Models derived from mice

John R. Steel

University of California, Berkeley

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A similar proof shows

Theorem 1 *Let* (M, λ, Σ) *be tractable, and* I *an* $\mathbb{R}$*-genericity iteration of* M *by* Σ*; then in the derived model* $D(M_\infty^I)$*, HOD is an* $L[\vec{E}]$*-model below* θ_0*.*

The following shows $< \lambda$-strong cardinals are needed to get $\theta_0 < \Theta$ in the derived model, at least for tractable M.

Theorem 2 *Let* (M, λ, Σ) *be tractable, and suppose no* $\kappa < \lambda$ *is* $< \lambda$*-strong in* M*. Then for any* $\mathbb{R}$*-genericity iteration* I *by* Σ*,*

$$\mathrm{Hom}_I^* = (\Delta_1^2)^{D(M_\infty^I)},$$

so that

$$D(M_\infty^I) \models \theta_0 = \Theta.$$

Moreover, $D(M_\infty^I)$ *is independent of* I*.*

Remark. Σ need not be a Wadge-minimal set not in $D(M_\infty^I)$ in this situation. An example is $M = M_{\mathrm{adr}}^\sharp$, and $\lambda =$ least limit of Woodin cardinals in $M_{\mathrm{adr}}^\sharp$.

Theorem 3 *Let Σ be a* Hom_∞ *iteration strategy for $M^\sharp_{adr}$, and let λ be the sup of all the Woodin cardinals of $M^\sharp_{adr}$. Let I be an $\mathbb{R}$ genericity iteration by Σ at λ. Then*

(a) $D((M^\sharp_{adr})^I_\infty) \models \theta_\omega = \Theta$.

(b) $D((M^\sharp_{adr})^I_\infty)$ *is independent of* I.

Proof. One can identify sets of Wadge rank θ_n in $D((M^\sharp_{\mathrm{adr}})^I_\infty)$ in a way that is independent of I, and show there sets are cofinal in the Wadge hierarchy of the model.

The following sets sit at θ_0:

(1) $M^\sharp_{\mathrm{adr}}(\mathbb{R})$,

(2) the function $x \mapsto M^\sharp_{\mathrm{adr}}(x)$, for $x \in \mathbb{R}$,

(3) Σ restricted to non-dropping trees based on $M^\sharp_{\mathrm{adr}} \restriction \delta_0$, where δ_0 is the least Woodin of $M^\sharp_{\mathrm{adr}}$.

The following sit at θ_1:

(1) $M^\sharp_{\mathrm{adr}}(M^\sharp_{\mathrm{adr}}\text{-closure})(\mathbb{R})$,

(2) the function $x \mapsto M^\sharp_{\mathrm{adr}}(M^\sharp_{\mathrm{adr}}\text{-closure})(x)$, for $x \in \mathbb{R}$,

(3) Σ restricted to non-dropping trees based between the first $< \lambda$-strong cardinal of $M^\sharp_{\mathrm{adr}}$ and the second Woodin cardinal above it.

Etc.

More generally, setting $M^{\sharp,0} = M^\sharp_{\mathrm{adr}}$, and $M^{\sharp,n} = M^\sharp_{\mathrm{adr}}(M^{\sharp,n}\text{-closure})$, we can show that $M^{\sharp,n}(\mathbb{R})$ sits at θ_n in $D((M^\sharp_{\mathrm{adr}})^I_\infty)$. The fact that they compute the fragments of Σ described in (3) above shows they are cofinal in the Wadge hierarchy of $D((M^\sharp_{\mathrm{adr}})^I_\infty)$. ■

Remarks.

- The same proof will probably show Σ is a Wadge minimal set not in $D((M^{\sharp}_{\mathrm{adr}})^{I}_{\infty})$. It does not show that this is the minimal model of $\mathsf{AD}_{\mathbb{R}}$, although that is probably true.
- Letting λ_0 be the least limit of Woodin cardinals of $M^{\sharp}_{\mathrm{adr}}$, and J an $\mathbb{R}$-genericity iteration at λ_0, we see that $D((M^{\sharp}_{\mathrm{adr}})^{J}_{\infty})$ is $L(A, \mathbb{R})$, where A is a complete Σ^2_1 set in $D((M^{\sharp}_{\mathrm{adr}})^{I}_{\infty})$. So Σ is not a Wadge minimal set above $D((M^{\sharp}_{\mathrm{adr}})^{J}_{\infty})$.

