The consistency strength of $\mathsf{AD}_{\mathbb{R}}$

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Plan:

I. The derived model theorem. The consistency strength of AD.

II. The consistency strength of $\mathsf{AD}_{\mathbb{R}}$.

III. Beyond $\mathsf{AD}_{\mathbb{R}}$.

I. The derived model theorem

Definition 1 Let T and T^* be trees on $\omega \times X$ and $\omega \times Y$ respectively; then T and T^* are *κ-absolute complements* iff whenever G is V-generic over a poset of size $\leq \kappa$, then

$$V[G] \models p[T] = \mathbb{R} \setminus p[T^*].$$

We say that $A \subseteq \mathbb{R}$ is *κ-universally Baire* (or κ-UB) iff $A = p[T]$ for some tree T for which there is a κ-absolute complement.

Now let λ be a limit of Woodin cardinals, and G be V-generic for $\mathrm{Col}(\omega, < \lambda)$, and set

$$\begin{aligned}
\mathbb{R}^*_G &= \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha], \\
\mathrm{Hom}^*_G &= \{p[T] \cap \mathbb{R}^* \mid \exists \alpha < \lambda (V[G \restriction \alpha] \models \\
&\qquad T \text{ is } < \lambda \text{ absolutely complemented})\}.
\end{aligned}$$

Theorem 2 (Derived model theorem I, Woodin 1987) *Assume* ZFC*, and λ is a limit of Woodin cardinals; then*

(1) $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$.

(2) $\mathrm{Hom}^* = \{A \subseteq \mathbb{R}^* \mid L(\mathbb{R}^*, \mathrm{Hom}^*) \models$ *A is Suslin and co-Suslin.*

Corollary 3 (Woodin 1986) *Con(*ZFC + *there are ω Woodin cardinals) $\Rightarrow$ Con(*$\mathsf{ZF} + \mathsf{AD}^+$*).*

Theorem 4 (Derived model theorem II, Woodin 1987) *Assume* $\mathsf{ZF} + \mathsf{AD}^+$*, and let* Γ *be the class of Suslin, co-Suslin sets. Then in some generic extension of* V*, there is*

$$M \models \mathsf{ZFC} + \lambda \textit{ is a limit of Woodin cardinals}$$

and G *generic over* M *for* $\mathrm{Col}(\omega, < \lambda)$ *such that*

$$\mathbb{R}^*_G = \mathbb{R}^V \textit{ and } \mathrm{Hom}^*_G = \Gamma.$$

Corollary 5 (Woodin 1986) *Con(*$\mathsf{ZF} + \mathsf{AD}^+$*)* $\Rightarrow$ *Con(*ZFC + *there are* ω *Woodin cardinals).*

Remark 6 (1) The model M which one forces looks like the HOD of the AD^+ world.

(2) One can obtain a premouse M_ω with ω Woodins from it.

(3) The derived model of the true M_ω satisfies $V = L(\mathbb{R})$, and the true $L(\mathbb{R})$ is the derived model of an iterate of M_ω.

How do mice, and their derived models, match up in general with models of AD^+?

Conjecture 7 ((Mouse Set Conjectures (MSC)**))** *Assume* AD^+*, and that there is no* ω_1*-iteration strategy for a mouse with a superstrong cardinal; then for any countable transitive set* y*,*

(I) if $x \subseteq y$*, and* x *is ordinal definable from parameters in* $y \cup \{y\}$*, then there is an* ω_1*-iterable mouse* $\mathcal{M}$ *over* y *such that* $x \in \mathcal{M}$*, and*

(II) One can force a premouse M*, and a* G*, such that* $\mathbb{R}^*_G = \mathbb{R}^V$ *and* Hom^*_G *is the class of Suslin-co-Suslin sets of* V*.*

Remark 8 (1) The converse of I is true.

(2) There are other forms of MSC.

(3) If we drop the hypothesis that there are no iterable mice with superstrongs, we will need a more general theory of mice; e.g., mice with supercompacts.

(4) Proving MSC involves using HOD-like “strategy mice” (such as appear in the proof of Derived Model Theorem II) to construct ordinary extender-based mice. The strategy mice have enough extenders to witness Woodinness with respect to complicated predicates (the strategies). They do not have extenders overlapping Woodin cardinals. Moving from such HOD-like models to ordinary extender-based mice is the key open problem for the core model induction. Most likely, one needs to prove MSC in order to construct mice at the level of measurable Woodin cardinals under any reasonable hypotheses.

II. The consistency strength of $\mathsf{AD}_{\mathbb{R}}$

Definition 9 (AD^+.) For $A \subseteq \mathbb{R}$, $\theta(A)$ is the least ordinal α such that there is no surjection of $\mathbb{R}$ onto α which is ordinal definable from A and a real. We set

$$\begin{aligned} \theta_0 &= \theta(\emptyset), \\ \theta_{\alpha+1} &= \theta(A), \text{ for any (all) } A \text{ of Wadge rank } \theta_\alpha, \\ \theta_\lambda &= \bigcup_{\alpha<\lambda} \theta_\alpha. \end{aligned}$$

$\theta_{\alpha+1}$ is defined iff $\theta_\alpha < \Theta$. Note $\theta(A) < \Theta$ iff there is some $B \subseteq \mathbb{R}$ such that $B \notin \mathrm{OD}(\mathbb{R} \cup \{A\})$. In this case, $\theta(A)$ is the least Wadge rank of such a B.

