A Weak Dodd-Jensen Lemma

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Abstract

We show that every sufficiently iterable countable mouse has a unique iteration strategy whose associated iteration maps are lexicographically minimal.This enables us to extend the results of [2] on the good behavior of the standard parameter from tame mice to arbitrary mice.

1 Introduction and Preliminaries

The Dodd-Jensen lemma on the minimality of iteration maps is a fundamental, often-used tool in inner model theory. Unfortunately, the proof of the lemma seems to require the hypothesis that the premouse involved has a *unique* iteration strategy, and this uniqueness assumption is sometimes a source of problems in applications.For example,in [1], [3], and [2], the Dodd-Jensen lemma is used heavily in showing that the standard parameters of a sufficiently iterable, tame mouse are solid and universal (and hence preserved by the appropriate embeddings), and in order to apply the lemma a fair amount of work must be done to reduce to the case of mice having unique iteration strategies.Worse still, this reduction only seems to work for tame mice. (See [2, section 3].) In this note we establish a version of the Dodd-Jensen lemma which has no uniqueness hypothesis, and use it to extend the solidity and universality results of [2] to arbitrary iterable mice.

We shall assume the reader is familiar with [1] and [3].

Let us recall some notions and facts concerning iterability. Let $\mathcal{M}$ be a k-sound premouse, where $k \leq \omega$, and let θ be an ordinal; then $\mathcal{G}_k(\mathcal{M}, \theta)$ is the full iteration game on $\mathcal{M}$ of length θ. In this game Players I and II cooperate to produce a k-maximal iteration tree on $\mathcal{M}$, with Player I extending the tree at successor steps by picking an extender from the current model. (k-maximality then requires that the extender be used to form an ultrapower of the longest possible initial segment of the earliest possible model, and determines the degree of the ultrapower. If this ultrapower is illfounded, then play is over and I has won.) At limit steps below θ, Player II must pick a cofinal wellfounded branch of the tree produced so far, thereby extending it one more step; if he fails to do so, then he loses. If after θ moves of $\mathcal{G}_k(\mathcal{M}, \theta)$, II has not lost, then he wins.

The Dodd-Jensen Lemma forces us to consider an elaboration of this game whose output is a linear composition of appropriately maximal iteration trees. Let θ be a successor ordinal. In $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$, there are α rounds, the β^{th} being played as follows. Let $\mathcal{Q}$ be the last model in the linear composition produced before round β; that is, let $\mathcal{Q} = \mathcal{M}$ if $\beta = 0$, $\mathcal{Q}$ be the last model of the tree played during round $\beta - 1$ if $\beta > 0$ is a successor, and $\mathcal{Q}$ the direct limit along the unique cofinal branch in the linear composition of trees produced before β, if β is a limit ordinal. (I wins if this branch is illfounded.) I begins round β by choosing an initial segment $\mathcal{P}$ of $\mathcal{Q}$, and an $i \leq \omega$ such that if $\mathcal{P} = \mathcal{Q}$ then $i \leq q$, where q is the degree of $\mathcal{Q}$. The rest of round β is a run of $\mathcal{G}_i(\mathcal{P}, \theta)$, except that we allow I to exit to round $\beta + 1$ before all θ moves have been played. (Since θ is a successor ordinal, when round β ends there will be in any case a last model to serve as $\mathcal{Q}$ for round $\beta + 1$.) II wins $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ just in case he does not lose any of the component games and, for $\beta \leq \alpha$ a limit ordinal, the unique cofinal branch in the composition of trees previously produced is wellfounded. A play of this game in which II has not yet lost is called a *k-bounded* iteration tree on $\mathcal{M}$. Notice that any winning strategy Γ for II in $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$ determines a winning strategy Σ for II in $\mathcal{G}_k(\mathcal{M}, \theta)$ in an obvious way: Σ calls for II to play as if he were using Γ in the first round of $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$, and I had not dropped to begin that round. We call Γ an *extension* of Σ from $\mathcal{G}_k(\mathcal{M}, \theta)$ to $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$.

Finally, $\mathcal{W}\mathcal{G}_k(\mathcal{M}, \alpha)$ is the weak iteration game on $\mathcal{M}$ of length α (see [3, section 1]), but modified as above so that I can drop in model or degree at the beginning of a round. In this game too I and II produce a k bounded composition of appropriately maximal iteration trees, but here I is given less freedom in choosing extenders.(At round β, I may drop in the model handed to him by the previous rounds; he then must play a countable, appropriately maximal, putative iteration tree on the current model. II can accept this tree if it has a last wellfounded model, and this model is then handed to the next round; otherwise II must play a maximal wellfounded branch of the tree, whose direct limit model is then handed to the next round.) We shall only consider this game in the case $\mathcal{M}$ is countable and $\alpha = \omega_1$. The weak game is important because our iterability theorems directly give winning strategies for II in it, rather than in the full game $\mathcal{G}_k(\mathcal{M}, \theta)$.

