# Some correctness results for extender models

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## Introduction

Gödel's legacy includes:

**Theorem 1 (Gödel 1937)** *Let $L$ be the universe of constructible sets; then*

*(1)* $L \models \mathsf{V} = \mathsf{HOD}$*, and in fact* $\mathsf{V{=}L}$*,*

*(2)* $L \models \mathbb{R}$ *admits a* $\Delta^1_2$ *wellorder.*

**Theorem 2 (Shoenfield 1967)** *$L$ is $\Sigma^1_2$-correct, that is,* $(HC, \in)^L \prec_{\Sigma_1} (HC, \in)$.

For the rest of the talk, we assume that there are arbitrarily large Woodin cardinals.

**Definition 3** For any $n \leq \omega$, $M_n$ is the canonical, minimal, proper class extender model satisfying "There are $n$ Woodin cardinals".

So $M_0 = L$.

**Theorem 4 (Martin, Mitchell, S.; 1986,1990)** *For any $n < \omega$,*

1. $M_n \models \mathsf{V} = \mathsf{HOD}$,
2. $M_n \models \mathbb{R}$ *admits a* $\Delta^1_{n+2}$ *wellorder,*
3. $M_n$ *is* $\Sigma^1_{n+1}$*-correct, and* $\Sigma^1_{n+2}$*- correct if* $n$ *is even.*

**Definition 5** $L(\mathbb{R})$ is the minimal model of $\mathsf{ZF}$ containing all ordinals and reals. $J_\alpha(\mathbb{R})$ is the $\alpha$-th level of the Jensen hierarchy on $L(\mathbb{R})$.

**Theorem 6 (S., early 90's)** 1. $M_\omega \models \mathsf{V} = \mathsf{HOD}$,

2. *Let* $\sigma$ *be least such that* $J_\sigma(\mathbb{R}) \prec_1 L(\mathbb{R})$*; then there are* $\tau, \pi$ *such that*
   (a) $\pi : J_\tau(\mathbb{R})^{M_\omega} \to J_\sigma(\mathbb{R})$ *is* $\Sigma_1$*-elementary,*
   (b) $J_\tau(\mathbb{R})^{M_\omega} \models \mathsf{AD}$*, and*
   (c) $J_\tau(\mathbb{R})^{M_\omega} \models$ *"there is a wellorder of* $\mathbb{R}$*".*

**Theorem 7 (Rudominer, S., 1996, 2005)** *Let $M$ be an initial segment of $M_\omega$ such that $M \models \mathsf{ZFC}$; then*

1. $M \models$ "$\mathsf{V = HOD(X)}$, for some set $\mathsf{X}$",
2. *there is a $\tau$ such that*
   (a) $J_\tau(\mathbb{R})^M \models \mathsf{AD}$,
   (b) $J_{\tau+1}(\mathbb{R})^M \models$ "*there is a wellorder of* $\mathbb{R}$", *and*
   (c) *for* $\theta = (\theta)^{J_\tau(\mathbb{R})^M}$, *there are $\pi$ and $\alpha$ such that* $\pi : J_\tau(\mathbb{R})^M \to J_\alpha(\mathbb{R})$ *is $\Sigma_1$-elementary.*

**Theorem 8 (Schindler, S. 2006)** *Let $M$ be an iterable, tame extender model; then for all cardinals $\kappa < \lambda$ of $M$, $M \models \diamond^*_{\kappa,\lambda}$.*

Behind both results is an analysis of how much $M$ knows about how to iterate itself.

$\diamond^*_{\kappa,\lambda}$**:** $\exists F : P_\kappa(\lambda) \to P(P_\kappa(\lambda))$ s.t.

1. $|F(X)| \leq |X|$ for all $X$, and
2. $\forall A \subseteq \lambda$, $A \cap X \in F(X)$ for club many $X$.

