THE COVERING LEMMA UP TO A WOODIN CARDINAL

W.J. MITCHELL, E. SCHIMMERLING, AND J.R. STEEL

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§1 Introduction

We say that a cardinal δ is countably closed if $\gamma^{\aleph_0} < \delta$ whenever $\gamma < \delta$. In this paper, we prove the following theorem.

Theorem 1.1. Suppose that Ω is a measurable cardinal, and that there is no inner model with a Woodin cardinal. Let K be the core model constructed in V_Ω. Suppose that $\kappa < \Omega$ is a K-cardinal such that $\operatorname{card}(\kappa)$ is countably closed. Let $\lambda = (\kappa^+)^K$. Then

$$\lambda < \kappa^+ \implies \operatorname{cf}(\lambda) = \operatorname{card}(\kappa).$$

In particular, K computes successors of countably closed, singular cardinals correctly.

In the theorem, K is the core model for one Woodin cardinal, as introduced in Steel [St1].

In order to trace the history behind the theorem, it will be convenient to make the following definitions. Consider an inner model M and an ordinal α. Let us say that

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CP(M,α) holds whenever

$$\forall X \subseteq \alpha \ \exists Y \in M \ (X \subseteq Y \ \& \ \mathrm{card}(Y) \leq \mathrm{card}(X) + \aleph_1)$$

and that WCP(M,α) holds whenever $(\alpha^+)^M = \alpha^+$. The theorem concludes that if κ is a singular, countably closed cardinal, then WCP(K,κ) holds.

Jensen's Covering Lemma for L says that either CP(L,α) holds for every ordinal α, or else $0^{\#}$ exists; see [De]. Dodd and Jensen introduced their core model in [DoJe1] and found an analogue of the result for L in [DoJe2]. A consequence of their work is that either CP(K,α) holds for every ordinal α, or there is a measurable cardinal in K, or $0^{\dagger}$ exists and is an element of K. The Dodd-Jensen result is extremely useful in proving that various set theoretic hypotheses have large cardinal consistency strength of at least one measurable cardinal: one merely has to violate CP(K,α). However, this method for obtaining lower bounds on consistency strength does not extend, at least without modification, to larger core models. This is because if μ is measurable, then CP(K,μ) fails in any Prikry generic forcing extension. Luckily, many of the applications of the Covering Property are really consequences of WCP(K,κ) for some singular cardinal κ. It is easy to see that if κ is a singular cardinal, then CP(M,$(\kappa^+)^M$) implies WCP(M,κ). The weak covering property that K computes successors of singular cardinals correctly, is expected to be true for much larger core models.

Indeed, above one measurable cardinal, weak covering properties are more a part of the core model theory, than a consequence thereof. In Mitchell [Mi1, Mi2], where the working assumption is that there is no inner model with a measurable cardinal μ of Mitchell order μ^{++}, Mitchell defines the countably complete core model, K^{cc}, and shows that for all countably closed, singular cardinals κ, WCP(K^{cc},κ) holds. Under the same working assumption, Mitchell goes on to define K as a transitive collapse of a certain elementary substructure of K^{cc} and to develop the basic properties of K using the iterability and weak covering properties of K^{cc}.

The working assumption in Steel [St1] is only that there is no inner model with a Woodin cardinal. Steel defines a preliminary model, K^c, the background certified core model, and shows that K^c is iterable in a sense involving iteration trees. But in order to establish a form of the weak covering property for K^c, Steel makes the *ad hoc* assumption that K^c is constructed inside some V_Ω with Ω a measurable cardinal in V. He shows that WCP(K^c,α) holds for almost every $\alpha < \Omega$, in the sense of any normal measure on Ω, and goes on to use this "cheapo" covering property and the iterability of K^c to isolate and develop the basic properties of K. Our Theorem 1.1 implies that K satisfies the Mitchell weak covering property in this setting.

Some important open questions regarding Theorem 1.1 remain. Can the assumption that there is a measurable cardinal be dropped? [1] What about the assumption that card(κ) is countably closed, can it be replaced by just $\kappa \geq \aleph_2$? [2] Towards a positive answer to these questions, Jensen showed in [Je2] that the assumption that there is a measurable cardinal above κ, and that κ is countably closed, are not needed if, instead, one assumes that $0^\P$ does not exist and $\kappa \geq \aleph_3$. Another question is whether Theorem 1.1 can be extended to larger core models. We expect that the argument given in the proof of Theorem 1.1, combined with the theory developed in [St3], [St4], and [SchSt] will yield extensions of Theorem 1.1 to core models with many Woodin cardinals.

This paper is organized as follows. Building on [MiSt], [St1], and [Sch1], we develop further the fine structure of K in §2. While we shall make some effort to remind the reader of the relevant definitions and facts, we shall, in the end, assume that the reader is quite familiar with these earlier works. The proof of Theorem 1.1 is given in §3. We encourage the reader to look ahead to §3 for additional motivation for the technical results in §2.

In proofs of covering properties for smaller core models, it is shown that for particular embeddings $\pi : N \longrightarrow V$, if we compare K^N with K and let $\mathcal{R}$ be the ultrapower

[1] See Schimmerling [Sch3] for some partial results.

[2] Mitchell and Schimmerling [MiSch] have since answered this question positively, by a proof that builds directly on the results presented here.

of the final model on the K-side by the long extender from π, then $\mathcal{R}$ is iterable in a way that guarantees that $\mathcal{R}$ is a level of K. The main new idea in the proof of Theorem 1.1 is to prove the required iterability, and more, by induction on the models occurring on the K-side of the comparison. The strengthening of the iterability hypothesis keeps the induction going.

The idea of proving more than the required iterability by induction, is due to Mitchell, and was first described by him in a note dated October, 1990. All three authors contributed to the work of converting that note into this paper. In the end, however, we were not able to show that the structure $\mathcal{R}$ of the previous paragraph is a level of K. As far as we know, $\mathcal{R}$ may itself be class size, or worse, $\mathcal{R}$ may fail to be a potential premouse, making it impossible that $\mathcal{R}$ be a level of K. The possibility that $\mathcal{R}$ is a class requires rearranging the proof in a way that is not particularly more complicated and leads to an interesting new perspective. Our method for dealing with the possibility that $\mathcal{R}$ is not a premouse, is due to Schimmerling, and was inspired by [Sch1].

§2 A Barrage of Fine Structure

In this section we extend the fine structure established in [MiSt], [St1], and [Sch1], and prove some facts that will be used in the proof of Theorem 1.1. We expect that the reader is quite familiar with those works, but we will try, whenever possible, to remind the reader of what things mean.

§2.1 The Dodd-squash

Suppose that F is a (κ, β)-extender (or pre-extender) over a potential premouse (ppm) $\mathcal{M}$. κ is the **critical point of** F and β is the **length of** F; we write $\mathrm{crit}(F) = \kappa$

and $\mathrm{lh}(F) = \beta$.

Suppose that $\xi \geq \kappa$ and $t \in [\beta]^{<\omega}$. We say that ξ is a **generator of F relative to** t iff $\xi \neq [a \cup t, f]^{\mathcal{M}}_F$ for any $a \in [\xi]^{<\omega}$ and $f \in |\mathcal{M}|$ with $f : [\kappa]^{|a \cup t|} \longrightarrow \kappa$. We say that ξ is **generator of F** iff ξ is a generator of F relative to $\emptyset$.

Note that whether or not ξ is a generator of F relative to t depends on ξ, t, F, and $\mathcal{P}(\kappa) \cap |\mathcal{M}|$. But since F determines $\mathcal{P}(\kappa) \cap |\mathcal{M}|$, the dependence on $\mathcal{M}$ is merely nominal. This will be true of many of the properties of F which we shall discuss below.

By the **strict supremum of the generators of F** we mean

$$\nu(F) = \sup(\{\ \xi + 1 \mid \xi \text{ is a generator of } F\ \} \cup (\kappa^+)^{\mathcal{M}}).$$

Let $\mathcal{P} = \mathrm{ult}_0(\mathcal{M}, F)$ and let $j : \mathcal{M} \longrightarrow \mathcal{P}$ be the ultrapower map. By the **trivial completion of** F, we mean the $(\kappa, (\nu(F)^+)^{\mathcal{P}})$-extender derived from j. Recall that if $\vec{E}$ is an extender from a good extender sequence, then every E_α is its own trivial completion. Assume for the rest of this subsection that the trivial completion of F is F again; in particular, $\beta = (\nu(F)^+)^{\mathcal{P}}$. By F^* we mean the 4-ary relation given by:

$$\begin{aligned}(\xi, \gamma, a, x) \in F^* \text{ if and only if } & \kappa < \xi < (\kappa^+)^{\mathcal{M}}, \\ & \gamma < \beta, \\ & F\ \cap\ ([\nu(F)]^{<\omega} \times J^{\mathcal{M}}_\xi) \in J^{\mathcal{P}}_\gamma, \text{ and} \\ & (a, x) \in F\ \cap\ ([\gamma]^{<\omega} \times J^{\mathcal{M}}_\xi).\end{aligned}$$

We purposely allow the possibility that $(\kappa^+)^{\mathcal{M}} < (\kappa^+)^{\mathcal{P}}$. F^* codes F by the fragments of F.

Varying slightly from [MiSt], our convention on ppm's is that if $\mathcal{M}$ is an active ppm, then $\dot{F}^{\mathcal{M}} = F^*$ for some extender F that is its own trivial completion. We call F (or F^*) the **last extender of $\mathcal{M}$** in this case. The advantage is that ppm's defined this way are amenable structures, and there is no need to consider *restricted* Σ_n formulas. The equivalence between this approach and the official approach taken in [MiSt] (which we avoid going into here) is explained in [MiSt].

The following definitions have their roots in some unpublished notes of A.J. Dodd. The reader should consult Section 3 of [Sch1].

We define the **Dodd-parameter of** $F = s(F)$ by induction. $s(F)(0)$ is the last generator of F above $(\kappa^+)^{\mathcal{M}}$, should it exist, and is left undefined otherwise. If $s(F)(0), \dots, s(F)(i)$ are defined, then $s(F)(i+1)$ is the last generator of F relative to $\{s(F)(0), \dots, s(F)(i)\}$ that is above $(\kappa^+)^{\mathcal{M}}$, should it exist, and is left undefined otherwise. Since $s(F)(i+1) < s(F)(i)$, we end up with $s(F) \in [\alpha - (\kappa^+)^{\mathcal{M}}]^{<\omega}$.

By the **Dodd-projectum** of F we mean the ordinal $\tau(F) =$ the strict supremum of the generators of F relative to $s(F)$, that is:

$$\tau(F) = \sup(\{ \xi + 1 \mid \xi \text{ is a generator of } F \text{ relative to } s(F) \} \cup (\kappa^+)^{\mathcal{M}}).$$

The **Dodd-squash** of F is $F \restriction (\tau(F) \cup s(F))$, usually viewed as coded by a subset of

$$[\tau \cup \{\check{s}_0, \check{s}_1, \dots\}]^{<\omega} \times J^{\mathcal{M}}_{(\kappa^+)^{\mathcal{M}}} ,$$

where each $\check{s}_i$ is a constant symbol in HF (the collection of hereditarily finite sets) representing s_i.

Suppose that $\mathcal{M}$ is an active ppm with $F^* = \dot{F}^{\mathcal{M}} =$ the last extender of $\mathcal{M}$. By the **Dodd-squash** of $\mathcal{M}$ we mean the structure

$$\mathcal{M}^{Ds} = \langle\ J^{\mathcal{M}}_{\tau(F)}\ ,\ \in\ ,\ \dot{E}^{\mathcal{M}} \restriction \tau(F)\ ,\ F \restriction (\tau(F) \cup s(F))\ \rangle.$$

In this situation, we also write $\dot{\tau}^{\mathcal{M}} = \tau(F)$ and $\dot{s}^{\mathcal{M}} = s(F)$. In [MiSt], active ppm are divided up into three types:

> $\mathcal{M}$ is type I iff F has only $\dot{\mu}^{\mathcal{M}} = \text{crit}(F)$ as a generator
>
> $\mathcal{M}$ is type II iff F has a last generator, but is not type I
>
> $\mathcal{M}$ is type III iff the generators of F have limit order type

Just to remind the reader:

> If $\mathcal{M}$ is type I, then $\dot{\tau}^{\mathcal{M}} = (\dot{\mu}^+)^{\mathcal{M}} = (\text{crit}(F)^+)^{\mathcal{M}} = \dot{\nu}^{\mathcal{M}}$ and $\dot{s}^{\mathcal{M}} = \emptyset$.
>
> If $\mathcal{M}$ is type II, then $\dot{\tau}^{\mathcal{M}} < \dot{\nu}^{\mathcal{M}} = \nu(F) = s(F)(0) + 1$.

If $\mathcal{M}$ is type III, then $\dot{\tau}^{\mathcal{M}} = \dot{\nu}^{\mathcal{M}} = \nu(F)$, $\dot{s}^{\mathcal{M}} = \emptyset$, and $\mathcal{M}^{Ds} = \mathcal{M}^{sq}$.

Some basic facts about the Dodd-squash are established in Section 3 of [Sch1]. Most notably, a proof of the following theorem of Steel is given.

THEOREM 2.1.1. Suppose that $\mathcal{M}$ is a 1-sound, 1-small, 0-iterable premouse. Let $F = \dot{F}^{\mathcal{M}}$, $s = \{s_0, \ldots, s_k\} = \dot{s}^{\mathcal{M}}$, and $\tau = \dot{\tau}^{\mathcal{M}}$. Then:

(A) τ is a cardinal in $\mathcal{M}$.
(B) For $i \leq k$, $F_i = F \restriction (s_i \cup (s \restriction i)) \in |\mathcal{M}|$.
(C) $\mathcal{M}^{Ds}$ is amenable.

The set F_i in (B) we call the i^{th} **Dodd-witness for** F (or $s(F)$ or $\mathcal{M}$ or $\mathcal{M}^{Ds}$), and (B) asserts that F (or $s(F)$ or $\mathcal{M}$ or $\mathcal{M}^{Ds}$) is **Dodd-solid**.

The next lemma extends Lemma 4.4 of [Sch1].

LEMMA 2.1.2. Suppose that $\mathcal{M}$ is a type II ppm. Let $\tau = \dot{\tau}^{\mathcal{M}}$, $s = \{s_0, \ldots, s_k\} = \dot{s}^{\mathcal{M}}$, $F = \dot{F}^{\mathcal{M}}$, $\mu = \dot{\mu}^{\mathcal{M}} = \mathrm{crit}(F)$. Suppose that $\mathcal{M}^{Ds}$ is amenable. Also, suppose that for some κ such that

$$(\mu^+)^{\mathcal{M}} \leq \kappa < \tau$$

and some parameter q from $\mathcal{M}$,

$$|\mathcal{M}| = H_1^{\mathcal{M}}(\kappa \cup q).$$

Then there is a parameter $x \in [\tau]^{<\omega}$ such that

$$\sup(\tau \cap H_1^{\mathcal{M}^{Ds}}(x)) = \tau.$$

PROOF. Choose $x \in [\tau]^{<\omega}$ with the following properties:

1. $\kappa \in x$
2. $s \in H_1^{\mathcal{M}}(x \cup q)$

3. There is some $f \in J^{\mathcal{M}}_{(\mu^+)^{\mathcal{M}}}$ such that $[x \cup s, f]^{\mathcal{M}}_F = \langle q, \dot\gamma^{\mathcal{M}} \rangle$ (recall that $\dot\gamma^{\mathcal{M}}$ codes the last initial segment of F).

Let $\sigma = \sup(\tau \cap H_1^{\mathcal{M}^{Ds}}(x))$. By 1, $\sigma > \kappa \geq (\mu^+)^{\mathcal{M}}$. Put

$$\tilde{\mathcal{Q}} = \mathcal{M}^{Ds} \restriction \sigma = \langle\ J^{\mathcal{M}}_\sigma\ ,\ \in\ ,\ \dot{E}^{\mathcal{M}} \restriction \sigma\ ,\ F \restriction (\sigma \cup s)\ \rangle.$$

Let $\mathcal{Q}$ be the **de-Dodd-squash of** $\tilde{\mathcal{Q}}$, that is, the structure

$$\langle\ J^{\mathcal{P}}_\beta\ ,\ \in\ ,\ \dot{E}^{\mathcal{P}} \restriction \beta\ ,\ G^*\ \rangle,$$

where $\mathcal{P} = \mathrm{ult}_0(\tilde{\mathcal{Q}}, F \restriction (\sigma \cup s))$,

$$\beta = \left[\{s_0\}, \{\xi\} \mapsto (\xi^+)^{\tilde{\mathcal{Q}}}\right]^{\tilde{\mathcal{Q}}}_{F \restriction (\sigma \cup s)}\ ,$$

and G is the (μ, β)-extender derived from the ultrapower embedding $\tilde{\mathcal{Q}} \longrightarrow \mathcal{P}$.

Let $\pi : \mathcal{Q} \longrightarrow \mathcal{M}$ be the canonical embedding. A straightforward calculation shows that π is a 0-embedding of $\mathcal{Q}$ into $\mathcal{M}$ with $\mathrm{crit}(\pi) \geq \sigma$. Note that

$$|\mathcal{Q}| = H_1^{\mathcal{Q}}(\sigma\ \cup\ \pi^{-1}(s)).$$

Using 2 and 3, we see that

$$|\mathcal{Q}| = H_1^{\mathcal{Q}}(\sigma\ \cup\ \pi^{-1}(q)).$$

But since

$$|\mathcal{M}| = H_1^{\mathcal{M}}(\sigma\ \cup\ q),$$

$\pi = \mathrm{id}$ and $\sigma = \tau$. $\square$ Lemma 2.1.2

From the lemma, standard fine structural calculations show:

Corollary 2.1.3. Under the same hypothesis, the ordinals τ, $(\kappa^+)^{\mathcal{M}}$, $\mathrm{OR}^{\mathcal{M}}$, and $(\mu^+)^{\mathcal{M}}$ have countable cofinality, definably over $\mathcal{M}$.

The next lemmas tell us how $\dot\tau$ and $\dot s$ change as we iterate.

LEMMA 2.1.4. Suppose that E is a (κ, β)-extender over a ppm $\mathcal{M}$, and $\kappa < \dot{\tau}^{\mathcal{M}}$. Suppose that $\mathcal{M}^{Ds}$ is amenable and Dodd-solid; say $\tau = \dot{\tau}^{\mathcal{M}}$ and $s = \{s_0, \dots, s_l\} = \dot{s}^{\mathcal{M}}$. Let $i : \mathcal{M} \longrightarrow \mathcal{P} = \mathrm{ult}_0(\mathcal{M}, E)$ be the ultrapower map. Then $\dot{s}^{\mathcal{P}} = i(s)$, $\dot{\tau}^{\mathcal{P}} = \sup(i''\tau)$, and $\mathcal{P}^{Ds}$ is amenable.

PROOF. The proof is essentially the same as that of Lemma 9.1 of [MiSt]. We have the commutative diagram:

$$\begin{array}{ccc} \mathrm{ult}_0(\mathcal{M}, \dot{F}^{\mathcal{M}}) & \xrightarrow{j} & \mathrm{ult}_0(\mathcal{P}, \dot{F}^{\mathcal{P}}) \\ \pi \uparrow & & \uparrow \sigma \\ \mathcal{M} & \xrightarrow{i} & \mathcal{P} = \mathrm{ult}_0(\mathcal{M}, E) \end{array}$$

with $i = j \restriction |\mathcal{M}|$. Since i is a 0-embedding and $\mathcal{M}$ is Dodd-solid,

$$i(\dot{F}^{\mathcal{M}} \restriction (s_n \cup (s|n))) = \dot{F}^{\mathcal{P}} \restriction (i(s_n) \cup i(s|n)) \in |\mathcal{P}|.$$

Since i is a 0-embedding and $\mathcal{M}^{Ds}$ is amenable, for each $\xi < \tau$,

$$i(\dot{F}^{\mathcal{M}} \restriction (\xi \cup s)) = \dot{F}^{\mathcal{P}} \restriction (i(\xi) \cup i(s)) \in |\mathcal{P}|,$$

and $\tau^* = \sup(i''\tau)$ is a limit of generators of $\dot{F}^{\mathcal{P}}$ relative to $i(s)$. So it is enough to see that no $\eta \geq \tau^*$ is a generator of $\dot{F}^{\mathcal{P}}$ relative to $i(s)$.

