN(u))$$

with χ compactly supported and equal to 1 near $t = 0$.

The classical approach for such problems uses the fact that

$$V : H^{s-1}_{comp} \to H^s_{loc};$$

then one can attempt to setup the fixed point argument in H^s_{comp}. This is, however, one derivative off the scaling and thus cannot possibly work for s close to $s_c = \frac{n}{2}$.

A better idea is to use the $X^{s,\theta}$ spaces (see e.g. [10], [11],[15]) associated to the wave equation and the mapping property

$$V : X^{s-1,-\frac{1}{2}} \to X^{s,\frac{1}{2}}$$

Although this is consistent with the scaling, the desired estimate

$$N : X^{s,\frac{1}{2}} \to X^{s-1,-\frac{1}{2}}$$

fails. However, the above estimate is true at fixed frequency; its failure in general is due to the interaction between the high and low frequencies.

Our approach is to find a suitable modification F^s of the $X^{s,\frac{1}{2}}$ space for which the appropriate estimates hold. A similar idea was successfully used in [11], [16] and [17] for other semilinear hyperbolic equations[3]. Before we describe the spaces F^s it is useful to collect the properties they should have. Suppose we want to use a mapping property for V of the form[4]

$$\chi V : \Box(F^s \cap H^s) + H^{s-1} \to F^s \cap H^s \tag{1.4}$$

to set up a fixed point argument in $F^s \cap H^s$. This mapping property follows if we know that

[2] with respect to the scaling associated to the equation

[3] Each of these articles uses different spaces, though.

[4] The reason why the H^s spaces appear below is that it is more convenient to work with homogeneous function spaces F^s

(i) The functions in F^s have bounded level s energy, $F^s \subset C(\dot{H}^s) \cap C^1(\dot{H}^{s-1})$

(ii) F^s contains all compactly supported $u \in H^s$ for which $Pu \in H^{s-1}$.

(iii) $F^s \cap H^s$ has the localization property, i.e. $\chi(F^s \cap H^s) \subset F^s \cap H^s$ for all smooth compactly supported χ.

Indeed, by **(ii)**, **(iii)** it suffices to look at the equation

$$\left\{ \begin{array}{l} \Box u = \Box v \\ u(0) = 0 \\ u_t(0) = 0 \end{array} \right. \tag{1.5}$$

for $v \in \Box(F^s \cap H^s)$. Then $u = v + w$ where w is the solution to the homogeneous wave equation with Cauchy data $(u(0), u_t(0))$. By **(i)** it follows that w is an H^s solution to the homogeneous wave equation, hence when truncated it belongs to $F^s \cap H^s$ by **(ii)**.

Finally, in order to use the fixed point argument we need to prove the appropriate multiplicative estimates for the nonlinear term $N.$, i.e.

(iv) $F^s \cap H^s$ is an algebra.

(v) $(F_s \cap H^s) \cdot (\Box(F^s \cap H^s) + H^{s-1}) \subset (\Box(F^s \cap H^s) + H^{s-1})$

(vi) $\nabla(F^s \cap H^s) \cdot \nabla(F^s \cap H^s) \subset \Box(F^s \cap H^s) + H^{s-1}$

The difficulty of this problem lies less with proving the appropriate estimates for the F^s space, and more with the construction of such a space. Thus, it is important to motivate the choices made below. After defining the F^s spaces, in the next section we proceed to prove the properties **(i-vi)**. If one tries to reproduce the same arguments for the $X^{s,\frac{1}{2}}$ spaces, the proofs are similar, but simpler. This works all the way to the last piece of **(vi)**, namely (2.27). However, it is not difficult to see that (2.27) fails. Then the natural question to ask is in what way does it fail. This question is answered in Lemmas 3.2,3.5, where we show that the "bad" part of the product is a sum of "outer atoms", which are $L^\infty(L^1)$ functions frequency localized in certain parallelepipeds which we call "outer blocks". This suggests that we might be able to overcome this difficulty if we shrink $X^{s,\frac{1}{2}}$ by adding some additional dual $L^1(L^\infty)$ bounds on these "outer blocks". The price to pay is that we need to recover this additional $L^1(L^\infty)$ norms in all the other estimates.

2 The F^s spaces

Denote the symbol of the d'Allembertian by

$$p(\xi) = \xi_0^2 - \xi'^2$$

First we describe the way we partition the Fourier space. Let

$$A_\lambda = \{\xi;\ \frac{\lambda}{4} \leq |\xi| \leq 4\lambda\} \qquad \tilde{A}_\lambda = \{\xi; |\xi| \leq 4\lambda\}$$

be the spherical dyadic regions of the Fourier space. For $d \leq \lambda$ set

$$A_{\lambda,d} = \{\xi \in A_\lambda; \frac{\lambda d}{4} \leq |p(\xi)| \leq 4\lambda d\}, \qquad \tilde{A}_{\lambda,d} = \{\xi \in A_\lambda; |p(\xi)| \leq 4\lambda\delta\}$$

i.e the region at frequency $\lambda$and at distance d from the cone, respectively the region at frequency λ and within distance d from the cone. We plan to cut these sets further into rectangular regions.

Definition 2.1 *A (λ, d) "outer block" is a rectangular region B in the Fourier space, of size*[5] $\lambda \times (\sqrt{\lambda d})^{n-1} \times d$*, contained in the exterior component of* $A_{\lambda,d}$.

Now consider a locally finite partition of the exterior of the cone

$$A_\lambda \cap \{p(\xi) < 0\} = \bigcup_{\delta,j} A^j_{\lambda,d},$$

where $A^j_{\lambda,d}$ are (λ, d) outer blocks, and a corresponding partition of unit

$$\chi_{A_\lambda} = \sum a^j_{\lambda,d}$$

where the derivatives of the symbols $a^j_{\lambda,d}$ satisfy the appropriate bounds relative to the size of the outer blocks.

