PFA **implies** $\mathsf{AD}^{L(\mathbb{R})}$

John R. Steel

University of California, Berkeley

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Introduction

Reference: PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$, available at http://www.math.berkeley.edu/∼steel.

We shall outline a proof of

Theorem 1 *Suppose there is a singular strong limit cardinal κ such that $\square_\kappa$ fails; then $\mathsf{AD}^{L(\mathbb{R})}$ holds.*

Todorcevic has shown that if the Proper Forcing Axiom (PFA) holds, then $\square_\kappa$ fails for all uncountable cardinals κ. Thus we get immediately:

Corollary 2 PFA *implies* $\mathsf{AD}^{L(\mathbb{R})}$.

The proof of the theorem uses the theory of core models below one Woodin cardinal in an inductive argument producing canonical inner models with many Woodin cardinals. This *core model induction* method is our focus here.

The core model induction method: Show $\mathsf{AD}^{L(\mathbb{R})}$ by constructing mice over reals which have Woodin cardinals, and are correct at higher and higher levels of $L(\mathbb{R})$. The induction on levels is guided by the way new Σ_1 statements about reals appear in $L(\mathbb{R})$, and in particular, by the way the verification of such statements is tied to the appearance of new guarantors of correctness, definable scales. It connects this to the appearance of new iteration strategies for mice which are correct at the given level of $L(\mathbb{R})$.

The induction on levels of $L(\mathbb{R})$ often takes place in some $V^{\mathrm{Col}(\omega,\kappa)}$, and it is often important to go back to V in the course of the argument. We can think of the argument as showing that in V, the mice over subsets of κ have certain closure properties.

Earlier results:

(1) (Woodin, early 90's) If PFA holds and κ is strongly inaccessible, then $V^{\mathrm{Col}(\omega,<\kappa)} \models \mathsf{AD}^{L(\mathbb{R})}$. (The proof is a *global* core model induction.)

(2) (Schimmerling, Steel, mid 90's) If κ is a singular strong limit, and $\square_\kappa$ fails, then $V^{\mathrm{Col}(\omega,\kappa)} \models \mathsf{PD}$. (The proof is a *local* core model induction.)

(3) (Steel, Zoble 2000) $\mathsf{SRP}^*(\omega_2)$ plus NS_{ω_1} is saturated implies $V^{\mathrm{Col}(\omega,\omega_2)} \models \mathsf{AD}^{L(\mathbb{R})}$, and hence Martin's Maximum (MM) implies the same. (Again, a local induction. Woodin showed the hypothesis consistent relative to ω^2 Woodins.)

The method also works beyond $L(\mathbb{R})$. The furthest it has been carried in print is in Ketchersid's thesis, where it is used to produce a nontame mouse, under a hypothesis asserting the existence of a certain ideal on ω_1.

Conjecture: If there is a singular strong limit κ such that $\square_\kappa$ fails, then there is a nontame mouse, and in fact there is a mouse with a superstrong cardinal.

Relativised mice

For A a transitive set equipped with a wellorder of length $o(A)$, an A-premouse is a model of the form $(J_\alpha[\vec{E}, A], \in, \vec{E})$, where $\vec{E}$ is a coherent sequence of extenders which is just like an ordinary premouse, except that its hierarchy begins with the rud-closure of $A \cup \{A\}$. All critical points on the sequence of an A-premouse are strictly greater than $o(A)$, so that iterations do not move A, and wellfounded iterates are A-premice. The hulls used to define soundness all contain $A \cup \{A\}$.

Definition 3 An A-premouse $\mathcal{M}$ is *countably iterable* iff whenever $\mathcal{N}$ is countable and elementarily embeddable in $\mathcal{M}$, then $\mathcal{N}$ is $\omega_1 + 1$-iterable.

Any two sound, countably iterable A-premice projecting to $o(A)$ are comparable (i.e. one is an initial segment of the other).

Definition 4

$$\begin{aligned}\mathrm{Lp}(A) \quad = \quad & \bigcup\{\mathcal{N} \mid \mathcal{N} \text{ is an } \omega\text{-sound, countably iterable} \\ & A\text{-premouse and } \rho_\omega(\mathcal{N}) = o(A)\}.\end{aligned}$$

Theorem 5 (Schimmerling, Zeman 1998) *Let $\mathcal{M}$ be a countably iterable A-premouse; then for all $\nu \geq o(A)$, $\mathcal{M} \models \square_\nu$.*

It is important in the last theorem that our mice are below (many) superstrongs.