**Conjecture.** Let $M$ be an $\omega_1 + 1$-iterable extender model satisfying $\mathsf{ZFC}$; then $M$ satisfies "for some real $x$, there is a wellorder of $\mathbb{R}$ which is ordinal definable from $x$."

**Remarks.**

1. Equivalently, $M \models \exists x \in \mathbb{R} \forall y \in \mathbb{R}(y \in \mathrm{OD}(x))$.
2. If $M$ is an $\omega_1 + 1$-iterable extender model, then every real in $M$ is ordinal definable. But $M$ may not realize this, because it may not know how to iterate itself.
3. There are initial segments of $M_2$ which satisfy $\mathsf{ZFC}$ plus "there is a real which is not ordinal definable".
4. Our result above proves the conjecture for $M$ which are initial segments of $M_\omega$. We believe the proof works for a wider class of $M$, the *tame* mice.

## Premice

**Definition 9** A *premouse* is a structure of the form $\mathcal{M} = (J_\gamma^{\vec{E}}, \in, \vec{E})$, where $\vec{E}$ is a *coherent sequence of extenders*.

**Coherence:** for all $\alpha \leq \gamma$, $E_\alpha = \emptyset$, or $E_\alpha$ is an extender (system of ultrafilters) with support $\alpha$ over $\mathcal{M}|\alpha = (J_\alpha^{\vec{E}\upharpoonright\alpha}, \in, \vec{E} \upharpoonright \alpha)$ coding

$$i : \mathcal{M}|\alpha \to \mathcal{N} = \mathrm{Ult}(\mathcal{M}|\alpha, E_\alpha)$$

such that

$$i(\vec{E} \upharpoonright \alpha) \upharpoonright \alpha = \vec{E} \upharpoonright \alpha \text{ and } i(\vec{E} \upharpoonright \alpha)_\alpha = \emptyset.$$

**Remark 10** The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

**Definition 11** $\rho_n(\mathcal{M})$ is the $\Sigma_n$-projectum of $\mathcal{M}$. We say $\mathcal{M}$ is *$n$-sound* iff

$$\mathcal{M} = \mathrm{Hull}_n^{\mathcal{M}}(\rho_n(\mathcal{M}) \cup r),$$

for $r$ the $n$-th standard parameter of $\mathcal{M}$.

We require of premice that proper initial segments be $n$-sound for all $n$. It follows that for all $\kappa$, $P(\kappa) \cap \mathcal{M} \subseteq \mathcal{M}|(\kappa^+)^{\mathcal{M}}$. This is a strong, local form of $\mathsf{GCH}$.

**Remark 12** If $\mathcal{M}$ is iterable, then the least nonempty $E_\alpha$ codes up a nontrivial elementary $j\colon L \to L$, and hence $0^\sharp$. In this case, $\mathcal{M}|(\alpha+1)$ has a new subset of $\mathrm{crit}(E_\alpha)$, not measured by $E_\alpha$. In fact, $\rho_1(\mathcal{M}|\alpha) = \omega$ in this case, so by soundness, $\alpha$ is countable in $\mathcal{M}|(\alpha+1)$. Further out in the sequence of $\mathcal{M}$ there may be extenders $E$ whose measures are total over all of $\mathcal{M}$, because no new subsets of $\mathrm{crit}(E)$ are constructed after $E$ has been added.

## The iteration game

Let $\mathcal{M}$ be a premouse. In $\mathcal{G}(\mathcal{M}, \theta)$, players I and II play for $\theta$ rounds, producing a tree $\mathcal{T}$ of models, with embeddings along its branches, and $\mathcal{M} = \mathcal{M}_0^{\mathcal{T}}$ at the base.