Next we introduce multipliers corresponding to all these partitions. Start with a smooth function ϕ supported in $|x| \leq 1$, which is 1 in $|x| \leq \frac{1}{2}$ and set

$$\phi_\lambda(x) = \phi(\frac{|x|}{\lambda})$$

Then define the following multipliers:

Operator	Symbol	Description
S_λ	$s_\lambda(\xi) = \phi_{2\lambda}(\|\xi\|) - \phi_\lambda(\|\xi\|)$	cutoff in A_λ
$\tilde{S}_\lambda$	$\tilde{s}_\lambda(\xi) = \phi_\lambda(\|\xi\|)$	cut off in $\tilde{A}_\lambda$
T_λ	$t_\lambda(\xi) = \phi_{4\lambda}(\|\xi\|) - \phi_{\lambda/2}(\|\xi\|)$	cuts off an enlargement of A_λ
$S_{\lambda,d}$	$s_{\lambda,d}(\xi) = (\phi_{2\lambda d}(p(\xi)) - \phi_{\lambda d}(p(\xi)))s_\lambda(\xi)$	cutoff in $A_{\lambda,d}$
$\tilde{S}_{\lambda,d}$	$\tilde{s}_{\lambda,d}(\xi) = \phi_{\lambda d}(p(\xi))s_\lambda(\xi)$	cutoff in $\tilde{A}_{\lambda,d}$
$T_{l,d}$	$t_{\lambda,d}(\xi) = (\phi_{4\lambda d}(p(\xi)) - \phi_{\lambda d/2}(p(\xi)))\tilde{s}_\lambda(\xi)$	cuts off an enlargement of $A_{\lambda,d}$
$S^j_{\lambda,d}$	$s^j_{\lambda,d}(\xi) = a^j_{\lambda,d}(\xi)s_{\lambda,d}(\xi)$	cutoff in $A^j_{\lambda,d}$
$T^j_{\lambda,d}$	$t^j_{\lambda,d} = (\sum_{\|j-i\|\leq 2} a^i_{\lambda,d})t_{\lambda,d}(\xi)$	cuts off an enlargement of $A^j_{\lambda,d}$

Denote by $\overset{\circ}{X}_\lambda$ the space of L^2 solutions to the homogeneous wave equation with Fourier transform supported in A_λ and norm

$$\|u\|_{\overset{\circ}{X}_\lambda} = \|u(0)\|_{L^2} + \lambda^{-1}\|u_t(0)\|_{L^2}$$

[5]Of course, the length of λ is achieved radially

Now define the spaces $X_\lambda^{\theta,p}$, Y_λ of temperate distributions with Fourier transform supported in A_λ, with norms

$$\|u\|^p_{X_\lambda^{\theta,p}} = \sum_d d^{\theta p} \|S_{\lambda,d} u\|^p_{L^2} \qquad -\frac{1}{2} \leq \theta \leq \frac{1}{2}, \quad 1 \leq p \leq \infty$$

$$\|u\|_{Y_\lambda} = \lambda^{-\frac{n-2}{2}} \sup_{d,j} \|S^j_{\lambda,d} u\|_{L^1(L^\infty)}$$

Note that the distributions in the spaces $X_\lambda^{\theta,p}$ are uniquely defined only modulo solutions to the homogeneous wave equation. However, we can make this choice unique (and save the scaling) if $\theta < \frac{1}{2}$ or $(\theta, p) = (\frac{1}{2}, 1)$ by assuming that the functions in $X_\lambda^{\theta,p}$ are the sum of their dyadic pieces (and therefore their Fourier transform has no component which is supported on the cone).

Adding these two norms we produce

$$F_\lambda = X_\lambda^{\frac{1}{2},1} \cap Y^s$$

Finally, we define the space of temperate distributions F^s as the square summable superposition of these dyadic pieces with the appropriate number of derivatives,

$$\|u\|^2_{F^s} = \sum_{\lambda = 2^j} \lambda^{2s} \|S_\lambda u\|^2_{F_\lambda}$$

The elements of F^s are uniquely determined as distributions modulo polynomials. However, since s is close to $\frac{n}{2}$, we can uniquely identify the derivatives of F^s functions with the sum of their dyadic parts. Then we can discard all the polynomials but the constants from F^s functions. Of course, this problem dissapears when we look at $F^s \cap H^s$ or at $\Box F^s$.

Recall now the Strichartz estimates for solutions to the homogeneous wave equation (see [3], and also the recent article [5] for the endpoint result):

$$\|u\|_{L^p(L^q)} \leq \|u(0)\|_{\dot{H}^s} + \|u_t(0)\|_{\dot{H}^{s-1}} \tag{2.6}$$

whenever

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - s, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \qquad 2 \leq p, q \leq \infty \tag{2.7}$$

(except for the case when both p and q are infinity.) By foliating the functions in $X_\lambda^{\frac{1}{2},1}$ in frequency with respect to translated cones, the above Strichartz estimates easily imply that

Lemma 2.2 *("Sobolev" embeddings) Let p, q be as in (2.7). Then the following embeddings hold uniformly in λ:*

$$X_\lambda^{\frac{1}{2},1} \subset \lambda^s L^p(L^q) \tag{2.8}$$

By duality this also implies that

$$S_\lambda L^{p'}(L^{q'}) \subset X_\lambda^{-\frac{1}{2},\infty} \tag{2.9}$$

In particular we get the energy estimate

$$F_\lambda \subset L^\infty(L^2)$$

which implies **(i)**.