Corollary 6 *If $\square_\kappa$ fails, then for all A bounded in κ^+, $o(\mathrm{Lp}(A)) < \kappa^+$.*

Mouse operators

Definition 7 Let $A \in H_\nu$; then a *mouse operator over A on H_ν* is a function J such that for some Q-formula ψ,

$$J(B) = \text{least } \mathcal{P} \trianglelefteq \text{Lp}(B) \text{ such that } \mathcal{P} \models \psi[A, B]$$

for all transitive $B \in H_\nu$ such that $A \in B$. (J must be defined at all such B.) We call J a (ν, A)-mo.

Example: For $A = \emptyset$, $\psi =$ "my last extender is nontrivial", the associated mo is essentially $J(B) = B^\sharp$.

Example: Letting J be a (ν, A)-mo, we define the "one J-Woodin" operator J^w by

$$\begin{aligned} J^w(B) &= M_1^J(B) = \text{ least } \mathcal{P} \trianglelefteq \mathrm{Lp}(B) \text{ s.t. } \exists\delta \\ & \quad \mathcal{P} \models \delta \text{ is Woodin, and } \mathcal{P} = J(\mathcal{P}|\delta). \end{aligned}$$

So if J_0 is the sharp operator, and $J_{n+1} = (J_n)^w$, then J_n is the $M_n^\sharp$ operator.

Lemma 8 *Let $\nu = \mu^+$, where $\mu \geq \omega_1$; then the following are equivalent:*

(1) for all $n < \omega$, J_n exists as a $(\nu, \emptyset)$-mo (that is, $J_n(B)$ is defined for all $B \in H_\nu$), where J_n is the $M_n^\sharp$-operator,

(2) PD *holds in $V^{\mathrm{Col}(\omega,\mu)}$.*

The lemma is also true for $\nu = \omega_1$, if we restrict $\omega + 1$-iterability so that the iteration trees involved are all projectively coded.

Core model theory for one J-Woodin

We assume in this section that Ω is a measurable cardinal, and that J is a (Ω, A)-mo. Let $B \in H_\Omega$ be transitive, with $A \in B$.

Lemma 9 *All tame levels of the $K^c(B)$-construction are countably iterable (above $o(B)$).*

We may and do assume that all levels of any $K^c(B)$-construction are tame, hence countably iterable. Hence any such construction converges to a model $K^c(B)$ of height Ω.

Lemma 10 *$K^c(B)$ is J-closed, in the sense that whenever η is a cardinal of $K^c(B)$, then $J(K^c(B)|\eta) \trianglelefteq K^c(B)$.*

In light of this lemma, we write $K^{c,J}(B)$ for $K^c(B)$. (When dealing later with *hybrid mouse operators* J, one must close the analog of $K^{c,J}$ under J by explicitly telling the model how J acts on its levels.)

Definition 11 A (ν, A)-mo J *relativises well* iff whenever $A \in B \in C$, with $B, C \in H_\nu$ transitive, then $J(B)$ belongs to all transitive models of ZFC^- to which $J(C)$ belongs.

Let us assume our J relativises well.

Theorem 12 *Either* $M_1^J(B) \trianglelefteq K^{c,J}(B)$, *or* $K^{c,J}(B)$ *is* $(\omega, \Omega + 1)$*-iterable.*

Theorem 13 *If* $M_1^J(B) \not\trianglelefteq K^{c,J}(B)$, *then there is a* true core model $K^J(B)$ *such that*

(a) *For all cardinals* η *such that* $o(B) < \eta < \Omega$, $K^J(B)|\eta$ *is definable over* $L(H_\eta)$ *from* B *(uniformly in* η*),*

(b) *if* G *is generic over* V_Ω, *and* J *extends to* $H_\Omega^{V[G]}$, *then* $K^J(B) = K^J(B)^{V[G]}$, *and*

(c) *for all singular cardinal* μ *such that* $o(B) < \mu < \Omega$, *we have* $(\mu^+)^{K^J(B)} = \mu^+$.

Remark: In practice, our J extend to $H_\Omega^{V[G]}$.

J-to-J^w, assuming $\neg\Box_\kappa$

Fix a singular strong limit κ such that $\neg\Box_\kappa$ for the rest of the talk.

Let J be a (κ^+, A)-mo, where $A \in H_\kappa$. We want to show that J^w exists as a (κ^+, A)-mo. That is, we want to show that $M_1^J(B)$ exists whenever $A \in B \in H_{\kappa^+}$, and B is a transitive set equipped with a wellorder.