Definition 1.1 *Let $\mathcal{M}$ be a k-sound premouse, where $k \leq \omega$; then a (k, θ) (respectively, (k, α, θ), or* **weak (k, α)) iteration strategy** *for $\mathcal{M}$ is a winning strategy for* II *in $\mathcal{G}_k(\mathcal{M}, \theta)$ (respectively, in $\mathcal{G}_k(\mathcal{M}, \alpha, \theta)$, or $\mathcal{WG}_k(\mathcal{M}, \alpha)$). $\mathcal{M}$ is* **(k, θ)-iterable** *(respectively,* **(k, α, θ)-iterable, or weakly (k, α)-iterable**) *just in case there is such a strategy.*

Suppose $\pi: \mathcal{M} \to \mathcal{N}$ is a weak k-embedding. If $\mathcal{T}$ is a k-bounded iteration tree on $\mathcal{M}$, then we can copy $\mathcal{T}$ with a tree $\pi\mathcal{T}$ on $\mathcal{N}$ having the same tree, drop, and degree structure as $\mathcal{T}$. In the inductive construction of $\pi\mathcal{T}$ we define, for each $\alpha < lh(\mathcal{T})$, a weak $\deg^{\mathcal{T}}(\alpha)$-embedding $\pi_\alpha : \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{N}_\alpha^{\pi\mathcal{T}}$. This copying construction enables us to pull back a strategy Σ in some game of the form $\mathcal{G}_k$ or $\mathcal{WG}_k$ on $\mathcal{N}$ to a strategy Σ^π in the same game on $\mathcal{M}$. Thus if $\mathcal{N}$ is (k, θ), (k, α, θ), or weakly (k, α) iterable, then so is $\mathcal{M}$. This pullback also makes sense if Σ is a strategy for II in a game of the form $\mathcal{G}_i(\mathcal{P}, \alpha, \theta)$ or $\mathcal{WG}_i(\mathcal{P}, \alpha)$, for some $\mathcal{P}$ of which $\mathcal{N}$ is an initial segment and such that $k \leq i$ if $\mathcal{N} = \mathcal{P}$, since $\mathcal{G}_i$ on $\mathcal{P}$ becomes $\mathcal{G}_k$ on $\mathcal{N}$ if I drops appropriately in round 0, and similarly for $\mathcal{WG}$.

$$\begin{tikzcd}
\mathcal{N} \arrow[r, dotted, "\pi\mathcal{T}"] & \mathcal{N}_\alpha^{\pi\mathcal{T}} \\
\mathcal{M} \arrow[u, "\pi"] \arrow[r, dotted, "\mathcal{T}"] & \mathcal{M}_\alpha^{\mathcal{T}} \arrow[u, "\pi_\alpha"']
\end{tikzcd}$$

Definition 1.2 *Let $\pi: \mathcal{M} \to \mathcal{N}$ be a weak k-embedding, and Σ a strategy for* II *in $\mathcal{G}_k(\mathcal{N}, \theta)$, or in $\mathcal{G}_i(\mathcal{P}, \alpha, \theta)$ or $\mathcal{WG}_i(\mathcal{P}, \alpha)$ for some $\mathcal{P}$ such that $\mathcal{N}$ is an initial segment of $\mathcal{P}$ and i such that $k \leq i$ if $\mathcal{N} = \mathcal{P}$; then the* **pullback of Σ under π** *is the strategy Σ^π in the corresponding game on $\mathcal{M}$ such that for any k-bounded T on $\mathcal{M}$,*

$$\mathcal{T} \text{ is by } \Sigma^\pi \iff \pi\mathcal{T} \text{ is by } \Sigma.$$

Finally, for the reader's convenience we state an abstract form of the full Dodd-Jensen Lemma. Suppose Σ is a (k, α, θ) or a weak (k, α) iteration strategy; then we call a position u which is according to Σ *transitional* if $u = (r, (\mathcal{P}, i))$ where r represents some number $\beta < \alpha$ of complete rounds of play according to Σ in which I has not lost, and $(\mathcal{P}, i)$ is a way I might legally begin round $\beta + 1$. Notice that in this situation, Σ determines an $(i, \alpha - \beta, \theta)$ or weak $(i, \alpha - \beta)$ iteration strategy for $\mathcal{P}$. We call this strategy Σ_u.