We continue with a second embedding, which is in a way similar to the dual energy estimate:

Lemma 2.3 *The following embedding holds uniformly in λ:*

$$L^1(L^2)_\lambda \subset \lambda^{-1}\Box Y_\lambda \tag{2.10}$$

Proof: We need to show that

$$S^j_{\lambda,d} L^1(L^2) \subset d\lambda^{\frac{n-2}{2}} L^1(L^\infty) \tag{2.11}$$

But this is a Sobolev embedding. The support of the symbol $s^j_{\lambda,d}$ with respect to ξ is contained in a cube of size $((\sqrt{\lambda d})^{n-1} \times \lambda)$. Then we get

$$\|S^j_{\lambda,d} u\|_{L^1(L^\infty)} \leq [\lambda(\sqrt{\lambda d}^{n-1})]^{\frac{1}{2}} \|u\|_{L^1(L^2)}$$

which implies (2.11) for $n \geq 5$. ♣

As a consequence of the previous two embedings we get

Lemma 2.4 *The following embeddings hold uniformly in $1 \leq \mu \leq \lambda$:*

$$S_\mu L^1(L^2) \subset \ln\mu(\mu^{-1}\Box F_\mu + L^2_\mu) \tag{2.12}$$

$$S_{\lambda,\mu} L^1(L^2) \subset \ln\mu(\lambda^{-1}\Box F_\lambda + L^2_\lambda) \tag{2.13}$$

Proof: We prove (2.12); the proof of (2.13) is similar. Note that

$$\mu^{-1}\Box F_\mu = X_\lambda^{-\frac{1}{2},1} \cap \mu^{-1}\Box Y_\mu$$

On the other hand, combining (2.9) and (2.10) we obtain

$$S_\mu L^1(L^2) \subset X_\mu^{-\frac{1}{2},\infty} \cap \mu^{-1}\Box Y_\mu$$

Then the component $\tilde{S}_{\mu,1} L^1(L^2)$ within distance 1 from the cone is in L^2 since

$$\tilde{S}_{\mu,1} X_\mu^{-\frac{1}{2},\infty} \subset L^2$$

while for outer component we use

$$(1 - S_{\mu,1}) X_\mu^{-\frac{1}{2},\infty} \subset \ln \mu X_\mu^{-\frac{1}{2},1}$$

The factor $\ln \mu$ is the number of dyadic regions whose distance to the cone is between 1 and μ. ♣

Proof of (ii): . It suffices to prove the estimate

$$\|u\|_{F^s} \leq \|u\|_{H^s} + \|tu\|_{H^s} + \|\Box u\|_{H^{s-1}} + \|\Box tu\|_{H^{s-1}}$$

If we use a Paley-Littlewood decomposition, together with the observation that

$$[t, S_\lambda] = \lambda^{-1} (\nabla S)_\lambda$$

then this reduces to the dyadic estimates

$$\|u\|_{F_\lambda} \leq \|u\|_{L^2} + \|tu\|_{L^2} + \lambda^{-1}(\|\Box u\|_{L^2} + \|\Box tu\|_{L^2}) \qquad \lambda \geq 1 \tag{2.14}$$

respectively

$$\|u\|_{F_\lambda} \leq \lambda^{1-n}(\|u\|_{L^2} + \|tu\|_{L^2}) \qquad \lambda \leq 1 \tag{2.15}$$

The L^2 estimate corresponding to (2.15) is trivial, while for (2.14) we need to show that

$$\|u\|_{X_\lambda^{\frac{1}{2},1}} \leq \|u\|_{L^2} + \lambda^{-1}\|\Box u\|_{L^2}$$

But $\lambda^{-1}\|\Box u\|_{L^2} = \|u\|_{X_\lambda^{1,2}}$ therefore the estimate follows if we use the L^2 norm of u within distance[6] 1 from the cone and the L^2 norm of $\Box u$ away from the cone.

To conclude the proof of **(ii)** it remains to prove the $L^1(L^\infty)$ estimates on outer cubes corresponding to (2.14), (2.15),

$$\|S^j_{\lambda,d} u\|_{L^1(L^\infty)} \leq \lambda^{\frac{n-4}{2}}(\|\Box u\|_{L^2} + \lambda^{\frac{n-4}{2}}\|t\Box u\|_{L^2}) \qquad \lambda \geq 1 \tag{2.16}$$

$$\|S^j_{\lambda,d} u\|_{L^1(L^\infty)} \leq \lambda^{\frac{n-2}{2}}(\|u\|_{L^2} + \|tu\|_{L^2}) \qquad \lambda \leq 1 \tag{2.17}$$

The important observation is that within an outer block $A^j_{\lambda,d}$ the operator $\Box$ behaves like λd times a zero order elliptic operator. Hence if $\lambda > 1$ then

$$\begin{aligned}
\|S^j_{\lambda,d} u\|_{L^1(L^\infty)} &\approx (\lambda d)^{-1}\|S^j_{\lambda,d}\Box u\|_{L^1(L^\infty)} && \text{(“ellipticity” of } \Box\text{)} \\
&\leq \lambda^{\frac{n-4}{2}}\|S^j_{\lambda,d}\Box u\|_{L^1(L^2)} && \text{(use the embedding (2.11))} \\
&\leq \lambda^{\frac{n-4}{2}}(\|S_\lambda \Box u\|_{L^2} + \|tS_\lambda \Box u\|_{L^2})
\end{aligned}$$

[6] in frequency

which implies (2.16).

If $\lambda < 1$, on the other hand, then

$$\begin{aligned}
\|S^j_{\lambda,d}u\|_{L^1(L^\infty)} &\leq \|S_\lambda u\|_{L^1(L^\infty)} \\
&\leq \lambda^{\frac{n}{2}}\|S_\lambda u\|_{L^1(L^2)} \qquad \text{(use the embedding (2.11))} \\
&\leq \lambda^{\frac{n}{2}}(\|S_\lambda u\|_{L^1(L^2)} + \|tS_\lambda u\|_{L^2})
\end{aligned}$$

i.e. (2.17). ♣

While the compactness assumption was essential in the proof of **(ii)**, it is no longer required for **(iii-vi)**. By **(ii)**, $F^s \cap H^s$ contains all the smooth compactly supported functions, therefore **(iii)** is a special case of **(iv)**.

To prove **(iv-vi)** we start with the partition of unit

$$1 = \tilde{S}_1 + \sum_{\lambda=2^j}^{j\geq 1} S_\lambda,$$

do the corresponding Paley-Littlewood decomposition for both factors and consider all possible cases. For all but the last case, we prove the appropriate dyadic estimates essentially using the embeddings in Lemma 2.4 and Lemma 2.2. The remaining estimate (2.26), which is the heart of the paper, corresponds to having a product of a high frequency factor with a low frequency factor in **(iv)**. The last section contains is devoted to its proof.