Lemma 14 *For any $C \in H_{\kappa^+}$ such that $A \in C$, there is a J-closed $\mathcal{P} \trianglelefteq \mathrm{Lp}(C)$ such that $\mathcal{P} \models \mathsf{ZFC}+$ "there is a measurable cardinal $> o(C)$".*

The lemma codifies the step J-to-$J^\dagger$. We need it because our one-J-Woodin core model theory needs a measurable. The lemma is proved using Dodd-Jensen core model theory (which does not need a measurable) in a way analogous to the proof we are about to give.

Let us write V_C^J for the $\mathcal{P}$ of this lemma.

Lemma 15 *If $A \in B \in H_\kappa$, and B is a transitive set with wellorder, then $M_1^J(B)$ exists.*

Proof sketch. Let $C \in H_{\kappa^+}$ code H_κ, together with a witness that κ is singular. Working in V_C^J, we use one-J-Woodin core model theory. We may assume $K^{c,J}(B)$ of V_C^J does not reach $M_1^J(B)$, otherwise we are done. Letting

$$K_C^J(B) = K^J(B)^{V_C^J},$$

we have, by weak covering:

$$(\kappa^+)^{K_C^J(B)} = (\kappa^+)^{V_C^J}.$$

The inductive definability of K gives that if $C \in D$, then for $\eta = (\kappa^+)^{K_C^J(B)}$,

$$K_C^J(B)|\eta = K_D^J(B)|\eta.$$

This implies that we can piece together the canonical $\Box_\kappa$- sequences of the $K_C^J(B)$, as $C \to \infty$, and obtain a full $\Box_\kappa$ sequence for V. Contradiction. $\blacksquare$

Lifting mouse-closure

We now remove the restriction $B \in H_\kappa$.

Lemma 16 *Let I be a (κ, D)-mo, where $D \in H_\kappa$; then I extends (uniquely) to a (κ^+, D)-mo.*

Proof. Let $B \in H_{\kappa^+}$ with $D \in B$. By $\neg\Box_\kappa$, we have

$$o(\mathrm{Lp}(B)) < \kappa^+.$$

As in the weak covering proof, let $N \in H_\kappa$ be transitive, closed under ω_1-sequences, and

$$\pi : N \to H_\theta,$$

where θ is large, $\mathrm{crit}(\pi) > o(D)$, everything relevant is in $\mathrm{ran}(\pi)$, and $\mathrm{ran}(\pi)$ is cofinal in $o(\mathrm{Lp}(B))$. As in weak covering

$$\mathrm{Lp}(\bar{B}) \subseteq N,$$

so $I(\bar{B}) \in N$. But then $N \models I(\bar{B})$ exists, so that $H_\theta \models I(B)$ exists , so $I(B)$ exists, as desired. ■

Remark. A method for lifting mouse-closure is crucial to local core model inductions.

A reflection point $\mu < \kappa$

For the rest of the talk, fix μ such that

$$\mathrm{cof}(o(\mathrm{Lp}(H_\kappa))) < \mu < \kappa \text{ and } \mu^{\omega_1} = \mu.$$

Lemma 17 *Let I be a (μ^+, D)-mo, where $D \in H_{\mu^+}$, which relativises well; then I extends (uniquely) to a (κ^+, D)-mo.*

Proof. By the last lemma, it is enough to see that I extends to a (κ, D)-mo. So fix $B \in H_\kappa$ transitive, with $D \in B$. Let $\pi : N \to H_\theta$ where N is transitive and closed under ω_1-sequences, $|N| = \mu$, $\mathrm{crit}(\pi) > o(D)$, and everything relevant is in $\mathrm{ran}(\pi)$, and

$$\mathrm{ran}(\pi) \text{ is cofinal in } o(\mathrm{Lp}(H_\kappa)).$$

As in weak covering,

$$\mathrm{Lp}(\bar{H}_\kappa) \subseteq N,$$

so $I(\bar{H}_\kappa) \in N$. Since I relativises well, we get $I(\bar{B}) \in N$, so $N \models I(\bar{B})$ exists, so $I(B)$ exists. ■

Mice over reals in $V^{\mathrm{Col}(\omega,\mu)}$

We want to inductively construct mouse operators on H_{μ^+}. Let g be V-generic for $\mathrm{Col}(\omega, \mu)$, and $\mathbb{R}^g$ be the reals of $V[g]$. So we want to construct mice over reals in $V[g]$. By induction on α, we construct mice over reals which capture Σ_1 truth in $J_\alpha(\mathbb{R}^g)$.