Theorem 10 (Woodin, mid 80's) *Assume* AD^+*, and suppose* A *and* $\mathbb{R} \setminus A$ *are Suslin; then*

(a) All $\Sigma_1^2(A)$ *sets of reals are Suslin, and*

(b) All $\Pi_1^2(A)$ *sets are Suslin iff all OD*(A) *sets are Suslin iff* $\theta(A) < \Theta$.

Theorem 11 (Martin, Woodin, mid 80's) *Assume* AD^+*; then the following are equivalent:*

(1) $\mathsf{AD}_{\mathbb{R}}$,

(2) Every set of reals is Suslin,

(3) $\Theta = \theta_\lambda$*, for some limit* λ.

Strong-to-λ cardinals yield more Suslin sets in the derived model:

Theorem 12 (Woodin, 1988, 2000) *Suppose λ is a limit of Woodins, and $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is a derived model at λ; then*

(a) $\exists\kappa < \lambda$(κ *is* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \theta_0 < \Theta$.

(b) λ *is a limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, for some limit* α*.*

(c) λ *is an inaccessible limit of* κ *which are* $< \lambda$*-strong* $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \Theta = \theta_\alpha$*, where* $\mathrm{cof}(\alpha) \geq \omega_1$.

(d) $\exists\kappa < \lambda$(κ *is* λ*-supercompact*) $\Rightarrow$
$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular.*

Remarks.

(1) The hypotheses of (a)-(c) follow from λ being a Woodin limit of Woodin cardinals.

(2) There is a big jump going from (c) to (d). How strong is $\mathsf{AD}_{\mathbb{R}} + \Theta$ regular ?

(3) There are other ways of calibrating the strength of determinacy models. One can look at the complexity of the games of length ω_1 which are determined in the model. One can look at the complexity of the mice which have iteration strategies in the model.

Definition 13 (1) The statement that there is a λ which is a limit of Woodins and some $\kappa < \lambda$ which is $< \lambda$-strong is called the $\theta_0 < \theta$ *hypothesis*. (Or *OD sets Suslin hypothesis*.)

(2) The statement that there is a λ which is a limit of Woodins and of $< \lambda$-strongs is called the $\mathsf{AD}_{\mathbb{R}}$ *hypothesis*. (Or *All sets Suslin hypothesis*.)

(3) The statement that there is a λ which is a limit of Woodins and such that the $< \lambda$-strongs have order type λ is called the $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ *hypothesis*.

Corollary 14 (Woodin, late 80's) *(1)* *Con*($\mathsf{ZFC} + \theta_0 < \theta$*-hypothesis*) $\Rightarrow$ *Con*($\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \theta$),

(2) *Con*($\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}}$*-hypothesis*) $\Rightarrow$ *Con*($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$),

(3) *Con*($\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$*-hypothesis*) $\Rightarrow$ *Con*($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$).

What about the other direction? Here one needs *mice*, because AD^+ implies HOD does not have cardinals which are strong past a Woodin. (Unpublished work of Woodin.)

Theorem 15 (Woodin, late 90's?) *The mouse set conjecture* $\mathsf{MSC\ I}$ *holds in the minimal model of* $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$.

($\mathsf{MSC\ I}$ said OD reals are captured by mice.)

Corollary 16 (to proof) *Con(*$\mathsf{ZF} + \mathsf{AD}^+ + \theta_0 < \theta$*)*$\Rightarrow$ *Con(*$\mathsf{ZFC} + \theta_0 < \theta$*-hypothesis).*

The proof doesn't quite give the appropriate relative consistency in the other two cases, and it does not give $\mathsf{MSC\ II}$. (I.e., that you can force a mouse whose derived model is the AD^+ model in question.)

In 2004, author and Itay Neeman found a somewhat different proof of $\mathsf{MSC\ I}$ which works a bit beyond the minimal model of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$. Pushing on it, one can show

Theorem 17 *Let V be the minimal model of AD^+ and one of: $\theta_0 < \theta$, $\mathsf{AD}_{\mathbb{R}}$, or $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$. Then $\mathsf{MSC\ II}$ holds.*

Corollary 18 *(1) Con($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$)$\Rightarrow$ Con($\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}}$-hypothesis),*

(2) Con($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$)$\Rightarrow$ Con($\mathsf{ZFC} + \mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$-hypothesis).

III. Beyond $\mathsf{AD}_{\mathbb{R}}$

Theorem 19 (S. 2004) *Let M be the minimal iterable mouse with a Woodin limit of Woodin cardinals; then the derived model of M (at its Woodin limit) satisfies* $\mathsf{AD}_{\mathbb{R}} + \theta = \theta_\theta$.

Using results of Itay Neeman on the existence of such M, one gets

Corollary 20 *Con(*ZFC + *there is a Woodin limit of Woodin cardinals)* $\Rightarrow$ *Con(*$\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \theta = \theta_\theta$*).*

Remark 21 (1) Erik Closson has identified the theory of the minimal mouse whose derived model satisfies $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}} + \theta_{\omega_1} < \theta$. (It is below a Woodin limit of Woodins.) So we know what the $\mathsf{AD}_{\mathbb{R}} + \theta_{\omega_1} < \theta$-hypothesis should be. We do not yet have either direction of the equiconsistency.

(2) Very recently, Grigor Sargsyan has proved $\mathsf{MSC\ I}$ in the minimal model of $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular, and somewhat beyond. His work also shows that $\mathsf{AD}_{\mathbb{R}} + \theta$ is regular is no stronger than a Woodin limit of Woodin cardinals. It should lead to equiconsistencies in this region, and beyond.