A. frequency $1 \times$ **frequency** $1 \to$ **frequency** 1.

We first need to see how do our function spaces look like at frequency 1. The following straightforward relations are sufficient:

$$\tilde{S}_1[\Box(F^s \cap H^s) + H^{s-1}] = \tilde{S}_1 L^2, \qquad \tilde{S}_1(L^1(L^\infty) \cap L^2) \subset \tilde{S}_1(F_s \cap H^s) \subset \tilde{S}_1 L^2 \tag{2.18}$$

Then all the estimates we need reduce to

$$\tilde{S}_1 L^2 \cdot \tilde{S}_1 L^2 \subset L^1(L^\infty) \cap L^2$$

which follows easily using Sobolev embeddings.

B. frequency $1 \times$ **frequency** $\lambda \to$ **frequency** λ**,** $1 \ll \lambda$

Given (2.18), it suffices to show that

$$\tilde{S}_1 L^2 \cdot (F_\lambda \cap L^2_\lambda) \subset (F_\lambda \cap L^2_\lambda)$$

and

$$\tilde{S}_1 L^2 \cdot (\lambda^{-1}\Box F_\lambda + L^2_\lambda) \subset \lambda^{-1}\Box F_\lambda + L^2_\lambda$$

(all the other cases are better). The summation with respect to λ causes no problems due to ortogonality.

To prove this we decompose the frequency λ factor in a part with Fourier transform supported within distance 10 from the cone, and a part supported away from the cone. To show that

$$\tilde{S}_1 L^2 \cdot \tilde{S}_{\lambda,10}(F_\lambda \cap L^2_\lambda) \subset (F_\lambda \cap L^2_\lambda)$$

and

$$\tilde{S}_1 L^2 \cdot \tilde{S}_{\lambda,10}(\lambda^{-1}\Box(F_\lambda \cap L^2) + L^2_\lambda) \subset \lambda^{-1}\Box F_\lambda + L^2_\lambda$$

we use an the analogue of (2.18),

$$\tilde{S}_{\lambda,10}[\lambda^{-1}\Box(F_\lambda \cap L^2) + H^{s-1}] = \tilde{S}_{\lambda,10}L^2, \ \ \tilde{S}_{\lambda,10}(L^1(L^2)\cap L^2) \subset \tilde{S}_{\lambda,10}(F_\lambda \cap L^2) \subset \tilde{S}_{\lambda,10}L^2 \quad (2.19)$$

where for the $L^1(L^2)$ part we use (2.11). Then both estimates follow from

$$\tilde{S}_1 L^2 \cdot \tilde{S}_{\lambda,10}L^2 \subset L^1(L^2) \cap L^2$$

For the outer part, it suffices to prove that

$$\tilde{S}_1 L^2 \cdot (1 - \tilde{S}_{\lambda,10})F_\lambda \subset F_\lambda$$

and

$$\tilde{S}_1 L^2 \cdot (1 - \tilde{S}_{\lambda,10})\Box F_\lambda \subset \Box F_\lambda$$

Both follow from the next Lemma, which we state in greater generality for later use.

Lemma 2.5 *Let $\mu \ll \lambda$. Then*

$$((1 - \tilde{S}_{\lambda,10\mu})F_\lambda) \cdot (\tilde{S}_\mu L^\infty) \subset F_\lambda,$$

$$((1 - \tilde{S}_{\lambda,10\mu})\Box F_\lambda) \cdot (\tilde{S}_\mu L^\infty) \subset \Box F_\lambda$$

and

$$((1 - \tilde{S}_{\lambda,10\mu})F_\lambda) \cdot (\tilde{S}_\mu L^\infty) \subset (\lambda\mu)^{-1}\Box F_\lambda,$$

Of course after multiplication by a frequency μ function the support of the Fourier transform increases by μ. This can be accounted for by adding $F_{2\lambda}$ and $F_{\lambda/2}$ to the result. However, for the sake of keeping the notations simple we choose to neglect this harmless imprecision here and in the sequel.

Proof: Start with the first product. For the $L^1(L^\infty)$ estimates observe that for $d > 10\mu$ the output (in frequency) in an outer block $A^j_{\lambda,d}$ is generated by a μ enlargement of it, therefore

$$S^j_{\lambda,d}(F_\lambda \cdot (\tilde{S}_\mu L^\infty)) = S^j_{\lambda,d}((T^j_{\lambda,d}F_\lambda) \cdot (\tilde{S}_\mu L^\infty)) \subset S^j_{\lambda,d}(\lambda^{\frac{n-2}{2}} L^1(L^\infty) \cdot L^\infty) \subset L^1(L^\infty)$$

For the L^2 estimate similarly observe that for $d > 10\mu$ the output generated by the $A_{\lambda,d}$ region in the first factor is contained in a μ enlargement of it, therefore it only affects the neighboring dyadic regions $A_{\lambda,d/2}, A_{\lambda,2d}$. Then the summability with respect to the (dyadic) distance to the cone is preserved, and by ortogonality it suffices to see that

$$S_{\lambda,d}(F_\lambda \cdot (\tilde{S}_\mu L^\infty)) = S_{\lambda,d}(T_{\lambda,d}F_\lambda) \cdot (\tilde{S}_\mu L^\infty)) \subset S_{\lambda,d}(d^{-\frac{1}{2}}L^2 \cdot L^\infty) \subset d^{-\frac{1}{2}}L^2$$

The last two parts of the lemma follow in a similar manner. ♣

C. frequency λ × frequency $\lambda \to$ frequency μ, $1 \leq \mu \leq \lambda$
In these cases we prove the dyadic estimates corresponding to $s = \frac{n}{2}$ in **(iv-vi)**,

$$\begin{aligned} S_\mu(F_\lambda \cdot F_\lambda) &\subset \lambda^n \mu^{-\frac{n}{2}}(F_\mu \cap L^2) && (2.20)\\ S_\mu(F_\lambda \cdot \Box F_\lambda) &\subset (\ln\mu)\lambda^n \mu^{-\frac{n}{2}}(\Box F_\mu + \mu L^2) && (2.21)\\ S_\mu(F_\lambda \cdot F_\lambda) &\subset (\ln\mu)\lambda^{n-2}\mu^{1-\frac{n}{2}}(\Box F_\mu + \mu L^2) && (2.22)\end{aligned}$$

but with the bad factor $\ln\mu$ in the last two estimates. However, if $s = \frac{n}{2} + \epsilon$ then we have an additional factor of $\lambda^{-2\epsilon}\mu^\epsilon$ which is sufficient to guarantee the summability for $1 \leq \mu \leq \lambda$.