So suppose that $\tau^* \leq \eta < \mathrm{OR}^{\mathcal{P}}$. Then $\eta = j(f)(b)$ for some $f \in |\mathcal{M}|$ and $b \in [\beta]^{<\omega}$ with $f : [\kappa]^{|b|} \longrightarrow \kappa$. Since $f \in |\mathcal{M}| \subset |\mathrm{ult}_0(\mathcal{M}, \dot{F}^{\mathcal{M}})|$, $f = \pi(g)(a \cup s)$ for some $g \in |\mathcal{M}|$ and $a \in [\tau]^{<\omega}$ with $g : [\dot{\mu}^{\mathcal{M}}]^{|a \cup s|} \longrightarrow |\mathcal{M}|$. Thus

$$\eta = j(f)(b) = j(\pi(g)(a \cup s))(b) = j(\pi(g))(j(a) \cup j(s))(b) = \sigma(i(g))(i(a) \cup i(s))(b).$$

So $\eta = [i(a) \cup b \cup i(s), h]^{\mathcal{P}}_{\dot{F}^{\mathcal{P}}}$ where

$$h(u) = i(g)(u^{i(a) \cup i(s)\ ,\ i(a) \cup b \cup i(s)})(u^{b\ ,\ i(a) \cup b \cup i(s)}).$$

Since $i(a) \in [\tau^*]^{<\omega}$ and $b \in [\beta]^{<\omega} \subseteq [\tau^*]^{<\omega}$, we are done proving Lemma 2.1.4.

$\square$ Lemma 2.1.4

We say that $\mathcal{M}$ is **Dodd-solid above** κ if the i'th Dodd-solidity witness for $\dot{F}^{\mathcal{M}}$ is an element of $|\mathcal{M}|$ whenever $\dot{s}_i^{\mathcal{M}} \geq \kappa$. An argument similar to that just given shows:

LEMMA 2.1.5. Suppose E is a (κ, β)-extender over $\mathcal{M}$ with $\dot{\tau}^{\mathcal{M}} \leq \kappa$. Suppose that $\mathcal{M}$ is Dodd-solid above κ. Let $j : \mathcal{M} \longrightarrow \mathcal{P} = \mathrm{ult}_0(\mathcal{M}, G)$ be the ultrapower map. Then $\dot{\tau}^{\mathcal{P}} \leq \nu(E)$,

$$j\left(\{\dot{s}_i^{\mathcal{M}} | \dot{s}_i^{\mathcal{M}} \geq \kappa\}\right) = \{\dot{s}_i^{\mathcal{P}} \mid \dot{s}_i^{\mathcal{P}} \geq \nu(E)\} ,$$

and $\mathcal{P}$ is Dodd-solid above $\nu(E)$.

Note that under the hypothesis of Lemma 2.1.5, it is possible that $\mathcal{P}^{Ds}$ is not amenable or that $\dot{s}^{\mathcal{P}}$ is not Dodd-solid; see Section 3 of [Sch1] for an examples using $0^{\P}$. The main corollary to the previous lemmas that we shall need is:

COROLLARY 2.1.6. Suppose that $\mathcal{T}$ is a padded ω-maximal iteration tree on a 1-small, Ω-iterable weasel W with models W_η for $\eta < \mathrm{lh}(\mathcal{T})$. Suppose that $\eta < \mathrm{lh}(\mathcal{T})$ and κ is a cardinal in W_η such that

$$\kappa \geq \sup\left(\{\nu_\beta^{\mathcal{T}} \mid \beta + 1 \leq_T \eta\}\right)$$

(recall that $\nu_\beta^{\mathcal{T}} = \nu(E_\beta^{\mathcal{T}})$ for such iteration trees). Suppose that $\mathcal{P}$ is an initial segment of W_η such that $\rho_1^{\mathcal{P}} \leq \kappa$ and $\kappa \geq (\dot{\mu}^+)^{\mathcal{P}}$. Then either $\dot{\tau}^{\mathcal{P}} \leq \kappa$ and $\mathcal{P}$ is Dodd-solid above κ, or else the ordinals $\dot{\tau}^{\mathcal{P}}$, $(\kappa^+)^{\mathcal{P}}$, $\mathrm{OR}^{\mathcal{P}}$, and $(\dot{\mu}^+)^{\mathcal{P}}$ all have countable cofinality.

§2.2 FOR WEASELS ONLY

By a **weasel**, we mean a premouse of height Ω. As we have seen in [Mi1], [Mi2], and [St1], the hull and definability properties for weasels play a role similar to the role that soundness plays for set premice. We take this analogy a bit further in this subsection.

DEFINITION 2.2.1. Let A_0 be the class defined in Section 1 of [St1]. Suppose that W be a weasel, $s \in [\Omega]^{<\omega}$, and $\kappa < \Omega$. We say that W has the **s-hull property at κ** if and only if

$$\mathcal{P}(\kappa) \subseteq \text{ the transitive collapse of } H^W(\kappa \cup s \cup \Gamma)$$

whenever Γ is A_0-thick in W. We say that W has the **s-definability property at κ** if and only if

$$\kappa \in H^W(\kappa \cup s \cup \Gamma)$$

whenever Γ is A_0-thick in W.

In the definitions that we are about to give, there is a somewhat suppressed dependence on our choice of $\Upsilon < \Omega$. In the proof of Theorem 1.1, Υ will be a cardinal $> \lambda$.

By analogy with both the standard parameter and the Dodd-parameter, we define the **class-parameter** $c(W)$ by induction. Let $c(W)(0)$ be the largest $\xi < \Upsilon$ such that W does not have the definability property at ξ; if no such ξ exists, then leave $c(W)(0)$ undefined. Suppose that we have defined $c(W)(0), \ldots, c(W)(i)$. Let $c(W)(i+1)$ be the largest $\xi < \Upsilon$ such that W does not have the $\{c(W)(0), \ldots, c(W)(i)\}$-definability property at ξ; if no such ξ exists, then leave $c(W)(i+1)$ undefined. Of course, $c(W)(i+1) < c(W)(i)$, so we are in fact getting $c(W) \in [\Upsilon]^{<\omega}$. We define the **class projectum** of W to be $\kappa(W)$ = the supremum of the ordinals ξ such that W does not have the $c(W)$-definability property at ξ. Notice that if Γ is A_0-thick in W, then we have the following analog to soundness:

$$H^W(\kappa(W) \cup c(W) \cup \Gamma) \cap J^W_\Upsilon = J^W_\Upsilon.$$

In particular, W has the $c(W)$-hull property at $\kappa(W)$ when $\kappa(W) < \Upsilon$.

§2.3 Protomice

DEFINITION 2.3.1. Suppose that $\mathcal{R}$ and $\mathcal{M}$ are a ppm's, F is a (κ, β)-extender over $\mathcal{M}$, $(\kappa^+)^{\mathcal{M}} \leq (\kappa^+)^{\mathcal{R}}$, and $\mathcal{M}$ and $\mathcal{R}$ agree below $(\kappa^+)^{\mathcal{M}}$. Then we say that F is a $(\kappa, \mathcal{M}, \beta)$-**extender fragment over** $\mathcal{R}$.

So extenders are extender fragments, but not necessarily the other way around. Note that if F is a $(\kappa, \mathcal{M}, \beta)$-extender fragment over $\mathcal{R}$, then F is a $(\kappa, \mathcal{N}, \beta)$-extender fragment over $\mathcal{R}$ for any initial segment $\mathcal{N}$ of $\mathcal{R}$ with $(\kappa^+)^{\mathcal{N}} = (\kappa^+)^{\mathcal{M}}$. Extender fragments arise naturally in the proof of weak square given in [Sch1]. They are a fine structural analogue of the background certificates used to build K^c in [St1]. In the proof of Theorem 1.1, we shall often encounter structures that look exactly like a premice, except that their last predicates code extender fragments that are not total extenders.

DEFINITION 2.3.2. A **protomouse** is a structure $\mathcal{R}$ in the language of ppm's which is of the form

$$\langle\ J_\alpha^{\vec{E}}\ ,\ \in\ ,\ \vec{E} \restriction \alpha\ ,\ F^*\ \rangle,$$

where

$$\mathcal{R}' = \langle\ J_\alpha^{\vec{E}}\ ,\ \in\ ,\ \vec{E} \restriction \alpha\ \rangle$$

is a passive premouse, and F is a $(\mu, \mathcal{M}, \alpha)$-extender fragment over $\mathcal{R}'$ for some ordinal μ and premouse $\mathcal{M}$ such that F is its own trivial completion and satisfies the coherence condition:

$$\dot{E}^{ult_0(\mathcal{M},F)} \restriction (\alpha + 1) = \vec{E} \restriction \alpha$$

and the initial segment condition: if F has a last generator, η, and the trivial completion of G of $F \restriction \eta$ has length γ, then either $G \subseteq \dot{E}_\gamma^{\mathcal{R}}$ or

$$G \subseteq \dot{E}_\gamma^{ult_0(J_\eta^{\mathcal{R}}, \dot{E}_\eta^{\mathcal{R}})}\ .$$

So a protomouse is a premouse if and only if its last predicate is a total extender over its universe, and not merely an extender fragment. The definitions and results in

Sections 1-4 of [MiSt] apply as well to protomice as to premice. The next definition uses notions from [MiSt] as applied to protomice. As is the case with premice, if $\mathcal{R}$ is a type III protomouse, then whenever we write $\mathcal{R}$, we almost always mean $\mathcal{R}^{sq}$.

DEFINITION 2.3.3. Suppose that $\mathcal{R}$ is a protomouse and κ is a cardinal in $\mathcal{R}$. If there is a largest integer n such that $\rho_n^{\mathcal{R}} > \kappa$, then we put $n(\mathcal{R}, \kappa) = n$; otherwise, set $n(\mathcal{R}, \kappa) = \omega$. Say $n(\mathcal{R}, \kappa) = n < \omega$. Suppose that $\mathcal{R}$ is n-sound; let $u = u_n(\mathcal{R}) =$ the solidity witness for $p_n(\mathcal{R})$ and $p = p_{n+1}^{(\mathcal{R},u)}$ = the $(n+1)$'st standard parameter of $(\mathcal{R}, u)$. Suppose that $p - \kappa = \{p_0 > \cdots > p_k\}$ and whenever $i \leq k$, the i'th solidity witness for p is in $\mathcal{R}$, that is,

$$w_i = Th_{n+1}(p_i \ \cup \ p|i) \in |\mathcal{R}|.$$

Then we say that $\mathcal{R}$ is **solid above** κ and we set $p(\mathcal{R}, \kappa) = \langle p, u, \langle w_i | i \leq k \rangle \rangle$. If, in addition,

$$|\mathcal{R}| = H_{n+1}^{\mathcal{R}}(\kappa \ \cup \ p(\mathcal{R}, \alpha)),$$

then we say that $\mathcal{R}$ is **κ-sound**.

§2.4 Iteration Trees on Phalanxes of Protomice

Our definition of a phalanx will differ from both definition 6.5 of [St1] and definition 9.6 of [St1], although the flavor is very much the same.

DEFINITION 2.4.1. Suppose that $\vec{\lambda} = \langle \lambda_\alpha \,|\, \alpha < \gamma \rangle$ is an increasing sequence of ordinals, and $\vec{\mathcal{R}} = \langle \mathcal{R}_\alpha \,|\, \alpha \leq \gamma \rangle$ is a sequence of protomice. Then $(\vec{\mathcal{R}}, \vec{\lambda})$ is a **phalanx of protomice of length** γ if and only if whenever $\alpha \leq \beta \leq \gamma$, $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ agree below λ_α.

To us, a phalanx will always be a phalanx of protomice. Note that in Definition

2.4.1, we do not require that $\mathcal{R}_\alpha$ be a premouse, nor that λ_α be a cardinal in $\mathcal{R}_\beta$. $\mathcal{R}_\gamma$ is called the **starting model**. We often write $((\vec{\mathcal{R}} \restriction \gamma, \mathcal{R}_\gamma), \vec{\lambda})$ for $(\vec{\mathcal{R}}, \vec{\lambda})$. Or, if $\gamma = 2$, we write $((\mathcal{R}_0, \mathcal{R}_1), \lambda_0)$.

Definition 2.4.2. Let G be anything and suppose that $\mathcal{R}$ is a protomouse such that

$$F = G \cap ([\Omega]^{<\omega} \times |\mathcal{R}|)$$

is an extender over $\mathcal{R}$ with $\mathrm{crit}(F) = \kappa$. Under these conditions, we define $\mathrm{ult}(\mathcal{R}, G)$ to be $\mathrm{ult}_n(\mathcal{R}, F)$, where $n = n(\mathcal{R}, \kappa)$.

Iteration trees on structures similar to phalanxes of protomice were defined in [MiSt] (where they were called "pseudo iteration trees"), and in Sections 6 and 9 of [St1]. Using Definition 2.4.2, we can modify these earlier definitions in the obvious way to allow a phalanx of protomice as an allowable base for an iteration tree.

Definition 2.4.3. By an **iteration tree on a phalanx** $(\mathcal{R}, \vec{\lambda})$ we mean a padded, ω-maximal pseudo-iteration tree on the phalanx of protomice $(\mathcal{R}, \vec{\lambda})$.

There are several implicit assumptions in Definition 2.4.3 which we clear up here. Suppose that $(\mathcal{R}, \vec{\lambda})$ is a phalanx of length γ on which $\mathcal{T}$ is an iteration tree. Suppose that $\gamma + \xi$ is least such that $E^{\mathcal{T}}_{\gamma+\xi} \neq \emptyset$. It is conceivable that $\nu^{\mathcal{T}}_{\gamma+\xi} = \nu(E^{\mathcal{T}}_{\gamma+\xi}) < \lambda_\alpha$ for some $\alpha < \gamma$; suppose that this is the case. Let α be minimal with this property. If $\gamma + \xi + 1 < \gamma + \eta + 1 < \mathrm{lh}(\mathcal{T})$ and $\nu(E^{\mathcal{T}}_{\gamma+\xi}) < \mathrm{crit}(E^{\mathcal{T}}_{\gamma+\eta}) < \lambda_\alpha$, then it is not entirely clear from Definition 2.4.3 whether it is α or γ that is the $\mathcal{T}$-predecessor of $\gamma + \eta + 1$. The answer is that $\gamma = \mathrm{pred}^{\mathcal{T}}(\gamma + \eta + 1)$. The relevant rule is that in applying $E^{\mathcal{T}}_{\gamma+\xi}$, we go back only as far as needed to avoid moving generators, and only then consider models before the starting model on the phalanx. We remark that in practise, $\nu^{\mathcal{T}}_{\gamma+\xi}$ will be $\geq \lambda_\alpha$ for every $\alpha < \gamma$.

Another implicit assumption in Definition 2.4.2 is that $(\mathcal{M}^*_{\gamma+\eta+1})^{\mathcal{T}}$ is

$\deg^{\mathcal{T}}(\gamma+\eta+1)$-sound whenever $E^{\mathcal{T}}_{\gamma+\xi} \neq \emptyset$. In practise, $(\mathcal{M}^*_{\gamma+\eta+1})^{\mathcal{T}}$ will also be solid above $\mathrm{crit}(E^{\mathcal{T}}_{\gamma+\eta+1})$.

Definition 2.4.4. If $(\vec{\mathcal{R}}, \vec{\lambda})$ is a phalanx of length γ on which $\mathcal{T}$ is an iteration tree, and $\eta < \mathrm{lh}(\mathcal{T})$, then by $\mathbf{root}^{\mathcal{T}}(\eta)$ we mean the unique $\alpha \leq \gamma$ such that $\alpha \leq_T \eta$.

The next definition is made relative to our choice of Υ. The notation in the last two sections should be recalled. A **set protomouse** is a protomouse of height $< \Omega$; of course, every protomouse of height Ω is passive, and hence a weasel.

Definition 2.4.5. A phalanx $(\vec{\mathcal{R}}, \vec{\lambda})$ of length γ is **special** if and only if there is a sequence $\vec{\kappa} = \langle\ \kappa_\alpha \mid \alpha < \gamma\ \rangle$ such that:

(i) If $\alpha \leq \beta < \gamma$, then κ_α is a cardinal in $\mathcal{R}_\beta$.
(ii) If $\alpha < \gamma$, then $\lambda_\alpha = (\kappa^+_\alpha)^{\mathcal{R}_\alpha}$
(iii) If $\alpha < \gamma$, and $\mathcal{R}_\alpha$ is a set protomouse, then $\mathcal{R}_\alpha$ is κ_α-sound.
(iv) If $\alpha < \gamma$, and $\mathcal{R}_\alpha$ is a weasel, then $\kappa(\mathcal{R}_\alpha) \leq \kappa_\alpha$.
(v) Suppose that $\beta \leq \gamma$ and $\mathcal{R}_\beta$ is not a premouse. Then there is a unique $\alpha < \beta$ $\dot{F}^{\mathcal{R}_\beta}$ is a $(\kappa_\alpha, \mathcal{R}_\alpha, \mathrm{OR}^{\mathcal{R}_\beta})$-extender fragment over $\mathcal{R}_\beta$.
(That is, a unique $\alpha < \beta$ such that $\dot{F}^{\mathcal{R}_\beta}$ is a $(\kappa_\alpha, \mathrm{OR}^{\mathcal{R}_\beta})$-extender over $\mathcal{R}_\alpha$.)

Definition 2.4.6. We say that $\mathcal{T}$ is a **special** iteration tree if and only if there is a phalanx $(\vec{\mathcal{R}}, \vec{\lambda})$ of length γ such that $\mathcal{T}$ is an iteration tree on $(\vec{\mathcal{R}}, \vec{\lambda})$, and:

- $(\vec{\mathcal{R}}, \vec{\lambda})$ is special.
- If $\gamma + \xi$ is least such that $E^{\mathcal{T}}_{\gamma+\xi} \neq \emptyset$, and $\alpha < \gamma$, then κ_α is a cardinal in $J^{\mathcal{R}_\gamma}_{lh(E^{\mathcal{T}}_{\gamma+\xi})}$.
- If $\alpha = \mathrm{pred}^{\mathcal{T}}(\gamma+\eta+1) < \gamma$, then $\mathrm{crit}(E^{\mathcal{T}}_{\gamma+\eta}) = \kappa_\alpha$.

Suppose that $(\vec{\mathcal{R}}, \vec{\lambda})$ is a special phalanx of length γ on which $\mathcal{T}$ is a special iteration tree. Suppose that $\beta < \gamma$ and $\mathcal{R}_\beta$ is not a premouse. By Definition 2.4.5(v), there is an ordinal $\alpha < \beta$ such that $\dot{F}^{\mathcal{R}_\beta}$ is a $(\kappa_\alpha, \mathrm{OR}^{\mathcal{R}_\beta})$-extender over $\mathcal{R}_\alpha$. Suppose that $\beta = \mathrm{root}^{\mathcal{T}}(\gamma+\eta)$ and that $\mathcal{D}^{\mathcal{T}} \cap (\beta, \gamma+\eta]_T = \emptyset$. Since $\kappa_\alpha < \kappa_\beta = \mathrm{crit}(i^{\mathcal{T}}_{\beta,\gamma+\eta})$, the iteration of $\mathcal{R}_\beta$ out to $\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}$ stretches the last extender fragment, without changing the critical point or the sets that it measures. So $\dot{F}^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}$ is also an extender over $\mathcal{R}_\alpha$ with critical point κ_α, and $\dot{F}^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}$ is a reasonable candidate for $E^{\mathcal{T}}_{\gamma+\eta}$ with $\alpha = \mathrm{pred}^{\mathcal{T}}(\gamma+\eta+1)$.

Continuing the same discussion, it is also possible that $E^{\mathcal{T}}_{\gamma+\eta}$ is an extender over an initial segment of $\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}$, but that it measures more subsets of its critical point than are in the model to which it is being applied. This happens, for example, when $\beta = \mathrm{root}^{\mathcal{T}}(\gamma+\eta)$, $E^{\mathcal{T}}_{\gamma+\eta}$ is an extender from the sequence $\dot{E}^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}$ with $\mathrm{crit}(E^{\mathcal{T}}_{\gamma+\eta}) = \kappa_\alpha$ for some $\alpha < \beta$, so that $\alpha = \mathrm{pred}^{\mathcal{T}}(\gamma+\eta+1)$, and $\lambda_\alpha < (\kappa_\alpha^+)^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}$. Notice that in this case,

$$\mathcal{M}^{\mathcal{T}}_{\gamma+\eta+1} \text{ and } \mathrm{ult}(J^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}_{lh(E^{\mathcal{T}}_{\gamma+\eta})}, E^{\mathcal{T}}_{\gamma+\eta}) \text{ agree below } (\nu(E^{\mathcal{T}}_{\gamma+\eta})^+)^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta+1}},$$

but it is possible that the successor of $\nu(E^{\mathcal{T}}_{\gamma+\eta})$ as computed in $\mathcal{M}^{\mathcal{T}}_{\gamma+\eta+1}$ is strictly less than the successor of $\nu(E^{\mathcal{T}}_{\gamma+\eta})$ as computed in

$$\mathrm{ult}(J^{\mathcal{M}^{\mathcal{T}}_{\gamma+\eta}}_{lh(E^{\mathcal{T}}_{\gamma+\eta})}, E^{\mathcal{T}}_{\gamma+\eta}).$$

This makes it unclear how we can compare two arbitrary special phalanxes (in the sense of Section 7 of [MiSt]).