Definition 1.3 *Let Σ be a (k, θ) iteration strategy for $\mathcal{M}$, let α be a limit ordinal, and let Γ be a (k, α, θ) iteration strategy for $\mathcal{M}$ which extends Σ; then we say Γ is a* **consistent extension of** Σ *iff whenever $u = (r, (\mathcal{P}, k))$ is a transitional position of Γ and $\pi: \mathcal{M} \to \mathcal{P}$ is a weak k-embedding, then we have that $(\Gamma_u)^\pi$ extends Σ.*

Theorem 1.4 (The Dodd-Jensen Lemma) *Let Σ be a (k, θ) iteration strategy for $\mathcal{M}$, and suppose there is a (k, ω, θ) iteration strategy for $\mathcal{M}$ which is a consistent extension of Σ. Let $\mathcal{T}$ be a k-maximal tree played according to Σ, and $\pi: \mathcal{N}$ a weak k-embedding, where $\mathcal{N}$ is an initial segment of $\mathcal{M}_\alpha^\mathcal{T}$. Then*

1. *$\mathcal{N} = \mathcal{M}_\alpha^\mathcal{T}$,*
2. *$[0, \alpha]_T$ does not drop (in model or degree), and*
3. *for all $x \in \mathcal{M}, i_{0,\alpha}^\mathcal{T}(x) \leq_L \pi(x)$, where $\leq_L$ is the order of construction.*

One could easily formulate 1.3 and 1.4 more generally, so that they would apply to (k, α, θ) and weak (k, α) iteration strategies Σ, in the case that α is additively closed, but we shall not bother to do this.

It is clear that if Σ is the unique (k, θ) iteration strategy for $\mathcal{M}$, then any extension of Σ to a (k, ω, θ) iteration strategy for $\mathcal{M}$ is a consistent extension of Σ. We do not know whether there can be more than one (k, θ) iteration strategy for $\mathcal{M}$, while at the same time some such strategy has a consistent extension to a (k, ω, θ) iteration strategy for $\mathcal{M}$. This question is most interesting when $\theta = \text{card}(\mathcal{M})^+ + 1)$, since then the iteration strategy suffices for comparison with other mice of the same size. For this value of θ, we do know that there is at most one (k, θ) iteration strategy for $\mathcal{M}$ which has such a consistent extension (see section 2). Also, we do not know whether there is a converse to 1.4, in the sense that the existence of consistent extensions of Σ follows from the minimality of the iteration maps to which it gives rise. (Again, $\theta = \mathrm{card}(\mathcal{M})^+ + 1$ is the interesting case.) Finally, and perhaps most importantly, we do not know whether every $(k, \omega_1, \mathrm{card}(\mathcal{M})^+ + 1)$ iterable premouse $\mathcal{M}$ has a $(k, \mathrm{card}(\mathcal{M})^+ + 1)$ iteration strategy Σ all of whose iteration maps are minimal in the sense of 1.4. We can, however, prove a useful result in this direction.

2 The Weak Dodd-Jensen Property

Let $\mathcal{M}$ and $\mathcal{P}$ be premice; then we say that $\mathcal{P}$ is $(\mathcal{M}, k)$-*large* just in case there is a near k-embedding from $\mathcal{M}$ to an initial segment of $\mathcal{P}$.(A near k-embedding is a weak k-embedding which is $r\Sigma_{k+1}$ elementary. See [2, 1.2,1.3], where it is shown that the copying construction gives rise to such embeddings. We could make do with weak k-embeddings throughout this paper, but it would be a bit awkward at one point.) Let $\vec{e} = \langle e_i \mid i < \omega \rangle$ enumerate the universe of a countable premouse $\mathcal{M}$, and $\pi\colon \mathcal{M} \to \mathcal{P}$ be a near k-embedding; then we say π is $(k, \vec{e})$-*minimal* iff whenever σ is a near k- embedding from $\mathcal{M}$ to an initial segment $\mathcal{N}$ of $\mathcal{P}$, then $\mathcal{N} = \mathcal{P}$ and either $\sigma = \pi$, or $\sigma(e_i) > \pi(e_i)$ where i is least such that $\sigma(e_i) \neq \pi(e_i)$. Notice that if $\mathcal{P}$ is $(\mathcal{M}, k)$-large but no proper initial segment of $\mathcal{P}$ is $(\mathcal{M}, k)$-large, then there is a $(k, \vec{e})$-minimal embedding from $\mathcal{M}$ to $\mathcal{P}$. This embedding is just the leftmost branch through a certain tree.