The product (2.20) requires only the "Sobolev" embeddings for $p = q = 4$ and for $p = 2,\ q = \infty$. We get

$$S_\mu(F_\lambda \cdot F_\lambda) \subset \lambda^{\frac{n-1}{2}} L^2_\mu \cap \lambda^{n-1} L^1(L^\infty)_\mu$$

which is more than enough. For (2.21), respectively (2.22) we compute in a similar fashion

$$S_\mu(F_\lambda \cdot \Box F_\lambda) \subset S_\mu(\lambda^{\frac{n-1}{2}} L^2(L^\infty) \cdot \lambda^{\frac{1}{2}} L^2) \subset \lambda^{\frac{n}{2}} S_\mu L^1(L^2)$$

respectively

$$S_\mu(F_\lambda \cdot F_\lambda) \subset \lambda^{\frac{n-2}{2}} S_\mu((L^2(L^4) \cdot L^2(L^4)) \subset \lambda^{\frac{n-2}{2}} S_\mu L^1(L^2)$$

Then both (2.21) and (2.22) follow from (2.12).

D. frequency λ × frequency $\mu \to$ frequency λ, $1 \leq \mu \ll \lambda$
Again, corresponding to $s = \frac{n}{2}$ in **(iv-vi)** we shall prove the dyadic estimates:

$$\begin{aligned} F_\lambda \cdot F_\mu &\subset \mu^{\frac{n}{2}}(F_\lambda \cap L^2) && (2.23)\\ F_\lambda \cdot (\mu^{-1}\Box F_\mu + L^2) &\subset (\ln\mu)\lambda\mu^{\frac{n}{2}-1}(\lambda^{-1}\Box F_\lambda + L^2) && (2.24)\\ F_\mu \cdot (\lambda^{-1}\Box F_\lambda + L^2) &\subset (\ln\mu)\mu^{\frac{n}{2}}(\lambda^{-1}\Box F_\lambda + L^2) && (2.25)\\ F_\lambda \cdot (F_\mu \cap L^2) &\subset (\ln\mu)^2\lambda\mu^{\frac{n}{2}-1}(\lambda^{-1}\Box F_\lambda + L^2) && (2.26)\end{aligned}$$

For $s = \frac{n}{2} + \epsilon$ we gain another μ^ϵ factor, which insures the summability with respect to $\mu \geq 1$. The summability with respect to λ is not an issue here since the L^2 Besov structure is transported from the frequency λ factor to the product.

To prove (2.23)-(2.26) observe first that the part of the frequency λ factor which is supported outside a 10μ neighbourhood of the cone can be treated using Lemma 2.5. Hence in order to prove (2.23) it suffices to use the "Sobolev" embeddings to prove the L^2 estimate

$$\tilde{S}_{\lambda,10\mu}F_\lambda \cdot F_\mu \subset F_\lambda \cdot F_\mu \subset L^\infty(L^2) \cdot \mu^{\frac{n-1}{2}} L^2(L^\infty) \subset \mu^{\frac{n-1}{2}} L^2$$

and the $L^1(L^\infty)$ estimate in an outer block $A^j_{\lambda,d}$, $d \leq 20\mu$:

$$\begin{aligned}
S^j_{\lambda,d}\left(\tilde{S}_{\lambda,\mu}F_\lambda \cdot F_\mu\right) &\subset S^j_{\lambda,d}\left(\lambda^{\frac{n-2}{4}}L^2(L^4)\cdot\mu^{\frac{n-1}{2}}L^2(L^\infty)\right) && (\text{“Sobolev” embeddings})\\
&\subset S^j_{\lambda,d}\left(\lambda^{\frac{n-2}{4}}\mu^{\frac{n-1}{2}}L^1(L^4)\right)\\
&\subset \lambda^{\frac{n-2}{4}}\mu^{\frac{n-1}{2}}\lambda^{\frac{n+1}{8}}d^{\frac{n-1}{8}}L^1(L^\infty) && (\text{Sobolev embeddings})\\
&\subset \lambda^{\frac{n-2}{2}}\mu^{\frac{n}{2}}L^1(L^\infty)
\end{aligned}$$

Since the product is supported within distance 20μ from the cone, (2.23) follows.

For (2.24), (2.25) we use the Sobolev and the “Sobolev” embeddings to estimate

$$(\tilde{S}_{\lambda,10\mu}F_\lambda)\cdot(\mu^{-1}\Box F_\mu + L^2) \subset \tilde{S}_{\lambda,20\mu}[\lambda L^2(L^{\frac{2n}{n-3}})\cdot\mu^{\frac{1}{2}}L^2] \subset \lambda\mu^{\frac{n}{2}}L^1(L^2)$$

respectively

$$F_\mu\cdot\tilde{S}_{\lambda,10\mu}(\lambda^{-1}\Box F_\lambda + L^2) \subset S_{\lambda,20\mu}[\mu^{\frac{n-1}{2}}L^2(L^\infty)\cdot\mu^{\frac{1}{2}}L^2] = \lambda\mu^{\frac{n}{2}}S_{\lambda,20\mu}L^1(L^2)$$

Then (2.24) and (2.25) follow from (2.13).

It is (2.26) which is causing all the difficulty. Observe that if one tries to prove the result with $F_\lambda = X_\lambda^{\frac{1}{2},1}$ then (2.26)'s replacement would roughly be

$$X_\lambda^{\frac{1}{2},1}\cdot X_\mu^{\frac{1}{2},1} \subset \mu^{\frac{n}{2}-1}X_\lambda^{-\frac{1}{2},\infty}$$

By duality this is equivalent to

$$S_\mu(X_\lambda^{\frac{1}{2},1}\cdot X_\lambda^{\frac{1}{2},1}) \subset \mu^{\frac{n}{2}-1}X_\mu^{-\frac{1}{2},\infty} \tag{2.27}$$

which cannot hold uniformly. Indeed, suppose we localize the two factors on opposite μ cubes on the cone. If we fix μ and let λ approach infinity then the two μ cubes become flat and in the limit we obtain a product which is concentrated in frequency on a hyperplane tangent to the cone. But this clearly fails to be square integrable.