Definition 18 Let $U \subseteq \mathbb{R}^g$, and $k < \omega$. Let N be countable and transitive, and suppose $\delta_0, \ldots, \delta_k$, S, and T are such that

(a) $N \models \mathsf{ZFC} \wedge \delta_0 < \ldots < \delta_k$ are Woodin cardinals,

(b) $N \models S, T$ are trees which project to complements after the collapse of δ_k to be countable, and

(c) there is an $\omega_1 + 1$-iteration strategy Σ for N such that whenever $i : N \to P$ is an iteration map by Σ and P is countable, then $p[i(S)] \subseteq U$ and $p[i(T)] \subseteq \mathbb{R}^g \setminus U$.

Then we say that N is a *coarse* (k, U)*-Woodin mouse*, as witnessed by $S, T, \Sigma, \delta_0, \ldots, \delta_k$.

Coarse mice capturing $\Sigma_1^{L(\mathbb{R}^g)}$-truth

Our inductive hypothesis is

(W_α^*) Let U be a subset of $\mathbb{R}^g$, and suppose there are scales $\vec{\varphi}$ and $\vec{\psi}$ on U and $\mathbb{R}^g \setminus U$ respectively such that $\vec{\varphi}^*, \vec{\psi}^* \in J_\alpha(\mathbb{R}^g)$, where $\vec{\varphi}^*$ and $\vec{\psi}^*$ are the sequences of prewellorders associated to the scales. Then for all $k < \omega$ and $x \in \mathbb{R}^g$ there are N, Σ such that

(1) $x \in N$, and N is a coarse (k, U)-Woodin mouse, as witnessed by Σ, and

(2) $\Sigma \restriction \mathrm{HC}^{V[g]} \in J_\alpha(\mathbb{R}^g)$.

Lemma 19 *If W_α^* holds, then $J_\alpha(\mathbb{R}^g) \models \mathsf{AD}$.*

Fine-structural mice capturing $\Sigma_1^{L(\mathbb{R}^g)}$ truth

Our fine-structural witnesses are as follows.

Definition 20 Suppose $\theta(v)$ is a Σ_1 formula (in the language of set theory expanded by a name for $\mathbb{R}$), and z is a real; then a $\langle\theta, z\rangle$*-witness* is an ω-sound, $(\omega, \omega_1, \omega_1 + 1)$-iterable z-mouse $\mathcal{N}$ in which there are $\delta_0 < \ldots < \delta_9$, S, and T such that $\mathcal{N}$ satisfies the formulae expressing

(a) ZFC,

(b) $\delta_0, \ldots, \delta_9$ are Woodin,

(c) S and T are trees on some $\omega \times \eta$ which are absolutely complementing in $V^{\mathrm{Col}(\omega,\delta_9)}$, and

(d) $p[T]$ is the Σ_1-theory (in the language with names for each real) of $J_\gamma(\mathbb{R})$, where γ is least such that $J_\gamma \models \theta[z]$.

Lemma 21 *If there is a* $\langle\theta, z\rangle$*-witness, then* $L(\mathbb{R}) \models \theta[z]$.

What we show in our core model induction is just that the converse of the last lemma holds for $L(\mathbb{R}^g)$, at least for α a limit ordinal. More precisely, we show that for α a limit,

(W_α) If $\theta(v)$ is Σ_1, $z \in R^g$, and $J_\alpha(\mathbb{R}^g) \models \theta[z]$, then there is a $\langle\theta, z\rangle$-witness $\mathcal{N}$ whose associated iteration strategy, when restricted to countable iteration trees, is in $J_\alpha(\mathbb{R}^g)$.

This follows from

Lemma 22 *Let α be a limit ordinal, and suppose that W_α^* holds; then W_α holds.*

This lemma is essentially Woodin's *mouse set theorem*. (See "A theorem of Woodin on mouse sets", at http://www.math.berkeley.edu/∼steel.)

Proof of $W^*_{\alpha+1}$

We are done unless new sets of reals get scales in $J_{\alpha+1}(\mathbb{R}^g)$. The way this can happen has been analyzed. We consider two typical cases.

Uncountable cofinality inadmissible case.