Definition 2.1 *Let Σ be a (k, α, θ) or weak (k, α) iteration strategy for $\mathcal{M}$, and $\vec{e}$ enumerate the universe of $\mathcal{M}$ in order type ω; then we say Σ has the* **weak Dodd-Jensen property (relative to $\vec{e}$)** *iff whenever $\mathcal{T}$ is an iteration tree on $\mathcal{M}$ played according to Σ, and $\beta < lh(\mathcal{T})$ is such that $\mathcal{M}_\beta^\mathcal{T}$ is $(\mathcal{M}, k)$-large, then $i_{0,\beta}^\mathcal{T}$ exists and is $(k, \vec{e})$-minimal.*

Theorem 2.2 (The Weak Dodd-Jensen Lemma) *Suppose $\mathcal{M}$ is (k, ω_1, θ) iterable (resp. weakly (k, ω_1) iterable), and that $\vec{e}$ enumerates the universe of $\mathcal{M}$ in order type ω; then there is a (k, ω_1, θ) iteration strategy (resp. weak (k, ω_1) iteration strategy) for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$.*

Proof. Let Σ be any (k, ω_1, θ) iteration strategy for $\mathcal{M}$ (the case Σ is a weak strategy is no different, so we omit it). We shall construct a transitional position $u = (r, (\mathcal{P}, k))$ of Σ and a $(k, \vec{e})$-minimal embedding $\pi: \mathcal{M} \to \mathcal{P}$ such that π is strongly $(k, \vec{e})$ minimal, in the sense that whenever $\mathcal{R}$ is an $(\mathcal{M}, k)$-large Σ_u-iterate of $\mathcal{P}$, then there is no dropping in the iteration from $\mathcal{P}$ to $\mathcal{R}$, and if $i: \mathcal{P} \to \mathcal{R}$ is the iteration map, then $i \circ \pi$ is $(k, \vec{e})$ minimal. It is then easy to see that the π- pullback of Σ_u has the weak Dodd-Jensen property.

Let us call a pair $(r, \mathcal{Q})$ *suitable* if $(r, (\mathcal{Q}, k))$ is transitional, and $\mathcal{Q}$ is $(\mathcal{M}, k)$-large but no proper initial segment of $\mathcal{Q}$ is $(\mathcal{M}, k)$-large. In order to obtain the desired u and π, we define by induction on $n < \omega$ suitable pairs $(r_n, \mathcal{P}_n)$. We maintain inductively that r_{n+1} extends $(r_n, (\mathcal{P}_n, k))$. We begin by letting r_0 be the empty position, and $\mathcal{P}_0 = \mathcal{M}$. Now suppose r_n and $\mathcal{P}_n$ have been defined.

Case 1. There is a suitable $(s, \mathcal{Q})$ such that s extends $(r_n, (\mathcal{P}_n, k)$ and the branch $\mathcal{P}_n - to - \mathcal{Q}$ in the iteration given by s has a drop.

In this case, we simply let $(r_{n+1}, \mathcal{P}_{n+1})$ be any such $(s, \mathcal{Q})$.

Case 2. Otherwise.

Let $\tau: \mathcal{M} \to \mathcal{P}_n$ be $(k, \vec{e})$-minimal.

Subcase 2a. There is a suitable $(s, \mathcal{Q})$ such that s extends $(r_n, (\mathcal{P}_n, k))$, and letting $i: \mathcal{P}_n \to \mathcal{Q}$ be the iteration map given by s, $i \circ \tau$ is not $(k, \vec{e})$-minimal.

In this case, let $m < \omega$ be least such that for some such s, $\mathcal{Q}$, and i we have, letting $\sigma: \mathcal{M} \to \mathcal{Q}$ be $(k, \vec{e})$ minimal, that $\sigma(e_m) \neq i \circ \tau(e_m)$ (and thus $\sigma(e_m) <_L i \circ \tau(e_m)$). We then let $(r_{n+1}, \mathcal{P}_{n+1})$ be a suitable pair $(s, \mathcal{Q})$ witnessing this property of m.

Subcase 2b. Otherwise.

In this case τ is strongly $(k, \vec{e})$-minimal in the sense advertised earlier, so we set $u = (r_n, (\mathcal{P}_n, k))$ and $\pi = \tau$, and stop the construction.

Now suppose the construction never stops. Notice that case 1 can only apply finitely often, since otherwise we get an iteration tree played according to Σ whose unique cofinal branch has infinitely many drops. Suppose then that case 2 applies at all $n \geq n_0$, so that for all $n_0 \leq n \leq m$ we have a k-embedding $i_{n,m}: \mathcal{P}_n \to \mathcal{P}_m$ given by r_m. For $n \geq n_0$, let $\pi_n: \mathcal{M} \to \mathcal{P}_n$ be $(k, \vec{e})$-minimal; then if $n_0 \leq n < m$, π_m is "to the left" of $i_{n,m} \circ \pi_n$. It follows that for any j, $i_{n,m}(\pi_n(e_j)) = \pi_m(e_j)$ for all sufficiently large n, m (by induction on j). Let

$$r = \bigcup_{n<\omega} r_n, \qquad \mathcal{P} = \lim_{n\to\infty} \mathcal{P}_n, \qquad u = (r, (\mathcal{P}, k),$$

let $i_{n,\infty} : \mathcal{P}_n \to \mathcal{P}$ be the direct limit map (a k-embedding), and define $\pi: \mathcal{M} \to \mathcal{P}$ by

$$\pi(e_j) = \text{ eventual value of } i_{n,\infty}(\pi_n(e_j)), as\ \ n \to \infty.$$

We claim that u and π are as advertised earlier.