If we use Lemma 2.5, then (2.26) would follow from

$$\tilde{S}_{\lambda,10\mu}F_\lambda\cdot(F_\mu\cap L^2) \subset (\ln\mu)^2\lambda^{\frac{n}{2}}(\lambda^{-1}\Box F_\lambda + L^2) \qquad \mu \ll \lambda \tag{2.28}$$

We want to reduce (2.28) to scale invariant estimates by localizing in frequency with respect to the distance to the cone. We first truncate the second factor into $\ln\mu$ terms

$$F_\mu\cap L^2 \subset \tilde{S}_{\mu,1}L^2 + \sum_{1\leq d\leq\mu}^{d=2^j} S_{\mu,d}F_\mu$$

The product has Fourier transform supported within distance 10μ from the cone, therefore we can perform a similar decomposition into $\ln\mu$ pieces. Then (2.28) would follow from the following estimates:

$$S_{\lambda,a}(\tilde{S}_{\lambda,\mu}F_\lambda\cdot S_{\mu,d}F_\mu) \subset \mu^{\frac{n}{2}-1}(X_\lambda^{-\frac{1}{2},\infty}\cap\lambda^{-1}\Box Y_\lambda) \qquad a,d\leq\mu\ll\lambda \tag{2.29}$$

$$\tilde{S}_{\lambda,1}(\tilde{S}_{\lambda,\mu}F_\lambda\cdot\tilde{S}_{\mu,1}L^2) \subset \mu^{\frac{n}{2}-1}L^2 \qquad \mu\ll\lambda \tag{2.30}$$

$$S_{\lambda,a}(\tilde{S}_{\lambda,\mu}F_\lambda\cdot\tilde{S}_{\mu,1}L^2) \subset \mu^{\frac{n}{2}-1}(X_\lambda^{-\frac{1}{2},\infty}\cap\lambda^{-1}\Box Y_\lambda) \qquad 1\leq a\leq\mu\ll\lambda \tag{2.31}$$

Since

$$T_{\lambda,a}\Box Y_\lambda = \lambda a T_{\lambda,a} Y_\lambda$$

we can further replace these by the stronger but simpler scale invariant estimates

$$S_{\lambda,a}(X_\lambda^{\frac{1}{2},1} \cdot (S_{\mu,d}F_\mu)) \subset \mu^{\frac{n}{2}-1}(X_\lambda^{-\frac{1}{2},\infty} \cap aY_\lambda) \qquad a, d \le \mu \ll \lambda \tag{2.32}$$

$$S_{\lambda,a}(X_\lambda^{\frac{1}{2},1} \cdot (S_{\mu,d}X_\mu^{\frac{1}{2},1})) \subset \mu^{\frac{n}{2}-1}(X_\lambda^{-\frac{1}{2},\infty} \cap aY_\lambda) \qquad d \le a \le \mu \ll \lambda \tag{2.33}$$

3 The geometry of outer blocks and the cone

It is easier to prove (2.32), (2.33) in a dual formulation. Hence we need to introduce some duals of the Y_λ spaces. Thus we define outer atoms as functions whose Fourier transform is supported in outer blocks and satisfies certain $L^\infty(L^1)$ bounds. More precisely,

Definition 3.1 *A function f is a (μ, d) outer atom iff*
i) There exists an outer block $A^j_{\mu,d}$ so that $\hat{f}$ is supported in $A^j_{\mu,d}$.
ii) $|f|_{L^\infty(L^1)} \le 1$

Now use these atoms to define the spaces of functions frequency localized in $A_{\mu,d}$,

$$\tilde{Y}_{\mu,d} = \{\sum a_j f_j;\ \sum |a_j| < \infty, \quad f_j \ \ (\mu, d) \text{ outer atoms}\} \tag{3.34}$$

with the corresponding norm.

Naively one can say that the following duality holds

$$\tilde{Y}_{\mu,d} = \lambda^{\frac{n}{2}-1}(S_{\mu,d}Y_\mu)'$$

While this is not correct as stated, it can easily be made precise. In effect, $\lambda^{\frac{n}{2}-1}(S_{\mu,d}Y_\mu)'$ is a quotient space of $\tilde{Y}_{\mu,d}$, since the support of the Fourier transform of $\tilde{Y}_{\mu,d}$ is larger. To get a converse, it suffices to reverse the relation between the supports. Then

$$\tilde{Y}_{\mu,d} \subset \lambda^{\frac{n}{2}-1}(T_{\mu,d}Y_\mu)'$$

Using these spaces one can easily see that by duality (2.32), (2.33) follow from the following four estimates:

$$S_{\mu,d}((S_{\lambda,a}X_\lambda^{\frac{1}{2},1}) \cdot (S_{\lambda,b}X_\lambda^{\frac{1}{2},1})) \subset \tilde{Y}_{\mu,d} \qquad a,\ b \ll d \tag{3.35}$$

$$S_{\mu,d}(X_\lambda^{\frac{1}{2},1} \cdot \lambda^{1-\frac{n}{2}} a^{-1} \tilde{Y}_{\lambda,a}) \subset \tilde{Y}_{\mu,d} + \mu^{\frac{n}{2}-1} X_\mu^{\frac{1}{2},\infty} \qquad a \ll d \tag{3.36}$$

$$S_{\mu,d}((S_{\lambda,a}X_\lambda^{\frac{1}{2},1}) \cdot (S_{\lambda,b}X_\lambda^{\frac{1}{2},1})) \subset \mu^{\frac{n}{2}-1} X_\mu^{\frac{1}{2},\infty} \qquad d \le a \tag{3.37}$$

$$S_{\mu,d}(X_\lambda^{\frac{1}{2},1} \cdot \lambda^{1-\frac{n}{2}} a^{-1} \tilde{Y}_{\lambda,a}) \subset \mu^{\frac{n}{2}-1} X_\mu^{\frac{1}{2},\infty} \qquad d \le a \tag{3.38}$$

As a first step in the proof of (3.35) we consider the product of two solutions to the homogeneous wave equations.

