Inner Model Theory

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Large Cardinal Hypotheses

The *ordinals* are
$0, 1, 2, ..., \omega, \omega + 1, \omega + 2, ..., \omega + \omega, ..., ...$ They are the possible stages in a definition or proof by transfinite induction.

Definition 1

$$V_0 = \emptyset,$$

$$V_{\alpha+1} = \{x \mid x \subseteq V_\alpha\},$$

and for λ a limit ordinal,

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha.$$

The universe V of all sets is the union of the V_α.

Definition 2

- The *language of set theory* has symbols $=$ and $\in$.
- ZFC is the theory in this language whose axioms are Extensionality, Infinity, Powerset, ...

All mathematical statements can be translated into the language of set theory, when this language is given its standard interpretation. All theorems proved before 1900 can be proved using only ZFC, and the same is true for the great majority of the theorems proved after 1900. But

ZFC is incomplete.

In fact

1. ZFC does not prove Con(ZFC).
2. ZFC does not decide whether all projective sets of reals are Lebesgue measurable.
3. ZFC does not decide the Continuum Hypothesis.

Definition 3 A set of reals is *projective* if it can be built up from a Borel set in $\mathbb{R}^n$ by repeated projection and complementation. Equivalently, B is projective iff B is definable from parameters over $(V_{\omega+1}, \in)$.

Definition 4

- A Σ_n *formula* is one of the form $\exists v_1 \forall v_2 \ldots \theta$ having n quantifiers prefixed to a matrix θ containing only bounded quantifiers (i.e. $\exists x \in y, \forall x \in y$).
- A set $B \subseteq V_{\omega+1}$ is Σ^1_n iff we can write

$$x \in B \text{ iff } (V_{\omega+1}, \in) \models \varphi[x],$$

where φ is a Σ_n formula.

ZFC proves that all Σ^1_1 sets of reals are Lebesgue measurable. But

Theorem 5 (Godel 1937, Solovay 1967) ZFC *does not decide whether all Σ^1_2 sets of reals are Lebesgue measurable.*

Informal Reflection Principle: Suitable properties of V are shared by, or "reflect to", some V_α.

Definition 6 • $j : N \to P$ is *elementary* iff $\forall x, \varphi$

$$N \models \varphi[x] \text{ iff } P \models \varphi[j(x)].$$

- $\mathrm{crit}(j)$ = least α such that $j(\alpha) \neq \alpha$.

Definition 7 A cardinal κ is *measurable* iff $\kappa = \mathrm{crit}(j)$ for some elementary $j : V \to M$. Moreover

- if $V_\alpha \subseteq M$, then κ is α-strong,
- if $V_{j(\kappa)} \subseteq M$, then κ is superstrong,
- if M is closed under λ-sequences, then κ is λ-supercompact.

κ is supercompact iff κ is λ-supercompact for all λ.

Large cardinals and Incompleteness

It seems that any natural consistency question can be decided by a large cardinal hypothesis. One of many examples:

Theorem 8 (Baumgartner, Shelah, early 80's) *Con(There is a supercompact cardinal)* $\Rightarrow$ *Con(Proper Forcing Axiom).*

Large cardinal hypotheses seem to decide all natural questions about projective sets. E.g.

Theorem 9 (Solovay 1967) *If there is a measurable cardinal, then all* Σ^1_2 *sets of reals are Lebesgue measurable.*

Theorem 10 (Shelah, Woodin 1984) *If there is a superstrong cardinal, then all projective sets of reals are Lebesgue measurable.*

None of the known large cardinal hypotheses decide the Continuum Hypothesis.

Determinacy

Let A be a set of infinite sequences of natural numbers. A is the *payoff set* for a game G_A:

$$\begin{array}{c|cccccccccc} I & n_0 & & n_2 & & \ldots & & n_{2i} & & \ldots & \\ \hline II & & n_1 & & n_3 & & \ldots & & n_{2i+1} & & \ldots \end{array}$$

where player I wins iff $\langle n_0, n_1, n_2, ...\rangle \in A$. We say G_A (or A) is *determined* iff one of the two players has a winning strategy.

Theorem 11 (Martin 1975) *All Borel games are determined.*

Theorem 12 (Martin 1968) *If there is a measurable cardinal, then all Σ^1_1 games are determined.*

Theorem 13 (Martin, Steel 1985) *If there is a superstrong cardinal, then all projective games are determined.*

Projective determinacy yields a “complete” structure theory of projective sets. In particular, it implies all projective sets are Lebesgue measurable (Mycielski, Swiercowski, early 60’s.)

Inner Model Theory

Slogan : Associate to each large cardinal hypothesis a canonical inner model which is in some sense minimal, and whose structure can be analyzed systematically in detail.