Clearly π is a near k-embedding, and so $\mathcal{P}$ is $(\mathcal{M}, k)$-large. No proper initial segment $\mathcal{R}$ of $\mathcal{P}$ can be $(\mathcal{M}, k)$-large, as then $(u, \mathcal{R})$ could serve as the $(s, \mathcal{Q})$ witnessing the occurrence of case 1 at a stage $n > n_0$. Similarly, π is $(k, \vec{e})$-minimal. For if σ is a near k-embedding of $\mathcal{M}$ into $\mathcal{P}$ which is to the left of π, then take m_0 to be the least j such that $\sigma(e_j) \neq \pi(e_j)$, and let $l < \omega$ be so large that $n_0 < l$ and $\pi(e_j) = i_{l,\infty}(\pi_l(e_j))$ for all $j \leq m_0$ (and so $m > m_0$, where m is as in case 2a at stage l). Then r,$\mathcal{P}$, and π could serve as the s,$\mathcal{Q}$, and τ witnessing $m \leq m_0$ at stage l, contradiction. Finally, let $\mathcal{R}$ be any $(\mathcal{M}, k)$-large iterate of $\mathcal{P}$ via Σ_u. Clearly, there is a transitional position $(v, (\mathcal{R}, k)$ such that v extends u. We can argue as above that there is no dropping in the iteration tree given by v from $\mathcal{P}$ to $\mathcal{R}$, and that if $i: \mathcal{P} \to \mathcal{R}$ is the iteration map, then $i \circ \pi$ is $(k, \vec{e})$-minimal. Thus u and π are as advertised.

We leave to the reader the easy verification that $(\Sigma_u)^\pi$ has the weak Dodd-Jensen property. □

The following is our uniqueness theorem for iteration strategies with the weak Dodd-Jensen property. Its proof is well known to inner model theorists.

Theorem 2.3 *1. Let $\vec{e}$ enumerate the universe of the k-sound premouse $\mathcal{M}$ in order type ω; then there is at most one $(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$;*

2. if in addition, $\mathcal{M}$ is tame, and there is a measurable Woodin cardinal, then there is at most one weak (k, ω_1) iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property relative to $\vec{e}$; moreover, if there is such a weak iteration strategy, then $\mathcal{M}$ is fully $(k, \omega_1 + 1)$-iterable.

Proof. For (1), suppose Σ and Γ are distinct such strategies. We can find a k-maximal iteration tree $\mathcal{T}$ on $\mathcal{M}$ such that $\mathcal{T}$ has limit length $\lambda < \omega_1$, $\mathcal{T}$ is played according to both Σ and Γ, and $\Sigma(\mathcal{T}) \neq \Gamma(\mathcal{T})$. Let $\mathcal{U}^*$ and $\mathcal{V}^*$ be the iteration trees of length $\lambda + 1$ extending $\mathcal{T}$ produced by Σ and Γ respectively. We now proceed as if we had produced $\mathcal{U}^*$ and $\mathcal{V}^*$ on the two sides of a coiteration, and continue "iterating the least disagreement". We thereby extend $\mathcal{U}^*$ and $\mathcal{V}^*$ to k-maximal trees $\mathcal{U}$ and $\mathcal{V}$, played according to Σ and Γ respectively, in such a way that the last model of one is an initial segment of the last model of the other. We may as well assume that $\mathcal{M}^{\mathcal{U}}_{\alpha}$ is an initial segment of $\mathcal{M}^{\mathcal{V}}_{\beta}$. We can also assume that $D^{\mathcal{U}} \cap [0, \alpha]_U = \emptyset$ and $\deg^{\mathcal{U}}(\alpha) = k$, so that $i^{\mathcal{U}}_{0,\alpha}$ exists and is a k-embedding. (See [1, Theorem 7.1].)

It follows that $\mathcal{M}^{\mathcal{V}}_{\beta}$ is $(\mathcal{M}, k)$-large. Since Γ has the weak Dodd-Jensen property relative to $\vec{e}$, $i^{\mathcal{V}}_{0,\beta}$ exists and is $(k, \vec{e})$-minimal. This implies that no proper initial segment of $\mathcal{M}^{\mathcal{V}}_{\beta}$ is $(\mathcal{M}, k)$-large, so $\mathcal{M}^{\mathcal{U}}_{\alpha} = \mathcal{M}^{\mathcal{V}}_{\beta}$. Because Σ also has the weak Dodd-Jensen property relative to $\vec{e}$, $i^{\mathcal{U}}_{0,\alpha}$ is also $(k, \vec{e})$-minimal. It follows that $i^{\mathcal{U}}_{0,\alpha} = i^{\mathcal{V}}_{0,\beta}$.