Lemma 3.2 *Let $d \leq \mu \ll \lambda$. Then*

$$S_{\mu,d}(\overset{\circ}{X}_\lambda \cdot \overset{\circ}{X}_\lambda) \subset \tilde{Y}_{\mu,d} \tag{3.39}$$

By foliating $X^{\frac{1}{2},1}$ functions in the Fourier space with respect to translated cones easily gets (3.35).

Proof: Let ξ, η be on different components of the characteristic cone, so that

$$|\xi|, |\eta| \approx \lambda, \qquad \angle(\xi, \eta) \approx \alpha$$

Then $\xi + \eta$ lies outside the cones and in the region

$$p(\xi + \eta) \approx \alpha^2 \lambda^2$$

At frequency μ this is roughly at distance

$$d = \frac{\alpha^2 \lambda^2}{\mu}$$

from the cone. Given d choose α as above. Let $u, v \in \overset{\circ}{X}_\lambda$ be solutions to the wave equation with initial data in L^2 and with frequency approximatively equal to λ. Then the above analysis suggests decomposing u, v in frequency with respect to sectors of angle $\frac{\alpha}{10}$,

$$u = \sum u_i, \qquad v = \sum v_j$$

Then

$$S_{\mu,d}(u \cdot v) = \sum_{i,j} S_{\mu,d}(u_i \cdot v_j)$$

But the right hand side terms are 0 unless the "distance" from i to j is of the order of 10. Since u_i, v_i are square summable in i, it then suffices to show that $S_{\mu,d}(u_i \cdot v_j)$ is an $Y_{\mu,d}$ atom. But the appropriate $L^\infty(L^1)$ estimate for the product follows from the energy estimates for u, v. ♣

To consider the remaining range for a, b, d and prov (3.37) we need to introduce of a second type of atoms.

Definition 3.3 *A (μ, d) inner block is a rectangular region B in the Fourier space of size $\mu \times (\sqrt{\mu d})^{n-1} \times d$ contained in $\tilde{A}_{\mu,d}$.*

Definition 3.4 *A function f is a (μ, d) inner atom iff*
i) There exists an inner block $\tilde{A}^i_{\mu,d}$ so that $\hat{f}$ is supported in $\tilde{A}^i_{\mu,d}$.
ii) $d^{1/2} |f|_{L^2(L^1)} \leq 1$

Now use these atoms to define the spaces

$$Z_{\mu,d} = \{\sum a_j f_j;\ \sum |a_j| < \infty,\quad f_j \text{ inner atoms supported in } B_{\mu,d}\} \tag{3.40}$$

with the corresponding norms. Then we can complete the above Corollary with

Lemma 3.5 *Suppose that $a \leq b \leq \mu \leq \lambda$. Then*

$$\tilde{S}_{\mu,b}(X^{\frac{1}{2},1}_{\lambda,a} \cdot X^{\frac{1}{2},1}_{\lambda,b}) \subset Z_{\mu,d} \tag{3.41}$$

Proof: In this case the output in the region $\tilde{A}_{\mu,b}$ is generated by pairs of symmetric α sectors where $\alpha^2\lambda^2 = \mu b$. Hence, by orthogonality, it suffices to assume that both factors are supported in symmetric α sectors. But then the product is supported in a (μ, b) inner cube. Since

$$S_{\lambda,a}X^{\frac{1}{2},1}_{\lambda,a} \cdot S_{\lambda,b}X^{\frac{1}{2},1} \subset L^\infty(L^2) \cdot b^{-f12}L^2 = b^{-\frac{1}{2}}L^1(L^2)$$

the conclusion follows. ♣

The spaces $Z_{\mu,d}$ play only an intermediate role in our considerations, as they relate well to the X spaces. The following Lemma completes the proof of (3.37).

Lemma 3.6 *The following embedding holds:*

$$Z_{\mu,d} \subset \mu^{\frac{n}{2}-1}X_\mu^{-\frac{1}{2},\infty}$$

Proof: It suffices to do this for a (μ, d) inner atom. Using the Sobolev embeddings within the corresponding inner block we have

$$d^{-\frac{1}{2}}L^2(L^1) \subset d^{-\frac{1}{2}}d^{\frac{n-8}{4}}\mu^{\frac{n+1}{8}}L^2(L^{\frac{4}{3}}) \subset \mu^{\frac{n-2}{4}}L^2(L^{\frac{4}{3}})$$

therefore the conclusion follows from the dual "Sobolev" embeddings. ♣

To prove (3.36) and (3.38) we need to take a closer look at the $\tilde{Y}_{\mu,d}$ spaces. While $Y_{\mu,d}$ is not contained in the same scale space $\mu^{\frac{n}{2}-1}X_\mu^{-\frac{1}{2},\infty}$, it nevertheless satisfies the analogue of the energy estimate, which is the formal dual to (2.10).

Lemma 3.7 *Assume that $n \geq 5$. Then we have*

$$\tilde{Y}_{\mu,d} \subset d\mu^{\frac{n-2}{2}}L^\infty(L^2)$$

Proof: It suffices to prove the result for a (μ, d) outer atom f. Then $\hat{f}$ is supported in an $\mu \times (\sqrt{\lambda d})^{n-1} \times d$ cube and

