Models derived from mice

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Homogeneously Suslin sets

Definition 1 A *homogeneity system with support Z* is a function $\bar{\mu}$ such that for all $s, t \in Y^{<\omega}$,

1. μ_t is a countably complete ultrafilter concentrating on $Z^{\mathrm{dom}(t)}$, and
2. $s \subseteq t \Rightarrow \mu_t$ projects to μ_s.

If all μ_t are κ-complete, then we say $\bar{\mu}$ is *κ-complete*.

Definition 2 If $\bar{\mu}$ is a homogeneity system, then for $x \in \omega^{<\omega}$,

$$x \in S_{\bar{\mu}} \quad \Leftrightarrow \quad \text{the tower of measures } \langle \mu_{x \restriction n} \mid n < \omega \rangle \text{ is countably complete.}$$

Let

$$\mathrm{Hom}_\kappa = \{S_{\bar{\mu}} \mid \bar{\mu} \text{ is } \kappa\text{-complete }\},$$

$$\mathrm{Hom}_{<\lambda} = \bigcap_{\kappa<\lambda} \mathrm{Hom}_\kappa,$$

$$\mathrm{Hom}_\infty = \bigcap_{\kappa \in \mathrm{OR}} \mathrm{Hom}_\kappa .$$

Theorem 3 *(a) [Martin 1968] Every* Hom_κ *set of reals is determined.*

(b) [Martin, S., 1985] If λ *is a limit of Woodin cardinals, then* $\mathrm{Hom}_{<\lambda}$ *is closed under complements and real quantifiers ; moreover every* $\mathrm{Hom}_{<\lambda}$ *set has a* $\mathrm{Hom}_{<\lambda}$ *scale.*

Question. Does any large cardinal hypothesis imply that every set of reals in $L(\mathbb{R}, \mathrm{Hom}_\infty)$ is in Hom_∞?

The inner model theory for such a large cardinal hypothesis would be significantly different from the one we have now.

Iteration strategies

Let $\mathcal{M}$ be a *premouse*: a model of the form $(J_\alpha[\vec{E}], \in, \vec{E})$, where $\vec{E}$ is a coherent sequence of extenders. In the *iteration game* $G(\mathcal{M}, \theta)$, I and II cooperate to build an iteration tree $\mathcal{T}$ on $\mathcal{M}$:

- I extends $\mathcal{T}$ at successor steps by applying an extender from the current model to some, possibly earlier, model. The resulting ultrapower is the new current model.
- At limit steps II picks a cofinal branch b, and $\lim_{\alpha \in b} \mathcal{M}_\alpha$ becomes the current model.

II wins if, after θ moves, no illlfounded model has been produced.

Definition 4 A *θ-iteration strategy for $\mathcal{M}$* is a winning strategy for II in $G(\mathcal{M}, \theta)$. $\mathcal{M}$ is *θ-iterable* if there is such an *iteration strategy*.

$\boxed{\text{Hom}_\infty \textbf{ iteration strategies}}$

Notice that if $\mathcal{M}$ is countable, then an ω_1-iteration strategy for $\mathcal{M}$ is essentially a set of reals. "Good" $\mathcal{M}$ (the "standard" ones), have ω_1-iteration strategies which are Hom_∞. This implies θ-iterability, for all θ.

For $L[\vec{E}]$-models in the region where we have a theory (can prove iterability),

$$x <^{L[\vec{E}]} y \quad \Leftrightarrow \quad \exists \mathcal{M}(\mathcal{M} \text{ has a } \text{Hom}_\infty \text{ iteration strategy and } \mathcal{M} \models x <^{L[\vec{E}]} y.$$

So the large cardinal hypotheses true in these models are compatible with there being a wellorder of $\mathbb{R}$ in $L(\mathbb{R}, \text{Hom}_\infty)$.