DEFINITION 2.4.7. Suppose that $(\vec{\mathcal{R}}, \vec{\lambda})$ is a special phalanx of length γ with the associated sequence $\vec{\kappa}$ as above. $(\vec{\mathcal{R}}, \vec{\lambda})$ is **very special** if and only if for $\alpha < \gamma$:

- If $\mathcal{R}_\alpha$ is a weasel, then $c(\mathcal{R}_\alpha) - \kappa_\alpha = \emptyset$.
- If $\mathcal{R}_\alpha$ is not a premouse, then $\mathcal{R}_\alpha$ is an active protomouse of type I or II and $n(\mathcal{R}_\alpha, \kappa_\alpha) = 0$.

Most of the phalanxes $(\vec{\mathcal{R}}, \vec{\lambda})$ that come up in our proof of Theorem 1.1 are special, and some of these are very special.

§2.5 <u>Long Extenders</u>

Throughout this section, X is an elementary substructure of $V_{\Omega+1}$ (together with the measure on Ω as a predicate) and $\pi : N \longrightarrow V_{\Omega+1}$ is the inverse of the transitive collapsing map for X. Let $\delta = \mathrm{crit}(\pi)$ and E_π be the (δ, Ω)-extender derived from π (see Steel [St2]). So

$$E_\pi = \langle (E_\pi)_a \mid a \in [\Omega]^{<\omega} \rangle,$$

where for any $a \in [\Omega]^{<\omega}$, if δ_a is the least $\widetilde{\delta} \geq \delta$ such that $a \subset \widetilde{\delta}$, then

$$(E_\pi)_a = \{x \subseteq [\delta_a]^{|a|} \mid a \in \pi(x)\}.$$

The ordinals δ_a we call the **critical points** of E_π; note that there may be many.

Suppose that $\mathcal{P}$ is a premouse and that κ is a cardinal in $\mathcal{P}$ such that $\mathcal{P}(\kappa) \cap |\mathcal{P}| \subset N$ and $\mathcal{P} \models \kappa^+$ exists. Let

$$F = E_\pi \ \cap \ ([\pi(\kappa)]^{<\omega} \times |\mathcal{P}|).$$

In this situation, we wish to define $\mathrm{ult}(\mathcal{P}, E_\pi \restriction \pi(\kappa))$.

First suppose that $\mathcal{P}$ is either passive or active of type I or II. Let $n = n(\mathcal{P}, \kappa)$. Then $\mathrm{ult}(\mathcal{P}, E_\pi \restriction \pi(\kappa))$ is $\mathrm{ult}_n(\mathcal{P}, F)$, the ultrapower of $\mathcal{P}$ by F using certain functions $f : [\kappa]^k \longrightarrow |\mathcal{P}|$, where $k < \omega$. In $n = 0$, we use only $f \in |\mathcal{P}|$. If $0 < n < \omega$, then functions f of the form $u \mapsto \tau_\varphi^{\mathcal{P}}[u, q]$ for a Σ_n-Skolem term τ_φ and parameter q from $\mathcal{P}$ are also allowed. If $n = \omega$, then all f that are first order definable over $\mathcal{P}$ are allowed.

Now suppose that $\mathcal{P}$ is active of type III. In this case, whenever we make reference to the fine structure of $\mathcal{P}$, we are really referring to the fine structure of $\mathcal{P}^{sq}$. So, for example, $\rho_n^{\mathcal{P}}$ is really $\rho_n^{\mathcal{P}^{sq}}$. With this interpretation, let n be as above. It is not difficult to extend the results in Section 3 of [MiSt] to see that $\mathrm{ult}_n(\mathcal{P}^{sq}, F)$ is itself a squashed potential premouse (sppm). By $\mathrm{ult}(\mathcal{P}, E_\pi \restriction \pi(\kappa))$, we mean the premouse $\mathcal{R}$ such that $\mathcal{R}^{sq} = \mathrm{ult}_n(\mathcal{P}^{sq}, F)$.

In either case, let $\mathcal{R} = \mathrm{ult}(\mathcal{P}, E_\pi \restriction \pi(\kappa))$. We assume throughout this section that $\mathcal{R}$ is well-founded. Let $\tilde{\pi}$ be the ultrapower map. So, except when $\mathcal{P}$ is active of type III, the map $\tilde{\pi} : \mathcal{P} \longrightarrow \mathcal{R}$ is given by

$$\tilde{\pi}(x) = [a, u \mapsto x]^{\mathcal{P}}_{E_\pi \restriction \pi(\kappa)} .$$

And when $\mathcal{P}$ is active of type III, the map $\tilde{\pi} : \mathcal{P}^{sq} \longrightarrow \mathcal{R}^{sq}$ is given by

$$\tilde{\pi}(x) = [a, u \mapsto x]^{\mathcal{P}^{sq}}_{E_\pi \restriction \pi(\kappa)} .$$

LEMMA 2.5.1. Suppose that $\mathcal{P}$ above is a κ-sound premouse and that $n = n(\mathcal{P}, \kappa) < \omega$. Suppose further that if $n = 0$, $\mathcal{P}$ is active of type I or II, and $\dot{\mu}^{\mathcal{P}} < \kappa$, then

$$\sup\left(\pi'' \left(\dot{\mu}^{+}\right)^{\mathcal{P}}\right) = \pi\left(\left(\dot{\mu}^{+}\right)^{\mathcal{P}}\right).$$

Then

1. $\tilde{\pi} \restriction (\kappa^+)^{\mathcal{P}} = \pi \restriction (\kappa^+)^{\mathcal{P}}$.
2. $\mathcal{R}$ is a premouse of the same type as $\mathcal{P}$.
3. $\tilde{\pi}$ is an n-embedding that is continuous at λ whenever $\kappa < \lambda < \rho_n^{\mathcal{P}}$ and $\mathcal{P} \models \mathrm{cf}(\lambda) > \kappa$.
4. $\mathcal{R}$ is $\pi(\kappa)$-sound.
5. $\tilde{\pi}(p(\mathcal{P}, \kappa)) = p(\mathcal{R}, \pi(\kappa))$.

LEMMA 2.5.2. Suppose that $\mathcal{P}$ above is a κ-sound premouse that is active of type I or II with $n(\mathcal{P}, \kappa) = 0$, $\dot{\mu}^{\mathcal{P}} < \kappa$, and

$$\sup\left(\pi'' \left(\dot{\mu}^{+}\right)^{\mathcal{P}}\right) < \pi\left(\left(\dot{\mu}^{+}\right)^{\mathcal{P}}\right).$$

Then

1. $\tilde{\pi} \restriction (\kappa^+)^{\mathcal{P}} = \pi \restriction (\kappa^+)^{\mathcal{P}}$.
2. $\mathcal{R}$ is *not* a premouse; it is a protomouse of the same type as $\mathcal{P}$.
3. $\tilde{\pi}$ is Σ_1-elementary, $\sup(\tilde{\pi}''\mathrm{OR}^{\mathcal{P}}) = \mathrm{OR}^{\mathcal{R}}$, and $\tilde{\pi}$ is continuous at λ whenever $\kappa < \lambda < \mathrm{OR}^{\mathcal{P}}$ and $\mathcal{P} \models \mathrm{cf}(\lambda) > \kappa$.

4. $\mathcal{R}$ is $\pi(\kappa)$-sound.
5. $\tilde{\pi}(p(\mathcal{P},\kappa)) = p(\mathcal{R},\pi(\kappa))$.

LEMMA 2.5.3. Suppose that $\mathcal{P}$ above is a weasel with $\kappa(\mathcal{P}) \leq \kappa$. Then

1. $\tilde{\pi} \upharpoonright (\kappa^+)^{\mathcal{P}} = \pi \upharpoonright (\kappa^+)^{\mathcal{P}}$.
2. $\mathcal{R}$ is a weasel.
3. $\tilde{\pi}$ is fully elementary and is continuous at λ whenever $\kappa < \lambda < \Omega$ and $\mathcal{P} \models \mathrm{cf}(\lambda) > \kappa$.
4. $\kappa(\mathcal{R}) \leq \pi(\kappa)$
5. $c(\mathcal{R}) = \tilde{\pi}(c(\mathcal{P}))$.

LEMMA 2.5.4. Suppose that $\mathcal{P}$ is a set premouse with $\dot{\tau}^{\mathcal{P}} \leq \kappa$ and that $\mathcal{P}$ is Dodd-solid above κ. Then $\dot{\tau}^{\mathcal{R}} \leq \pi(\kappa)$, $\mathcal{R}$ is Dodd-solid above $\pi(\kappa)$, and

$$\{\dot{s}_i^{\mathcal{R}} \mid \dot{s}_i^{\mathcal{R}} \geq \pi(\kappa)\} = \tilde{\pi}\left(\{\dot{s}_i^{\mathcal{P}} \mid \dot{s}_i^{\mathcal{P}} \geq \kappa\}\right) .$$

LEMMA 2.5.5. Suppose that $\mathcal{P}'$ is a protomouse such that $\mathcal{P}$ and $\mathcal{P}'$ agree below $(\kappa^+)^{\mathcal{P}}$, and $\mathcal{R}' = \mathrm{ult}(\mathcal{P}', E_\pi \upharpoonright \pi(\kappa))$ is well-founded. Then $\mathcal{R}'$ and $\mathcal{R}$ agree below

$$(\pi(\kappa)^+)^{\mathcal{R}} = \pi((\kappa^+)^{\mathcal{P}}) = \sup(\pi''(\kappa^+)^{\mathcal{P}})$$
$$= \{\pi(f)(a) \mid a \in [\pi(\kappa)]^{<\omega} \ \&\ f : [\kappa]^{|a|} \to (\kappa^+)^{\mathcal{P}} \ \&\ f \in |\mathcal{P}|\}.$$

§2.6 Lifting by a Dodd-Squashed Extender Fragment

The proof of the next lemma is implicit in the proof of weak square in Section 5 of [Sch1].

LEMMA 2.6.1. Let $\mathcal{R}$ be a protomouse, $\mu = \dot{\mu}^{\mathcal{R}}$, $F^* = \dot{F}^{\mathcal{R}}$, $\alpha = \mathrm{lh}(F) = \mathrm{OR}^{\mathcal{R}}$, $s = \dot{s}^{\mathcal{R}}$, and $\tau = \dot{\tau}^{\mathcal{R}}$. Let $\mathcal{M}$ be a protomouse such that F is a (μ, α)-extender over $\mathcal{M}$. Suppose that κ is a cardinal in $\mathcal{R}$ such that $\tau \leq \kappa$ and $\mathcal{R}$ is Dodd-solid above κ. Let $\mathcal{S} = \text{ult}(\mathcal{M}, F)$ and let $j : \mathcal{M} \longrightarrow \mathcal{S}$ be the ultrapower embedding. So $\mathcal{S} = \text{ult}(\mathcal{M}, F \restriction (\kappa \cup s))$, decap$(\mathcal{R})$ is a proper initial segment of $\mathcal{S}$, and $\text{OR}^{\mathcal{R}}$ is the cardinal successor of $\dot{\nu}^{\mathcal{R}}$ in $\mathcal{S}$.

Given the above, the following hold.

A. Suppose that $\mathcal{M}$ is μ-sound with $p(\mathcal{M}, \mu) = \langle p, u, w \rangle$. Then $\mathcal{S}$ is κ-sound with $p(\mathcal{S}, \kappa) = \langle j(p) \cup s, j(u), w' \rangle$ for the right w'.

B. Suppose that $\mathcal{M}$ is a weasel with $\kappa(\mathcal{M}) \leq \mu$. Then $\mathcal{S}$ is a weasel with $\kappa(\mathcal{S}) \leq \kappa$ and $c(\mathcal{S}) - \kappa = j(c(\mathcal{M})) \cup s$. In particular, $\mathcal{S}$ has the $c(\mathcal{S})$-hull property and the $c(\mathcal{S})$-definability property above κ.

§3 Proof of the Weak Covering Property

Assume the hypothesis of Theorem 1.1. So $\kappa < \Omega$ is a cardinal of K, card(κ) is countably closed, Ω is measurable, and there is no inner model with a Woodin cardinal. Construct K and K^c as subsets of V_Ω using a fixed measure on $\mathcal{P}(\Omega)$. Let $\lambda = (\kappa^+)^K$ and let Υ be any cardinal $> \lambda$. Let X be an elementary substructure of $V_{\Omega+1}$ (together with a predicate for the measure on Ω) such that:

$$^{\omega}(X \cap V_\Omega) \subseteq X$$

$$\text{card}(X) < \text{card}(\kappa)$$

$$\kappa, \lambda, \Upsilon, \Omega \in X \ .$$

Such X exist because card(κ) is countably closed. Note that if cf$(\lambda) <$ card(λ), then there would be such X with the additional property that $\lambda \cap X$ is cofinal in λ; in fact, this could be achieved with

$$\text{card}(X) = (\text{cf}(\lambda))^{\omega} \cdot \omega_1 < \text{card}(\kappa) \ .$$

But we prefer to argue directly, rather than by contradiction, so we do not assume, at least for now, that $\mathrm{cf}(\lambda) < \mathrm{card}(\lambda)$. We shall develop properties of X (which are of independent interest), and then somewhat later argue by contradiction that Theorem 1.1 holds (just after Corollary 3.12).

Let $\pi : N \longrightarrow V_{\Omega+1}$ be the inverse of the transitive collapse of X and let $\delta = \mathrm{crit}(\pi)$. Let W be the canonical very soundness witness for $\mathcal{J}^K_\Upsilon$. and let $\overline{W} = \pi^{-1}(W)$. Compare W and $\overline{W}$ as in Section 7 of [MiSt]. The result is an ordinal $\theta \leq \Omega$ and ω-maximal, padded iteration trees $\mathcal{T}$ on W and $\overline{\mathcal{T}}$ on $\overline{W}$, both of length $\theta + 1$. Note that $\overline{W}$ is Ω-iterable, as any putative iteration tree on $\overline{W}$ can be copied to a putative iteration tree on W via π, and, of course, W is Ω-iterable. For $\eta \leq \theta$, put $W_\eta = \mathcal{M}^{\mathcal{T}}_\eta$ and $\overline{W}_\eta = \mathcal{M}^{\overline{\mathcal{T}}}_\eta$. Because W is universal, $\overline{W}_\theta$ is an initial segment of W_θ and there is no dropping along the main branch of $\overline{\mathcal{T}}$, i.e.,

$$[0, \theta]_{\overline{\mathcal{T}}} \cap \mathcal{D}^{\overline{\mathcal{T}}} = \emptyset.$$

We shall see, in fact, that $\overline{W} = \overline{W}_\theta$.

Let $\overline{\kappa} = \pi^{-1}(\kappa)$, $\overline{\lambda} = \pi^{-1}(\lambda)$, $\overline{\Upsilon} = \pi^{-1}(\Upsilon)$, and $\overline{\Omega} = \pi^{-1}(\Omega)$. For $\eta \leq \theta$, put $E_\eta = E^{\mathcal{T}}_\eta$ and $\overline{E}_\eta = E^{\overline{\mathcal{T}}}_\eta$. Choose $\vec{\kappa}$ and γ so that

$$\vec{\kappa} = \langle \aleph^{\overline{W}_\theta}_\alpha \,|\, \alpha \leq \gamma \rangle$$

lists the cardinals of $\overline{W}_\theta$ that are $\leq \overline{\Upsilon}$ and let

$$\vec{\lambda} = \langle \kappa_{\alpha+1} \,|\, \alpha < \gamma \rangle \ .$$

We define premice $\mathcal{P}_\alpha$ and ordinals $\eta(\alpha)$ for $\alpha \leq \gamma$ as follows If there is an η such that $\kappa_\alpha < \nu^{\mathcal{T}}_\eta$, then let $\eta(\alpha)$ be the least such η; otherwise, let $\eta(\alpha) = \theta$. If there is an initial segment $\mathcal{P}$ of $W_{\eta(\alpha)}$ with $\mathrm{OR}^{\mathcal{P}} \geq \lambda_\alpha$ and $\rho^{\mathcal{P}}_\omega \leq \kappa_\alpha$, then let $\mathcal{P}_\alpha$ be the shortest such $\mathcal{P}$; otherwise, let $\mathcal{P}_\alpha = W_{\eta(\alpha)}$. Observe that $\mathcal{P}_\alpha$ is exactly the premouse to which an extender with critical point κ_α would be applied in an iteration tree extending $\mathcal{T}$. When $\lambda_\alpha < \delta$, $\mathcal{P}_\alpha$ is just W. Let $\vec{\mathcal{P}} = \langle \mathcal{P}_\alpha \,|\, \alpha \leq \gamma \rangle$. Then $(\vec{\mathcal{P}}, \vec{\lambda})$ is a very special phalanx of premice. Moreover, any iteration tree on $(\vec{\mathcal{P}}, \vec{\lambda})$ can be construed as an iteration tree extending $\mathcal{T}$, so $(\vec{\mathcal{P}}, \vec{\lambda})$ is iterable.

For $\alpha \leq \gamma$, let $\mathcal{R}_\alpha = \mathrm{ult}(\mathcal{P}_\alpha, E_\pi \restriction \pi(\kappa_\alpha))$, as long as this ultrapower is wellfounded. Also, let $\pi_\alpha : \mathcal{P}_\alpha \longrightarrow \mathcal{R}_\alpha$ be the ultrapower map and $m_\alpha = n(\mathcal{P}_\alpha, \kappa_\alpha)$. For $\alpha < \gamma$, put $\Lambda_\alpha = \sup(\pi_\alpha{}''\lambda_\alpha) = (\pi_\alpha(\kappa_\alpha)^+)^{\mathcal{R}_\alpha}$. So $\vec{\mathcal{R}} = \langle \mathcal{R}_\alpha \mid \alpha \leq \gamma \rangle$ and $\vec{\Lambda} = \langle \Lambda_\alpha \mid \alpha < \gamma \rangle$. Our first essential use of countable closure is the following lemma; the other will be Lemmas 3.3 and 3.13.

Lemma 3.1. For every $\alpha \leq \gamma$, the ultrapower defining $\mathcal{R}_\alpha$ is wellfounded. Moreover, $(\vec{\mathcal{R}}, \vec{\Lambda})$ is an iterable, very special phalanx.

Proof sketch. The well-foundedness and iterability follow from the countable completeness of E_π in the standard way. The argument is very similar to the one we shall give in the proof of Lemma 3.13. The rest follows from the lemmas in §2.5.

$\square$ Lemma 3.1

We give some motivation for what is to come. The basic idea behind earlier proofs that K satisfies the weak covering property lower down in the large cardinal hierarchy (i.e., those in [DoJe2], [Mi2], and [Je2]) is as follows. First, show that $\overline{W}$ does not move in its comparison with W; this would imply that $\overline{\kappa} = \kappa_{\gamma_0}$ for some $\gamma_0 < \gamma$, and that $(\kappa^+)^{\mathcal{R}_{\gamma_0}} = \lambda$ Second, show that $\mathcal{R}_{\gamma_0}$ is a set premouse; then $\mathcal{R}_{\gamma_0}$ would be κ-sound. Third, show that $\mathcal{R}_{\gamma_0}$ is iterable in a way that implies that $\mathcal{R}_{\gamma_0}$ is an element of K, which would immediately give a contradiction.

We shall be able to carry out a version of the first step. By induction on $\alpha \leq \gamma$, we shall show the following; we remark that this is the first of several induction hypotheses to come.

(1)$_\alpha$ If $\overline{E}_\eta \neq \emptyset$, then $\mathrm{lh}(\overline{E}_\eta) > \lambda_\alpha$.

However, there does not seem to be much hope of carrying out the second step, that is, in showing that $\mathcal{R}_{\gamma_0}$ is a set premouse. But if $\mathcal{R}_{\gamma_0}$ is a weasel, then clearly it is not an element of K. And if $\mathcal{R}_{\gamma_0}$ is merely a protomouse that is not a premouse, then it is not clear what sort of iterability of $\mathcal{R}_{\gamma_0}$ would imply that it is in K. So our next step will be to identify a phalanx of premice $(\vec{\mathcal{S}}, \vec{\Lambda})$ with properties very similar to those of $(\vec{\mathcal{R}}, \vec{\Lambda})$. Later, in order to establish some connection between $\mathcal{S}_{\gamma_0}$ and K, we shall show inductively that $\mathcal{S}_\alpha$ is iterable in a strong sense. In the end, we do not show that $\mathcal{S}_{\gamma_0}$ is a set premouse, but rather just deal with the possibility that it is a weasel.

We define $\mathcal{S}_\beta$ by induction on $\beta \leq \gamma$ as follows. If $\mathcal{R}_\beta$ is a premouse, then $\mathcal{S}_\beta = \mathcal{R}_\beta$. Suppose that $\mathcal{R}_\beta$ is not a premouse. Then, since $(\vec{\mathcal{R}}, \vec{\Lambda})$ is a special phalanx, there is an ordinal $\alpha < \beta$ such that $\pi(\kappa_\alpha) = \dot{\mu}^{\mathcal{R}_\beta} = \mathrm{crit}(\dot{F}^{\mathcal{R}_\beta})$. Let $\mathcal{S}_\beta = \mathrm{ult}(\mathcal{S}_\alpha, \dot{F}^{\mathcal{R}_\beta})$, so long as this ultrapower is wellfounded. Before showing that this ultrapower is always wellfounded, we define some related premice $\mathcal{Q}_\beta$ by induction on $\beta \leq \gamma$. If $\mathcal{S}_\beta = \mathcal{R}_\beta$, then $\mathcal{Q}_\beta = \mathcal{P}_\beta$. Suppose $\mathcal{S}_\beta \neq \mathcal{R}_\beta$. Then there is an ordinal $\alpha < \beta$ such that $\kappa_\alpha = \dot{\mu}^{\mathcal{P}_\beta}$. Let $\mathcal{Q}_\beta = \mathrm{ult}(\mathcal{Q}_\alpha, \dot{F}^{\mathcal{P}_\beta})$, so long as this ultrapower is wellfounded. The proof of the following fact is is similar to the proof of the main result in Section 9 of [St1]; we omit the details.