$$|f|_{L^\infty(L^1)} \leq 1$$

Using the Sobolev embeddings

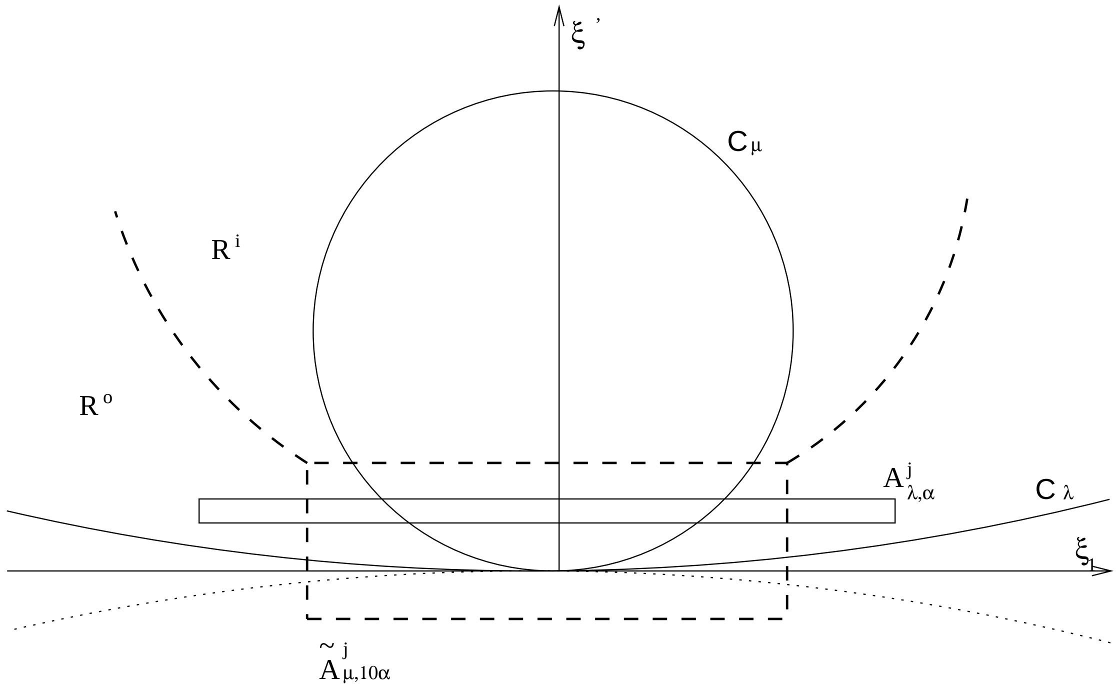
$$|f|_{L^\infty(L^2)} \leq (\mu(\sqrt{\mu d})^{n-1})^{1/2} \leq \mu^{\frac{n-2}{2}}$$

q.e.d. ♣

This plays a role in the proof of (3.36) and (3.38) where we need to look at the product

$$S_{\mu,d}(X_\lambda^{\frac{1}{2},1} \cdot \lambda^{1-\frac{n}{2}} a^{-1} \tilde{Y}_{\lambda,a})$$

Due to the atomic structure of the $\tilde{Y}$ spaces it suffices to consider the case when the second factor is an (λ, a) atom, supported in a $\lambda \times \sqrt{\lambda a}^{n-1} \times a$ cube $A^j_{\lambda,a}$. To fix the geometry suppose that $A^j_{\lambda,a}$ is parallel to the plane $\xi_0 = \xi_0$ and at distance a from it. It is now useful to visualize the geometry. To achieve that, it is simpler to look at sections $\xi_0 = const$. Then we translate the two λ sections to the frequency μ section which is their sum, along the ray $(\xi_0, \xi_0, 0, ...0)$. This is represented in the picture below. The large circles are the translated λ circles, while the small one is the radius μ circle. The section of the translated cube $A^j_{\lambda,a}$ corresponds to the dotted circle. The μ circle could be on either side of the plane $\{\xi' = 0\}$; however, this makes no difference in the proof.



We split the μ dyadic region into three components (see the interrupted lines in the picture):

$$A_\mu = \tilde{A}^i_{\mu,10a} \cup R^i \cup R^o$$

where $\tilde{A}^i_{\mu,10a}$ is the $(\mu, 10a)$ inner block parallel to Q and

$$R^i = \{10\mu\xi_1 > \tilde{\xi}^2\}$$

To prove both (3.36) and (3.38) it suffices to obtain an $L^\infty(L^1)$ estimate in R^o and L^2 estimates in R^i and $\tilde{A}^i_{\mu,10a}$.

By Lemma 3.7 we obtain

$$X_\lambda^{\frac{1}{2},1} \cdot (\lambda^{\frac{2-n}{2}} a^{-1} \tilde{Y}_{\lambda,a}) \subset L^\infty(L^1)$$

This estimate suffices in the outer region R^o since it contains only a limited number of (μ, d) outer blocks.

It remains to estimate the product in the regions $\tilde{A}^i_{\mu,10a}$ and R^i in the space $\mu^{\frac{n}{2}-1} X_\mu^{-\frac{1}{2},\infty}$. If we just use naively the "Sobolev" embeddings and the Sobolev embeddings then we obtain

$$\begin{aligned} X_\lambda^{\frac{1}{2},1} \cdot (\lambda^{\frac{2-n}{2}} a^{-1} \tilde{Y}_{\lambda,a}) &\subset \lambda^{\frac{n+1}{2(n-1)}} L^2(L^{\frac{2(n-1)}{n-3}}) \cdot \lambda^{\frac{2-n}{2}} a^{-1} (\lambda \sqrt{a\lambda}^{n-1})^{\frac{n-3}{2(n-1)}} L^\infty(L^{\frac{2(n-1)}{n+1}}) \\ &= \lambda^{\frac{5-n}{4}} a^{\frac{n-7}{4}} L^2(L^1) \\ &\subset a^{-\frac{1}{2}} L^2(L^1) \end{aligned}$$

Within $\tilde{A}^i_{\mu,10a}$ this is a $Z_{\mu,10a}$ atom and the conclusion follows from Theorem 3.6. This is, however, not strong enough in the region R^i.

The dyadic subset R^i_b of R^i at distance b from the plane $\{\xi' = 0\}$ has size $\mu \times \sqrt{\mu b}^{n-1} \times b$. To get output in R^i_b the Fourier variable in the first factor must be roughly at distance b from the frequency λ piece of the cone. Then we only need to use Lemma 3.7 to estimate

$$b^{-\frac{1}{2}} L^2 \cdot (\lambda^{\frac{2-n}{2}} a^{-1} \tilde{Y}_{\lambda,a}) \subset b^{-\frac{1}{2}} L^2(L^1)$$

Within R^i_b this is a $Y_{\mu,b}$ atom and the conclusion again follows.

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