From this we get:

1. Evidence for the consistency of the large cardinal hypothesis in question.
2. Consistency strength lower bounds. For example

   Theorem 14 (Dodd, Jensen, Todorcevic, 75-85) *Con(Proper Forcing Axiom)* $\Rightarrow$ *Con(There is a measurable cardinal).*

   Conjecture 15 *Con(Proper Forcing Axiom)* $\Rightarrow$ *Con(There is a supercompact cardinal).*

3. Consequences in the theory of $V_{\omega+1}$ from theories of high consistency strength. For example

   Theorem 16 (Woodin 1992) *The Proper Forcing Axiom implies that all projective games are determined (hence all projective sets of reals are Lebesgue measurable).*

   The theorem rests upon inner model theory developed by Dodd, Jensen, Kunen, Martin, Mitchell, Schimmerling, Steel, and Silver in the period 1967-1994.

4. An important tool for using large cardinal hypotheses, especially in Descriptive Set Theory.

The Constructible Sets

Definition 17

$$L_0 = \emptyset,$$

$$L_{\alpha+1} = \{x \subseteq L_\alpha \mid x \text{ is definable over } (L_\alpha, \in)\},$$

and for λ a limit

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha.$$

L is the union of the L_α's.

Theorem 18 (Godel 1937) *L is the minimum model of* ZFC *containing all the ordinals. The following are true in L:*

- *The Generalized Continuum Hypothesis (GCH),*
- *there is a Σ^1_2 wellorder of $\mathbb{R}$, and hence a non-Lebesgue-measurable Σ^1_2 set of reals.*

Set Theory in L

Infinitary combinatorics goes far beyond cardinal arithmetic.

Theorem 19 (Jensen, ca. 1967) *L satisfies "There is a Suslin line". In fact, $L \models \diamondsuit$.*

Jensen et. al. developed in great detail the general set theory of L. The key is "condensation":

Theorem 20 (Godel 1937, Jensen ca. 1970) *Let $X \subseteq L_\alpha$; then there is an L_β of cardinality at most that of X plus ω and an elementary $\pi : L_\beta \to L_\alpha$ whose range includes X.*

So for example, there are countable L_β which satisfy "ω_1 exists".

Fine Structure Theory

Jensen (ca. 1970) realized that it is important to look carefully at how new sets appear in L, step-by-step and quantifier-by-quantifier.

Theorem 21 (Jensen, ca. 1970) *The following are equivalent, for any $n > 0$:*

- *There is an $A \subseteq \rho$ which is Σ_n definable from parameters over L_α but not a member of L_α,*
- *There is a partial map from ρ onto L_α which is Σ_n definable over L_α from parameters.*

Corollary 22 *Every L_α satisfies GCH.*

Definition 23 $\Box_\kappa$ is the statement: there is a sequence $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit } \rangle$ such that for all α

1. C_α is closed and cofinal in α, and has order type $\leq \kappa$,
2. if λ is a limit point of C_α, then $C_\alpha \cap \lambda = C_\lambda$.

Theorem 24 (Jensen ca. 1970) $L \models \forall\kappa\Box_\kappa$.

Corollary 25 (Jensen ca. 1970) *L satisfies "There is a Suslin tree on ω_2".*

How close is L to V?

Theorem 26 (Shoenfield, late 60's) *L is Σ^1_2-correct.*

Theorem 27 (Kechris, Moschovakis ca. 1971) *The reals in L are precisely those which are Σ^1_2 in a countable ordinal.*

Are there canonical inner models having more complicated reals, and a greater degree of correctness?

Inner Models with a measurable

Let κ be measurable, and U a normal ultrafilter on κ witnessing this.

Theorem 28 (Silver 1966) *$L[U]$ satisfies "κ is measurable". It also satisfies GCH, and "there is a Σ^1_3 wellorder of $\mathbb{R}$". Every real in $L[U]$ is Σ^1_3 in a countable ordinal.*

Theorem 29 (Kunen ca. 1968) *$L[U]$ depends only on κ, not on which U on κ one constructs from. If $\kappa_1 < \kappa_2$ and U_i is on κ_i, then there is an elementary*

$$j : L[U_1] \to L[U_2]$$

with crit(j)=κ_1.

Kunen's method of *iterated ultrapowers* is used throughout inner model theory.

Theorem 30 (Gaifman, Rowbottom 1967) *$\mathbb{R} \cap L$ is countable in $L[U]$.*

$\boxed{0^\sharp}$

What is the simplest canonical real past the constructible ones? Let U be a normal ultrafilter on κ, and set

$$\mathcal{M} = (L_{(\kappa^+)^L}, \in, U \cap L_{(\kappa^+)^L}),$$

and

$$\mathcal{H} = \Sigma_1 \text{ hull } of\, \mathcal{M}.$$

$\mathcal{H}$ can be identified with its Σ_1 theory, so it is essentially a real. We call it $0^\sharp$. Every real in L is recursive in $0^\sharp$.

$0^\sharp$ is the unique *iterable* model of a certain sentence θ. Since iterability is a Π^1_2 condition, $\{0^\sharp\}$ is Π^1_2, and hence $0^\sharp$ is itself Σ^1_3.

Definition 31 "$0^\sharp$ exists" is the assertion that there is an iterable model of θ.