Fact 3.2. For every $\beta \leq \gamma$, the ultrapower defining $\mathcal{Q}_\beta$ is wellfounded.

Lemma 3.3. For every $\beta \leq \gamma$, the ultrapower defining $\mathcal{S}_\beta$ is wellfounded.

Proof sketch. The proof of Lemma 3.3 is essentially included in the proof of Lemma 3.13 to come. Very briefly, the idea is as follows. First show that $\mathcal{S}_\beta$ is the ultrapower of $\mathcal{Q}_\beta$ by $E_\pi \restriction \pi(\kappa_\beta)$. And then use the countable completeness of E_π in the usual way to see that $\mathcal{S}_\beta$ is wellfounded. $\square$ Lemma 3.3

We want to show that $(\vec{\mathcal{S}}, \vec{\Lambda})$ is a special phalanx of premice. But first we need some preliminary lemmas.

LEMMA 3.4. Suppose that $\mathcal{R}_\alpha$ is not a premouse. Then $\dot{\tau}^{\mathcal{P}_\alpha} \leq \kappa_\alpha$ and $\mathcal{P}_\alpha$ is Dodd-solid above κ_α.

PROOF. Since $\mathcal{R}_\alpha$ is not a premouse, the lemmas in §2.5 imply that $\mathcal{P}_\alpha$ is a κ_α-sound premouse that is active of type I or II, $n(\mathcal{P}_\alpha, \kappa_\alpha) = 0$, $\dot{\mu}^{\mathcal{P}_\alpha} < \kappa_\alpha$ and π is discontinuous at $(\dot{\mu}^+)^{\mathcal{P}_\alpha}$. If $\mathcal{P}_\alpha$ is type I, then $\kappa_\alpha = (\dot{\mu}^+)^{\mathcal{P}_\alpha} = \dot{\tau}^{\mathcal{P}_\alpha}$ and $\dot{s}^{\mathcal{P}_\alpha} = \emptyset$, so we are done. Suppose that $\mathcal{P}_\alpha$ is type II, but that the conclusion of Lemma 3.4 fails. Then, by Corollary 2.1.6, $(\dot{\mu}^+)^{\mathcal{P}_\alpha}$ has countable cofinality. But ${}^\omega(X \cap V_\Omega) \subset X$, so π is continuous at $(\dot{\mu}^+)^{\mathcal{P}_\alpha}$. Contradiction. $\square$ Lemma 3.4

We do not consider Lemma 3.4 and essential use of countable closure, since the proof only uses the continuity of π at ordinals of countable cofinality, which is easy to arrange. By combining Lemmas 3.4 and 2.5.4, we get the following corollary.

COROLLARY 3.5. If $\mathcal{R}_\alpha$ is not a premouse, then $\dot{\tau}^{\mathcal{R}_\alpha} \leq \pi(\kappa_\alpha)$ and $\mathcal{R}_\alpha$ is Dodd-solid above $\pi(\kappa_\alpha)$.

Recall that $m_\alpha = n(\mathcal{P}_\alpha, \kappa_\alpha) = n(\mathcal{R}, \pi(\kappa_\alpha))$. Define $n_\alpha = n(\mathcal{Q}_\alpha, \kappa_\alpha)$. Then also $n_\alpha = n(\mathcal{S}_\alpha, \pi(\kappa_\alpha))$. The next lemma follows by an easy induction on $\alpha \leq \gamma$, using Lemma 2.6.1. The analogous statement (really part of the proof) holds for the $\mathcal{P}_\alpha$'s, $\mathcal{Q}_\alpha$'s, and κ_α's.

LEMMA 3.6. Consider any $\alpha \leq \gamma$. There is a unique parameter $\{\alpha_0 > \alpha_1 > \cdots > \alpha_k\}$ with

1. $\alpha_0 = \alpha$,
2. $\pi(\kappa_{\alpha_i}) = \dot{\mu}^{\mathcal{R}_{\alpha_i}}$ whenever $0 \leq i \leq k$, and
3. $\mathcal{R}_{\alpha_k}$ is a premouse.

We have the sequence of ultrapower maps:

$$\mathcal{R}_{\alpha_k} = \mathcal{S}_{\alpha_k} \longrightarrow \mathcal{S}_{\alpha_{k-1}} \longrightarrow \cdots \longrightarrow \mathcal{S}_{\alpha_0} = \mathcal{S}_\alpha$$

which we call the **decomposition** of $\mathcal{S}_\alpha$. Let $\sigma_i : \mathcal{S}_{\alpha_i} \longrightarrow \mathcal{S}_\alpha$ whenever $0 \le i \le k$, with $\sigma_0 = \mathrm{id}$. Then the following hold.

(a) Suppose that $\mathcal{R}_{\alpha_k}$ is a weasel. Then $\mathcal{S}_\alpha$ is a weasel with $\kappa(\mathcal{S}_\alpha) \le \pi(\kappa_\alpha)$ and

$$c(\mathcal{S}_\alpha) - \pi(\kappa_\alpha) = \bigcup_{0<i\le k} \sigma_i(\dot{s}^{\mathcal{R}_{\alpha_i}}) \ .$$

In particular, $n_\alpha = m_{\alpha_k} = \omega$.

(b) Suppose that $\mathcal{R}_{\alpha_k}$ is a set premouse with

$$p(\mathcal{R}_{\alpha_k}, \pi(\kappa_{\alpha_k})) = \langle p, u, w \rangle \ .$$

Then $\mathcal{S}_\alpha$ is a $\pi(\kappa_\alpha)$-sound set premouse with $n_\alpha = m_{\alpha_k}$, and for the right u' and w',

$$p(\mathcal{S}_\alpha, \pi(\kappa_\alpha)) = \langle p', u', w' \rangle$$

with

$$p' = \sigma_k(p) \cup \bigcup_{0<i\le k} \sigma_i(\dot{s}^{\mathcal{R}_{\alpha_i}}) \ .$$

Definition 3.6.1. If $\mathcal{M}$ is a protomouse, then $\mathbf{decap}(\mathcal{M}) = \langle\, |\mathcal{M}|\, , \in, \dot{E}^{\mathcal{M}} \,\rangle$. If $\xi \le \mathrm{OR}^{\mathcal{M}}$, then $\mathcal{M}\|\xi = \mathrm{decap}(\mathcal{J}^{\mathcal{M}}_\xi)$.

An easy induction using, among other things, the coherence condition on $\dot{F}^{\mathcal{R}_\alpha}$ gives the following lemma. Again, the analogous statement is true of the $\mathcal{P}_\alpha$'s, $\mathcal{Q}_\alpha$'s and κ_α's.

Lemma 3.7. The notation here is as in Lemma 3.6. Suppose that $\mathcal{R}_\alpha \ne \mathcal{S}_\alpha$. Then:

- $n_\alpha = n_{\alpha_0} = \cdots = n_{\alpha_k} = m_{\alpha_k} \le \omega$
- $m_\alpha = m_{\alpha_0} = \cdots = m_{\alpha_{k-1}} = 0$

- $\mathrm{OR}^{\mathcal{R}_\alpha}$ is a successor cardinal in $\mathcal{S}_\alpha$ and $\mathrm{decap}(\mathcal{R}_\alpha)$ is a proper initial segment of $\mathcal{S}_\alpha$.
- For $i = 0, \ldots, k-1$,

$$\mathrm{OR}^{\mathcal{S}_\alpha} = \sup(\sigma_k{}''\mathrm{OR}^{\mathcal{R}_{\alpha_k}}) \geq \sigma_k(\rho_{n_{\alpha_k}}(\mathcal{R}_{\alpha_k})) = \rho_{n_\alpha}(\mathcal{S}_\alpha) > \sigma_i(\mathrm{OR}^{\mathcal{R}_{\alpha_i}}) \geq \mathrm{OR}^{\mathcal{R}_\alpha}$$

COROLLARY 3.8. Both $(\vec{\mathcal{S}}, \vec{\Lambda})$ and $(\vec{\mathcal{Q}}, \vec{\lambda})$ are special phalanxes of premice.

We remark that $(\vec{\mathcal{S}}, \vec{\Lambda})$ is an iterable phalanx; this will follow from the proof of Theorem 1.1, but will not be isolated, nor used. Next, we list induction hypotheses $(2)_\alpha$ through $(6)_\alpha$. The reader should note the similarity between $(2)_\alpha$ and the definition of $\pi(\kappa_\alpha)$-strong given in Section 6 of [St1].

(2)$_\alpha$ $((W, \mathcal{S}_\alpha), \pi(\kappa_\alpha))$ is an iterable phalanx of premice.

(3)$_\alpha$ $((\overline{W}, \mathcal{Q}_\alpha), \kappa_\alpha)$ is an iterable phalanx of premice.

(4)$_\alpha$ $((\vec{\mathcal{P}} \restriction \alpha, \overline{W}), \vec{\lambda} \restriction \alpha)$ is an iterable, very special phalanx of premice.

(5)$_\alpha$ $((\vec{\mathcal{R}} \restriction \alpha, W), \vec{\Lambda} \restriction \alpha)$ is a very special phalanx, that is iterable with respect to special iteration trees.

(6)$_\alpha$ $(\vec{\mathcal{S}} \restriction \alpha, W), \vec{\Lambda} \restriction \alpha)$ is a very special phalanx, that is iterable with respect to special iteration trees.

Assuming that $(1)_\beta$ holds for every $\beta < \alpha$, we shall show:

$$(3)_\alpha \Longrightarrow (2)_\alpha$$

$$(6)_\alpha \Longrightarrow (5)_\alpha \Longrightarrow (4)_\alpha \Longrightarrow (1)_\alpha$$

$$\forall \beta < \alpha\ (4)_\beta \Longrightarrow (3)_\alpha$$

$$\forall \beta < \alpha\ (2)_\beta \Longrightarrow (6)_\alpha$$

Thus, by induction, $(1)_\alpha$ through $(6)_\alpha$ hold for all $\alpha < \gamma$. What this buys us is explained by Corollary 3.12 below.

LEMMA 3.9. Let $\alpha \leq \gamma$ be given. Suppose that $(2)_\alpha$ holds. Then there is an iteration tree $\mathcal{U}$ on W with the following properties:

(a) $\mathcal{U}$ has successor length, $\varphi + 1$.
(b) $\mathrm{lh}(E^{\mathcal{U}}_\eta) \geq \pi(\kappa_\alpha)$ for every $\eta \leq \varphi$.
(c) There is an initial segment $\mathcal{N}$ of $\mathcal{M}^{\mathcal{U}}_\varphi$ and an n_α-embedding $j : \mathcal{S}_\alpha \longrightarrow \mathcal{N}$ such that $\mathrm{crit}(j) \geq \pi(\kappa_\alpha)$.

LEMMA 3.10. Let $\alpha \leq \gamma$ be given. Suppose that $(1)_\alpha$ and $(2)_\alpha$ hold and that $\mathcal{S}_\alpha$ is a set premouse. If $E^W_{\Lambda_\alpha} = \emptyset$, then $\mathcal{S}_\alpha$ is a proper initial segment of W. Otherwise, $\mathcal{S}_\alpha = \mathrm{ult}(\mathcal{N}, E^W_{\Lambda_\alpha})$ for the longest initial segment $\mathcal{N}$ of W over which $E^W_{\Lambda_\alpha}$ is an extender.

LEMMA 3.11. Let $\alpha \leq \gamma$ be given. Suppose that $(1)_\alpha$ and $(2)_\alpha$ hold and that $\mathcal{S}_\alpha$ is a weasel different from W. Let $\mathcal{U}$, φ, j, and $\mathcal{N}$ be as in Lemma 3.9. Then

(a) j is an elementary embedding from $\mathcal{S}_\alpha$ into $\mathcal{N} = \mathcal{M}^{\mathcal{U}}_\varphi$.
(b) $\mathrm{lh}(E^{\mathcal{U}}_\eta) \geq \Lambda_\alpha = (\pi(\kappa_\alpha)^+)^{\mathcal{S}_\alpha}$ whenever $\eta \leq \varphi$.
(c) $1 \leq_U \varphi$
(d) The Dodd-projectum of $E^{\mathcal{U}}_0$, $\tau(E^{\mathcal{U}}_0)$, is $\leq \pi(\kappa_\alpha)$;
in particular, $\Lambda_\alpha < (\pi(\kappa_\alpha)^+)^W$.

COROLLARY 3.12. Suppose that $\alpha \leq \gamma$, $\delta \leq \kappa_\alpha$, and both $(1)_\alpha$ and $(2)_\alpha$ hold. Then $\Lambda_\alpha < (\pi(\kappa_\alpha)^+)^W$. Also, there is an iteration tree $\mathcal{U}$ on W satisfying 3.9(a) and 3.9(b) and an elementary embedding $j : \mathcal{S}_\alpha \longrightarrow \mathcal{M}^{\mathcal{U}}_\varphi$ with $\mathrm{crit}(j) \geq \pi(\kappa_\alpha)$ (if $\mathcal{S}^\alpha$ is a set, then j is the identity).

Theorem 1.1 will follow from Corollary 3.12 once we show that $(1)_\alpha$ – $(6)_\alpha$ hold for all $\alpha < \gamma$. Here is the argument. Suppose that $(1)_\alpha$ – $(6)_\alpha$ hold for all $\alpha < \gamma$. Assume, for contradiction, that $\mathrm{cf}(\lambda) < \mathrm{card}(\kappa)$. Then we may assume without loss of generality that π is continuous at $\overline{\lambda}$. Since $(1)_\alpha$ holds for all $\alpha < \gamma$, there is some $\alpha < \gamma$ such that $\overline{\kappa} = \kappa_\alpha$ and $\overline{\lambda} = \lambda_\alpha$. By the lemmas in §2.5, $\pi \restriction \overline{\lambda} = \pi_\alpha \restriction \lambda_\alpha$. Hence,

$$\lambda = \pi(\overline{\lambda}) = \sup(\pi\, ''\overline{\lambda}) = \sup(\pi_\alpha\, ''\lambda_\alpha) = \Lambda_\alpha\ .$$

Therefore $\lambda < (\kappa^+)^W = (\kappa^+)^K$, by Corollary 3.12. But this is a contradiction, since $\lambda = (\kappa^+)^K$ by definition.

Just to be explicit, through the remainder of the proof, we are not assuming that $(1)_\alpha$ – $(6)_\alpha$ hold, but rather proving them by induction on $\alpha < \gamma$.

Here are some further corollaries to Lemmas 3.9 and 3.10. If $(1)_\alpha$ and $(2)_\alpha$ hold, $\mathcal{S}_\alpha$ is a set premouse, and $\pi(\kappa_\alpha)$ is a cardinal in W, then $\mathcal{S}_\alpha$ is a proper initial segment of W. So if $(2)_\alpha$ holds and $(1)_\beta$ holds for some $\beta \geq \alpha$ such that κ_β is a cardinal in $\overline{W}$ (rather than just in $\overline{W}_\theta$), and $\mathcal{S}_\alpha$ is a set premouse, then $\mathcal{S}_\alpha$ is a proper initial segment of W.

Proof of 3.9. Compare W versus $((W, \mathcal{S}_\alpha), \pi(\kappa))$. This can be done since we are assuming $(2)_\alpha$. This results in iteration trees $\mathcal{U}$ on W and $\mathcal{V}$ on $((W, \mathcal{S}_\alpha), \pi(\kappa))$. For some φ, $\mathrm{lh}(\mathcal{U}) = \varphi + 1$ and $\mathrm{lh}(\mathcal{V}) = 1 + \varphi + 1$. The comparison ends with either $\mathcal{M}^{\mathcal{U}}_\varphi$ an initial segment of $\mathcal{M}^{\mathcal{V}}_{1+\varphi}$ or vice-versa. Note that all extenders used on either iteration tree have length at least $\pi(\kappa_\alpha)$ since by $(2)_\alpha$, $\mathcal{S}_\alpha$ and W agree below $\pi(\kappa_\alpha)$. The usual arguments using the universality, hull, and definability properties of W (as in Section 3 of [St1]) show that $1 = \mathrm{root}^{\,V}(1+\varphi)$. That is, the last model on $\mathcal{V}$ lies above $\mathcal{S}_\alpha$ and not above W. Similarly, we see that $\mathcal{M}^{\mathcal{V}}_{1+\varphi}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_\varphi$, that $[1, 1+\varphi]_V \cap \mathcal{D}^{\mathcal{V}} = \emptyset$, and that $\deg^{\mathcal{V}}(1+\varphi) = n_\alpha$. Let $j = i^{\mathcal{V}}_{1,1+\varphi}$ and $\mathcal{N} = \mathcal{M}^{\mathcal{V}}_{1+\varphi}$. □ Lemma 3.9

Proof of 3.10. We pick up where we left off in the proof of Lemma 3.9, only now we assume that $(1)_\alpha$ holds. Then $\mathcal{S}_\alpha$ and W agree below Λ_α, so all extenders used on either of $\mathcal{U}$ and $\mathcal{V}$ have length at least Λ_α. We also assume that $\mathcal{S}_\alpha$ is a set premouse.

Claim 1. $\mathcal{S}_\alpha$ does not move, that is, $\mathcal{N} = \mathcal{S}_\alpha$.

Proof. Suppose otherwise. Then $\mathcal{N}$ is not $\pi(\kappa_\alpha)$-sound, so $[0,\varphi]_U \cap \mathcal{D}^{\mathcal{U}} \neq \emptyset$ and $\mathcal{M}^{\mathcal{U}}_\varphi = \mathcal{N}$. Let $\eta+1$ be the last drop (of any kind) along $[0,\varphi]_U$. By this we mean that $\eta+1$ is largest in $[0,\varphi]_U$ such that either $\eta+1 \in \mathcal{D}^{\mathcal{U}}$, or $\deg^{\mathcal{U}}(\eta+1) < \deg^{\mathcal{U}}(\operatorname{pred}^{\mathcal{U}}(\eta+1))$, where we take $\deg^{\mathcal{U}}(0) = \deg^{\mathcal{U}}(1) = \omega$. Notice that, by construction, $\xi+1$ is a drop whenever $1 = \operatorname{pred}^{\mathcal{U}}(\xi+1)$. Let $\eta^* = \operatorname{pred}^{\mathcal{U}}(\eta+1)$.

If $\operatorname{crit}(E^{\mathcal{U}}_\eta) \geq \pi(\kappa_\alpha)$, then both j and $i^{\mathcal{U}}_{\eta+1,\varphi} \circ (i^*_{\eta+1})^{\mathcal{U}}$ are the inverse of the transitive collapsing map for

$$H^{\mathcal{N}}_{n_\alpha+1}(\pi(\kappa_\alpha) \;\cup\; p(\mathcal{N},\pi(\kappa_\alpha))) \;.$$

This gives the usual contradiction as in the proof of the Comparison Lemma 7.1 of [MiSt].

So $\operatorname{crit}(E^{\mathcal{U}}_\eta) < \pi(\kappa_\alpha)$. Therefore $\eta^* = 0$ and

$$n_\alpha = n(\mathcal{N},\pi(\kappa_\alpha)) = n(\mathcal{M}^{\mathcal{U}}_{\eta+1},\pi(\kappa_\alpha)) = n((\mathcal{M}^*_{\eta+1})^{\mathcal{U}},\operatorname{crit}(E^{\mathcal{U}}_\eta)),$$

where $(\mathcal{M}^*_{\eta+1})^{\mathcal{U}}$ is the proper initial segment of W to which $E^{\mathcal{U}}_\eta$ is applied. Since $\mathcal{S}_\alpha$ and $\mathcal{N}$ agree beyond $\Lambda_\alpha = (\pi(\kappa_\alpha)^+)^{\mathcal{S}_\alpha}$ and

$$\mathcal{S}_\alpha = \mathcal{H}^{\mathcal{N}}_{n_\alpha+1}\left(\pi\left(\kappa_\alpha\right) \cup p\left(\mathcal{N},\pi\left(\kappa_\alpha\right)\right)\right) \;,$$

we have that $\mathcal{P}(\pi(\kappa_\alpha)) \cap |\mathcal{N}| \subseteq$ the transitive collapse of

$$H^{\mathcal{N}}_{n_\alpha+1}\left(\pi\left(\kappa_\alpha\right) \cup p\left(\mathcal{N},\pi\left(\kappa_\alpha\right)\right)\right) \;.$$

But this is only possible if there are no generators of $E^{\mathcal{U}}_\eta \geq \pi(\kappa_\alpha)$. So $\nu(E^{\mathcal{U}}_\eta) = \pi(\kappa_\alpha)$, which in turn is only possible if $\eta = 0$ and $E^{\mathcal{U}}_\eta = E^W_{\Lambda_\alpha}$. But then $\mathcal{M}^{\mathcal{U}}_{\eta+1}$ is $\pi(\kappa_\alpha)$-sound, so both $i^{\mathcal{U}}_{\eta+1,\varphi}$ and j are the inverse of the transitive collapsing map for

$$H^{\mathcal{N}}_{n_\alpha+1}\left(\pi\left(\kappa_\alpha\right) \;\cup\; p\left(\mathcal{N},\pi\left(\kappa_\alpha\right)\right)\right) \;.$$

This gives a contradiction as in the proof of the Comparison Lemma 7.1 of [MiSt].