*Round $\beta + 1$:* I picks an extender $E_\beta^{\mathcal{T}}$ from the sequence of $\mathcal{M}_\beta^{\mathcal{T}}$ with support $\geq$ the supports of all earlier extenders chosen . Let $\xi$ be least such that $\mathrm{crit}(E_\beta^{\mathcal{T}}) <$ support $(E_\xi^{\mathcal{T}})$. We set

$$\mathcal{M}_{\beta+1}^{\mathcal{T}} = \mathrm{Ult}_k(\mathcal{M}_\xi^{\mathcal{T}} | \eta, E_\beta^{\mathcal{T}}),$$

where $\langle \eta, k \rangle$ is as large as possible.

*Round $\lambda$, for $\lambda$ limit:* II picks a branch $b$ of $\mathcal{T}$ which is cofinal in $\lambda$, and we set

$$\mathcal{M}_\lambda^{\mathcal{T}} = \text{ dirlim }_{\alpha \in b} \mathcal{M}_\alpha^{\mathcal{T}}.$$

As soon as an illfounded model $\mathcal{M}_\alpha^{\mathcal{T}}$ arises, player I wins. If this has not happened after $\theta$ rounds, then II wins.

**Definition 13** A $\theta$*-iteration strategy for* $\mathcal{M}$ is a winning strategy for II in $\mathcal{G}(\mathcal{M}, \theta)$. We say $\mathcal{M}$ is $\theta$*-iterable* just in case there is such a strategy. If $\Sigma$ is a strategy for II in $\mathcal{G}(\mathcal{M}, \theta)$, and $\mathcal{P} = \mathcal{M}_\alpha^{\mathcal{T}}$ for some $\mathcal{T}$ played by $\Sigma$, then we call $\mathcal{P}$ a $\Sigma$*-iterate* of $\mathcal{M}$.

**Theorem 14 (Comparison Lemma)** *Let* $\Sigma$ *and* $\Gamma$ *be* $\theta + 1$ *iteration strategies for* $\mathcal{M}$ *and* $\mathcal{N}$ *respectively, where* $\theta =$ *max* $(|\mathcal{M}|, |\mathcal{N}|)^+$. *Then either*

*(a) there is a* $\Gamma$*-iterate* $\mathcal{P}$ *of* $\mathcal{N}$*, and a map* $j\colon \mathcal{M} \to \mathcal{P}|\eta$ *produced by* $\Sigma$*-iteration, or*

*(b) there is a* $\Sigma$*-iterate* $\mathcal{P}$ *of* $\mathcal{M}$*, and a map* $j\colon \mathcal{N} \to \mathcal{P}|\eta$ *produced by* $\Gamma$*-iteration* $\mathcal{M}$*.*

**Corollary 15** *If $\mathcal{M}$ and $\mathcal{N}$ are $\omega_1 + 1$-iterable, then $\mathcal{M}|\alpha = \mathcal{N}|\alpha$, where $\alpha = \inf(\omega_1^{\mathcal{M}}, \omega_1^{cN})$. That is, their canonical wellorders of the reals by stage-of-construction are compatible.*

*Proof.* If $j : \mathcal{M} \to \mathcal{P}$ is produced by iteration, then $\mathcal{P}|\omega_1^{\mathcal{P}} = \mathcal{M}|\omega_1^{\mathcal{M}}$. So we can apply the comparison lemma. ■

**Corollary 16** *If $\mathcal{M}$ is an $\omega_1 + 1$-iterable premouse, and $x \in \mathbb{R} \cap \mathcal{M}$, then $x$ is ordinal definable.*

*Proof.* Let $x$ be the $\alpha$-th real in $\mathcal{M}$. Then $y = x$ iff $y$ is the $\alpha$-th real in some $\omega_1 + 1$- iterable premouse. ■

## The existence of iterable mice

Here are some well-known Holy Grails of inner model theory:

**Conjecture 17** *Each of the following statements implies the existence of an* $\omega_1 + 1$*-iterable mouse with a superstrong cardinal:*

*(1)* *There is a supercompact cardinal,*

*(2)* *There is a strongly compact cardinal,*

*(3)* $\mathsf{PFA}$,

*(4)* $\mathsf{CH}$ *holds, and there is a homogeneous pre-saturated ideal on* $\omega_1$,

*(5)* $\mathsf{AD}_{\mathbb{R}}$ *holds, and* $\theta$ *is regular.*

This list could be lengthened substantially. In all cases, we can prove the statement in question implies that there is an iterable mouse with infinitely many Woodin cardinals.