Now let $\gamma < \lambda$ be largest such that $\gamma \in [0, \alpha]_U \cap [0, \beta]_V$. Let $\nu = \sup\{\, \nu(E^{\mathcal{T}}_{\xi}) \mid \xi T \gamma\}$. Every member of $\mathcal{M}^{\mathcal{T}}_{\gamma}$ is of the form $i^{\mathcal{T}}_{0,\gamma}(f)(a)$, for some $f \in \mathcal{M}$ and $a \in [\nu]^{<\omega}$. (We take $k = 0$ for notational simplicity; otherwise we have f $r\Sigma_k$ over $\mathcal{M}$.) Since $i^{\mathcal{U}}_{\gamma,\alpha}$ and $i^{\mathcal{V}}_{\gamma,\beta}$ have critical point at least ν, this representation of $\mathcal{M}^{\mathcal{T}}_{\gamma}$ and the fact that $i^{\mathcal{U}}_{0,\alpha} = i^{\mathcal{V}}_{0,\beta}$ yield that $i^{\mathcal{U}}_{\gamma,\alpha} = i^{\mathcal{V}}_{\gamma,\beta}$.

Let $(\xi+1) \in (\gamma, \alpha]_U$ be such that U-pred$(\xi+1) = \gamma$. Let $(\sigma+1) \in (\gamma, \beta]_V$ be such that V-pred$(\sigma + 1) = \gamma$. Since $i^{\mathcal{U}}_{\gamma,\alpha} = i^{\mathcal{V}}_{\gamma,\beta}$, the extenders $E^{\mathcal{U}}_{\xi}$ and $E^{\mathcal{V}}_{\sigma}$ are compatible, that is, they agree up to the inf of the sups of their generators. If $\xi < \lambda$ or $\sigma < \lambda$, this is impossible as no extender used in an iteration tree is compatible with any extender used later in the same tree. If $\lambda \leq \xi$ and $\lambda \leq \sigma$, this is impossible as no two extenders used in a coiteration are compatible. We have reached a contradiction, and thereby proved (1).

The same proof gives the first clause of (2), since by [3, 1.10], weak iteration strategies suffice for the comparison of tame mice. We need the measurable Woodin cardinal here in order to have enough generic absoluteness to get a branch at ω_1. (We can avoid the measurable Woodin if our weak strategy is a "realization strategy"; see below.) This then yields the following full (k, ω_1) strategy for a weakly (k, ω_1)-iterable $\mathcal{M}$: choose the unique cofinal branch b of $\mathcal{T}$ such that $(\mathcal{T}, b)$ is the first round of play according to the unique weak (k, ω_1) strategy with the weak Dodd-Jensen property. The measurable Woodin cardinal guarantees that this extends to a full $(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$. $\square$

3 An Application and a Conjecture

In [1], the Dodd-Jensen lemma is used heavily in the proof that the $k + 1^{st}$ standard parameter of a sufficiently iterable, k-sound, 1-small premouse is k+1-solid and k+1-universal. (See [1, 8.1].) This result is basic to the theory, as it guarantees that the standard parameter is preserved by the appropriate embeddings. The result was extended from 1-small to meek mice in [3], and then to tame mice in [2, 3.10]. In these papers, the authors prove a "strong uniqueness" property of the relevant iteration strategies ([1, 6.2],[2, 3.7]), and thereby obtain the full Dodd-Jensen property for these strategies. Unfortunately, it is not at all clear how push the strong uniqueness proof past the tame mice. Fortunately, we don't need to do so, as we can make do with the weak Dodd-Jensen property.

The following is just Theorem 8.1 of [1], with its 1-smallness hypothesis removed and its iterability hypothesis appropriately strengthened.

Theorem 3.1 *Let $k < \omega$, and let $\mathcal{M}$ be a k-sound, $(k, \omega_1, \omega_1 + 1)$-iterable premouse. Let r be the $k + 1^{st}$ standard parameter of $(\mathcal{M}, u_k(\mathcal{M}))$; then r is $k + 1$-solid and $k + 1$-universal over $(\mathcal{M}, u_k(\mathcal{M}))$.*

Proof. The solidity and universality of r are statements in the first order theory of $\mathcal{M}$, so if they fail, they fail in some countable fully elementary submodel of $\mathcal{M}$. Any countable elementary submodel of $\mathcal{M}$ inherits its $(k, \omega, \omega_1 + 1)$-iterability. Thus we may assume without loss of generality that $\mathcal{M}$ is countable.