$\square$ Claim 1

Thus $\mathcal{S}_\alpha$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_\varphi$.

CLAIM 2. If $\mathcal{S}_\alpha$ is a proper initial segment of $\mathcal{M}^{\mathcal{U}}_\varphi$, then $\mathcal{M}^{\mathcal{U}}_\varphi = W$.

PROOF. Suppose that $\mathcal{S}_\alpha$ is a proper initial segment of $\mathcal{M}^{\mathcal{U}}_\varphi$. Since $\mathcal{S}_\alpha$ is $\pi(\kappa_\alpha)$-sound, $\mathcal{S}_\alpha = \mathcal{J}_\varepsilon^{\mathcal{M}^{\mathcal{U}}_\varphi}$ for some $\varepsilon < (\pi(\kappa_\alpha)^+)^{\mathcal{M}^{\mathcal{U}}_\varphi}$. But $(\pi(\kappa_\alpha)^+)^{\mathcal{M}^{\mathcal{U}}_\varphi} \leq \mathrm{lh}(E^{\mathcal{U}}_\eta)$ for the least η such that $(\eta+1) \leq_U \varphi$. So $\mathcal{S}_\alpha$ is an initial segment of W. $\square$ Claim 2

So we may assume that $\mathcal{S}_\alpha = \mathcal{M}^{\mathcal{U}}_\varphi$, and therefore, $[0,\varphi]_U \cap \mathcal{D}^{\mathcal{U}} \neq \emptyset$. Since $\mathcal{S}_\alpha$ is $\pi(\kappa_\alpha)$-sound, this is only possible if $\mathcal{S}_\alpha = \mathrm{ult}((\mathcal{M}^*_{\eta+1})^{\mathcal{U}}, E^{\mathcal{U}}_\eta)$, for the least η such that $(\eta+1) \leq_U \varphi$. Reasons as before show that $E^{\mathcal{U}}_\eta$ cannot have generators $\geq \pi(\kappa_\alpha)$. Therefore $\eta = 0$ and $\mathrm{lh}(E^{\mathcal{U}}_\eta) = \Lambda_\alpha$. $\square$ Lemma 3.10

PROOF OF 3.11. Again, we continue the comparison begun in the proof of Lemma 3.9. Since we are assuming $(1)_\alpha$, $\mathcal{S}_\alpha$ and W agree below Λ_α, so all extenders used on either of $\mathcal{U}$ and $\mathcal{V}$ have length at least Λ_α. We are also assuming also that $\mathcal{S}_\alpha$ is a weasel different from W. So $\mathcal{S}_\alpha$ is a universal weasel with $\kappa(\mathcal{S}) \leq \pi(\kappa_\alpha)$. By the usual arguments using the universality, hull, and definability properties of W and $\mathcal{S}_\alpha$, $\mathcal{M}^{\mathcal{U}}_\varphi = \mathcal{N}$, $[0,\varphi]_U \cap \mathcal{D}^{\mathcal{U}} = \emptyset$, and $[1, 1+\varphi]_V \cap \mathcal{D}^{\mathcal{V}} = \emptyset$, and $j : \mathcal{S}_\alpha \longrightarrow \mathcal{N}$ is elementary. Note that $\mathcal{N}$ has the $j(c(\mathcal{S}_\alpha))$-hull property at $\pi(\kappa_\alpha)$. Let η be least such that $(\eta+1) \leq_U \varphi$.

CLAIM 1. There is a parameter t such that $\mathcal{M}^{\mathcal{U}}_{\eta+1}$ has the t-hull property at $\pi(\kappa_\alpha)$.

PROOF. Suppose that for every t, the t-hull property fails at $\pi(\kappa_\alpha)$ in $\mathcal{M}^{\mathcal{U}}_{\eta+1}$.

SUBCLAIM. For every $\xi_0 \in [\eta+1, \varphi]_U$, the t-hull property fails at $\pi(\kappa_\alpha)$ in $\mathcal{M}^{\mathcal{U}}_{\xi_0}$.

Proof. By induction on ξ_0. Let ξ_0 be the least counterexample, and let t be a witness. Clearly $\xi_0 = \xi + 1$ for some $\xi > \gamma$. Let $\xi^* = \text{pred}^{\mathcal{U}}(\xi + 1)$. Say $t = i^{\mathcal{U}}_{\xi^*,\xi+1}(f)(a)$ with $a \in [\nu(E^{\mathcal{U}}_\xi)]^{<\omega}$, $f : [\text{crit}(E^{\mathcal{U}}_\xi)]^{|a|} \longrightarrow |\mathcal{M}^{\mathcal{U}}_{\xi^*}|$, and $f \in |\mathcal{M}^{\mathcal{U}}_{\xi^*}|$. By Section 4 of [MiSt], $E^{\mathcal{U}}_\xi$ is close to $\mathcal{M}^{\mathcal{U}}_{\xi^*}$. In particular, $(E^{\mathcal{U}}_\xi)_a$ is Σ_1 over $\mathcal{M}^{\mathcal{U}}_{\xi^*}$. Since $\mathcal{M}^{\mathcal{U}}_{\xi^*}$ is a weasel, $(E^{\mathcal{U}}_\xi)_a \in |\mathcal{M}^{\mathcal{U}}_{\xi^*}|$. By the induction hypothesis, there is an $A \subseteq \pi(\kappa_\alpha)$ such that $A \notin$ the transitive collapse of

$$H^{\mathcal{M}^{\mathcal{U}}_{\xi^*}}_\omega \left(\pi(\kappa_\alpha) \cup \{f, (E^{\mathcal{U}}_\xi)_a\} \cup \Gamma\right) \ ,$$

where $\Gamma =$ the fixed points of $i^{\mathcal{U}}_{\xi^*,\xi+1}$. On the other hand, there is a term τ_ψ and $c \in [\Gamma]^{<\omega}$ such that

$$A = \tau_\psi^{\mathcal{M}^{\mathcal{U}}_{\xi+1}}[t, c] \cap \pi(\kappa_\alpha).$$

By Łoś' Theorem, for $\zeta < \pi(\kappa_\alpha)$,

$$\zeta \in A \iff \zeta \in \tau_\psi^{\mathcal{M}^{\mathcal{U}}_{\xi^*}}[f(u), c] \text{ for } (E^{\mathcal{U}}_\xi)_a \text{ a.e. } u\ .$$

But then A is not as it was chosen to be, a witness that the $\{f, (E^{\mathcal{U}}_\xi)_a\}$-hull property fails at $\pi(\kappa_\alpha)$ in $\mathcal{M}^{\mathcal{U}}_{\xi^*}$. □ Subclaim

By the Subclaim, the t-hull property fails at $\pi(\kappa_\alpha)$ in $\mathcal{N} = \mathcal{M}^{\mathcal{U}}_\varphi$ for every t. But this is a contradiction with $t = j(c(\mathcal{S}_\alpha))$. □ Claim 1

Claim 2. $\text{crit}(E^{\mathcal{U}}_\eta) < \pi(\kappa_\alpha)$.

Proof. We have two "iterations" from W into $\mathcal{N}$. The first is $i^{\mathcal{U}}_{0,\varphi}$. The second is

$$W \xrightarrow{i^{\mathcal{T}}_{0,\eta(\alpha_k)}} W_{\eta(\alpha_k)} = \mathcal{P}_{\alpha_k} \xrightarrow{\pi_{\alpha_k}} \mathcal{R}_{\alpha_k} = \mathcal{S}_{\alpha_k} \xrightarrow{\sigma_k} \mathcal{S}_{\alpha_0} = \mathcal{S}_\alpha \xrightarrow{j} \mathcal{M}^{\mathcal{V}}_{1+\varphi} = \mathcal{N}$$

where α_k and σ_k are as in Lemma 3.6 (note that all the models above are weasels). Since W has the definability property at all ordinals $< \Upsilon$, the critical points of these two maps are the same. But the critical point of the latter is $\leq \pi(\kappa_\alpha)$ since we are assuming that $\mathcal{S}_\alpha \neq W$, and $\text{crit}(i^{\mathcal{U}}_{0,\varphi}) = \text{crit}(E^{\mathcal{U}}_\eta)$. □ Claim 2

Pick t as in Claim 1. Say $t = i^{\mathcal{U}}_{0,\eta+1}(f)(a)$ with $f \in W$, $a \in [\nu(E^{\mathcal{U}}_\eta)]^{<\omega}$, and $f : [\mathrm{crit}(E^{\mathcal{U}}_\eta)]^{|a|} \longrightarrow W$.

Claim 3. $\eta = 0$.

Proof. Suppose that $\eta > 0$. Then Λ_α is a cardinal in $\mathcal{M}^{\mathcal{U}}_\eta$ and $\mathrm{lh}(E^{\mathcal{U}}_\eta) > \Lambda_\alpha$. By strong acceptability, for every $\zeta < \Lambda_\alpha$,

$$E^{\mathcal{U}}_\gamma \restriction (\zeta \cup a) \in J^{\mathcal{M}^{\mathcal{U}}_\eta}_{\Lambda_\alpha} \ .$$

Also, the a-generators of $E^{\mathcal{U}}_\eta$ are unbounded in Λ_α. Let $G = E^{\mathcal{U}}_\eta \restriction (\pi(\kappa_\alpha) \cup a)$ and let A be a subset of of $\pi(\kappa_\alpha)$ which codes G. Then

$$A \notin |\mathrm{ult}(W, G)|$$

and

$$A \in |\mathcal{M}^{\mathcal{U}}_{\eta+1}| \ .$$

Let k be the map from $\mathrm{ult}(W, G)$ to $\mathcal{M}^{\mathcal{U}}_{\eta+1}$. By the choice of t, there is a term τ_ψ and parameters $b \in [\pi(\kappa_\alpha)]^{<\omega}$ and $c \in$ [fixed points of $i^{\mathcal{U}}_{0,\eta+1}]^{<\omega}$, such that for $\zeta < \pi(\kappa_\alpha)$,

$$\zeta \in A \iff \tau_\psi^{\mathcal{M}^{\mathcal{U}}_{\eta+1}}[b, c, t] \iff \tau_\psi^{ult(W,G)}[b, c, k^{-1}(t)] \ .$$

But then $A \in \mathrm{ult}(W, G)$, a contradiction. □ Claim 3

It remains to see that $\tau(E^{\mathcal{U}}_0)$, the Dodd-projectum of $E^{\mathcal{U}}_0$, is $\leq \pi(\kappa_\alpha)$. It is enough to see that there are no a-generators of $E^{\mathcal{U}}_0 \geq \pi(\kappa_\alpha)$.

Suppose, to the contrary, that there is an a-generator of $E^{\mathcal{U}}_0 \geq \pi(\kappa_\alpha)$. Let $G = E^{\mathcal{U}}_0 \restriction (\pi(\kappa_\alpha) \cup a)$. Then, from Theorem 2.1.1 it follows that $G \in J^W_{lh(E^{\mathcal{U}}_0)}$. By strong acceptability, $G \in J^W_{\Lambda_\alpha}$. But now an argument as in the proof of Claim 3 gives a contradiction. □ Lemma 3.11

The main use of countable closure comes in the proof of the following lemma.

LEMMA 3.13. Suppose that $\alpha \leq \gamma$ and $(3)_\alpha$ holds. Then $(2)_\alpha$ holds.

PROOF. Suppose $(3)_\alpha$ holds and that $\mathcal{U}$ is a simple iteration tree of limit length on $((W, \mathcal{S}_\alpha), \pi(\kappa_\alpha))$. We shall show that $\mathcal{U}$ is well-behaved.

For contradiction, suppose that $\mathcal{U}$ is ill-behaved. Take a countable elementary submodel of $V_{\Omega+1}$ that has as elements $\mathcal{U}$, W, $\mathcal{S}_\alpha$, κ_α, and $\pi(\kappa_\alpha)$. Let $\psi : M \longrightarrow V_{\Omega+1}$ be the inverse of the transitive collapse of this submodel; say $\psi(\mathcal{U}') = \mathcal{U}$, $\psi(W') = W$, $\psi(\mathcal{S}') = \mathcal{S}_\alpha$, and $\psi(\kappa') = \pi(\kappa_\alpha)$. By the elementarity of ψ and absoluteness as in Section 2 of [St1], $\mathcal{U}'$ is a simple, countable, ill-behaved iteration tree on the countable phalanx $((W', \mathcal{S}'), \kappa')$.

Recall that $\pi : N \longrightarrow V_{\Omega+1}$. The following fact, we probably should have established earlier.

CLAIM. $\mathcal{S}_\alpha = \mathrm{ult}(\mathcal{Q}_\alpha, E_\pi \restriction \pi(\kappa_\alpha))$.

PROOF. The iteration

$$\mathcal{R}_{\alpha_k} = \mathcal{S}_{\alpha_k} \longrightarrow \cdots \longrightarrow \mathcal{S}_{\alpha_0} = \mathcal{S}_\alpha$$

is the result of copying the iteration

$$\mathcal{P}_{\alpha_k} = \mathcal{Q}_{\alpha_k} \longrightarrow \cdots \longrightarrow \mathcal{Q}_{\alpha_0} = \mathcal{Q}_\alpha$$

using the maps $\pi_{\alpha_k}, \ldots, \pi_{\alpha_0} = \pi_\alpha$. For $i = k, \ldots 0$, let $\chi_i : \mathcal{Q}_{\alpha_i} \longrightarrow \mathcal{S}_{\alpha_i}$ be the corresponding copying map. So $\chi_k = \pi_{\alpha_k}$ and by induction using the Shift Lemma, $\chi_i \restriction |\mathcal{P}_{\alpha_i}| = \pi_{\alpha_i}$ for $i = k, \ldots 0$. Set $\chi = \chi_0$; then χ is a map from $\mathcal{Q}_\alpha$ into $\mathcal{S}_\alpha$ and $\chi \restriction |\mathcal{P}_\alpha| = \pi_\alpha$. Hence

$$E_\chi = E_\pi \cap \left([\pi(\kappa_\alpha)]^{<\omega} \times |\mathcal{P}_\alpha| \right) .$$

So there is a natural n_α-embedding

$$h : \mathrm{ult}(\mathcal{Q}_\alpha, E_\pi \restriction \pi(\kappa_\alpha)) \longrightarrow \mathcal{S}_\alpha$$

such that $\mathrm{crit}(h) > \pi(\kappa_\alpha)$. If $\mathcal{S}_\alpha$ is a set premouse, then since $\mathcal{S}_\alpha$ is $\pi(\kappa_\alpha)$-sound, equality must hold. If $\mathcal{S}_\alpha$ is a weasel, then since $\kappa(\mathcal{S}_\alpha) \leq \pi(\kappa_\alpha)$ and $c(\mathcal{S}_\alpha) \in \mathrm{ran}(h)$, and the fixed points of h are thick, again we may conclude that h is the identity.

$\square$ Claim

Let $\{x_0, \ldots, x_m, \ldots\}$ and $\{y_0, \ldots, y_n, \ldots\}$ be enumerations of $|W'|$ and $|\mathcal{S}'|$ respectively. For each $m < \omega$, choose $a_m \in [\Omega]^{<\omega}$ and $f_m \in N$ such that $\psi(x_m) = \pi(f_m)(a_m)$. For each $n < \omega$, choose $b_n \in [\pi(\kappa_\alpha)]^{<\omega}$ and a function g_n such that $\mathrm{dom}(g_n) = [\kappa_\alpha]^{|b_n|}$ and $\psi(y_n) = [b_n, g_n]^{\mathcal{Q}_\alpha}_{E_\pi \restriction \pi(\kappa_\alpha)}$. Depending on whether $n_\alpha = n(\mathcal{Q}_\alpha, \kappa_\alpha) = 0$ or not, g_n is either an element of $\mathcal{Q}_\alpha$ or is given by a Σ_{n_α}-Skolem term and parameter in $\mathcal{Q}_\alpha$. If $x_m = y_n < ((\kappa')^+)^{\mathcal{S}'}$, then we can and do select $f_m = g_m \in |\mathcal{Q}_\alpha|$ and $a_m = b_n$. For any $m, n < \omega$, put $c_m = a_0 \cup \cdots \cup a_m$ and $d_n = b_0 \cup \cdots \cup b_n$.

Recall the definition, given at the beginning of §2.5, of the critical point δ_a corresponding to some $a \in [\Omega]^{<\omega}$. Given a first-order formula $\sigma(z_0, \ldots, z_m)$, let

$$A_{\sigma,m} = \{\ u \in [\delta_{c_m}]^{|c_m|} \mid \overline{W} \models \sigma(f_0^{a_0,c_m}(u), \ldots, f_m^{a_m,c_m}(u))\ \}\ .$$

Similarly, for any Σ_{n_α}-formula $\tau(z_0, \ldots, z_n)$, let

$$B_{\tau,n} = \{\ u \in [\delta_{d_n}]^{|d_n|} \mid \mathcal{Q}_\alpha \models \tau(g_0^{b_0,d_n}(u), \ldots, g_n^{b_n,d_n}(u))\ \}\ .$$

Since ${}^\omega(X \cap V_\Omega) \subseteq X \overset{\pi}{\prec} V_{\Omega+1}$, there are order preserving maps $s : \bigcup_{m<\omega} a_m \longrightarrow \overline{\Omega}$ and $t : \bigcup_{n<\omega} b_n \longrightarrow \kappa_\alpha$ such that

$$A_{\sigma,m} \in (E_\pi)_{c_m} \implies s''c_m \in A_{\sigma,m}\ ,$$

$$B_{\tau,n} \in (E_\pi)_{d_n} \implies t''d_n \in B_{\tau,n}\ ,$$

and

$$x_m = y_n < ((\kappa')^+))^{\mathcal{S}'} \implies s''a_m = t''b_n\ .$$

Define $\varphi_0 : W' \longrightarrow \overline{W}$ and $\varphi_1 : \mathcal{S}' \longrightarrow \mathcal{Q}_\alpha$ by

$$\varphi_0(x_m) = f_m(s''a_m)$$

and

$$\varphi_1(y_n) = g_n(t\,''\,b_n)$$

Then

$$\varphi_0 \restriction ((\kappa')^+)^{\mathcal{S}'} = \varphi_1 \restriction ((\kappa')^+)^{\mathcal{S}'} \ .$$

Using the Loś Theorem, we see that φ_0 is an elementary embedding of W' into $\overline{W}$, and φ_1 is a weak n_α-embedding of $\mathcal{S}'$ into $\mathcal{Q}_\alpha$. Let $\chi : \mathcal{Q}_\alpha \longrightarrow \mathcal{S}_\alpha$ be the natural n_α-embedding, as in the proof of the Claim. We also know that $\chi \circ \varphi_1 = \psi \restriction |\mathcal{S}'|$. Since χ is $\Sigma_{n_\alpha+1}$-elementary, φ_1 is Σ_{n_α}-elementary, and $\psi \restriction |\mathcal{S}'|$ is fully elementary, it follows that, in fact, φ_1 is $\Sigma_{n_\alpha+1}$-elementary. We also know that $\pi \circ \varphi_0 = \psi \restriction |W'|$.

Now copy $\mathcal{U}'$ using the pair of maps (φ_0, φ_1); the result is an ill-behaved iteration tree on $((\overline{W}, \mathcal{Q}_\alpha), \kappa_\alpha)$. This contradicts $(3)_\alpha$. $\square$ Lemma 3.13

We next work towards proving $(3)_\alpha$ and $(1)_\alpha$ using instances of $(4)_\beta$.

Lemma 3.14. Suppose that $\alpha \leq \gamma$, $(1)_\beta$ holds for every $\beta < \alpha$, and $(4)_\alpha$ holds. Then there is an iteration tree $\widetilde{\mathcal{T}}$ on W and an elementary embedding $k : \overline{W} \longrightarrow \mathcal{N}$ such that:

(a) $\widetilde{\mathcal{T}}$ extends $\mathcal{T} \restriction (\eta(\alpha) + 1)$.
(b) $\widetilde{\mathcal{T}}$ has a final model and $\mathcal{N}$ is an initial segment of the final model of $\widetilde{\mathcal{T}}$.
(c) $\mathrm{lh}(E_\eta^{\widetilde{\mathcal{T}}}) \geq \lambda_\alpha$ whenever $\eta(\alpha) \leq \eta < \mathrm{lh}(\widetilde{\mathcal{T}})$.
(d) $\mathrm{crit}(k) \geq \kappa_\alpha$.