In cases (2)-(5), we cannot yet produce a mouse with a Woodin limit of Woodin cardinals. (Case (1) is special; here Itay Neeman has partial results which reach a bit past a Woodin limit of Woodins.)

The partial results we have in cases (2)-(5) come from the *core model induction* method. This method uses core model theory to construct new mice, while using descriptive set theory to keep track of the degree of correctness and the complexity of the iteration strategies of the mice constructed already.

An $\omega_1$-iteration strategy for a countable $\mathcal{M}$ can be coded by a set of reals. One gets $\omega_1 + 1$ strategies from *sufficiently absolutely definable* $\omega_1$ strategies.

**Mouse set conjecture:** Assume $\mathsf{AD}$ and that there is no $\omega_1$- iteration strategy for a premouse with a superstrong cardinal. Then for $x, y \in \mathbb{R}$, equivalent are

(a) $x \in \mathrm{OD}(y)$,

(b) $x \in \mathcal{M}$, for some $\omega_1$-iterable $y$-premouse $\mathcal{M}$.

# Woodin cardinals and iteration strategies

## Definability of strategies

**Theorem 18** *Let $\mathcal{M}$ be an $n$-sound premouse such that $\rho_n(\mathcal{M}) = \omega$, for some $n$. Then $\mathcal{M}$ has at most one $\omega_1 + 1$-iteration strategy.*

**Definition 19** Let $\kappa < \delta$, and $A \subseteq \delta$; then

- $\kappa$ is $A$-reflecting in $\delta$ iff for all $\eta < \delta$ there is $j \colon V \to M$ such that $\mathrm{crit}(j) = \kappa$, and $j(A) \cap \eta = A \cap \eta$.
- $\delta$ is Woodin with respect to $A$ iff some $\kappa < \delta$ is $A$-reflecting in $\delta$.
- $\delta$ is Woodin iff $\forall A \subseteq \delta$($\delta$ is Woodin with respect to $A$.)

**Definition 20** A premouse $\mathcal{M}$ is *tame* iff whenever $\mathrm{crit}(E_\alpha^{\mathcal{M}}) \leq \delta < \alpha$, then $\mathcal{M}|\alpha \models \delta$ is not Woodin.

**Remark 21** Tame mice can have many Woodin cardinals. They cannot satisfy "there is a Woodin limit of Woodin cardinals".

**Definition 22** If $\mathcal{T}$ is an iteration tree of limit length, then $\delta(\mathcal{T})$ is the supremum of the supports of the $E_\beta^{\mathcal{T}}$.

**Remark 23** If $b$ and $c$ are cofinal branches of $\mathcal{T}$, then $\mathcal{M}_b^{\mathcal{T}}|\delta(\mathcal{T}) = \mathcal{M}_c^{\mathcal{T}}|\delta(\mathcal{T}) = \bigcup_\beta \mathcal{M}_\beta^{\mathcal{T}}|\text{ support }(E_\beta^{\mathcal{T}})$. We call this common lined up part of all branch models "$\mathcal{M}(\mathcal{T})$".