The derived model

Given $A \in \mathrm{Hom}_\kappa$, as witnessed by both $\bar{\mu}$ and $\bar{\nu}$, and G V-generic for a poset of size $< \kappa$, we have

$$(S_{\bar{\mu}})^{V[G]} = (S_{\bar{\nu}})^{V[G]}.$$

So we write $A^{V[G]}$ for the common value of all $(S_{\bar{\mu}})^{V[G]}$ such that $A = S_{\bar{\mu}}$.

Now let λ be a limit of Woodin cardinals, and G be V-generic for $\mathrm{Col}(\omega, < \lambda)$, and set

$$\begin{aligned} \mathbb{R}^*_G &= \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha], \\ \mathrm{Hom}^*_G &= \{A^* \mid \exists \alpha < \lambda (A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]})\}, \end{aligned}$$

where $A^* = \bigcup_{\beta<\lambda} A^{V[G\restriction\beta]}$.

Theorem 5 (Derived model theorem, Woodin 1987)

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$.

(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid$
$A, \mathbb{R} \setminus A$ *have scales in* $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

Woodin also showed that all models of AD^+ arise this way. Moreover, if $\Gamma \subsetneq \mathrm{Hom}_\infty$, then $L(\mathbb{R}, \Gamma) \models \mathsf{AD}^+$.

Note the theory of $L(\mathbb{R}^*_G, \mathrm{Hom}^*_G)$ is independent of G. It is possible that the derived model is just $L(\mathbb{R}^*)$, but if there are enough large cardinals below λ, it is larger.

Definition 6 (AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set

$$\begin{aligned} \theta_0 &= \theta(\emptyset), \\ \theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\ \theta_\lambda &= \bigcup_{\alpha<\lambda} \theta_\alpha. \end{aligned}$$

$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$. Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B.

Theorem 7 (Woodin, mid 80's) *Assume* AD^+*, and suppose* A *and* $\mathbb{R} \setminus A$ *admit scales; then*

(a) All $\Sigma^2_1(A)$ *sets of reals admit scales, and*

(b) All $\Pi^2_1(A)$ *sets admit scales iff* $\theta(A) < \Theta$.

Theorem 8 (Martin, Woodin, mid 80's) *Assume* AD^+*; then the following are equivalent:*

(1) $\mathsf{AD}_{\mathbb{R}}$,

(2) Every set of reals admits a scale,

(3) $\Theta = \theta_\lambda$*, for some limit* λ.

Theorem 9 (Woodin, 1988, 2000) *Suppose λ is a limit of Woodins, and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is a derived model at λ; then*

(a) $\exists\kappa < \lambda(\kappa$ *is* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \theta_0 < \Theta$.

(b) λ *is a limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, for some limit* α*.*

(c) λ *is an inaccessible limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, where* $\mathrm{cof}(\alpha) \geq \omega_1$.

(d) $\exists\kappa < \lambda(\kappa$ *is* λ*-supercompact* $) \Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular.*

Remarks.

- The hypotheses of (a)-(c) follow from λ being a Woodin limit of Woodin cardinals.
- The hypothesis of (b), that there is a λ which is a limit of Woodins and of $< \lambda$-strongs, is called the $\mathsf{AD}_{\mathbb{R}}$ *hypothesis*. Woodin has shown the $\mathsf{AD}_{\mathbb{R}}$ hypothesis is equiconsistent with $\mathsf{AD}_{\mathbb{R}}$. One direction is (b) above.
- There is a big jump going from (c) to (d). How strong is $\mathsf{AD}_{\mathbb{R}} + \Theta$ regular ? It might be weaker than a Woodin limit of Woodins. One the other hand, $\mathsf{AD}_{\mathbb{R}} + \theta_{\omega_1} < \Theta$ might be stronger than a supercompact.
- There are other ways of calibrating the strength of determinacy models. One can look at the complexity of the games of length ω_1 which are determined in the model. One can look at the complexity of the mice which have iteration strategies in the model.