Assume $J_\alpha(\mathbb{R}^g)$ is inadmissible, and α has uncountable cofinality in $V[g]$, and new Σ_1 formulae are verified to hold of reals cofinally below α. Let $\varphi(v_0, v_1)$ and $x \in \mathbb{R}^g$ determine the failure of admissibility, so that φ is Σ_1,

$$\forall y \in \mathbb{R}^g \exists \beta < \alpha \, J_\beta(\mathbb{R}^g) \models \varphi[x, y],$$

and letting $\beta(x, y)$ be the least such β,

$$\alpha = \sup\{\beta(x, y) \mid y \in \mathbb{R}^g\}.$$

(Since α begins a gap, $J_\alpha(\mathbb{R}^g)$ is the Σ_1 hull of its reals, so the parameter from which a failure of admissibility is defined can be taken to be a real.) Let

$$x = \tau^g.$$

Let U be the universal $\Sigma_1^{J_\alpha(\mathbb{R}^g)}$ set of reals. We want to construct in V a (μ^+, τ)-mo which "knows U".

Let $p \in g$ force over V all the properties of τ which we have listed as properties of x in $V[g]$ so far. Let $\tau \in B \in H_{\mu^+}$, where B is transitive wellordered, and $\nu = o(B)$.

Notice that if $\mathcal{M}$ is an B-premouse, and $G \times H$ is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$, then $\mathcal{M}[G][H]$ can be regarded as a z-premouse, where $z = z(G, H)$ is a real obtained in some simple fashion from G, H, and B, and which in turn codes G, H, and B in some simple fashion. Also, there is a term $\sigma = \sigma_B$ defined uniformly from B in $\mathcal{M}$ such that whenever $G \times H$ is generic as above, then $\sigma^{(G\times H} \in \mathbb{R}$ and

$$(\sigma^{G\times H})_0 = \tau^G,$$

and

$$\{(\sigma^{G\times H})_i \mid i > 0\} = \{\rho^{G\times H} \mid \rho \in L_1(A) \text{ and } \rho^{G\times H} \in \mathbb{R}\}.$$

Here $(w)_i$ is the i^{th} real coded into the real w.

Let φ^* be the Σ_1 formula

$$\varphi^*(v) = \forall i \in \omega(i > 0 \Rightarrow \varphi((v)_0, (v)_i)).$$

Now let ψ be the natural sentence in the language of B-premice (having therefore a name for B) such that for any B-premouse $\mathcal{M}$:

$$\mathcal{M} \models \psi$$

iff whenever $G \times H$ is $\mathcal{M}$-generic over $\mathrm{Col}(\omega, \mu) \times \mathrm{Col}(\omega, \nu)$ and $p \in G$, then for some γ

$$(\mathcal{M}[z(G, H)]||\gamma \text{ is a } \langle\varphi^*, \sigma_B^{G \times H}\rangle\text{- witness.}$$

Definition 23 For any B as above $J(B)$ is the shortest initial segment of $\mathrm{Lp}(B)$ which satisfies ψ, if it exists, and is undefined otherwise.

Lemma 24 *J is a (μ^+, τ)-mo, and J relativises well.*

Since we stated W_α as a closure condition on the mice over reals in $V[g]$, there is some work to be done in going back to V, as is done in this lemma.

From our work in the section on lifting closure at μ to closure at κ, we get immediately

Corollary 25 *J extends to a (κ^+, τ)-mo.*

This gives us J^w, $(J^w)^w$, etc.

Finally, note that for $B \in H_{\mu^+}$, $M_1^J(B)[g]$ knows U on reals z having $\mathrm{Col}(\omega, o(B))$-names. For letting δ be its Woodin cardinal, and letting $\mathcal{P} = J(M_1^J(B)|\delta$, and letting $z \in \mathcal{P}[g][h]$ be such a real, we look for a (U, z) witness among the initial segments of $\mathcal{P}[g][h]$. Note any real y can be made generic over an iterate of $\mathcal{P}$ for the collapse of the image of δ, and hence this iterate has a witness capturing the theory of $J_{\beta(x,y)}(\mathbb{R}^g)$. Since the $\beta(x, y)$ are cofinal in α, this means $\mathcal{P}[g][h]$ has enough initial segments to provide $\langle U, z\rangle$-witnesses when they exist.

The proper weak gap case

Suppose $[\alpha, \beta]$ is a weak Σ_1-gap in $L(\mathbb{R}^g)$, and $\alpha < \beta$. We are given W^*_β, and we want $W^*_{\beta+1}$.

Definition 26 For any transitive $A \in H_{\mu^+}$, let $\mathrm{Lp}^\alpha(A)$ be the "union" of all A-mice $\mathcal{N}$ projecting to $o(A)$ such that $J_\alpha(\mathbb{R}^g) \models \mathcal{N}$ is ω_1-iterable.