**Definition 24** Let $\mathcal{T}$ be an iteration tree of limit length on an $n$-sound premouse $\mathcal{M}$ such that $\rho_n(\mathcal{M}) = \omega$, and let $b$ be a cofinal branch of $\mathcal{T}$. Then

$$\begin{aligned} \mathcal{Q}(b, \mathcal{T}) \quad = \quad & \text{least initial segment } \mathcal{P} \text{ of } \mathcal{M}_b^{\mathcal{T}} \\ & \text{such that } \delta(\mathcal{T}) \text{ is not Woodin over } \mathcal{M}(\mathcal{T}) \\ & \text{with respect to some } \mathcal{P}\text{-definable } A. \end{aligned}$$

**Theorem 25 (Martin, Mitchell, S. 1986–1990)** *Let $\mathcal{M}$ be an $n$-sound premouse such that $\rho_n(\mathcal{M}) = \omega$, for some $n$, and $\Sigma$ be an $\omega_1 + 1$-iteration strategy for $\mathcal{M}$. Let $\mathcal{T}$ be an iteration tree of countable limit length played by $\Sigma$; then*

- *$\Sigma(\mathcal{T})$ is the unique cofinal branch $b$ of $\mathcal{T}$ such that $\mathcal{T}^\frown\mathcal{Q}(b, \mathcal{T})$ is $\omega_1 + 1$-iterable, and*
- *if $\mathcal{M}$ is tame, then $\Sigma(\mathcal{T})$ is the unique cofinal branch $b$ of $\mathcal{T}$ such that $\mathcal{Q}(b, \mathcal{T})$ is $\omega_1 + 1$-iterable above $\delta(\mathcal{T})$.*

So canonical failures of Woodinness guide the unique iteration strategy on an $n$-sound mouse such that $\rho_n(\mathcal{M}) = \omega$.

**Correctness from Woodinness**

**Definition 26** $\mathcal{B}_\delta$ is Woodin's *extender algebra* below $\delta$.

**Remark 27** If $\delta$ is Woodin, then $\mathcal{B}_\delta$ is a $\delta$-cc complete Boolean algebra, and $\mathcal{B}_\delta \subseteq V_\delta$.

**Theorem 28 (Woodin 1988)** *Let $\mathcal{M}$ be countable, and $\mathcal{M} \models \delta$ is Woodin. Let $\Sigma$ be an $\omega_1 + 1$ iteration strategy for $\mathcal{M}$; for any real $x$, there is a $\Sigma$-iteration map $i \colon \mathcal{M} \to \mathcal{N}$, with $\mathcal{N}$ countable, such that $x$ is $i(\mathcal{B}_\delta^{\mathcal{M}})$-generic over $\mathcal{N}$.*

**Remark 29** It is not hard to get the correctness properties of the $M_n$'s from this. For example, $M_1$ "knows" $\Sigma^1_3$ truth, since for $\varphi(v)$ a $\Pi^1_2$ formula,

$$\exists x \in \mathbb{R}\varphi(x) \text{ iff } M_1 \models (\mathrm{Col}(\omega, \delta) \Vdash \exists x \varphi(x)),$$

where $\delta$ is the Woodin of $M_1$.

An induction shows that $M_n$ knows $\Sigma^1_{n+2}$-truth.

Correctness imposes lower bounds on the complexity of iteration strategies. One can also get such results directly from genericity iterations.

*Example:* We claim $L[M_1^\sharp] \models M_1^\sharp$ is not $\omega_1 + 1$-iterable. For otherwise, let $\delta$ be the Woodin cardinal of $M_1$ and $\mathcal{B} = \mathcal{B}_\delta^{M_1}$. Work in $L[M_1^\sharp]$. We get $i\colon M_1 \to \mathcal{N}$ such that $M_1^\sharp$ is $\mathcal{N}$ generic over $i(\mathcal{B})$, and $i(\delta)$ is countable. By the $i(\delta)$-cc, we have that $i(\delta)$ is uncountable in $\mathcal{N}[M_1^\sharp]$. But this is $V$!.

One can actually show that $M_1^\sharp$ is not OD in $L[M_1^\sharp]$.

**Remark 30** $M_1^\sharp$ is set-length iterable in the "sharp-closed" model $L^\sharp[M_1^\sharp]$. This is because the $\mathcal{Q}$-structure needed to identify the good branch of an iteration tree $\mathcal{T}$ on $M_1^\sharp$ is (at worst) $\mathcal{M}(\mathcal{T})^\sharp$).