Proof. Since $(1)_\beta$ holds for every $\beta < \alpha$, $\mathcal{P}_\alpha$ and $\overline{W}$ agree below λ_α. We must allow for the possibility, though, that they disagree at λ_α. Both $(\vec{\mathcal{P}} \restriction (\alpha+1), \vec{\lambda} \restriction \alpha)$ and and $((\vec{\mathcal{P}} \restriction \alpha, \overline{W}), \vec{\lambda} \restriction \alpha)$ are iterable, very special phalanxes of premice, since we are assuming $(4)_\alpha$. Compare these two phalanxes. This results in iteration trees $\mathcal{U}$ on $(\vec{\mathcal{P}} \restriction (\alpha+1), \vec{\lambda} \restriction \alpha)$ and $\mathcal{V}$ on $((\vec{\mathcal{P}} \restriction \alpha, \overline{W}), \vec{\lambda} \restriction \alpha)$ with $\mathrm{lh}(\mathcal{U}) = \alpha + \varphi + 1 = \mathrm{lh}(\mathcal{V})$ for some φ, and either $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi}$ is and initial segment of $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ or vice-versa. All extenders used on $\mathcal{U}$ and $\mathcal{V}$ have length $\geq \lambda_\alpha$. The iteration tree $\widetilde{\mathcal{T}}$ whose existence is asserted by Lemma 3.14 is just $\mathcal{U}$ reorganized in the obvious way. We shall show that $\alpha = \text{root}^{\,V}(\alpha+\varphi)$ and that $[\alpha, \alpha+\varphi]_U \cap \mathcal{D}^{\mathcal{U}} = \emptyset$. We will also show that $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi}$. Once this is accomplished, we can set $\mathcal{N} = \mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ and $k = i^{\mathcal{V}}_{\alpha,\alpha+\varphi}$, and Lemma 3.14 follows.

Claim 1. Let

$$\tilde{\eta} = \sup\left(\{\eta < \eta(\alpha) \mid E_\eta \neq \emptyset\}\right) \ .$$

Then $\overline{W} = \overline{W}_{\tilde{\eta}}$ and $W_{\tilde{\eta}} = W_{\eta(\alpha)}$. If $\overline{E}_{\tilde{\eta}} \neq \emptyset$ (that is, if we are not padding at stage $\tilde{\eta}$ in the construction of $\overline{\mathcal{T}}$), then $E^{\mathcal{V}}_\alpha = \overline{E}_{\tilde{\eta}}$. If $\tilde{\eta} = \eta(\alpha)$, then $E^{\mathcal{U}}_\alpha = E_{\eta(\alpha)}$.

Proof. If $\tilde{\eta} < \eta(\alpha)$, then $\tilde{\eta} T \eta(\alpha)$ and $(\tilde{\eta}, \eta(\alpha)]_T$ consists only of padding. That is, $W_{\tilde{\eta}} = W_{\eta(\alpha)}$.

Suppose that $\eta < \tilde{\eta}$ and $\overline{E}_{\tilde{\eta}} \neq \emptyset$. as we noted, $\text{lh}(\overline{E}_\eta) \geq \lambda_\alpha$. This implies that $\nu(E_\xi) \geq \lambda_\alpha$ whenever $\xi > \eta$ and $E_\xi \neq \emptyset$. Thus $E_\xi = \emptyset$ whenever $\eta < \xi < \eta(\alpha)$, so $\tilde{\eta} \leq \eta$, a contradiction. Therefore $\overline{W}_{\tilde{\eta}} = \overline{W}$.

If $\overline{E}_{\tilde{\eta}} \neq \emptyset$, then $\overline{E}_{\tilde{\eta}}$ is part of the least disagreement between $\overline{W}_{\tilde{\eta}}$ and $W_{\tilde{\eta}}$, hence between $\overline{W}$ and $W_{\eta(\alpha)}$, hence between $\overline{W}$ and $\mathcal{P}_\alpha$.

If $\tilde{\eta} = \eta(\alpha)$, then $E_{\eta(\alpha)}$ is part of the least disagreement between $\overline{W}_{\tilde{\eta}}$ and $W_{\eta(\alpha)}$, hence between $\overline{W}$ and $\mathcal{P}_\alpha$. $\square$ Claim 1

Say $\beta_0 = \text{root}^{\,U}(\alpha+\varphi)$ and $\beta_1 = \text{root}^{\,V}(\alpha+\varphi)$. Then $\beta_0, \beta_1 \leq \alpha$.

Claim 2. $\beta_1 = \alpha$, that is, the last model on $\mathcal{V}$ follows $\overline{W}$.

Assume to the contrary that $\beta_1 < \alpha$.

Subclaim A. $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi}$ is an initial segment of $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$.

PROOF. Either because $\mathcal{P}_{\beta_1}$ is κ_{β_1}-sound and $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ is not, or because $\mathcal{P}_{\beta_1}$ is a universal weasel. $\square$ Subclaim A

SUBCLAIM B. $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi} = \mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$.

PROOF. This is clear unless $\beta_0 = \alpha$ and $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi} = \mathcal{P}_\alpha$. So suppose that $\mathcal{P}_\alpha$ is a proper initial segment of $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$. Then $\mathcal{P}_\alpha$ is a sound, set premouse that projects to κ_α, so

$$\mathrm{OR}^{\mathcal{P}_\alpha} < (\kappa_\alpha^+)^{\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}}$$

But by Claim 1, the right hand side is λ_α. Thus

$$\mathrm{OR}^{\mathcal{P}_\alpha} \leq (\kappa_\alpha^+)^{\overline{W}_\theta} \leq (\kappa_\alpha^+)^{W_\theta} < (\kappa_\alpha^+)^{W_{\eta(\alpha)+1}} \leq \mathrm{lh}(E_{\eta(\alpha)}) \leq \mathrm{OR}^{\mathcal{P}_\alpha} \ ,$$

which is absurd. $\square$ Subclaim B

Put $\mathcal{M} = \mathcal{M}^{\mathcal{U}}_{\alpha+\varphi} = \mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$. Let $\sigma = \max(\ [0,\eta(\beta_0)]_T \ \cap\ [0,\eta(\beta_1)]_T\ )$. We have two different ways of iterating W to $\mathcal{M}$:

$$\begin{array}{ccccccc} & & & & W_{\eta(\beta_0)} \supseteq \mathcal{P}_{\beta_0} & \longrightarrow & \mathcal{M}^{\mathcal{U}}_{\alpha+\varphi} = \mathcal{M} \\ & & & \nearrow & & & \\ W & \longrightarrow & W_\sigma & & & & \\ & & & \searrow & & & \\ & & & & W_{\eta(\beta_1)} \supseteq \mathcal{P}_{\beta_1} & \longrightarrow & \mathcal{M}^{\mathcal{V}}_{\alpha+\varphi} = \mathcal{M} \end{array}$$

The arrows do not necessarily indicate maps, as we may have *drops* across the top or bottom. For the purposes of this proof only, by a drop across the top we mean one of the following:

I. an ordinal $\eta+1 \leq_T \eta(\beta_0)$ such that $\eta+1$ is a drop of any kind in the sense of $\mathcal{T}$, or

II. the ordinal $\alpha+\xi+1$ such that $\alpha+\xi+1$ is least in $(\beta_0, \alpha+\varphi]_U$, on the condition that $\mathcal{P}_{\beta_0}$ is a proper initial segment of $W_{\eta(\beta_0)}$, or

III. an ordinal $\alpha+\xi+1$ that is a drop of any kind in the sense of $\mathcal{U}$, so long as $\alpha+\xi+1 \leq_U \alpha+\varphi$ and $\beta_0 \neq \mathrm{pred}^{\mathcal{U}}(\alpha+\xi+1)$.

And replacing β_0 and $\mathcal{U}$ by β_1 and $\mathcal{V}$ defines a drop across the bottom. There are many cases, depending on whether or not there are drops, and, if there are, where the last drops across the top and bottom occur. In all these cases, the contradiction is roughly like in the Comparison Lemma 7.1 of [MiSt]. We shall give the argument in a single illustrative case, and leave the other cases to the reader.

Illustrative Case. Suppose that the last drop across the top is $\eta+1$, a drop by condition I, and the last drop across the bottom is $\alpha+\xi+1$, a drop by condition II or III. Let $G_0 = E_\eta$ and $G_1 = E^{\mathcal{V}}_{\alpha+\xi}$. If $\nu = \min(\nu(G_0), \nu(G_1))$, then by a standard argument, $G_0 \restriction \nu = G_1 \restriction \nu$. Since $\eta < \eta(\beta_0) \leq \eta(\alpha)$, $\nu(G_0) \leq \kappa_\alpha$. $\mathrm{lh}(G_0) \leq \lambda_\alpha$, as otherwise, $\mathrm{lh}(G_0) > \lambda_\alpha$, so λ_α is a cardinal in $J^{W_\eta}_{lh(G_0)}$, so $\nu(E_\eta) \geq \lambda_\alpha$, so $\eta \geq \eta(\alpha)$, but it is not. If $\nu(G_0) = \kappa_\alpha$, then $\mathrm{lh}(G_0) = \lambda_\alpha$, as there are no cardinals between κ_α and λ_α in W_θ. Therefore

$$\mathrm{lh}(G_0) \leq \lambda_\alpha \leq \mathrm{lh}(G_1)$$

and

$$\nu(G_0) \leq \kappa_\alpha \leq \nu(G_1) \ .$$

Since $\eta < \eta(\alpha)$, $G_0 \notin |W_{\eta(\alpha)}|$, so $G_0 \notin |\mathcal{P}_\alpha|$. Suppose first that $\nu(G_0) < \nu(G_1)$. Then, by the initial segment condition on $\mathcal{M}^{\mathcal{V}}_{\alpha+\xi}$, G_0 is an element of $\mathcal{M}^{\mathcal{V}}_{\alpha+\xi}$ constructed at a level before

$$\lambda_\alpha = (\kappa_\alpha^+)^{J^{\mathcal{M}^{\mathcal{V}}_{\alpha+\xi}}_{lh(G_1)}} \ .$$

But then

$$G_0 \in J^{\overline{W}}_{\lambda_\alpha} = J^{\mathcal{P}_\alpha}_{\lambda_\alpha} \ ,$$

a contradiction. So $\nu(G_0) = \nu(G_1) = \kappa_\alpha$. But then $G_0 = G_1$, and this is an extender of length λ_α. So $\xi = 0$ and G_1 is an extender on the $\overline{W}$-sequence. Note that $\eta < \tilde{\eta}$, so $\overline{W}_\eta = \overline{W}$ by Claim 1. Therefore

$$E_\eta = E^{W_\eta}_{\lambda_\alpha} = G_0 = G_1 = E^{\overline{W}_\eta}_{\lambda_\alpha} = \overline{E}_\eta$$

which contradicts the rules for comparison by which we formed $\mathcal{T}$ and $\overline{\mathcal{T}}$.

$\square$ Illustrative Case $\square$ Claim 2

By an almost identical argument, we see that there is no dropping along the main branch of $\mathcal{V}$. Again, we leave the details to the reader.

CLAIM 3. $[\alpha, \alpha + \varphi]_U \cap \mathcal{D}^{\mathcal{U}} = \emptyset$.

CLAIM 4. $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ is an initial segment of $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi}$.

PROOF. This is clear unless $\mathcal{M}^{\mathcal{U}}_{\alpha+\varphi} = \mathcal{P}_\alpha$. But we can show that $\mathcal{P}_\alpha$ is not a proper initial segment of $\mathcal{M}^{\mathcal{V}}_{\alpha+\varphi}$ just as in the proof of Subclaim A. $\square$ Claim 4 $\square$ Lemma 3.14

LEMMA 3.15. Suppose that $\alpha \leq \gamma$, $(1)_\beta$ holds for every $\beta < \alpha$, and $(4)_\alpha$ holds. Then $(1)_\alpha$ holds.

PROOF. Suppose that Lemma 3.15 fails. Then there is an η such that $\overline{E}_\eta \neq \emptyset$ and $\mathrm{lh}(\overline{E}_\eta) \leq \lambda_\alpha$. Since $(1)_\beta$ holds for every $\beta < \alpha$, $\mathrm{lh}(\overline{E}_\eta) = \lambda_\alpha$. Therefore

$$\mathrm{crit}(\overline{E}_\eta) < \kappa_\alpha \leq \nu(\overline{E}_\eta) < \lambda_\alpha = \mathrm{lh}(\overline{E}_\eta).$$

Now continue the comparison begun in the proof of Lemma 3.14. Then $\eta \leq \eta(\alpha)$, $E^{\mathcal{V}}_\alpha = \overline{E}_\eta$, and $\mathrm{pred}^{\,\mathcal{V}}(\alpha + 1) < \alpha = \mathrm{root}^{\,V}(\alpha + \varphi)$. Let $\alpha + \xi + 1$ be least in $(\alpha, \alpha + \varphi]_V$. Then $\lambda_\beta \leq \mathrm{crit}(E^{\mathcal{V}}_{\alpha+\xi}) < \nu(\overline{E}_\eta)$ for every $\beta < \alpha$, which can mean only that $\mathrm{crit}(E^{\mathcal{V}}_{\alpha+\xi}) = \kappa_\alpha$.

Since we hit $\overline{E}_\eta$, an extender of length λ_α, at the first step in building $\mathcal{V}$, $\lambda_\alpha = (\kappa^+_\alpha)^{\mathcal{M}^{\mathcal{V}}_{\alpha+\xi}} < (\kappa^+_\alpha)^{\overline{W}}$. So $\alpha + \xi + 1 \in \mathcal{D}^{\mathcal{U}}$, contradicting Lemma 3.14. $\square$ Lemma 3.15

LEMMA 3.16. Suppose that $(4)_\beta$ holds for every $\beta < \alpha$. Then $(3)_\alpha$ holds.

PROOF. By Lemma 3.15, we know that $(1)_\beta$ holds for every $\beta < \alpha$. So by Lemma

3.14, for every $\beta < \alpha$, there is an iteration tree $\widetilde{\mathcal{T}}_\beta$ on W with $\mathrm{lh}(\widetilde{\mathcal{T}}_\beta) = \varphi_\beta + 1$ such that $\widetilde{\mathcal{T}}_\beta$ extends $\mathcal{T} \restriction (\eta(\beta)+1)$ and there is an elementary embedding k_β with critical point at least κ_β from $\overline{W}$ into an initial segment $\mathcal{N}_\beta$ of the last model $\mathcal{M}^{\widetilde{\mathcal{T}}_\beta}_{\varphi_\beta}$ of $\widetilde{\mathcal{T}}_\beta$. All extenders $E^{\widetilde{\mathcal{T}}_\beta}_\eta$ have length at least λ_β whenever $\eta(\beta) \leq \eta < \mathrm{lh}(\widetilde{\mathcal{T}}_\beta)$. Let $k_\alpha = \mathrm{id} \restriction |\mathcal{Q}_\alpha|$, $\vec{\mathcal{N}} = \langle\ \mathcal{N}_\beta \mid \beta < \alpha\ \rangle$, and $\vec{k} = \langle\ k_\beta \mid \beta \leq \alpha\ \rangle$. The proof of the following fact is is similar to the proof of the main result in Section 9 of [St1]; we omit the details.

Fact 3.16.1. $((\vec{\mathcal{N}}, \mathcal{Q}_\alpha), \vec{\kappa} \restriction \alpha)$ is an iterable phalanx.

Suppose that $\mathcal{U}$ is an iteration tree on $((\overline{W}, \mathcal{Q}_\alpha), \kappa_\alpha)$. We wish to copy $\mathcal{U}$ to an iteration tree $\mathcal{V}$ on $((\vec{\mathcal{N}}, \mathcal{Q}_\alpha), \vec{\kappa} \restriction \alpha)$ using the system $\vec{k}$. By Fact 3.16.1, this is enough. Note that there are two base models on $\mathcal{U}$, while there will be α-many base models on $\mathcal{V}$.

To avoid some triple superscripts, we shall use the notation:

$$\mathcal{M}^*(\mathcal{U}, 1+\eta+1) = (\mathcal{M}^*_{1+\eta+1})^{\mathcal{U}}$$

and

$$\mathcal{M}^*(\mathcal{V}, \alpha+\eta+1) = (\mathcal{M}^*_{\alpha+\eta+1})^{\mathcal{V}}$$

in the discussion immediately below.

There is really only one subtlety in the copying construction giving $\mathcal{V}$. Suppose that for some $1+\eta+1 < \mathrm{lh}(\mathcal{U})$, $E^{\mathcal{U}}_{1+\eta}$ has critical point κ_β for some $\beta < \alpha$. Then $E^{\mathcal{V}}_{\alpha+\eta} = k_{1+\eta}(E^{\mathcal{U}}_{1+\eta})$, with the usual understanding that $k_{1+\eta}(\dot{F}^{\mathcal{M}^{\mathcal{U}}_{1+\eta}}) = \dot{F}^{\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}}$. Since $k_{1+\eta} \restriction \kappa_\alpha = k_\alpha \restriction \kappa_\alpha = \mathrm{id} \restriction \kappa_\alpha$, $\mathrm{crit}(E^{\mathcal{V}}_{\alpha+\eta}) = \kappa_\beta$. But it is possible that $\mathrm{crit}(k_\beta) = \kappa_\beta$, in which case the Shift Lemma 5.2 of [MiSt] does not literally apply. But the proof works if we copy as follows. In this situation, $(\mathcal{M}^*_{1+\eta+1})^{\mathcal{U}}$ is an initial segment of $\overline{W}$ and $(\mathcal{M}^*_{\alpha+\eta+1})^{\mathcal{V}}$ is an initial segment of $\mathcal{N}_\beta$. Given coordinates $a \in [\mathrm{lh}(E^{\mathcal{U}}_{1+\eta})]^{<\omega}$ and a function $f : [\mathrm{crit}(E^{\mathcal{U}}_{1+\eta})]^{|a|} \longrightarrow |(\mathcal{M}^*_{1+\eta+1})^{\mathcal{U}}|$ with either $f \in |(\mathcal{M}^*_{1+\eta+1})^{\mathcal{U}}|$ or given by a Skolem term and parameter in $(\mathcal{M}^*_{1+\eta+1})^{\mathcal{U}}$, let

$$k_{1+\eta+1}\left(\ [a,f]^{\mathcal{M}^*(\mathcal{U},1+\eta+1)}_{E^{\mathcal{U}}_{1+\eta}} \ \right) = \left[\ k_{1+\eta}(a) \ , \ k_\beta(f) \restriction [\kappa_\beta]^{|a|} \ \right]^{\mathcal{M}^*(\mathcal{V},\alpha+\eta+1)}_{E^{\mathcal{V}}_{\alpha+\eta}} .$$

The proof of the Shift Lemma still works. For example, given

$$x = [a,f]^{\mathcal{M}^*(\mathcal{U},1+\eta+1)}_{E^{\mathcal{U}}_{1+\eta}}$$

and

$$y = [a,g]^{\mathcal{M}^*(\mathcal{U},1+\eta+1)}_{E^{\mathcal{U}}_{1+\eta}} ,$$

if $X = \{ \ u \in [\kappa_\beta]^{|a|} \mid f(u) = g(u) \ \}$, then $x = y$ if and only if $X \in (E^{\mathcal{U}}_{1+\eta})_a$ if and only if

$$k_{1+\eta}(X) = X = k_\beta(X) \cap \kappa_\beta \in (E^{\mathcal{V}}_{1+\eta})_{k_{1+\eta}(a)}$$

if and only if $k_{1+\eta+1}(x) = k_{1+\eta+1}(y)$.

The rest of the details are left to the reader. Note that the above variation on the usual copying construction is used in Section 6 of [St1]. $\square$ Lemma 3.16

LEMMA 3.17. Suppose that $(5)_\alpha$ holds. Then $(4)_\alpha$ holds.

PROOF. We can use the system of maps $\vec{\sigma} = (\langle \ \pi_\beta \mid \beta < \alpha \ \rangle), \pi)$ to copy a given iteration tree $\mathcal{U}$ on the phalanx $((\vec{\mathcal{P}} \restriction \alpha, \overline{W}), \vec{\lambda} \restriction \alpha)$ to an iteration tree $\vec{\sigma}''\mathcal{U}$ on the phalanx $((\vec{\mathcal{R}} \restriction \alpha, W), \vec{\Lambda} \restriction \alpha)$. Notice that $\vec{\sigma}''\mathcal{U}$ is a special iteration tree, since every extender used on it has critical point in the range of π. So $\vec{\sigma}''\mathcal{U}$ is well-behaved. The usual argument now shows that $\mathcal{U}$ is well-behaved. $\square$ Lemma 3.17

LEMMA 3.18. Suppose that $(6)_\alpha$ holds. Then $(5)_\alpha$ holds.

