Iterations with long extenders

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Abstract

We look at the comparison process involving extenders that have generators above the images of their critical points. One main difference here is that we allow those large generators to moved along branches of an iteration tree. After discussing some examples to motivate this, we prove the comparison lemma for premice having extenders that are not too long.

1. Long extenders and large generators

Definition *(large generators and long extenders)*
Let M be a transitive and rud-closed model, and let E be an extender over M. A generator ξ of E is *large* if $i_E(\mathrm{crit}(E)) < \xi$, where $i_E : M \to \mathrm{ult}(M, E)$ is the canonical embedding.
We say that E is *long* if E has at least one large generator.

We denote the strict supremum of the generators of E by $\nu(E)$. Thus E is long iff $i_E(\mathrm{crit}(E)) < \nu(E)$. If $a \in [\mathrm{lh}(E)]^{<\omega}$, then we let $\sigma_E(a) =$ "least σ s.t. $a \subseteq i_E(\sigma)$". Thus E_a is an ultrafilter on $[\sigma_E(a)]^{|a|}$. E is long iff there is a finite $a \subseteq \nu(E)$ s.t. $\sigma_E(a) > \mathrm{crit}(E)$.

When we want to include long extenders in our setup for inner model theory, we run into problems when we try to compare them in the usual way, that is, avoiding overlapping of extenders along branches of the resulting iteration tree. We illustrate this in the examples given below in section 2. It turns out that for fine-structural models M, i.e. premice, we cannot avoid overlapping even when there is only one generator of E above $i_E(\mathrm{crit}(E))$. Concerning the fine-structural context, we have to deal with the following questions:

How to compare premice having long extenders?
What is iterability in this situation?
What kind of iterations are needed to compare premice?
What is the correct definition of premice (initial segment condition)?

We give partial answers to these questions below (namely for the special case when there are not too many large generators).

2. Three cautionary examples

A. Moving spaces problem

Suppose we have defined an iteration tree up to β. Now if $\kappa = \mathrm{crit}(E_\beta) < \nu(E_\alpha)$, $\alpha < \beta$, then the usual application of E_β to M_α (or an initial segment of M_α) doesn't make sense in general: There could be an $(E_\beta)_a$ which concentrates on $[\sigma_{E_\beta}(a)]^{|a|}$, where $\sigma_{E_\beta}(a)$ is bigger than $\mathrm{On} \cap M_\alpha$.

To produce this state of affairs, let E be a κ,λ-extender in $M_{\alpha+1}$, and suppose that there is a measurable μ and a finite $a \subseteq \lambda$ s.t. $\kappa < \mu < \min(a)$ and $\mu = \sigma_a$. Now we iterate $M_{\alpha+1}$ to M_β using the measure on μ until $\mu^* = i_{\alpha+1,\beta}(\mu)$ gets larger than $\mathrm{On} \cap M_\alpha$. But then $\mu^* = \sigma_{E_\beta}(a^*)$, where a^* is the image of a, and we cannot apply $E_\beta := i_{\alpha+1,\beta}(E)$ to M_α.

This problem doesn't occur until past 2^κ-supercompact. But there is a fine-structural version of this problem, to which we now turn.

B. One generator above the image of the critical point

Let M_α be a premouse, and let E_α be a κ,λ-extender over M_α.
Let $i_E : M_\alpha \to \mathrm{ult}(M_\alpha, E_\alpha) =: M_{\alpha+1}$. Suppose that there is a unique generator ξ of E_α s.t. $\xi > i_E(\kappa)$. Then $\nu(E_\alpha) = \xi + 1$.
We have $|\xi| = |i_E(\kappa)|^{M_\alpha}$ and therefore

$$\lambda = (i_{E_\alpha}(\kappa)^+)^{M_{\alpha+1}}.$$

Let $E_{\alpha+1} = i(E_\alpha)$, and let

$$M_{\alpha+2} = \mathrm{ult}(M_{\alpha+2}{}^*, E_{\alpha+1}),$$

where $M_{\alpha+2}{}^* = M_\alpha|\lambda$.
(We have to drop to λ when we want to apply $E_{\alpha+1}$ to M_α.)
But here $\mathrm{ult}(M_{\alpha+2}{}^*, E_{\alpha+1})$ doesn't make obvious sense, since $(E_{\alpha+1})_{\{i(\xi)\}}$ concentrates on unbounded subsets of λ.

So even with only one generator past the image of the critical point there are problems.