Let $r = \langle \alpha_0, ..., \alpha_S \rangle$, where the ordinals α_i are listed in decreasing order. Let $\vec{e}$ be an enumeration of the universe of $\mathcal{M}$ such that $e_i = \alpha_i$ for all $i \leq S$. Let Σ be a $(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ having the weak Dodd-Jensen property relative to $\vec{e}$. We can now simply repeat, almost verbatim, the proof of 8.1 in [1]. Where that proof uses the full Dodd-Jensen property of the unique iteration strategy for its $\mathcal{M}$, we use the weak Dodd-Jensen property of Σ relative to $\vec{e}$. Since the reader with [1] in hand can easily check that this works, we shall give no further detail, except by pointing out that our requirement that $e_i = \alpha_i$ for all $i \leq S$ comes up in the proof of Claim 6 ([1, p.84]), and only there. $\square$

The proof of Theorem 8.1 in [1] also gives some useful condensation results for 1-small mice. Once again, by substituting the weak Dodd-Jensen property for the full one, we can extend these results to arbitrary mice. The following is such an extension of [1, Theorem 8.2].

Theorem 3.2 *Let $\mathcal{M}$ be ω-sound and $(\omega, \omega_1, \omega_1{+}1)$-iterable. Suppose $\pi\colon \mathcal{H} \to \mathcal{M}$ is fully elementary, and crit(π)=$\rho_\omega^{\mathcal{H}}$; then either*

1. *$\mathcal{H}$ is a proper initial segment of $\mathcal{M}$, or*
2. *there is an extender E on the $\mathcal{M}$ sequence such that lh(E)= $\rho_\omega^{\mathcal{H}}$, and $\mathcal{H}$ is a proper initial segment of Ult$_0(\mathcal{M}, E)$.*

The proof of 3.2 is a nearly verbatim copy of the proof in [1]. The version of 3.2 for $n+1$-sound, 1-small mice stated on p. 87 of [1] also extends routinely to arbitrary $n+1$-sound mice.

We turn now to the conjecture.

Our original construction of iteration strategies with the weak Dodd-Jensen property was organized somewhat differently from that presented in Theorem 2.2. Rather than transform an arbitrary strategy into one with the weak Dodd-Jensen property, as is done in 2.2, we observed that the standard methods by which iteration strategies are constructed can be easily modified so as to yield the weak Dodd-Jensen property. This approach has some independent interest, as it leads to a form of the Unique Branches Hypothesis which is not refuted by Woodin's counterexample, yet is strong enough to yield the iterability required to develop inner model theory. So let us look at this approach more closely.

There are two closely related constructions of iterable premice, the $L[\vec{E}]$ construction of [1] and [3], and the K^c construction of [4]. The former requires full background extenders with a certain strength for the extenders put on the coherent sequence of the current approximation $\mathcal{N}_\xi$ to the eventual $L[\vec{E}]$, the latter requires fragments of such background extenders. Let us consider first the construction of [1] and [3], but without any smallness or tameness restriction on the premice $\mathcal{N}_\xi$ constructed. In order to show that the $\mathcal{N}_\xi$ converge to a model of height OR which, for example, satisfies "there is a superstrong cardinal" if V does, it seems necessary and sufficient to prove:

cSBH for $L[\vec{E}]$: Let $\mathcal{M}$ be a countable, fully elementary submodel of $\mathcal{C}_k(\mathcal{N}_\xi)$, for some k, ξ; then $\mathcal{M}$ is $(k, \omega_1 + 1)$-iterable.

In the case that $\mathcal{N}_\xi$ is tame, and one demands only weak (k, ω_1)-iterability, cSBH is known ([1, section 12]). In fact, for tame $\mathcal{N}_\xi$, a slight embellishment on this proof gives a full $(k, \omega_1 + 1)$ iteration strategy for $\mathcal{M}$ which has the weak Dodd-Jensen property. To see this, let $\mathcal{M}$ be tame, and let $\vec{e}$ enumerate the universe of $\mathcal{M}$ in order type ω. Let ξ be least such that there is a near k-embedding from $\mathcal{M}$ into $\mathcal{C}_k(\mathcal{N}_\xi)$. It is easy to see that no proper initial segment of $\mathcal{C}_k(\mathcal{N}_\xi)$ is $(\mathcal{M}, k)$-large, so we have a unique $(k, \vec{e})$-minimal embedding $\pi\colon \mathcal{M} \to \mathcal{C}_k(\mathcal{N}_\xi)$. The proof in [1, section 12] produces a weak (k, ω_1) iteration strategy Σ for $\mathcal{M}$ such that if $\mathcal{T}$ is an almost normal tree played by Σ and $\alpha < \mathrm{lh}(\mathcal{T})$, then

1. if $D^{\mathcal{T}} \cap [0, \alpha]_T = \emptyset$ and $\deg^{\mathcal{T}}(\alpha) = k$, then there is a near k-embedding $\sigma\colon \mathcal{M}^{\mathcal{T}}_\alpha \to \mathcal{C}_k(\mathcal{N}_\xi)$ such that $\pi = \sigma \circ i^{\mathcal{T}}_{0,\alpha}$, and
2. otherwise, there is a near $\deg^{\mathcal{T}}(\alpha)$-embedding of $\mathcal{M}^{\mathcal{T}}_\alpha$ into some $\mathcal{C}_j(\mathcal{N}_\eta)$, where $(\eta, j) <_{\mathrm{lex}} (\xi, k)$.