Iterations to make $\mathbb{R}^V = \mathbb{R}^*$

Let M be countable and sufficiently iterable, and $M \models \lambda$ is a limit of Woodin cardinals. In $V^{\mathrm{Col}(\omega,\mathbb{R})}$, we can form an iteration I of the form

$$M = M_0 \to M_1 \to \ldots \to M_n \to \ldots M_\infty^I$$

so that for some G generic over M_∞^I for $\mathrm{Col}(\omega, < i_{0,\infty}(\lambda))$,

$$\mathbb{R}^*_G = \mathbb{R}^V.$$

We call I an $\mathbb{R}$-*genericity iteration*, and write $\mathrm{Hom}^*_G = \mathrm{Hom}^*_I$, and

$$D(M_\infty^I) = L(\mathbb{R}^V, \mathrm{Hom}^*_I)$$

for the derived model determined by I and $\mathbb{R}^V$.

Models derived from mice

Lemma 10 *Suppose $\mathcal{M}$ is a sound mouse projecting to ω, and $\mathcal{M} \models \lambda$ is a limit of Woodins, and Σ is a (necessarily unique)* Hom_∞ *iteration strategy for $\mathcal{M}$. Let I be an $\mathbb{R}$-genericity iteration according to Σ. Then*

(a) $\mathrm{Hom}^*_I \subseteq \mathrm{Hom}_\infty$,

(b) $\Sigma \notin \mathrm{Hom}^*_I$.

Example. Let $\mathcal{M} = M^\sharp_\omega$ be the sharp of the minimal model with ω Woodin cardinals, let Σ be its unique ω_1- iteration strategy, and I be an $\mathbb{R}$-genericity iteration by Σ. Then

$$\begin{aligned} D(M^I_\infty) &= L(\mathbb{R}), \\ \mathrm{Hom}^*_I &= (\Delta^2_1)^{L(\mathbb{R})}, \\ \Sigma &\equiv_W \mathbb{R}^\sharp = \text{ Wadge mimimal set not in } L(\mathbb{R}). \end{aligned}$$

Theorem 11 (Woodin, S.) *If* $M_\omega^\sharp$ *exists and has a* Hom_∞ *iteration strategy, then every set of reals in* $L(\mathbb{R})$ *is* Hom_∞*, so that* $L(\mathbb{R}) \models \mathsf{AD}^+$*. Moreover,* $L(\mathbb{R}) \equiv L(\mathbb{R})^{V[G]}$*, for all set-generic* G*.*

The existence of stronger mice with Hom_∞ iteration strategies implies more generic absoluteness.

Definition 12 $M^\sharp_{\mathrm{adr}}$ is the sharp of the minimal $\omega_1 + 1$- iterable mouse satisfying the $\mathsf{AD}_\mathbb{R}$ hypothesis.

Theorem 13 *If* $M^\sharp_{adr}$ *exists and has a (necessarily unique)* Hom_∞ *iteration strategy, then there is a model of* $\mathsf{AD}_\mathbb{R}$ *of the form* $L(\Gamma, \mathbb{R})$*, where* $\Gamma \subsetneq \mathrm{Hom}_\infty$*. Moreover, the theory of the minimal such model is generically absolute.*

Remark. If there is a Hom_∞ iteration strategy for M satisfying "there is a measurable Woodin cardinal", then Σ^2_1 statements are absolute between models of CH.

Questions. Let $M = M^{\sharp}_{\text{adr}}$, and let Σ be the unique ω_1-iteration strategy for M. Let I be an $\mathbb{R}$-genericity iteration by Σ.

(1) Is $D(M^I_\infty)$ the minimal model of $\mathsf{AD}_\mathbb{R}$ containing all reals and ordinals?

(2) Is $D(M^I_\infty)$ independent of I? Is Σ a Wadge-minimal set of reals not in $D(M^I_\infty)$?

(3) How do the θ_α of $D(M^I_\infty)$ correspond to the $< \lambda$-strong cardinals of M?

(4) What is the cofinality of Θ in $D(M^I_\infty)$? What is its first order theory?

(5) If $\Gamma \subseteq P(\mathbb{R}) \cap D(M^I_\infty)$, then must $L(\Gamma, \mathbb{R})$ be of the form $D(N^J_\infty)$, for some mouse N?