Theorem 4 *Let Σ be a* Hom_∞ *iteration strategy for $M^\sharp_{in.lim}$, and let λ be the sup of the Woodin cardinals of $M^\sharp_{in.lim}$. Let I be an $\mathbb{R}$-genericity iteration at λ by Σ. Then*

- $D((M^\sharp_{in.lim})^I_\infty) \models \Theta = \theta_{\omega_1}$.
- $D((M^\sharp_{in.lim})^I_\infty)$ *is independent of I.*

Proof. One can define $M^{\sharp,\alpha}(\mathbb{R})$ for $\alpha < \omega_1$ as was done above (but replacing $M^\sharp_{\mathrm{adr}}$ with $M^\sharp_{\mathrm{in.lim}}$ everywhere). These sets sit at the θ_α's, and are Wadge cofinal in the derived model. ■

Again, the proof will probably show that Σ is Wadge minimal above $D((M^\sharp_{\mathrm{in.lim}})^I_\infty)$.

Remark It is open whether the results above generalize to $M^\sharp_{\mathrm{wdn.lim}}$.

An application

The following shows there are complicated iteration trees on $M_{\mathrm{adr}}^{\sharp}$.

Theorem 5 *Let Σ be the unique* Hom_∞ *iteration strategy for $M_{adr}^{\sharp}$; then there is a nondropping tree $\mathcal{T}$ on $M_{adr}^{\sharp}$ of limit length, and a cofinal branch c of $\mathcal{T}$ such that $\mathcal{M}_c^{\mathcal{T}}$ is ω_1+1-iterable, but $c \neq \Sigma(\mathcal{T})$.*

Remarks

- $\Sigma(\mathcal{T})$ is the unique cofinal b such that $\mathcal{T}^\frown b$ is a winning position in the iteration game.
- One needs at least two cardinals strong past a Woodin in the base model to generate such a $\mathcal{T}$.
- The idea of the proof is that if there were no such $\mathcal{T}$, some $(M_{\mathrm{adr}}^{\sharp})_\infty^I[G]$ would know too much about how to iterate $M_{\mathrm{adr}}^{\sharp}$.

- It is open whether if $\mathcal{T}$ is a non-dropping, normal iteration tree on a *Jensen indexed* mouse projecting to ω which is according to its unique Hom_∞ iteration strategy Σ, then $\Sigma(\mathcal{T})$ is the unique cofinal iterable branch of $\mathcal{T}$. A Jensen-indexed $\mathcal{M}$ supporting such a ambiguous $\mathcal{T}$ would have to be stronger that $M^\sharp_{\text{in.lim}}$, though probably not as strong as $M^\sharp_{\text{wdn.lim}}$. It seems one must understand derived models at this level to get an example.

- An example in the Jensen indexing is essentially the same as a "strongly nonoverlapping" tree on an ordinary mouse: one demands that if $\alpha T \beta T \gamma$, then $i_{\alpha,\beta}(\mathrm{crit}(i_{\alpha,\beta})) \leq \mathrm{crit}(i_{\beta,\gamma})$.

Questions

(1) Is $D(M_\infty^I)$ independent of I?

(2) Is every Wadge initial segment of $D(M_\infty^I)$ of the form $D(N_\infty^J)$?

(3) How strong is $\mathsf{AD}_\mathbb{R} + \theta_{\omega_1} < \Theta$? Does it hold in any $D(M_\infty^I)$?

(4) Do any of the following imply that $M^\sharp_{\text{wdn.lim}}$ exists:

  (a) PFA.

  (2) "There is an ω_1-dense ideal on ω_1" plus CH.

  (3) $\mathsf{AD}_\mathbb{R} + \Theta$ is regular.