C. Trees with short branches

This example is due to Itay Neeman. Let E be an extender on V with critical point κ, and assume that $V_{i(\kappa)+\omega} \subseteq M := \mathrm{ult}(V, E)$, and that $\mathrm{support}(E) = V_{i(\kappa)+\omega}$, where $i = i_E$. Then all the component measures of E concentrate on $V_{\kappa+\omega}$.
We form an iteration tree $\langle T, \langle M_n, E_n : n \in \omega\rangle\rangle$ of length ω on V just using E and its images:
Set $E_0 = E$, $M_0 = V$. Given M_n, E_n, let

$$k = \text{"least k s.t. } \mathrm{crit}(E_n) < \nu(E_k)\text{"}.$$

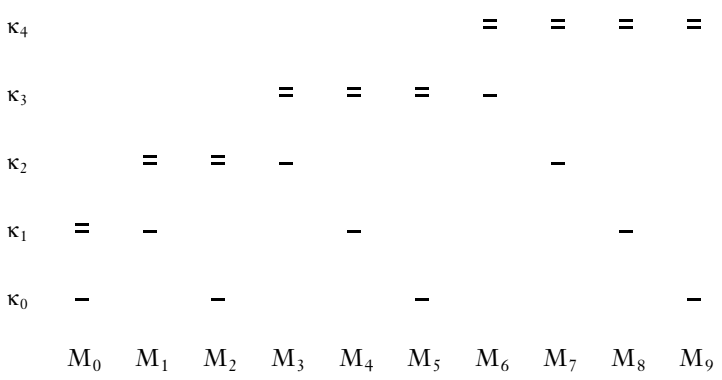
We then set

$M_{n+1} = \mathrm{ult}(M_k, E_n)$,
$E_{n+1} = i_{E_n}(E_k)$.

All these ultrapowers make sense. But:

(+) T has height 3 (i.e., at most two extenders are used along each branch).

It is easy to see this. The following diagram displays the extenders E_n over M_n with $\kappa_0 = \kappa$, $\kappa_{n+1} = i(\kappa_n)$, where $i = i_E : V \to \mathrm{ult}(V, E)$).



The tree order T is:

0 :
1 : 4, 8, 13, 19, 26, ...
2
3 : 7, 12, 18, 25, ...
5
6 : 11, 17, 24, ...
9
10 : 16, 23, ...
14
15 : 22, ...
20
21 : ...
...

(Here a colon indicates T-successors. The successors of 0 are 1, 2, 3, 5, …, and, e.g., the successors of 6 are 11, 17, 24, …)

For conditions on the extender sequence which imply the existence of cofinal branches see [Steel 1993, *The well-foundedness of the Mitchell order*] and [Neeman, 2002].

3. A comparison lemma

We generalize the definition of a premouse allowing (moderately) long extenders to live on the sequence of a premouse (the initial segment condition is as before). The smallness condition we impose is specified by the following definition:

Definition *(a smallness condition for long extenders)*
Let G be a (κ, λ)-extender on the sequence of a premouse M, and let $G = E^M_\alpha$. We say that G satisfies the smallness condition (S) if the following holds:

(S) $L[\vec{E} \restriction (\alpha + 1)] \models |lh(G)| \leq i_G(\kappa)$,

where i_G is the canonical embedding.
We say that *M satisfies (S)*, or that *M has (S)-extenders*, if all long extenders on the M-sequence satisfy (S).

Thus for an extender G satisfying (S) we can collapse the length of G to the image of the critical point by just adding ordinals above the stage where we activated G.

When dealing with large generators, the following modification of the iteration game seems to be natural (in order to avoid the problems discussed in the previous section).

The iteration game for premice with large generators
We define the iteration game as before, but now we allow large generators to be moved. That is, we define the premouse M_β the extender E_α is applied to by:

β = "least $\beta \leq \alpha$ s.t. $crit(E_\alpha) < \min(\nu(E_\beta), i_{E_\beta}(crit(E_\beta)))$".

We now prove the comparison lemma for mice satisfying the smallness condition above.

Theorem *(comparison lemma for premice with (S)-extenders)*

Let M, N be k-sound premice of size $\leq \theta$ s.t. the extenders E on the sequences of M and N satisfy the smallness condition (S).

Suppose that Σ, Γ are $(k, \theta^+ + 1)$-iteration strategies for M and N, resp. Then the Σ, Γ-coiteration of M, N terminates before step θ^+ (and one side of the comparison does not drop in model or degree).

Proof

We do the well-known reflection argument of the proof of the comparison lemma, cf. [Steel 2000]. Let $\mathcal{T}$ and $\mathcal{U}$ be the trees on M and N the coiteration produces, and assume that $\mathcal{T}$ and $\mathcal{U}$ are of size θ^+. Let η be large enough and let $X \prec V_\eta$ s.t.