PROOF. Let $\mathcal{U}$ be a special iteration tree on $((\vec{\mathcal{R}}, W), \vec{\Lambda} \restriction \alpha)$. We wish to *enlarge* $\mathcal{U}$ to a special iteration tree $\mathcal{V}$ on $((\vec{\mathcal{S}}, W), \vec{\Lambda} \restriction \alpha)$. This will differ significantly from constructions that we are familiar with. For example, although we will have $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{V})$, it will be possible that the tree structures U and V are different. Also, it will be possible that for some $\alpha+\eta+1<\mathrm{lh}(\mathcal{U})$, $E^{\mathcal{U}}_{\alpha+\eta}\neq\emptyset$ while $E^{\mathcal{V}}_{\alpha+\eta}=\emptyset$. Before we describe the construction of $\mathcal{V}$, let us make some observations about $\mathcal{U}$. Suppose that $\alpha\leq\alpha+\eta+1<\mathrm{lh}(\mathcal{U})$. Let $\beta=\mathrm{root}^{\mathcal{U}}(\alpha+\eta)$. We identify two cases:

Case A. $\dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$ is not an extender over $\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}$.

Case B. Otherwise.

Case A occurs exactly when the following three conditions hold:

1. $\beta<\alpha$
2. $\mathcal{R}_\beta$ is not a premouse; equivalently, $\mathcal{R}_\beta\neq\mathcal{S}_\beta$
3. $(\beta,\alpha+\eta]_U\cap\mathcal{D}^{\mathcal{U}}=\emptyset$

In Case A, we also have that:

4. $\mathcal{R}_\beta$ is an active protomouse that is active of type I or II
5. $m_\beta=0$
6. $\deg^{\mathcal{U}}(\alpha+\eta)=0$
7. There is a unique $\widetilde{\beta}<\beta$ such that $\dot{\mu}^{\mathcal{R}_\beta}=\pi(\kappa_{\widetilde{\beta}})$
8. $\mathrm{crit}(i^{\mathcal{U}}_{\beta,\alpha+\eta})>(\dot{\mu}^+)^{\mathcal{R}_\beta}$
9. $\dot{\mu}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}=\dot{\mu}^{\mathcal{R}_\beta}$
10. $\dot{F}^{\mathcal{R}_\beta}$ is an extender over $\mathcal{R}_{\widetilde{\beta}}$

Moreover, the relationship between $\mathcal{R}_\beta$ and $\mathcal{S}_\beta$ is as described in Lemmas 3.6 and 3.7.

Note that in Case A, if $E^{\mathcal{U}}_{\alpha+\eta}=\dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$, then $\mathrm{pred}^{\mathcal{U}}(\alpha+\eta+1)=\widetilde{\beta}$,

$$\alpha+\eta+1\notin\mathcal{D}^{\mathcal{U}}\ ,$$

and $\deg^{\mathcal{U}}(\alpha+\eta+1)=m_{\widetilde{\beta}}$. In this case, $E^{\mathcal{U}}_{\alpha+\eta}$ measures precisely those subsets of its critical point that are in $\mathcal{R}_{\widetilde{\beta}}=(\mathcal{M}^*_{\alpha+\eta+1})^{\mathcal{U}}$. Note that in other cases, it is possible that $E^{\mathcal{U}}_{\alpha+\eta}$ measures more subsets of its critical point than are present in

$(\mathcal{M}^*_{\alpha+\eta+1})^{\mathcal{U}}$. This happens, for example, if $E^{\mathcal{U}}_{\alpha+\eta}$ is an extender over $J^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}_{lh(E^{\mathcal{U}}_{\alpha+\eta})}$ and $\mathrm{crit}(E^{\mathcal{U}}_{\alpha+\eta}) = \pi(\kappa_\beta)$ for some $\beta < \alpha$ such that $\mathcal{R}_\beta$ is not a premouse. In such cases, we are using the full force of Definition 2.4.2.

Consider any $\beta < \alpha$. If $\mathcal{R}_\beta = \mathcal{S}_\beta$, then let $\mathcal{L}_\beta$ be this premouse. If, on the other hand, $\mathcal{R}_\beta \neq \mathcal{S}_\beta$, then let $\mathcal{L}_\beta = \mathrm{decap}(\mathcal{R}_\beta)$. Let $\mathcal{L}_\alpha = W$. For all $\beta \leq \alpha$, let e_α be the identity map on $|\mathcal{L}_\beta|$. These are the starting maps for our enlargement. Either $\mathcal{L}_\beta$ equals $\mathcal{S}_\beta$, or else $\mathcal{L}_\beta$ is $\mathcal{S}_\beta$ cut off at a successor cardinal of $\mathcal{S}_\beta$.

Suppose, for the moment, that $\mathcal{R}_\beta \neq \mathcal{S}_\beta$. Let

$$\mathcal{R}_{\beta_k} = \mathcal{S}_{\beta_k} \longrightarrow \cdots \longrightarrow \mathcal{S}_{\beta_0} = \mathcal{S}_\beta$$

be the decomposition of $\mathcal{S}_\beta$ as in Lemma 3.6, with $\sigma_i : \mathcal{S}_{\beta_i} \longrightarrow \mathcal{S}_\beta$ for $i = 0, 1, ..., k$ the natural maps. Note that $n_\beta = n_{\beta_0} = n_{\beta_1} = \cdots = n_{\beta_k} \leq \omega$; call this ordinal n. Also suppose that $n < \omega$. Then

$$\rho^{\mathcal{S}_\beta}_{n+1} \leq \pi(\kappa_\beta) < \rho_n(\mathcal{S}_{\beta_i})$$

and for $i = 0, 1, \ldots, k$,

$$\rho_n(\mathcal{S}_\beta) = \sigma_i(\rho_n(\mathcal{S}_{\beta_i})) > \sigma_i(\pi(\kappa_{\beta_i})) \ .$$

Also, for $i = 1, \ldots, k$,

$$\sigma_i(\pi(\kappa_{\beta_i})) > \sigma_{i-1}(\mathrm{OR} \cap \mathcal{R}_{\beta_{i-1}}) = \mathrm{OR} \cap \sigma_{i-1}(\mathcal{L}_{\beta_{i-1}}) \ .$$

In particular,

$$\rho^{\mathcal{S}_\beta}_{n+1} < \pi(\kappa_\beta) < \mathrm{OR}^{\mathcal{L}_\beta} < \rho_n(\mathcal{S}_\beta)$$

whenever $\mathcal{L}_\beta$ is a proper initial segment of $\mathcal{S}_\beta$.

By induction on $\alpha + \eta < \mathrm{lh}(\mathcal{U})$, we wish to define the initial segments $\mathcal{V} \restriction (\alpha+\eta)$ of a special iteration tree $\mathcal{V}$ on $((\vec{\mathcal{S}} \restriction \alpha, W), \vec{\Lambda} \restriction \alpha)$ together with initial segments $\mathcal{L}_{\alpha+\eta}$ of $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$ and maps $e_{\alpha+\eta} : \mathcal{M}^{\mathcal{U}}_{\alpha+\eta} \longrightarrow \mathcal{L}_{\alpha+\eta}$. The maps will satisfy certain inductively maintained agreement, elementarity, and commutativity properties. In order to state these properties, we first make the following definition.

Remark: There is a slight lie here. $\mathcal{V}$ will involve a generalization of padding, to be explained at the beginning of Case 4 below.

Suppose that $\alpha+\eta < \mathrm{lh}(\mathcal{U})$ and that $\beta = \mathrm{root}^{\mathcal{U}}(\alpha+\eta)$. There is a unique pair of (finite) sequences $\beta_0 > \cdots > \beta_\ell$ and $\eta_0 < \cdots < \eta_{\ell-1} < \eta_\ell$ that are of maximal length with the following properties.

- $\eta_\ell = \eta$.
- If $0 \le i \le \ell$, then $\beta_i = \mathrm{root}^{\mathcal{U}}(\alpha+\eta_i)$; so $\beta_\ell = \beta$.
- If $0 < i \le \ell$, then $\beta_i = \mathrm{pred}^{\mathcal{U}}(\alpha+\eta_{i-1}+1) \le_U (\alpha+\eta_i)$.
- If $0 < i \le \ell$, then $\alpha+\eta_{i-1}$ falls under Case A and $E^{\mathcal{U}}_{\alpha+\eta_{i-1}} = \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta_{i-1}}}$.

We shall call $(\vec{\beta},\vec{\eta})$ the **trace** of $\alpha+\eta$. It is not difficult to see that the decomposition of $\mathcal{S}_{\beta_0}$ in the sense of Lemma 3.6 is

$$\mathcal{R}_{\beta_k} = \mathcal{S}_{\beta_k} \longrightarrow \cdots \longrightarrow \mathcal{S}_{\beta_\ell} \longrightarrow \cdots \longrightarrow \mathcal{S}_{\beta_0}$$

for some $k \ge \ell$. The following diagram may serve as a useful reference in what is to come.

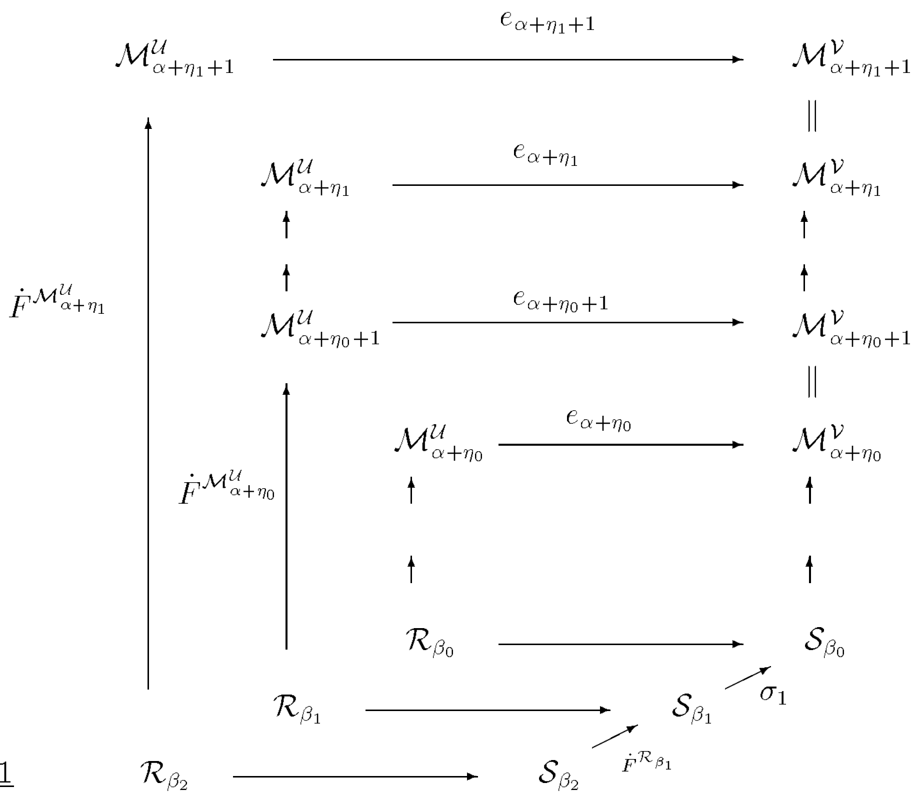

FIGURE 1

Let $\alpha+\eta < \mathrm{lh}(\mathcal{U})$ be given, and suppose it has trace $(\vec{\beta}, \vec{\eta})$. Put $\beta = \mathrm{root}^{\,\mathcal{U}}(\alpha+\eta) = \beta_\ell$. We inductively maintain the following properties.

The tree, drop, and degree structure of $\mathcal{V}$

For any $\xi \leq \alpha+\eta$, $\xi \leq_V (\alpha+\eta)$ if and only if $\xi \leq_U (\alpha+\eta_i)$ for some $i \in \{0,\ldots,\ell\}$. This determines the tree structure V of $\mathcal{V}$.

$\mathcal{D}^{\mathcal{V}} \cap (\alpha+\eta+1) = \mathcal{D}^{\mathcal{U}} \cap (\alpha+\eta+1)$, so in the end, we shall have that the drop structures for $\mathcal{U}$ and $\mathcal{V}$ are the same.

Suppose that $\alpha+\eta$ falls under Case A. Then

$$\deg^{\mathcal{V}}(\alpha+\eta) = n_{\beta_0} = \cdots = n_{\beta_\ell} = n_\beta .$$

Recall that $\deg^{\mathcal{U}}(\alpha+\eta) = 0$ in Case A; so it is possible that $\deg^{\mathcal{V}}(\alpha+\eta)$ is different from $\deg^{\mathcal{V}}(\alpha+\eta)$.

On the other hand, suppose that $\alpha+\eta$ falls under Case B. Then $\deg^{\mathcal{V}}(\alpha+\eta) = \deg^{\mathcal{U}}(\alpha+\eta)$.

The premouse $\mathcal{L}_{\alpha+\eta}$

Suppose that $\alpha+\eta$ falls under Case A. Then $\mathcal{L}_{\alpha+\eta}$ is a proper initial segment of $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$ and $\mathrm{OR}^{\mathcal{L}_{\alpha+\eta}}$ is a successor cardinal in $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$. If $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$ is a set premouse, then

$$\rho_{n_{\beta_0}+1}^{\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}} \leq \pi(\kappa_{\beta_0}) < \mathrm{OR}^{\mathcal{L}_{\alpha+\eta}} < \rho_{n_{\beta_0}}(\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}) \ .$$

More precisely,

$$\mathcal{L}_{\alpha+\eta} = (i^{\mathcal{V}}_{\beta_0,\alpha+\eta} \circ \sigma_\ell)(\mathcal{L}_\beta)$$

where, as before, $\sigma_\ell : \mathcal{S}_{\beta_\ell} \longrightarrow \mathcal{S}_{\beta_0}$ is the ℓ'th decomposition map for $\mathcal{S}_{\beta_0}$.

On the other hand, suppose that $\alpha+\eta$ falls under Case B. Then $\mathcal{L}_{\alpha+\eta} = \mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$.

Agreement

Suppose that $\alpha+\xi \leq_U \alpha+\eta_i$ for some $i \in \{0,\ldots,\ell\}$. Then

$$e_{\alpha+\eta} \restriction \nu^{\mathcal{U}}_\xi = e_{\alpha+\xi} \restriction \nu^{\mathcal{U}}_\xi .$$

Elementarity

Suppose that $\alpha+\eta$ falls under Case A. Then $e_{\alpha+\eta}$ is an elementary embedding of decap($\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}$) into $\mathcal{L}_{\alpha+\eta}$ that is cofinal:

$$\sup(e_{\alpha+\eta}\ ''\ \mathrm{OR}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}) = \mathrm{OR}^{\mathcal{L}_{\alpha+\eta}}.$$

On the other hand, suppose that $\alpha+\eta$ falls under Case B. Then $e_{\alpha+\eta}$ is a weak $\deg^{\mathcal{U}}(\alpha+\eta)$-embedding of $\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}$ into $\mathcal{L}_{\alpha+\eta}$. Somewhat more elementarity will follow from the commutativity property below. For example, suppose that $\alpha+\xi+1$ is either the last drop along $(\beta,\alpha+\eta]_U$, or the least ordinal in $(\beta,\alpha+\eta]_U$ if no such drops exist. Let

$$Y(\mathcal{U},\alpha+\eta) = (i^{\mathcal{U}}_{\alpha+\xi+1,\alpha+\eta}\circ(i^{*}_{\alpha+\xi+1})^{\mathcal{U}})\ ''\ |(\mathcal{M}^{*}_{\alpha+\xi+1})^{\mathcal{U}}|\ .$$

Then $e_{\alpha+\eta}$ is $\Sigma_{deg(\alpha+\eta)+1}$-elementary on points in $Y(\mathcal{U},\alpha+\eta)$.

Commutativity

Suppose that $(\beta,\alpha+\eta]_U\cap\mathcal{D}^{\mathcal{U}}=\emptyset$. Then both $i^{\mathcal{U}}_{\beta,\alpha+\eta}$ and $i^{\mathcal{V}}_{\beta_\ell,\alpha+\eta}$ are defined, and

$$e_{\alpha+\eta}\circ i^{\mathcal{U}}_{\beta,\alpha+\eta} = i^{\mathcal{U}}_{\beta_\ell,\alpha+\eta}\circ\sigma_\ell\circ e_\beta = i^{\mathcal{V}}_{\beta_\ell,\alpha+\eta}\circ\sigma_\ell\restriction|\mathcal{R}_\beta|$$

Now suppose that $\beta<_U(\alpha+\xi)<_U(\alpha+\eta)$ and that $(\alpha+\xi,\alpha+\eta]_U\cap\mathcal{D}^{\mathcal{U}}=\emptyset$. Then both $i^{U}_{\alpha+\xi,\alpha+\eta}$ and $i^{V}_{\alpha+\xi,\alpha+\eta}$ are defined, and

$$e_{\alpha+\eta}\circ i^{U}_{\alpha+\xi,\alpha+\eta} = i^{V}_{\alpha+\xi,\alpha+\eta}\circ e_{\alpha+\xi}.$$

Finally, suppose that $\alpha+\eta \in \mathcal{D}^{\mathcal{U}}$ and that $\xi = \text{pred}^{\mathcal{U}}(\alpha+\eta)$. Then $\xi = \text{pred}^{\mathcal{V}}(\alpha+\eta)$ and

$$e_{\alpha+\eta} \circ (i^*_{\alpha+\eta})^{\mathcal{U}} = (i^*_{\alpha+\eta})^{\mathcal{V}} \circ e_{\alpha+\xi} \ .$$

We now describe the construction of $\mathcal{V}$ and the enlargement maps. The last case is a bit trickier than the first three.

Case 1. Defining $\mathcal{V} \restriction (\alpha+\eta)$ when $\alpha+\eta$ is a limit ordinal.

Here $\mathcal{V} \restriction (\alpha+\eta)$ is just the limit of $\mathcal{V} \restriction (\alpha+\xi)$ for $\alpha+\xi < \alpha+\eta$.

Case 2. Defining $\mathcal{V} \restriction (\alpha+\eta+1)$ when $\alpha+\eta$ is a limit ordinal.

Let $\beta = \text{root}^{\mathcal{U}}(\alpha+\eta)$ and $\beta_0 > \cdots \beta_\ell = \beta$, $\eta_0 < \cdots < \eta_\ell = \eta$ be the trace of η. By a straightforward generalization of the Strong Uniqueness Theorem 6.2 of [MiSt], we see that $[\beta, \alpha+\eta)_U$ is the sole cofinal wellfounded branch in $\mathcal{U} \restriction (\alpha+\eta)$. The inductively maintained properties of $\mathcal{V} \restriction (\alpha+\xi)$ and $e_{\alpha+\xi}$ for $\alpha+\xi < \alpha+\eta$ guarantee that

$$\{ \ \alpha+\zeta \mid \text{there is some } (\alpha+\xi) <_U (\alpha+\eta) \text{ s.t. } (\alpha+\zeta) <_V (\alpha+\xi) \ \}$$

$$= [\beta_\ell, \alpha+\eta_{\ell-1}+1]_V \ \cup \ (\alpha+\eta_{\ell-1}+1, \alpha+\eta)_U$$

is the unique cofinal wellfounded branch in $\mathcal{V} \restriction (\alpha+\eta)$. This determines $\mathcal{V} \restriction (\alpha+\eta)$, $\mathcal{L}_{\alpha+\eta}$, and $e_{\alpha+\eta}$ as limits in the usual way.

Case 3. Defining $\mathcal{V} \restriction (\alpha+\eta+2)$ when either $\alpha+\eta$ falls under Case B or $E^{\mathcal{U}}_{\alpha+\eta} \neq \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$.

Let $\eta^* = \mathrm{pred}^{\mathcal{U}}(\alpha+\eta+1)$. Here we use the usual copying construction, using the maps e_{η^*} and $e_{\alpha+\eta}$ to define $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta+1}$ as the model following $\mathcal{M}^{\mathcal{V}}_{\eta^*}$ in $\mathcal{V}$. If $E^{\mathcal{U}}_{\alpha+\eta} \neq \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$, then we put $E^{\mathcal{V}}_{\alpha+\eta} = e_{\alpha+\eta}(E^{\mathcal{U}}_{\alpha+\eta})$. Suppose, on the other hand, that $E^{\mathcal{U}}_{\alpha+\eta} = \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$. Then our case hypothesis implies that $\alpha+\eta$ falls under Case B, so $e_{\alpha+\eta}$ is a weak $\deg^{\mathcal{U}}(\alpha+\eta)$-embedding, and more, of $\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}$ into $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$; in particular, it has some elementarity with respect to the last predicate of $\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}$. Thus, it is reasonable that we set $E^{\mathcal{V}}_{\alpha+\eta} = \dot{F}^{\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}}$.

Having defined $E^{\mathcal{V}}_{\alpha+\eta}$, the rules for forming iteration trees determine $\mathcal{V} \restriction (\eta+2)$. The map $e_{\alpha+\eta+1} : \mathcal{M}^{\mathcal{V}}_{\alpha+\eta} \longrightarrow \mathcal{L}_{\alpha+\eta}$ is as given by the Shift Lemma 5.2 of [MiSt]. It is relatively easy to check that the inductive properties continue to hold at $\alpha+\eta+1$.

