Gödel's legacy in set theory

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In 1930-37, Kurt Gödel proved the first substantial theorems in the metamathematics of set theory.

Theorem 1 (Gödel 1931) *If T is a consistent, axiomatizable extension of* ZF*, then T does not prove Con(T).*

Theorem 2 (Gödel 1937) *If* ZF *is consistent, then so is* ZFC + GCH + *"there is a Δ^1_2 wellorder of the reals".*

In his 1947 paper "What is Cantor's Continuum Hypothesis" he suggested a response to the incompleteness these results reveal.

We shall trace some subsequent developments, under the two headings *Consistency* and *Truth*.

Consistency

We know of no further use of L in relative consistency results before Cohen, except for Addison 1959.

I. Forcing

Theorem 3 (Cohen 1963) *If* ZF *is consistent, then so are*

(a) ZFC + ¬CH,

(b) ZF + *there is no wellorder of* $\mathbb{R}$.

Theorem 4 (Solovay (1967?)) *If* ZFC + *"there is an inaccessible cardinal" is consistent, then so is* ZF + DC + *"every set of reals is Lebesgue measurable".*

Theorem 5 (Solovay, 1971) *If* ZFC + *"there is a measurable cardinal" is consistent, so is* ZFC + *"there is a total extension of Lebesgue measure".*

As a tool for proving some T is consistent, granted the consistency of ZFC plus perhaps some large cardinal

hypotheses, forcing is far more powerful than inner model theory. One calls such relative consistency results *consistency strength upper bounds*.

II. Inner model theory

But L and the larger canonical inner models seem to be indispensable for *consistency-strength lower bounds*.

Theorem 6 (Solovay, late 60's (?)) *If* ZFC + *Kurepa's Hypothesis is consistent, so is* ZFC + *"there is an inaccessible cardinal".*

Theorem 7 (Shelah 1984) *If* ZF + DC+ *"every set of reals is Lebesgue measurable" is consistent, then so is* ZFC + *"there is an inaccessible cardinal"* .

Theorem 8 (Solovay 1971) *If* ZFC + *"there is a total extension of Lebesgue measure" is consistent, so is* ZFC + *"there is a measurable cardinal".*

L and larger canonical minimal inner models are interesting in their own right. They admit a systematic, detailed analysis, or *fine structure theory*.

Theorem 9 (Jensen, 1970) *The following are equivalent*

1. *There is a* $\Sigma_n^{L_\alpha}$ *subset of* ρ *which is not in* L_α,
2. *There is a* $\Sigma_n^{L_\alpha}$ *map from a subset of* ρ *onto* L_α.

Corollary 10 (Jensen 1970) *In L, for regular kappa, there is a* κ*-Suslin tree iff* κ *is not weakly compact.*

Theorem 11 (Jensen 1975) *The following are equivalent:*

1. $0^\sharp$ *exists,*
2. *there is an uncountable* $X \subseteq L$ *such that if* $X \subseteq Y$ *with* $|Y| = |X|$, $Y \notin L$.

A theory of canonical inner models for large cardinal hypotheses as strong as "there are many Woodin cardinals", which generalizes the theory of L, has been developed. (Silver, Kunen, Mitchell, Dodd, Jensen, Martin, Steel, Woodin, ...1966-present.)

Besides its technical uses, this theory provides evidence for the consistency of the large cardinal hypotheses in question.

III. The consistency strength hierarchy

Large cardinal hypotheses enable us to survey, organize, and calibrate the strengths of possible extensions of ZFC.

The interpretability hierarchy It seems that every natural extension of ZFC is equiconsistent with an extension axiomatized by something like large cardinal axioms. These natural consistency strengths seem to be wellordered! This wellorder corresponds to the inclusion order on the set of Π^0_1 (or in fact, arithmetical, or even Σ^1_2) consequences of the theories in question.

At the level of Σ^1_2 sentences, we know of only one road upward, and the large cardinal hypotheses are its central markers.