(i) $|X| = \theta$,
(ii) $\alpha := X \cap \theta^+ \in On$,
(iii) $\mathcal{T}, \mathcal{U} \in X$.

Let $\pi : H \rightarrow V_\eta$, H transitive, be the collapsing embedding. Note that $crit(\pi) = \alpha$, $\pi(\alpha) = \theta^+$. As usual, we get

$i^{\mathcal{T}}_{\alpha,\theta^+} = \pi | M^{\mathcal{T}}_\alpha$,
$i^{\mathcal{U}}_{\alpha,\theta^+} = \pi | M^{\mathcal{U}}_\alpha$,

(We ignore here and in the following the difference between M and $\mathcal{C}_0(M)$ for notational simplicity.) Thus the mappings $i^{\mathcal{T}}_{\alpha,\theta^+}$ and $i^{\mathcal{U}}_{\alpha,\theta^+}$ agree wherever they are defined, and they are defined on the same subsets of α. In particular we have $crit(i^{\mathcal{T}}_{\alpha,\theta^+}) = crit(i^{\mathcal{U}}_{\alpha,\theta^+}) = \alpha$. Let

$\xi + 1$ = "the T-successor of α in $[\alpha, \theta^+]_T$",
$\gamma + 1$ = "the U-successor of α in $[\alpha, \theta^+]_U$",
α^* = $i^{\mathcal{T}}_{\alpha,\xi+1}(\alpha)$,
α' = $i^{\mathcal{U}}_{\alpha,\gamma+1}(\alpha)$.
G = $E^{\mathcal{T}}_\xi$, i.e., the extender on M_ξ used to go from M_α to $M_{\xi+1}$,
H = $E^{\mathcal{U}}_\gamma$, i.e., the extender on M_γ used to go from M_α to $M_{\gamma+1}$,

We shall first look at the $\mathcal{T}$-side, and therefore drop $\mathcal{T}$-superscripts.
Assume that G has a large generator and let

E = "the (α^*, θ^+)-extender derived from $i_{\xi+1,\theta^+}$",
M_ξ^* = "the longest initial segment N of M_ξ s.t. $\mathcal{P}(\alpha^*) \cap N \subseteq M_{\xi+1}$",
$k : M_\xi^* \rightarrow ult_n(M_\xi^*, E)$, where n is least such that $\rho_{n+1}(M_\xi^*) = \alpha^*$.

Then:

(#) k(G) is an initial segment of E_π.

Proof of (#)

We shall need the natural embedding $\sigma : \mathrm{ult}(M_{\xi+1}, E) \to M_{\theta^+}$. Let θ^* = "the cardinal successor of θ^+ in $\mathrm{ult}(M_{\xi+1}, E)$". Notice that $\mathrm{crit}(\sigma) \geq \theta^*$. Now let

- $a \subseteq \theta^+$ be finite,
- $f : [\alpha^*]^{|a|} \to [\nu(G)]^{|a|}$, $f \in M_\xi^*$,
- $z \subseteq [\alpha^+]^{|a|}$, where $\alpha^+ := (\alpha^+)^{M_\alpha}$ $(= (\alpha^+)^{M_\beta}$ for all $\beta \geq \alpha)$.

(We can write elements of k(G) in the form $([a, f]_E^{M_{\xi^*}}, z)$: Note that k(G) is an $(\alpha, k(\mathrm{lh}(G)))$-extender on $\mathrm{ult}(M_\xi^*, E)$. Thus every finite $b \subseteq \nu(k(G))$ can be written as $[a, f]_E^{M_{\xi^*}}$ with a, f as above, since E has length θ^+ and $\sigma_E = \alpha^*$ holds because $i_{\xi+1,\theta^+}(\alpha^*) = \theta^+$. Moreover, we have $\sigma_{k(G)} \leq \alpha^+$ because $\sigma_G \leq \alpha^+$ and $k(\alpha^+) = \alpha^+$.)

Then:

$([a, f]_E^{M_{\xi^*}}, z) \in k(G)$ *iff* $[k(z) = z]$

$(f(u), z) \in G$ for E_a a.e. u *iff*

$f(u) \in i_G(z)$ for E_a a.e. u *iff* $[i_E^{M_{x+1}}(z) = z]$

$[a, f]_E^{M_{\xi+1}} \in i_E^{M_{\xi+1}}(i_G(z))$ *iff* $[\mathrm{crit}(\sigma) \geq \theta^* > [a, f]_E^{M_{\xi+1}}]$

$[a, f]_E^{M_{\xi+1}} \in \sigma(i_E^{M_{\xi+1}}(i_G(z)))$ *iff* $[i_G(z) = i_{\alpha,\xi+1}(z)]$

$[a, f]_E^{M_{\xi+1}} \in \pi(z)$ *iff*

$([a, f]_E^{M_{\xi+1}}, z) \in E_\pi$.