Note $\mathrm{Lp}^\alpha(A)$ is an initial segment of $\mathrm{Lp}(A)$, since the iteration strategy witnessing $\mathcal{N} \in \mathrm{Lp}^\alpha$ is unique, so that its restriction to V is in V.

Definition 27 Let $A \in H_{\mu^+}$. An A-premouse $\mathcal{N}$ is *suitable* iff card $(\mathcal{N}) = \mu$ and

(a) $\mathcal{N} \models$ there is exactly one Woodin cardinal. We write $\delta^{\mathcal{N}}$ for the unique Woodin cardinal of $\mathcal{N}$.

(b) Letting $\mathcal{M}_0 = \mathcal{N}|\delta^{\mathcal{N}}$, and $\mathcal{M}_{i+1} = \mathrm{Lp}^\alpha(\mathcal{M}_i)$, we have that $\mathcal{N} = \bigcup_{i<\omega} \mathcal{M}_i$. That is, $\mathcal{N}$ is the Lp^α closure of $\mathcal{N}|\delta^{\mathcal{N}}$, up to its ω^{th} cardinal above $\delta^{\mathcal{N}}$.

(c) If $\xi < \delta^{\mathcal{N}}$ is a cardinal of $\mathcal{N}$, then $\mathrm{Lp}^\alpha(\mathcal{N}|\xi) \models \xi$ is not Woodin.

Definition 28 Let Σ be a (μ^+, μ^+)-iteration strategy on a suitable $\mathcal{N}$; then Σ is *fullness-preserving* iff whenever $\mathcal{P}$ is an iterate of $\mathcal{N}$ by Σ, then

(1) if $\mathcal{N}$-to-$\mathcal{P}$ does not drop, then $\mathcal{P}$ is suitable, and

(2) if $\mathcal{N}$-to-$\mathcal{P}$ drops, then $J_\alpha(\mathbb{R}) \models \mathcal{P}$ is ω_1-iterable.

Definition 29 An iteration strategy Σ *condenses well* iff whenever $\mathcal{T}$ is an iteration tree played according to Σ, and $\mathcal{S}$ is a hull of $\mathcal{T}$, then $\mathcal{S}$ is according to Σ.

Lemma 30 *There is, in V, a suitable $\mathcal{N}$ and a fullness-preserving μ^+-iteration strategy Σ for $\mathcal{N}$ such that Σ condenses well.*

Lemma 31 *In V: let $\mathcal{N}$ be a suitable premouse (over some A bounded in μ^+), and let Σ be a fullness-preserving (μ^+, μ^+)-iteration strategy for $\mathcal{N}$ which condenses well. Then Σ has a unique extension Γ to a (κ^+, κ^+)- iteration strategy which condenses well.*

Definition 32 Let $B \in H_{\kappa^+}$ and $\mathcal{N} \in B$; then $P_n^{\Sigma}(B)^{\sharp}$ is the minimal iterable Σ-hybrid mouse over B which is active, and satisfies "there are n Woodin cardinals".

The map $J_n(B) = P_n^{\Sigma}(B)^{\sharp}$ is a *hybrid* $(\mu^+, \mathcal{N})$-*mouse operator*. Our one-J-Woodin core model theory extends to such hybrid J, as does our lifting of mouse closure from H_{μ^+} to H_{κ^+}. We get

Lemma 33 *For all $n < \omega$ and all $B \in H_{\kappa^+}$ such that $\mathcal{N} \in B$, $P_n^{\Sigma}(B)^{\sharp}$ exists, and is (κ^+, κ^+)-iterable.*

The $P_n(A)^{\sharp}$-operators can be used to verify $W^*_{\beta+1}$. (There is enough information coded in Σ that all sets of reals in $J_{\beta+1}(\mathbb{R}^g)$ are projective in Σ; moreover, and these hybrid mice have been told enough about Σ. It is important here that $P_n^{\Sigma}(B)^{\sharp}$ knows Σ on its generic extensions.)

Questions

(1) $\neg\Box_\kappa$, for κ singular, implies $\mathsf{AD}^{L(\mathbb{R})}$?

(2) Can one get strength from PFA without going through $\neg\Box$?

(3) $\neg\Box_\kappa$, for κ singular strong limit, implies there is a nontame mouse?

(4) Does the scale-analysis for $L(\mathbb{R})$ generalize fully to $K(\mathbb{R})$?

(5) How far does Woodin's mouse set theorem go?