For T a theory and Γ a set of sentences, let

$$(\Gamma)_T = \{\varphi \in \Gamma \mid T \vdash \varphi\}.$$

The interpretability hierarchy II

For T and U natural theories of consistency strength at least that of "There are n Woodin cardinals with a measurable above", we have

$$(\Pi^0_1)^T \subseteq (\Pi^0_1)^U \leftrightarrow (\Sigma^1_n)^T \subseteq (\Sigma^1_n)^U.$$

For example, any natural theory of consistency strength at least that of PD actually implies PD. For example, the Proper Forcing Axiom implies PD. So does the existence of a homogeneous saturated ideal on ω_1.

At the level of Σ^1_n sentences, we know of only one road upward, and large cardinals are its central markers; moreover this road goes through PD.

Theorem 12 (Martin, Woodin, Steel 1986-1990)
The following are equivalent:

1. *PD ,*
2. *For all $n < \omega$, every Σ^1_n consequence of ZFC + "there are n Woodin cardinals" is true.*

The limitations to independence techniques

For inner model theory: our canonical inner models with arbitrarily large Woodin cardinals, or more, are projectively correct.

For forcing:

Theorem 13 (Woodin) *Suppose that over every set belongs to an iterable inner model satisfying "there are ω Woodin cardinals"; then if M and N are set-generic extensions of V, we have*

$$L(\mathbb{R})^M \equiv L(\mathbb{R})^N.$$

There is only one theory with this kind of "generic completeness":

Theorem 14 (Woodin, S.) *Suppose that whenever M and N are set generic extensions of V, we have $L(\mathbb{R})^M \equiv L(\mathbb{R})^N$; then every set belongs to an iterable inner model satisfying "there are ω Woodin cardinals".*

Generic Absoluteness and CH

There are no generic absoluteness theorems at the Σ^2_1 level:

Theorem 15 (Levy, Solovay) *1. Nothing like the current large cardinal axioms decides CH.*

2. Let A be one of the current large cardinal axioms, and suppose $V \models A$. Then there are set generic extensions M and N of V which satisfy A, but are not Σ^2_1-equivalent.

There are, however, *conditional* generic absoluteness theorems at the Σ^2_1 level.

Theorem 16 (Woodin) *Suppose $V \models$ "There are arbitrarily large measurable Woodin cardinals". Let M and N be set-generic extensions of V satisfying CH; then M and N are Σ^2_1-equivalent.*

Truth

Godel believed that the axioms of ZFC describe a "well-determined reality" in which CH is either true or false. He also believed that strong axioms of infinity are implicit in the concept of set. He knew such axioms had new consequences for small sets, even natural numbers. This led to his 1947 statement in "What is Cantor's Continuum Hypothesis" of a program, which we formulate as:

Gödel's Program: Decide mathematically interesting questions (such as CH) which not decided by ZFC in well-justified extensions of ZFC (perhaps by large cardinal hypotheses).

Remark. "Well-justified" leads us into Philosophy. Underlying Gödel's program is a faith in reason, and a belief in the objectivity of mathematics.

Results to date:

1. Nothing like the large cardinal hypotheses we know of can decide CH. (Levy, Solovay 1967)
2. There are strong axioms of infinity which decide in ea natural, systematic way all the independent-of-ZFC questions of descriptive set theory. (1962–present, many people.)

Three possible futures

1. There is a natural extension of our current large cardinal notions, some cntral markers of higher consistency strengths, which lead to a "complete" theory at the Σ^2_n level (and beyond).
2. All large cardinals are preserved by small forcing, but there are conditional generic absoluteness theorems at the Σ^2_n level, for all n.
3. Neither of the above.

Future (1) is the closest to a full realization of Godel's program.

In future (2), we discover an ambiguity in the language of set theory, along with the means to remove it–perhaps in more than one way.

Godel's Legacy

(1) Discovered the consistency strength hierarchy, and recognized the central role of large cardinal hypotheses.

(2) Proved the first major results of inner model theory.

(3) Advocated the proper response to incompleteness.