But $[a, f]_E^{M_{\xi+1}} = [a, f]_E^{M_{\xi^*}}$, and this proves (#).

Now assume that H has a large generator, and let, dropping $\mathcal{U}$-superscripts,

F = "the (α', θ^+)-extender derived from $i_{\gamma+1,\theta^+}$",

M_γ^* = "the longest initial segment N of M_γ s.t. $\mathcal{P}(\alpha') \cap N \subseteq M_{\gamma+1}$",

$\ell : M_\xi^* \to \mathrm{ult}_{n'}(M_\gamma^*, F)$, where n' is least such that $\rho_{n'+1}(M_\gamma^*) = \alpha'$.

As in the proof of (#) above we get that $\ell(H)$ is an initial segment of E_π. Thus k(G) is an initial segment of $\ell(H)$ or vice versa. We show:

(+) $k(G) = \ell(H)$.

Proof of (+)

Assume otherwise. We may then assume w.l.o.g. that k(G) is a proper initial segment of $\ell(H)$. Let

$P = \mathrm{ult}_n(M_\xi^*, E)$,
$Q = \mathrm{ult}_{n'}(M_\gamma^*, F)$.

Notice that $M_{\theta^{\bar{g}}} | \theta^+ = M_{\theta^{\bar{u}}} | \theta^+$ is an initial segment of both P and Q. The idea is to use the fact that M_ξ^* collapses lh(G) to α^* to get the desired contradiction (α^* is a cardinal in Q). By our smallness assumption, k(G) and $\ell(H)$ are the last active predicates of P and Q, respectively. It seems that we can not directly infer that $P \in Q$, but we can show that P is an element of ult(Q, $\ell(H)$): We have that $k(G) \in Q$, ult(Q, lh(H)) by assumption and the initial segment condition. But P can be computed from k(G) and $Q | (\alpha^{+++})^Q = P | (\alpha^{+++})^P$ (by taking the ultrapower). Thus $P \in$ ult(Q, $\ell(H)$). Now let

$f : \alpha^* \to M_\xi^*$ onto, $f \in \Sigma_{n+1}(M_\xi^*, p)$ for a parameter p.

Then $k \circ f$ maps α^* onto rng(k), and $k \circ f \in \Sigma_{n+1}(P, k(p))$. Therefore rng(k) $\in$ ult(Q, $\ell(H)$). But M_ξ^* is the transitive collapse of rng(k), and thus M_ξ^* $\in$ ult(Q, $\ell(H)$). Thus lh(G) has cardinality α^* in Q, a contradiction, since lh(G) is a cardinal in $M_{\theta^{\bar{g}}} | \theta^+$, hence a cardinal in Q. This proves (+).

Thus we have $k(G) = \ell(H)$. But then

(++) $P = Q$,

because we can recover P and Q from k(g) and the model up to θ^+.
Finally we will show that $M_\xi^* = M_\gamma^*$. From this and the fact that G and H are the last active predicates in the models it follows that G = H, contradiction!

We have that E is close to M_ξ^*, i.e.,

(i) if $A \in M$, $M \models |A| \leq \alpha^*$, then $E_\alpha \cap A \in M$,
(ii) E_α is $\Sigma_1(M)$ in parameters for every finite $a \subseteq \theta^+$.

(cf. [Mitchell, Steel 1994, 6.1.5])
It follows from this that $M_\xi^* = \mathscr{C}_{n+1}(P)$, $P = \mathscr{C}_n(P)$. In the same way we get that $M_\gamma^* = \mathscr{C}_{n'+1}(Q)$, $Q = \mathscr{C}_{n'}(Q)$. But P = Q by (++), and therefore n = n' and M_ξ^* $= M_\gamma^*$, as desired.

This finishes the proof in the case where both G and H have large generators. If both G and H are short, then we argue as in the proof of the comparison lemma for mice without long extenders, i.e., we show that G and H are compatible. So let as assume w.l.o.g. that G is long and that H has no large generators.

1. case $\alpha' < \alpha^*$

In this case we don't move generators of H on both sides of the iteration, and the usual argument yields that H is an initial segment of G, which is a contradiction.

2. case $\alpha^* \leq \alpha'$

Then H is an initial segment of E_π, and we get that H is an initial segment of k(G). If follows by the initial segment condition that H is on the sequence of P, so it collapses its length. But lh(H) is a cardinal in P, contradiction.

This finishes the proof of the comparison lemma.

4. References

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