Questions (2)-(5) make sense for any M with a limit of Woodin cardinals.

Hello.

Goodbye.

A similar proof shows

Theorem 14 *Let* (M, λ, Σ) *be tractable, and* I *an* $\mathbb{R}$*-genericity iteration of* M *by* Σ*; then in the derived model* $D(M^I_\infty)$*, HOD is an* $L[\vec{E}]$*-model below* θ_0.

The following shows $< \lambda$-strong cardinals are needed to get $\theta_0 < \Theta$ in the derived model, at least for tractable M.

Theorem 15 *Let* (M, λ, Σ) *be tractable, and suppose no* $\kappa < \lambda$ *is* $< \lambda$*-strong in* M*. Then for any* $\mathbb{R}$*-genericity iteration* I *by* Σ*,*

$$\mathrm{Hom}^*_I = (\Delta^2_1)^{D(M^I_\infty)},$$

so that

$$D(M^I_\infty) \models \theta_0 = \Theta.$$

Moreover, $D(M^I_\infty)$ *is independent of* I.

Remark. Σ need not be a Wadge-minimal set not in $D(M^I_\infty)$ in this situation. An example is $M = M^\sharp_{\text{adr}}$, and λ = least limit of Woodin cardinals in $M^\sharp_{\text{adr}}$.

Theorem 16 *Let Σ be a* Hom_∞ *iteration strategy for $M^\sharp_{adr}$, and let λ be the sup of all the Woodin cardinals of $M^\sharp_{adr}$. Let I be an $\mathbb{R}$ genericity iteration by Σ at λ. Then*

(a) $D((M^\sharp_{adr})^I_\infty) \models \theta_\omega = \Theta$.

(b) $D((M^\sharp_{adr})^I_\infty)$ *is independent of I.*

Proof. One can identify sets of Wadge rank θ_n in $D((M^\sharp_{\mathrm{adr}})^I_\infty)$ in a way that is independent of I, and show there sets are cofinal in the Wadge hierarchy of the model.

The following sets sit at θ_0:

(1) $M^\sharp_{\mathrm{adr}}(\mathbb{R})$,

(2) the function $x \mapsto M^\sharp_{\mathrm{adr}}(x)$, for $x \in \mathbb{R}$,

(3) Σ restricted to non-dropping trees based on $M^\sharp_{\mathrm{adr}} \restriction \delta_0$, where δ_0 is the least Woodin of $M^\sharp_{\mathrm{adr}}$.

The following sit at θ_1:

(1) $M^{\sharp}_{\text{adr}}(M^{\sharp}_{\text{adr}}\text{-closure })(\mathbb{R})$,

(2) the function $x \mapsto M^{\sharp}_{\text{adr}}(M^{\sharp}_{\text{adr}}\text{-closure })(x)$, for $x \in \mathbb{R}$,

(3) Σ restricted to non-dropping trees based between the first $< \lambda$-strong cardinal of $M^{\sharp}_{\text{adr}}$ and the second Woodin cardinal above it.

Etc.

More generally, setting $M^{\sharp,0} = M^{\sharp}_{\text{adr}}$, and $M^{\sharp,n} = M^{\sharp}_{\text{adr}}(M^{\sharp,n}\text{-closure})$, we can show that $M^{\sharp,n}(\mathbb{R})$ sits at θ_n in $D((M^{\sharp}_{\text{adr}})^I_\infty)$. The fact that they compute the fragments of Σ described in (3) above shows they are cofinal in the Wadge hierarchy of $D((M^{\sharp}_{\text{adr}})^I_\infty)$. ■

Remarks.