Case 4. Defining $\mathcal{V} \restriction (\alpha+\eta+2)$ when $\alpha+\eta$ falls under Case A and $E^{\mathcal{U}}_{\alpha+\eta} = \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$.

We let $\beta_0 > \cdots > \beta_\ell$, $\eta_0 < \cdots < \eta_\ell$ be the trace of $\alpha+\eta+1$. If $\beta = \mathrm{pred}^{\mathcal{U}}(\alpha+\eta+1)$, then $\beta = \beta_0 < \alpha$, $\alpha+\eta = \alpha+\eta_{\ell-1}$, and $\alpha+\eta+1 = \alpha+\eta_\ell$.

Here we set $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta+1} = \mathcal{M}^{\mathcal{V}}_{\alpha+\eta}$ and $\alpha+\eta = \mathrm{pred}^{\mathcal{V}}(\alpha+\eta+1)$. So it seems that we are merely padding from $\alpha+\eta$ to $\alpha+\eta+1$ in the formation of $\mathcal{V}$. This is not exactly true, for we also set $\nu^{\mathcal{V}}_{\alpha+\eta} = e_{\alpha+\eta}(\nu^{\mathcal{U}}_{\alpha+\eta})$. Technically, this is not padding in the sense of [MiSt], since we have given a condition by which it is possible that $\alpha+\eta = \mathrm{pred}^{\mathcal{V}}(\alpha+\zeta)$ for some $\alpha+\zeta > \alpha+\eta+1$; the condition is that $\nu^{\mathcal{V}}_{\alpha+\xi} \leq \mathrm{crit}(E^{\mathcal{U}}_{\alpha+\zeta}) < \nu^{\mathcal{V}}_{\alpha+\eta}$ for every $\alpha+\xi < \alpha+\eta$. But clearly this is just a question of indexing; any phalanx iterable in the usual sense is iterable in a sense involving this generalized form of padding.

Given this definition of $\mathcal{M}^{\mathcal{V}}_{\alpha+\eta+1}$, we must next determine a map

$$e_{\alpha+\eta+1} : |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta+1}| \longrightarrow |\mathcal{M}^{\mathcal{V}}_{\alpha+\eta}| \; .$$

But first some motivation. Consider the very simple situation in which $\eta_0 = 1$, $\ell = 1$, and m_{β_0}, m_{β_1}, n_{β_0}, and n_{β_0} are all equal to 0. Put $G = E^{\mathcal{U}}_{\alpha}$ and $F = \dot{F}^{\mathcal{R}_{\beta_0}}$. Compare the sequence of extenders used along:

$$|\mathcal{R}_{\beta_0}| \xrightarrow{G} |\mathcal{M}^{\mathcal{U}}_{\alpha+1}| \subset |\mathcal{M}^{\mathcal{U}}_{\alpha+2}| = |\mathrm{ult}(\mathcal{R}_\beta, i_G{}''F)|$$

with those used along:

$$\mathcal{S}_{\beta_1} \xrightarrow{F} \mathcal{S}_{\beta_0} \xrightarrow{G} \mathcal{M}^{\mathcal{V}}_{\alpha+1}$$

The sequence of extenders used in the bottom iteration consists of images of those used along the top system applied in a different order. Note that $\mathrm{crit}(i_G{}''F) = \mathrm{crit}(F) = \pi(\kappa_{\beta_1})$. Suppose that $x = i_{i_G{}''F}(f)(a)$ for some $a \in [\mathrm{lh}(i_G{}''F)]^{<\omega}$ and $f \in |\mathcal{R}_{\beta_1}|$ with $\mathrm{dom}(f) = [\mathrm{crit}(F)]^{|a|}$. There are $b \in [\mathrm{lh}(G)]^{<\omega}$ and $g \in |\mathcal{R}_{\beta_1}|$ with $\mathrm{dom}(g) = [\mathrm{crit}(G)]^{|b|}$ such that $a = i_G(g)(b)$. Then $i_G(i_F(f) \circ g)(b) \in |\mathcal{M}^{\mathcal{V}}_{\alpha+1}|$. This would be our value for $e_{\alpha+2}(x)$.

A bit more motivation for what is to come can be gotten by considering Figure 1, which illustrates the situation when $\ell = 2$. Compare the sequence of extenders that are used along

$$|\mathcal{R}_{\beta_0}| \longrightarrow \overset{i^{\mathcal{U}}_{\beta_0,\alpha+\eta_0}}{\cdots} \longrightarrow |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta_0}|$$

$$\subset |\ \mathrm{ult}\,(\,\mathcal{R}_{\beta_1}\,,\ i^{\mathcal{U}}_{\beta_0}{}''\dot{F}^{\mathcal{R}_{\beta_0}}\ )\ | = |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta_0+1}| \longrightarrow \overset{i^{\mathcal{U}}_{\alpha+\eta_0+1,\alpha+\eta_1}}{\cdots} \longrightarrow |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta_1}|$$

$$\subset |\ \mathrm{ult}\,(\,\mathcal{R}_{\beta_2}\,,\ i^{\mathcal{U}}_{\beta_1,\alpha+\eta_1}{}''\dot{F}^{\mathcal{R}_{\beta_1}}\ )\ | = |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta_1+1}|$$

with those used along

$$\mathcal{S}_{\beta_2} \overset{\dot{F}^{\mathcal{R}_{\beta_1}}}{\to} \mathcal{S}_{\beta_1} \overset{\dot{F}^{\mathcal{R}_{\beta_0}}}{\to} \mathcal{S}_{\beta_0} \to \overset{i^{\mathcal{V}}_{\beta_0,\alpha+\eta_0}}{\cdots} \to \mathcal{M}^{\mathcal{V}}_{\alpha+\eta_0} = \mathcal{M}^{\mathcal{V}}_{\alpha+\eta_0+1} \to \overset{i^{\mathcal{V}}_{\alpha+\eta_0+1,\alpha+\eta_1}}{\cdots} \to \mathcal{M}^{\mathcal{V}}_{\alpha+\eta_1}$$

Again, the second iteration is just the first done in a different order, and then pushed over using the maps $e_{\alpha+\xi}$.

Now we give the general definition of $e_{\alpha+\eta+1}$ for Case 4. Let $x \in |\mathcal{M}^{\mathcal{U}}_{\alpha+\eta+1}|$. We may write x as

$$i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(f)(a)$$

for some $a \in [\nu^{\mathcal{U}}_{\alpha+\eta}]^{<\omega}$ and function f with $\mathrm{dom}(f) = [\pi(\kappa_\beta)]^{|a|}$. If $m_\beta = 0$, then we take $f \in |\mathcal{R}_\beta|$. If $m_\beta > 0$, then $f : u \mapsto \tau_\psi^{\mathcal{R}_\beta}[u, q]$ for some Σ_{m_β}-Skolem term τ_ψ and parameter $q \in |\mathcal{R}_\beta|$, and by

$$i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(f)(a)$$

we really mean

$$[a, f]^{\mathcal{R}_\beta}_{E^{\mathcal{U}}_{\alpha+\eta}} = \tau_\psi^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta+1}}\left[a, i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(q)\right]$$

(we maintain this convention in what follows).

Next, consider any sequence with $(f_\ell, \ldots, f_0)$, $(a_\ell, \ldots, a_1)$, and $(b_{\ell-1}, \ldots, b_0)$ such that:

1. $f_\ell = f$ and $a_\ell = a$.
2. If $\ell \geq 2$, then for $i = \ell - 1, \ldots, 1$,

$$a_i \in [\nu^{\mathcal{U}}_{\alpha+\eta_{i-1}}]^{<\omega}$$

   and

$$b_i \in \left[\sup\left(\{\ \nu^{\mathcal{U}}_\xi \mid \xi \leq_U \alpha + \eta_i\ \}\right) - \nu^{\mathcal{U}}_{\alpha+\eta_{i-1}}\right]^{<\omega} .$$

   Also,

$$b_0 \in \left[\sup\left(\{\ \nu^{\mathcal{U}}_\xi \mid \xi \leq_U \alpha + \eta_0\ \}\right)\right]^{<\omega} .$$

3. For $i = \ell - 1, \ldots, 0$, f_i is a function, $f_i \in |\mathcal{R}_{\beta_i}|$, and $\mathrm{dom}(f_i) = [\pi(\kappa_{\beta_i})]^{|a_i|}$.
4. If $\ell \geq 2$, then for $i = \ell, \ldots, 2$,

$$a_i = i^{\mathcal{U}}_{\beta_{i-1},\alpha+\eta_{i-1}}(f_{i-1})(a_{i-1})(b_{i-1}) .$$

5. $a_1 = i^{\mathcal{U}}_{\beta_0,\alpha+\eta_0}(f_0)(b_0)$

When $i = \ell - 1, \dots, 0$ (as in 3 above) we are able to choose $f_i \in |\mathcal{R}_{\beta_i}|$, since, as we already noted, $\deg^{\mathcal{U}}(\alpha + \eta_i) = 0$. Recall that we took σ_i to be the embedding of $\mathcal{S}_{\beta_i}$ into $\mathcal{S}_{\beta_0}$. Define

$$g = (\sigma_\ell(f_\ell)) \circ \cdots \circ (\sigma_1(f_1)) \circ f_0$$

Note that if $\deg^{\mathcal{V}}(\alpha + \eta) = n_{\beta_0} = 0$, then g is an element of $\mathcal{S}_{\beta_0}$; otherwise, $\deg^{\mathcal{V}}(\alpha + \eta) = n_{\beta_0} > 0$, and g is given by a $\Sigma_{n_{\beta_0}}$-Skolem term and parameter in $\mathcal{S}_{\beta_0}$. For $i = \ell - 1, \dots 0$, let

$$c_i = e_{\alpha+\eta_i}(b_i) = e_{\alpha+\eta_{\ell-1}}(b_i) \ .$$

Finally, we set

$$e_{\alpha+\eta+1}(x) = i^{\mathcal{V}}_{\beta_0,\alpha+\eta}(g)(c_0)\cdots(c_{\ell-1}) \ .$$

The argument that $e_{\alpha+\eta+1}$ is well-defined is straightforward. Our rules determine that

$$\mathcal{L}_{\alpha+\eta+1} = i^{\mathcal{V}}_{\beta_0,\alpha+\eta}(\sigma_\ell(\mathcal{L}_\beta))$$

and a similar straightforward calculation shows that $e_{\alpha+\eta+1}$ has the required elementarity (almost by design). The agreement property follows as usual, using the fact that if $\xi < \mathrm{lh}(E^{\mathcal{U}}_{\alpha+\eta})$, then $\xi = i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(u \mapsto u_0)(\{\xi\})$. The commutativity property also follows as usual, using the fact that $i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(x) = i^{\mathcal{U}}_{\beta,\alpha+\eta+1}(u \mapsto x)(a)$. This completes the definition of our enlargement.

Now suppose, for contradiction, that $\mathcal{U}$ has limit length, but that there is no cofinal, wellfounded branch through $\mathcal{U}$. By $(6)_\alpha$, there is a cofinal, wellfounded branch b through $\mathcal{V}$. Let $\mathcal{G}$ be the collection of $\alpha + \eta + 1 \in b$ such that $\alpha + \eta$ falls under Case A and $E^{\mathcal{U}}_{\alpha+\eta} = \dot{F}^{\mathcal{M}^{\mathcal{U}}_{\alpha+\eta}}$ (i.e., the definition of $\mathcal{V} \restriction (\alpha + \eta + 2)$ is by case 4). Let us suppose that $\mathcal{G} \neq \emptyset$. If $\alpha + \xi + 1 <_V \alpha + \eta + 1$ are both in $\mathcal{G}$, then $\mathrm{root}^{\,\mathcal{U}}(\alpha + \xi + 1) > \mathrm{root}^{\,\mathcal{U}}(\alpha + \eta + 1)$. Therefore $\mathcal{G}$ is finite; let $\alpha + \eta + 1$ be the largest element of $\mathcal{G}$. Suppose that the trace of $\alpha + \eta + 1$ is $\beta_0 > \cdots > \beta_\ell$, $\eta_0 < \cdots < \eta_\ell$. Then $\beta_0 = \min(b)$,

$$\mathcal{G} = \{\ \alpha + \eta_{\ell-1} + 1, \dots, \alpha + \eta_0 + 1\ \} \ ,$$

and

$$c = \{\beta_0\} \cup (b - (\alpha + \eta + 1))$$

is a cofinal branch through $\mathcal{U}$. Let e be the direct limit of the maps e_ξ for $\xi \in c$. Then e is an embedding of $\mathcal{M}_c^{\mathcal{U}}$ into $\mathcal{M}_b^{\mathcal{V}}$; since the latter structure is wellfounded, both are. This is a contradiction. Similar arguments show that $\mathcal{U}$ is well-behaved in the remaining cases as well. $\square$ Lemma 3.18

LEMMA 3.19. Suppose that $(1)_\beta$ and $(2)_\beta$ holds for every $\beta < \alpha$. Then $(6)_\alpha$ holds.

PROOF. Consider any $\beta < \alpha$. By Corollary 3.12, there is an iteration tree $\mathcal{U}_\beta$ on W of length $\varphi_\beta + 1$ such that all extenders used on $\mathcal{U}_\beta$ have length at least Λ_β, and there is an elementary embedding $j_\beta : \mathcal{S}_\beta \longrightarrow \mathcal{M}_{\varphi_\beta}^{\mathcal{U}_\beta}$ with critical point at least $\pi(\kappa_\beta)$. It follows that $((\langle\, \mathcal{M}_{\varphi_\beta}^{\mathcal{U}_\beta} \mid \beta < \alpha\,\rangle, W), \vec{\Lambda} \restriction \alpha)$ is a phalanx. The proof of the following fact is is similar to the proof of the main result in Section 9 of [St1]; we omit the details.

FACT 3.19.1. $((\langle\, \mathcal{M}_{\varphi_\beta}^{\mathcal{U}_\beta} \mid \beta < \alpha\,\rangle, W), \vec{\Lambda} \restriction \alpha)$ is an iterable phalanx.

The system of maps $(\langle\, j_\beta \mid \beta < \alpha\,\rangle, \mathrm{id} \restriction W)$ now guarantees that $((\vec{\mathcal{S}} \restriction \alpha, W), \vec{\Lambda} \restriction \alpha)$ is also iterable in the usual way. $\square$ Lemma 3.19

This completes the proof that the implications listed before Lemma 3.9 hold, and consequently that $(1)_\alpha$ through $(6)_\alpha$ hold for every $\alpha < \gamma$. As already explained (just after the statement of Corollary 3.12), Theorem 1.1 follows. $\square$ Theorem 1.1

§4 ADDENDUM

Woodin [W] observed the following easy corollary to the proof of Theorem 1.1, and used it to prove that AD holds in $L(\mathbb{R})$ from various hypotheses.

Corollary 4.1 ("Baby covering lemma"). Let W be an iterable weasel such that no extender from the W-sequence is total on W. Let κ be a singular, countably closed cardinal. Suppose that $(\kappa^+)^W < \kappa^+$. Then, one of the following holds:

(1) For arbitrarily large ordinals $\delta < \kappa$, there is a countably complete, normal ultrafilter U on $\mathcal{P}(\delta) \cap W$ such that U is weakly amenable to W.
(2) There is an active, iterable premouse $\mathcal{M}$ with $\kappa \leq \dot{\mu}^{\mathcal{M}}$, such that if $\mathcal{I}$ is the class length, linear iteration of $\mathcal{M}$ obtained by setting $E_\zeta^{\mathcal{I}} = \dot{F}^{\mathcal{M}_\zeta^{\mathcal{I}}}$ for all ordinals ζ, then W is the lined up model of $\mathcal{I}$.

In case (1), for a given δ, one can use U to get a reasonable interpretation of $K^W \parallel \delta$, just as the normal measure on $\mathcal{P}(\Omega)$ was used to develop the theory of $K \parallel \Omega$ in Steel [St1]. Similarly, in case (2), the last extender of $\mathcal{M}$ codes a measure over $\mathcal{P}(\dot{\mu}^{\mathcal{M}}) \cap W$, which can be used to understand $K^W \parallel \dot{\mu}^{\mathcal{M}}$. In either case, if W is closed under a suitable inner model operator (as is typical in Woodin's applications), then the core model of W is appropriately iterable, not just in W, but in V as well.

Proof sketch. We begin just as in the proof of Theorem 1.1, and use the notation established there. The proof will simplify substantially, and there will be no need for an induction. So we have $X \prec V_{\Omega+1}$ with ${}^\omega X \cap V_\Omega \subset X$. The inverse collapse of X is $\pi : N \longrightarrow V_{\Omega+1}$. We have $\pi(\overline{\kappa}) = \kappa$, $\pi(\overline{\lambda}) = \lambda = (\kappa^+)^W$, $\pi(\overline{W}) = W$, and $\delta = \mathrm{crit}(\pi) < \kappa$. We assume from the start that X is cofinal in λ.

By the condensation results in Section 8 of [MiSt], $\overline{W}$ and W agree below $(\delta^+)^{\overline{W}}$. First, consider the easy case in which no matter which embedding π we use, it always happens that $(\delta^+)^{\overline{W}} = (\delta^+)^W$, so that $\overline{W}$ and W have the same subsets of δ. Since δ can be taken arbitrarily large below κ, and the measure U derived from π satisfies (1) for any given δ, we are done. Thus, we may assume without loss of generality that $(\delta^+)^{\overline{W}} < (\delta^+)^W$.

Since there are no total-on-W extenders on the W-sequence, all non-trivial iterations of W are linear and involve a drop (a trivial iteration is one that only involves padding). Similarly for $\overline{W}$. As in the proof of Theorem 1.1, $(\overline{\mathcal{T}}, \mathcal{T})$ is the coiteration of $(\overline{W}, W)$ and has length $\theta + 1$. By standard arguments, $\overline{\mathcal{T}}$ is a trivial iteration, and $\mathcal{T}$ is a linear iteration that drops immediately to a level of W below $(\delta^+)^W$. Moreover, for every $\eta < \theta$, $E^{\mathcal{T}}_\eta$ has only one generator, its critical point, and there are no extenders on the sequence of $\mathcal{M}^{\mathcal{T}}_\eta \parallel \mathrm{lh}(E^{\mathcal{T}}_\eta)$ that are total on $\mathcal{M}^{\mathcal{T}}_\eta \parallel \mathrm{lh}(E^{\mathcal{T}}_\eta)$.

Let η be least such that $\mathrm{crit}(E^{\mathcal{T}}_\eta) \geq \overline{\kappa}$, and let $\mathcal{P}$ be the longest initial segment of $\mathcal{M}^{\mathcal{T}}_\eta$ with exactly the subsets of $\overline{\kappa}$ that are in $\overline{W}$. Note that $\mathcal{P}$ is what we were calling $\mathcal{P}_\gamma$ in the proof of Theorem 1.1. Our situation here is simple enough to determine that $\mathcal{P}$ is a set premouse, and that if $\mathcal{P}$ is active, then $\dot{\mu}^{\mathcal{P}} \geq \overline{\kappa}$. Let $\mathcal{R} = \mathrm{ult}(\mathcal{P}, E_\pi \restriction \kappa)$. Countable completeness is enough to guarantee that $\mathcal{R}$ is wellfounded, and so $\mathcal{R}$ is a κ-sound premouse which agrees with W up to λ and satisfies $(\kappa^+)^{\mathcal{R}} = \lambda$. Countable completeness can also be used to show that $\mathcal{R}$ is iterable.

Now let $(\mathcal{U}, \mathcal{V})$ be the coiteration of $(\mathcal{R}, W)$. Then $\mathcal{V}$ is a trivial iteration. Thus, the coiteration has length OR, and W is the lined up model of $\mathcal{U}$. It is also easy to see that $\mathcal{U}$ is a linear iteration such that for every ordinal ξ, there is no extenders on the sequence of $\mathcal{M}^{\mathcal{U}}_\xi \parallel \mathrm{lh}(E^{\mathcal{U}}_\xi)$ that are total on $\mathcal{M}^{\mathcal{U}}_\xi \parallel \mathrm{lh}(E^{\mathcal{U}}_\xi)$. Let $\xi + 1$ be the last drop of any kind on $\mathcal{U}$; if there are no drops, then set $\xi = 0$. Let $\mathcal{M}$ be the initial segment of $\mathcal{M}^{\mathcal{U}}_\xi$ whose last extender is $E^{\mathcal{U}}_\xi$. Then, the iteration $\mathcal{I}$ described in (2) of the corollary is essentially the tail of $\mathcal{U}$ starting from $\mathcal{M}$.

This completes the sketch of the corollary. We remark that $\mathcal{M}$ is a type I premouse. And, also, that if $\mathcal{M}$ is not sound, then $\mathcal{M} = \mathcal{R}$ and $\mathcal{M}$ is κ-sound. $\square$ Corollary 4.1

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Department of Mathematics, University of Florida, Gainesville, FL 32611-2082

Department of Mathematics, MIT, Cambridge, MA 02139-4307

Department of Mathematics, UCLA, Los Angeles, CA 90024-1555

E-mail addresses: mitchell@math.ufl.edu, ernest@math.mit.edu, steel@math.ucla.edu