Let us call an iteration strategy (weak or full) with these properties a π-*realization strategy* for $\mathcal{M}$. It is simply a matter of unravelling definitions to show that, if ξ is least such that $\mathcal{C}_k(\mathcal{N}_\xi)$ is $(\mathcal{M}, k)$-large and $\pi\colon \mathcal{M} \to \mathcal{C}_k(\mathcal{N}_\xi)$ is $(k, \vec{e})$-minimal, then any π realization strategy for $\mathcal{M}$ has the weak Dodd-Jensen property relative to $\vec{e}$. Now we can apply the proof of 2.3 (2) to get a $(k, \omega_1 + 1)$-iteration strategy for $\mathcal{M}$. (We don't need the measurable Woodin because we are working with a realization strategy; cf. [3].) This strategy is simply: at limit α, choose the unique cofinal branch b such that setting $b = [0, \alpha]_T$, items (1) and (2) above hold. It is a π realization strategy, and therefore has the weak Dodd-Jensen property relative to $\vec{e}$.

cSBH is also known for some mice slightly beyond the tame mice. Again, the proof gives π realization strategies, so that by taking π to be minimal, we get the weak Dodd-Jensen property. We expect this pattern to hold true for all mice, and indeed it does if V is $\omega_1 + 1$-iterable. For let Σ be an

ω_1+1 iteration strategy for V. Let $\pi\colon \mathcal{M}\to \mathcal{C}_k(\mathcal{N}_\xi)$, where $\mathcal{M}$ is countable. Any iteration tree $\mathcal{T}$ on $\mathcal{M}$ lifts to a tree $\mathcal{U}(\pi,\mathcal{T})$ on V via the techniques of [1, section 12].(One resurrects background extenders, then applies them. The tree $\mathcal{U}(\pi,\mathcal{T})$ may not actually lift all of $\mathcal{T}$, as a branch chosen by $\mathcal{T}$ may lift to an illfounded branch of $\mathcal{U}(\pi,\mathcal{T})$.) Thus we can pull back Σ to a (k,ω_1+1) iteration strategy Σ^π for $\mathcal{M}$. The argument of [1] shows that Σ^π is a π realization strategy. It follows that if we choose ξ minimal so that $\mathcal{C}_k(\mathcal{N}_\xi)$ is $(\mathcal{M},k)$-large, and choose π to be $(k,\vec{e})$-minimal, then Σ^π has the weak Dodd-Jensen property relative to $\vec{e}$.

Notice that in the situation just described, we have by Theorem 2.3 that Σ^π is independent of Σ; that is, if Σ and Γ are any two ω_1+1 iteration strategies for V, and ξ,π are minimal in the sense of the last paragraph, then $\Sigma^\pi=\Gamma^\pi$. Put another way, in this situation Σ and Γ must agree on all trees of the form $\mathcal{U}(\pi,\mathcal{T})$. This leads us to our conjecture.

Conjecture. Let ξ be least such that $\mathcal{C}_k(\mathcal{N}_\xi)$ is $(\mathcal{M},k)$-large, and let $\pi\colon\mathcal{M}\to \mathcal{C}_k(\mathcal{N}_\xi)$ be $(k,\vec{e})$ minimal, for some $\vec{e}$. Let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$ such that lh($\mathcal{T}$)=lh($\mathcal{U}(\pi,\mathcal{T})$) and lh($\mathcal{T}$) is a limit ordinal $\leq\omega_1$; then $\mathcal{U}(\pi,\mathcal{T})$ has at most one cofinal wellfounded branch.

Let us call this conjecture "cUBH for $L[\vec{E}]$". Let "cCBH for $L[\vec{E}]$" be the variant with the words "at most one" replaced by "at least one". The arguments above show that cUBH for $L[\vec{E}]$ follows from cCBH for $L[\vec{E}]$. On the other hand, essentially by [5], cUBH for $L[\vec{E}]$ implies cCBH for $L[\vec{E}]$.(One must choose the background extenders for the $L[\vec{E}]$ construction to be $2^{\aleph_0}$ closed.) Thus for $L[\vec{E}]$, cUBH and cCBH are equivalent, and they imply cSBH. There is at least a faint hope that some reasonably direct proof will establish cUBH.

References

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