- The same proof will probably show Σ is a Wadge minimal set not in $D((M^\sharp_{\text{adr}})^I_\infty)$. It does not show that this is the minimal model of $\mathsf{AD}_{\mathbb{R}}$, although that is probably true.
- Letting λ_0 be the least limit of Woodin cardinals of $M^\sharp_{\text{adr}}$, and
  J an $\mathbb{R}$-genericity iteration at λ_0, we see that $D((M^\sharp_{\text{adr}})^J_\infty)$ is $L(A, \mathbb{R})$, where A is a complete Σ^2_1 set in $D((M^\sharp_{\text{adr}})^I_\infty)$. So Σ is not a Wadge minimal set above $D((M^\sharp_{\text{adr}})^J_\infty)$.

Theorem 17 *Let Σ be a* Hom_∞ *iteration strategy for* $M^\sharp_{in.lim}$*, and let λ be the sup of the Woodin cardinals of* $M^\sharp_{in.lim}$*. Let I be an $\mathbb{R}$-genericity iteration at λ by Σ. Then*

- $D((M^\sharp_{in.lim})^I_\infty) \models \Theta = \theta_{\omega_1}$.
- $D((M^\sharp_{in.lim})^I_\infty)$ *is independent of I.*

Proof. One can define $M^{\sharp,\alpha}(\mathbb{R})$ for $\alpha < \omega_1$ as was done above (but replacing $M^\sharp_{\mathrm{adr}}$ with $M^\sharp_{\mathrm{in.lim}}$ everywhere). These sets sit at the θ_α's, and are Wadge cofinal in the derived model. ■

Again, the proof will probably show that Σ is Wadge minimal above $D((M^\sharp_{\mathrm{in.lim}})^I_\infty)$.

Remark It is open whether the results above generalize to $M^\sharp_{\mathrm{wdn.lim}}$.

An application

The following shows there are complicated iteration trees on $M^\sharp_{\mathrm{adr}}$.

Theorem 18 *Let Σ be the unique* Hom_∞ *iteration strategy for $M^\sharp_{adr}$; then there is a nondropping tree $\mathcal{T}$ on $M^\sharp_{adr}$ of limit length, and a cofinal branch c of $\mathcal{T}$ such that $\mathcal{M}^{\mathcal{T}}_c$ is $\omega_1 + 1$-iterable, but $c \neq \Sigma(\mathcal{T})$.*

Remarks

- $\Sigma(\mathcal{T})$ is the unique cofinal b such that $\mathcal{T}^\frown b$ is a winning position in the iteration game.
- One needs at least two cardinals strong past a Woodin in the base model to generate such a $\mathcal{T}$.
- The idea of the proof is that if there were no such $\mathcal{T}$, some $(M^\sharp_{\mathrm{adr}})^I_\infty[G]$ would know too much about how to iterate $M^\sharp_{\mathrm{adr}}$.

- It is open whether if $\mathcal{T}$ is a non-dropping, normal iteration tree on a *Jensen indexed* mouse projecting to ω which is according to its unique Hom_∞ iteration strategy Σ, then $\Sigma(\mathcal{T})$ is the unique cofinal iterable branch of $\mathcal{T}$. A Jensen-indexed $\mathcal{M}$ supporting such a ambiguous $\mathcal{T}$ would have to be stronger that $M^\sharp_{\text{in.lim}}$, though probably not as strong as $M^\sharp_{\text{wdn.lim}}$. It seems one must understand derived models at this level to get an example.
- An example in the Jensen indexing is essentially the same as a "strongly nonoverlapping" tree on an ordinary mouse: one demands that if $\alpha T \beta T \gamma$, then $i_{\alpha,\beta}(\text{crit}(i_{\alpha,\beta})) \leq \text{crit}(i_{\beta,\gamma})$.

Questions

(1) Is $D(M^I_\infty)$ independent of I?

(2) Is every Wadge initial segment of $D(M^I_\infty)$ of the form $D(N^J_\infty)$?

(3) How strong is $\mathsf{AD}_{\mathbb{R}} + \theta_{\omega_1} < \Theta$? Does it hold in any $D(M^I_\infty)$?

(4) Do any of the following imply that $M^\sharp_{\text{wdn.lim}}$ exists:

   (a) PFA.

   (2) "There is an ω_1-dense ideal on ω_1" plus CH.

   (3) $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.