Theorem 32 (Kunen, late 60's) $0^\sharp$ *exists if and only if there is a nontrivial, elementary* $j : L \to L$.

The Covering Theorem

$0^\sharp$ is the simplest real beyond L in various senses. By far the deepest and most useful result in this vein is the Covering Theorem:

Theorem 33 (Jensen 1975) *Exactly one of the following holds:*

- $0^\sharp$ *exists,*
- *For any uncountable $X \subseteq L$, there is a $Y \in L$ such that $X \subseteq Y$ and Y has the same cardinality as X.*

The proof makes use of the fine structure of L!

Corollary 34 (Jensen 1975) *Exactly one of the following holds:*

- $0^\sharp$ *exists,*
- *whenever κ is a singular cardinal (of V), we have $(\kappa^+)^L = \kappa^+$.*

Corollary 35 (Jensen 1975) *If the GCH fails at a singular strong limit cardinal, then* $0^\sharp$ *exists.*

Theorem 36 (Todorcevic, 80's) *The Proper Forcing Axiom implies that* $\Box_\kappa$ *fails, for all* κ.

Corollary 37 *The Proper Forcing Axiom implies that* $0^\sharp$ *exists.*

(Proof sketch: Let κ be a singular cardinal. By the theorem, $\Box_\kappa$ fails. But $\Box_\kappa$ holds in L. Since the $\Box$-sequence of L does not witness $\Box$ in V, we must have $(\kappa^+)^L < \kappa^+$. By the corollary to the Covering Theorem, $0^\sharp$ exists.)

Between $0^\sharp$ and $L[U]$

$L[0^\sharp]$ has a fine structure theory, hence satisfies $\forall\kappa\Box_\kappa$. There is an analog of the covering theorem for $L[0^\sharp]$. So from the Proper Forcing Axiom we get $(0^\sharp)^\sharp$. This can be considered a structure of the form $(L_\alpha[U_0], \in, U_1 \cap L_\alpha[U_0])$. Repeating this process, we form a universe K_{DJ} of the form $L[\langle U_\alpha \mid \alpha < \beta\rangle]$ which is as large as possible, *modulo* its not reaching an inner model with a measurable cardinal.

Theorem 38 (Dodd, Jensen 1978) *Either K_{DJ} covers V (as in the Covering Theorem), or there is an inner model with a measurable cardinal.*

Theorem 39 (Solovay, Welch 1978?) $K_{DJ} \models \forall\kappa\Box_\kappa$.

Corollary 40 *The Proper Forcing Axiom implies there is an inner model with a measurable cardinal, and hence all Σ^1_1 games are determined.*

Extender Models and Woodin Cardinals

We summarize the progress of inner model theory from 1978 to the present.

Mitchell (1974,1978) discovered what is probably the general form of the canonical inner models up through models with supercompacts. The models are (or will be) of the form $L[\vec{E}]$, where $\vec{E}$ is a *coherent* sequence of *extenders*.

Martin and Steel (1986) found the generalization of iterability which is appropriate for $L[\vec{E}]$ models below a superstrong. (But probably not below a supercompact!) They were able to actually *prove* iterability for such models up through models with infinitely many Woodin cardinals. Our inability to prove iterability higher up is the main obstacle to further progress.

Mitchell and Steel (1989) developed the basic fine structure theory of iterable $L[\vec{E}]$ models below a superstrong. Schimmerling and Zeman (1998) showed that such models satisfy $\forall\kappa\,\Box_\kappa$.

Steel (1990), building on Mitchell (1981) extended the Dodd-Jensen construction of K_{DJ} to the level of Woodin cardinals. Mitchell, Schimmerling, and Steel extended the Covering Theorem to this level.

Theorem 41 *Suppose that $x^\sharp$ exists for all sets x, but there is no proper class model with a Woodin cardinal. Then there is an $L[\vec{E}]$-model K such that*

1. *K has a fine structure theory; in particular $K \models \forall\mu\Box_\mu$, and*
2. *for all singular μ, $(\mu^+)^K = \mu^+$.*

Definition 42 $M_n(x)$ is the minimal iterable $L[\vec{E}]$-model built over x which contains all ordinals, and satisfies "there are n Woodin cardinals".

Theorem 43 (Woodin) *If $M_n(x)^\sharp$ exists for all x, but there is an x such that $M_{n+1}(x)^\sharp$ does not exist, then there is a K as in the last theorem.*

Corollary 44 (Woodin) *The Proper Forcing Axiom implies that for all n and x, $M_n(x)$ exists.*

The next theorem explains as well as anything why Woodin cardinals are a natural landmark for inner model theory.

Theorem 45 (Woodin) *If it exists, then $M_n(x)$ is Σ^1_{n+1}-correct.*

Theorem 46 (Woodin) *The Proper Forcing Axiom implies that all projective sets of reals are Lebesgue measurable.*

Beyond Woodin cardinals

Conjecture 47 *Con(Proper Forcing Axiom)* $\Rightarrow$ *Con(There is a supercompact cardinal).*