A slight elaboration of a result of Woodin yields

**Theorem 31** *Let $\mathcal{M}$ be a premouse, and $\mathcal{M} \models \mathsf{ZFC} + \delta$ is Woodin. Then $\mathcal{M} \models$ "I am not $\delta^+ + 1$-iterable".*

**Theorem 32 (S., 2006)** *Let $\mathcal{M}$ be a premouse, and $\mathcal{M} \models \mathsf{ZFC} + \delta$ is Woodin. Suppose $\mathcal{M}$ is $\kappa^+ + 1$-iterable, where $\kappa > \delta$ is a cutpoint of the extender sequence of $\mathcal{M}$; then $(\kappa^+)^{\mathcal{M}} < \kappa^+$.*

**Remark 33** Woodin has shown that the restriction to $\kappa$ which are cutpoints is necessary; otherwise, it is consistent that there are counterexamples $\mathcal{M}$ weaker than $M_2$.

## A proof-component

**Theorem 34 (Schindler, S. 2006)** *Let $\mathcal{N} \models \mathsf{ZFC}$ be tame, and $\Sigma$ an $\omega_1 + 1$ iteration strategy for $\mathcal{N}$. Then for some $\xi < \omega_1^{\mathcal{N}}$, the restriction of $\Sigma$ to iteration trees on $\mathcal{N}|\omega_1^{\mathcal{N}}$ which are countable in $\mathcal{N}$, and have all critical points above $\xi$, is in $\mathcal{N}$.*

**Remark 35**

1. $\omega_1$-iterability is not enough for comparison, so we don't directly get an $\mathrm{OD}(x)$ wellorder of $\mathbb{R}$ in $\mathcal{N}$.
2. By melding the proof into an argument which keeps track of the correctness of $L(\mathbb{R})^{\mathcal{N}}$ inside $L(\mathbb{R})$, we get an $L(\mathbb{R})$-definable-from-$x$ wellorder of $\mathbb{R}$ in $\mathcal{N}$, in the case $\mathcal{N}$ is an initial segment of $M_\omega$.
3. Probably: we get an $K(\mathbb{R})$-definable-from-$x$ wellorder of $\mathbb{R}$ in $\mathcal{N}$, in the case $\mathcal{N}$ is tame.

*Proof.* Work in $\mathcal{N}$ for a minute. Given $\xi < \omega_1$, we define a partial strategy $\Gamma = \Gamma_\xi$ for countable trees $\mathcal{T}$ on $\mathcal{N}|\omega_1$ which are above $\xi$. (It will turn out that $\Gamma \subseteq \Sigma$.)

Let $\mathcal{T}$ be by $\Gamma$, and have limit length. Our goal is to find $\mathcal{Q}(b, \mathcal{T})$, where $b = \Sigma(\mathcal{T})$.

Pick any $\nu < \omega_1$ such that $\mathcal{T} \in \mathcal{N}|\nu$ and $\nu$ is a cutpoint of $\mathcal{N}$. We try to compare $\mathcal{Q}(b, \mathcal{T})$, as a mouse over $\mathcal{M}(\mathcal{T})$, with $\mathcal{N}|\omega_1$, as a mouse over $\mathcal{N}|\nu$. To do this, we iterate $\mathcal{M}(\mathcal{T})$, and thus implicitly $\mathcal{Q}(b, \mathcal{T})$, so as to make initial segments of $\mathcal{N}$ beyond $\nu$ generic over the image of the extender algebra at $\delta(\mathcal{T})$ of $\mathcal{M}(\mathcal{T})$.

Let $\mathcal{U}$ be the genericity-iteration tree on $\mathcal{M}(\mathcal{T})$ we are building. At limit stages in the construction of $\mathcal{U}$, the situation is: