AN INTRODUCTION TO CORE MODEL THEORY

BENEDIKT LÖWE, JOHN R. STEEL

Abstract. In this paper we give an informal introduction to core model theory at the level of Woodin cardinals.

1. Introduction

Zermelo–Fraenkel set theory with choice, or ZFC, is the commonly accepted system of axioms for set theory, and hence for all of mathematics. Most of the axioms of ZFC express closure properties of the universe of sets. (The exceptions are Extensionality and Foundation, which in effect limit the objects under consideration.) Although all mathematical assertions can be expressed in the language of ZFC, and "most" of them can be decided using only the axioms of ZFC, there are nevertheless interesting mathematical assertions which cannot be decided using ZFC alone. The most famous of these is the Continuum Hypothesis.

Gödel's response to the incompleteness of ZFC with respect to assertions like the Continuum Hypothesis was that one should seek well–justified extensions of ZFC which decide these assertions (*cf.* [Gö47]). This is known as "Gödel's Program" and is still one of the most important tasks of higher set theory. Gödel suggested *strong axioms of infinity*, now more commonly known as *large cardinal axioms*, as candidates for basic principles to be added to the foundation provided by ZFC. In the years since [Gö47], large cardinal axioms have been extensively investigated, and have proved very fruitful in deciding in natural ways propositions about the real numbers left undecided by ZFC. They do not decide the Continuum Hypothesis, however.

There are many other natural extensions of ZFC which have been studied; for example, there are the forcing axioms MA, PFA, and MM. We do not think any of these extensions are currently well–justified in the way the large cardinal axioms are, but they are certainly interesting and useful. Indeed, the development of any consistent theory can be justified in a Hilbertian vein: one

Date: Submitted Version after Refereeing — April 18, 2005.

1991 *Mathematics Subject Classification.* **03E15 03E45** 01A65 03–03 03E55 03E60.

The first author was partially supported by DAAD–Grant Ref.316–D/96/20969 in the program HSPII/AUFE and the Studienstiftung des Deutschen Volkes.

The authors would like to thank Martin Zeman (Berlin) for his comments and Robert Schmidt (Potsdam) for his TEXnical support.

can regard such a theory as a tool for proving true Π^0_1 statements.[1] In this connection, the following remarkable phenomenon has emerged over the past thirty or forty years: the family of all remotely natural extensions of ZFC seems to be prewellordered by consistency strength[2] moreover, it seems that each theory in this family is equivalent in consistency strength to a theory whose axioms are large cardinal axioms. One of our goals in this paper is to exhibit some of the mathematical structure underlying this remarkable phenomenon.[3]

One can think of the large cardinal axioms as extrapolating from, and strengthening, the closure principles on the universe of sets which are inherent in ZFC. One way to understand such principles is to study minimal universes satisfying them. It seems to be the case that for each large cardinal axiom A there is a canonical minimal universe satisfying A, and that the structure of this universe can be analyzed in detail. The consistency strength order on large cardinal axioms corresponds to the inclusion order on the canonical minimal universes satisfying them. We shall call these canonical minimal universes, and their "iterates", *core models*.[4]

In general, one "computes" the consistency strength of a theory T extending ZFC by proving $T \equiv_{\mathrm{Cons}} \mathsf{A}$ for some large cardinal axiom A. Thinking of A as providing the standard measure, one says that $T \leq_{\mathrm{Cons}} \mathsf{A}$ yields an upper bound, and $\mathsf{A} \leq_{\mathrm{Cons}} T$ a lower bound, on the consistency strength of T. Upper bounds are generally proved by forcing over a model of A. Lower bounds are nearly always proved by constructing a core model satisfying A inside an arbitrary model of T. In this paper we shall give an introduction to the core model techniques.[5]

The simplest core model is Gödel's universe $\mathbf{L}$ of constructible sets (*cf.* [Gö40]). Many of the important features of core model theory are already

---

[1]Of course, not all consistent theories are worthy of development; the point here is just that a theory need not be true to be worthy of development.

[2]Perhaps the most natural way to define the consistency strength order is by

$$S \leq_{\mathrm{Cons}} T \iff P_S \subseteq P_T,$$

where P_S (resp. P_T) is the set of Π^0_1 consequences of S (resp.T).

[3]In fact, for "remotely natural" extensions S and T of ZFC, it seems to be the case that $S \leq_{\mathrm{Cons}} T$ if and only if every Σ^1_3 consequence of S is a consequence of T. One can replace Σ^1_3 by Σ^1_{n+3} in this last statement if one restricts attention to theories of consistency strength greater than the large cardinal hypothesis "there are n Woodin cardinals". Thus as we climb the consistency strength hierarchy of "natural" theories, our theories *converge*, not just in their Π^0_1 consequences, but in their consequences of ever greater complexity. The mathematical structure underlying this aspect of the remarkable phenomenon has to do with the correctness properties of core models.

[4]This usage of "core model" differs somewhat from the original one of Dodd and Jensen ([Do82], [DoJen81], [Jen95]).

[5]For a less technical introduction directed at a broader audience, see the survey in [Jen95]. The introduction to [MaSt94] also contains a general description of core model theory, and some of its history. Further history can be found in the introduction to [MiSt94].

present in the theory of $\mathbf{L}$: the existence of a fine–grained stratification of the model with strong condensation properties, the existence of a definable wellorder of the reals in the model, the absoluteness of the construction of the model (between universes resembling each other sufficiently, which in the case of $\mathbf{L}$ means having the same ordinals), and the existence of covering theorems. Since the work of Gödel and Jensen (*cf.* [Gö40], [Jen72] and [DevJen75]) on $\mathbf{L}$, many larger core models have been constructed and studied. Although significant new ideas have appeared, these basic themes remain. There is a great deal of unity of method in core model theory, and a great deal of resemblance among core models.

At the foundation of core model theory is the existence of a constructive *comparison process*, a method for *iterating* two approximations (called *mice*) $\mathfrak{M}$ and $\mathfrak{N}$ to the model in question so as to produce elementary embeddings $i\colon \mathfrak{M} \to \mathfrak{P}$ and $j\colon \mathfrak{N} \to \mathfrak{Q}$ such that the hierarchy on $\mathfrak{P}$ is an initial segment of that on $\mathfrak{Q}$ or *vice versa*.[6] The problem of extending core model theory to stronger large cardinal axioms comes down to the problem of extending the comparison process to more complicated mice. There are at least two ways to set goals and measure progress here. One can aim at mice satisfying certain large cardinal axioms, or one can aim at mice containing reals of a certain complexity. This latter measure is closely related to the logical complexity of the wellorder of the reals satisfied to exist by the core model in question. It also corresponds closely to the constructivity of the comparison process, a fact which lends it special technical importance.

In the period between 1966 and 1980 core models larger than $\mathbf{L}$ were constructed by Silver, Kunen, Dodd, Jensen, Mitchell and Baldwin[7]. These models can satisfy large cardinal hypotheses as strong as "there are many strong cardinals", and certain aspects of their form seem suitable for models satisfying much stronger large cardinal hypotheses.[8] However, the models themselves cannot satisfy hypotheses much stronger than the existence of many strong cardinals, and in the descriptive–set–theoretic measure of progress they do not go very far at all: each of these core models satisfies "there is a Δ^1_3 wellorder of the reals", and under reasonable large cardinal assumptions, there is a single Δ^1_3 real enumerating all the reals occurring in any of them. Behind these limitations is the fact that the mice approximating these core models can all be compared using only linear iteration, which is

[6]Gödel's constructible universe $\mathbf{L}$ is something of a degenerate case here: the mice are simply transitive models of $\mathbf{V} = \mathbf{L}$, and we can take $\mathfrak{M} = \mathfrak{P}$, $\mathfrak{N} = \mathfrak{Q}$, and $i = j = \mathrm{id}$.

[7]*Cf.* [Si71a], [Si71b], [Ku70], [Mi74], [Mi78], [Do81], [Bal83], [Bal86], [DoJen81], [Mi84], [Jen88] and [Jen90]. A very thorough introduction to the theory of these smaller core models will appear in [Ze9?].

[8]The models are constructed from *coherent sequences of extenders* (*cf.* Section 2.2 below). This framework, due mainly to [Mi74] and [Mi78], seems adequate for models satisfying "there is a superstrong cardinal".

a particularly simple comparison process.[9] In [MaSt94], [MiSt94], [St93] and [St96] the authors make use of a more complicated comparison process involving iterations with a nonlinear tree structure. This enables them to develop a good theory of core models satisfying large cardinal hypotheses as strong as "there are many Woodin cardinals" (*cf.* Definition 2.27), and containing all reals which are Δ^1_n for some natural number n. These models still cannot satisfy "there is a superstrong cardinal", and there is still an upper bound on the complexity of the reals which occur in them.[10] This work represents more or less the frontier of current core model theory.

The purpose of this introductory article is to give the interested reader an informal introduction to the concepts behind the construction of these models. We will be very sketchy at times, and this paper is not to be understood as a means to learn core model theory, but merely as a means to acquire a rough idea of what it is, and what its central open problems are. The word "Proof" is to be understood as an abbreviation for "Proof Sketch" throughout this paper.

In particular, we hope to convey the important features of these ideas without sinking into the morass of finestructural detail in which they are embedded. The level–by–level finestructural analysis of the models one constructs is the locus of some of the most important ideas of core model theory, and so we shall not to able to avoid it entirely. We shall, however, suppress as much detail as possible. We hope that the resulting outline will be useful to anyone acquainted with the theory of iterated ultrapowers contained in Kunen's paper [Ku70] on $\mathbf{L}[U]$, and the finestructure theory for $\mathbf{L}$ contained in Jensen's paper [Jen72]. The knowledge about $\mathbf{L}[U]$ can be easily obtained from [Kan94, §§12, 19, 20].

One hypothesis whose consistency strength can be bounded from below using the core model theory of this paper is the hypothesis of $\mathbf{L}(\mathbb{R})$–*generic absoluteness*. The fact that this instance of generic absoluteness can be proved from large cardinal hypotheses[11] is strong evidence that the theory of $\mathbf{L}(\mathbb{R})$ which follows from these hypotheses is complete for natural statements, and hence has a privileged position. As an application of the core model theory

[9] More precisely, certain artificial limitations on the "overlapping" of extenders on the coherent sequence from which the model is constructed are imposed. These limitations make comparison via linear iteration possible. They make the appearance of extenders witnessing large cardinal hypotheses stronger than the existence of many strong cardinals impossible.

[10] This is again due to an artificial limitation on the degree of overlapping in the coherent sequence from which the model is constructed; *cf.* Definition 2.31.

The precise upper bound on the logical complexity of the reals in these models involves the game quantifier for certain clopen games of length ω_1; it goes well beyond notions of definability familiar from descriptive set theory, like definability over $\mathbf{L}(\mathbb{R})$. *Cf.* [Nee98].

[11] The existence of arbitrarily large Woodin cardinals is sufficient.

we outline here, we shall sketch a proof of the following theorem of Hugh Woodin and the second author:

Theorem 1.1. If for every set–sized forcing notion $\mathbb{P}$ and every G which is $\mathbb{P}$–generic over $\mathbf{V}$ we have

$$(\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}} \equiv (\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}[G]},$$

then there is an inner model with a Woodin cardinal.

In other words, the consistency strength of generic absoluteness is at least the consistency strength of "There is a Woodin cardinal".[12]

Of course, we hope that this paper sparks an interest in core model theory in some readers. The constructions described in this paper can be found in much greater detail in [MiSt94] and [St96], and will be described in somewhat more detail in the survey article [St9?a].

2. Preliminaries

We will introduce the main notions and some notation. Readers familiar with extenders and premice can skip the first two subsections.

2.1. **Extenders.** Many large cardinal axioms can be written in some form of an elementary embedding axiom. In fact, researchers in the field of large cardinals tend to use elementary embedding properties to define the cardinals they study. If j is an elementary embedding from the universe $\mathbf{V}$ to a transitive class M which is nontrivial (*i.e.* not the identity), then we denote the least ordinal κ such that $j(\kappa) \neq \kappa$ by $\mathrm{crit}(j)$ and call it *the critical point of j*.

To see the reflection inherent in the existence of such an embedding, notice that if $\mathbf{V} \models \varphi[\kappa]$, and M resembles $\mathbf{V}$ enough that $M \models \varphi[\kappa]$, then $M \models \exists\alpha < j(\kappa)(\varphi(\alpha))$, so there is an $\alpha < \kappa$ such that $\mathbf{V} \models \varphi[\alpha]$. Thus the more M resembles $\mathbf{V}$, the stronger the reflection in the cumulative hierarchy present at κ, and, it turns out, the greater the consistency strength of the axiom asserting that such an embedding exists. The "ultimate" resemblance $M = \mathbf{V}$ is impossible by a result of Kunen (*cf.* [Ku71]).

The large cardinal corresponding to an elementary embedding is its critical point. So we have:

- κ is a *measurable* cardinal iff there is an elementary embedding $j : \mathbf{V} \to M$ with $\mathrm{crit}(j) = \kappa$.
- κ is an *α–strong* cardinal iff there is an elementary embedding $j : \mathbf{V} \to M$ with $\mathrm{crit}(j) = \kappa$ and $\mathbf{V}_\alpha \subseteq M$ (*cf.* Figure 1).
- κ is a *strong* cardinal iff κ is α–strong for all $\alpha \in \mathrm{Ord}$.

[12]This instance of generic absoluteness actually implies that for each $n < \omega$, there is an inner model with n Woodin cardinals. Hugh Woodin has shown that generic absoluteness

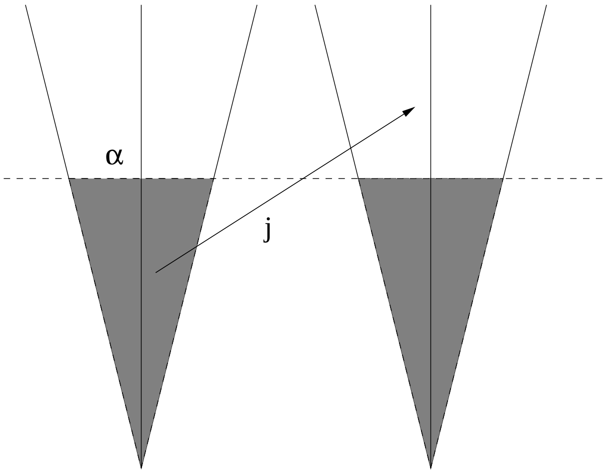


FIGURE 1. An α–strong embedding

- κ is a *superstrong* cardinal iff there is an elementary embedding $j : \mathbf{V} \to M$ with $\mathrm{crit}(j) = \kappa$ and $\mathbf{V}_{j(\kappa)} \subseteq M$.
- κ is an *α–supercompact* cardinal iff there is an elementary embedding $j : \mathbf{V} \to M$ with $\mathrm{crit}(j) = \kappa$ and $M^{\alpha} \subseteq M$.
- κ is a *supercompact* cardinal iff κ is α–supercompact for all $\alpha \in \mathrm{Ord}$.
- κ is a *huge* cardinal iff there is an elementary embedding $j : \mathbf{V} \to M$ with $\mathrm{crit}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$.

The axioms corresponding to the embedding properties above, namely, that the cardinals (or embeddings) in question exist, are listed in order of increasing consistency strength (for any nontrivial choice of α), and in most cases in order of least cardinal having the property in question.[13] The cardinals with which we shall be mainly concerned in this paper, *Woodin* cardinals, lie between strong and superstrong cardinals in consistency strength. We shall define them in Section 2.7 (*cf.* Definition 2.27).

Asserting the existence of elementary embeddings with certain additional properties is an elegant and powerful way to obtain systems transcending ZFC. But, as elementary embeddings from a proper class model into another proper class model are proper classes, we need some coding device to talk about them properly in our formal language. Readers familiar with measurable cardinals

for arbitrary statements about $\mathbf{L}(\mathbb{R})$ implies that there is an inner model with ω Woodin cardinals.

[13]The resemblance of M to $\mathbf{V}$ suffices in many cases to reflect all the weaker properties to smaller cardinals. This is not obvious in some cases, however; the argument requires the use of something like extenders representing the weaker embeddings.

The notable exceptions are: The least superstrong is smaller than the least strong, and the least huge is smaller than the least supercompact. In these cases the fact that an extender from the weaker cardinal overlaps the stronger cardinal allows downward reflection.

know that ultrafilters do this job to a certain extent: A nonprincipal κ–complete ultrafilter U on κ induces the ultrapower embedding $j_U : \mathbf{V} \to M := \mathrm{Ult}(\mathbf{V}, U)$ with critical point κ, and conversely if we have any embedding j with $\mathrm{crit}(j) = \kappa$ then we can define an ultrafilter on κ by setting

$$X \in U \iff \kappa \in j(X).$$

So ultrafilters are one way to witness the existence of an embedding. However, one can easily show that $U \notin \mathrm{Ult}(\mathbf{V}, U)$, and thus such an ultrapower embedding can never have the property $\mathbf{V}_{\kappa+2} \subseteq M$, because the ultrafilter (an element of $\mathfrak{P}(\mathfrak{P}(\kappa))$) is not in $\mathfrak{M}$. So to represent embeddings whose target models have the stronger closure properties described above, we need a more powerful coding device.

Given $j : M \to N$ with $\kappa = \mathrm{crit}(j)$ and $\lambda < j(\kappa)$, put

$$X \in E_a :\iff a \in j(X)$$

for $a \in [\lambda]^{<\omega}$ and $X \subseteq [\kappa]^{|a|}$, $X \in M$. We call the system

$$E := \langle E_a : a \in [\lambda]^{<\omega} \rangle$$

the $\langle \kappa, \lambda \rangle$*–extender over* M derived from j. We call κ the *critical point of* E and λ the *length of* E, in symbols: $\kappa = \mathrm{crit}(E)$, $\lambda = \mathrm{lh}(E)$.

If $a \subseteq b$, then there is a natural embedding from $\mathrm{Ult}(M, E_a)$ into $\mathrm{Ult}(M, E_b)$ (send $[f]$ to $[\tilde{f}]$, where $\tilde{f}$ comes from f by adding dummy variables at spots corresponding to ordinals in $b \setminus a$). Thus we can set

$$\mathrm{Ult}(M, E) := \mathrm{dirlim}_{a \in [\lambda]^{<\omega}} \mathrm{Ult}(M, E_a).$$

We write $[a, f]$ for the image of the element $[f]$ of $\mathrm{Ult}(M, E_a)$ under the natural map from $\mathrm{Ult}(M, E_a)$ into $\mathrm{Ult}(M, E)$. Thus the universe of $\mathrm{Ult}(M, E)$ consists of all $[a, f]$ such that $a \in [\mathrm{lh}(E)]^{<\omega}$ and $f \in M$. We have a natural embedding

$$i_E : M \to \mathrm{Ult}(M, E)$$

given by $i_E(x) = [\{\kappa\}, c_x]$, where c_x is the constant function $\alpha \mapsto x$. Since there is no danger of confusion, we will call the embedding i_E the *ultrapower embedding* and the model $\mathrm{Ult}(M, E)$ the *ultrapower of* M *by* E.

We have

$$\begin{tikzcd} M \arrow[r, "j"] \arrow[d, "i_E"'] & N \\ \mathrm{Ult}(M, E) \arrow[ur, "k"'] & \end{tikzcd}$$

with $k([a, f]) = j(f)(a)$ and $k \upharpoonright \lambda = \mathrm{id}$. So

$$X \in E_a \iff a \in i_E(X),$$

and hence E is the extender derived from i_E.

Because of the diagram above, any embedding can be fully represented as an extender ultrapower.[14] In particular, if $j\colon M \to N$ with $\mathbf{V}_\lambda \subseteq N$, $|\mathbf{V}_\lambda| = \lambda$, and E is the $\langle\kappa,\lambda\rangle$-extender derived from j, then $\mathbf{V}_\lambda \subseteq \mathrm{Ult}(M,E)$.

2.2. **Coherent Sequences and Premice.** The models of set theory we consider will be of the form $\mathbf{L}[\vec{E}]$ where $\vec{E} = \langle E_\alpha : \alpha \in \mathrm{Ord}\rangle$ is a coherent sequence of extenders. Roughly speaking, this means that the αth extender on the sequence is an extender for the αth level of the model, and that the extenders on the sequence are listed in order of increasing strength and without leaving gaps. This setup is due mainly to Mitchell (*cf.* [Mi74], [Mi78]), and it seems adequate to provide models satisfying any of the known large cardinal axioms. The limitations on present–day core model theory probably do not lie in the coherent–sequence–of–extenders framework.

As usual we put[15]:

$$\mathbf{J}_0[\vec{E}] := \emptyset,$$

$$\mathbf{J}_\lambda[\vec{E}] := \bigcup_{\beta<\lambda} \mathbf{J}_\beta[\vec{E}]$$

for λ a limit ordinal, and

$$\mathbf{J}_{\alpha+1}[\vec{E}] := \mathrm{rud}(\mathbf{J}_\alpha[\vec{E}]).$$

Here is an elliptical definition of the sort of extender sequences from which our core models are constructed:

Definition 2.1. A *good extender sequence* is a sequence $\vec{E}$ such that for all α we have either that $E_\alpha = \emptyset$ or that E_α is an extender over $\mathbf{J}_\alpha[\vec{E}]$ such that $(\mathrm{crit}(E_\alpha)^+)^{\mathbf{J}_\alpha[\vec{E}]} \leq \mathrm{lh}(E_\alpha)$, and

(1) $i_{E_\alpha}(\vec{E}) \restriction \alpha = \vec{E} \restriction \alpha$,
(2) $\left(i_{E_\alpha}(\vec{E})\right)_\alpha = \emptyset$,
(3) $\alpha = (\mathrm{lh}(E_\alpha)^+)^{\mathrm{Ult}(\mathbf{J}_\alpha[\vec{E}],E_\alpha)}$, and
(4) $\mathbf{J}_\alpha[\vec{E}]$ satisfies the first order conditions enumerated in [MiSt94, p.7].

[14] We ignore here the limitation $\mathrm{lh}(E) < i_E(\mathrm{crit}(E))$, which we imposed only to simplify matters. To represent superstrong, supercompact, or huge embeddings by extenders one would need to drop this limitation.

However, core model theory does not reach these cardinals anyway for other reasons, so for our purposes we may as well assume $\mathrm{lh}(E) < i_E(\mathrm{crit}(E))$. We also ignore the possibility that one might need a proper class extender to capture j.

[15] The reader who is not familiar with Jensen's $\mathbf{J}$–hierarchy can replace every $\mathbf{J}$ by an $\mathbf{L}$ and the rudimentary closure with the definable closure without great loss. Anyway we have that

$$\mathfrak{P}(\mathbf{J}_\alpha[\vec{E}]) \cap \mathbf{J}_{\alpha+1}[\vec{E}] = \mathrm{def}(\mathbf{J}_\alpha[\vec{E}])$$

where $\mathrm{def}(X)$ denotes the first order definable subsets of X. For this *cf.* [Jen72, Section 2.4].

The most important of the first order conditions from [MiSt94] which we have not listed is the *initial segment condition*. If E is an extender and $\mathrm{crit}(E) < \nu \le \mathrm{lh}(E)$, then we set $E \restriction \nu = \{\langle a, x\rangle \in E \colon a \in [\nu]^{<\omega}\}$, and call $E \restriction \nu$ an *initial segment* of E. The initial segment condition of [MiSt94] has as a consequence that every proper initial segment of E_α is a member of $\mathbf{J}_\alpha^{\vec{E}}$.[16]

Conditions (1.) and (2.) are the coherence conditions. Condition (3.) guarantees that the ordinal α at which $E = E_\alpha$ is indexed is determined completely by E, and therefore must not be used to code random information into the model.[17] We are aiming to construct canonical models, so coding in random information cannot be allowed. There are variant indexing schemes possible; one sometimes convenient scheme involves indexing E at $i_E(\mathrm{crit}(E))$, or at its successor in the E-ultrapower.[18] Such variant schemes lead to hierarchies which are level–by–level intertranslatable with ours.[19]

One important feature of Definition 2.1 is that we have only required E_α to be an extender over $\mathbf{J}_\alpha[\vec{E}]$. E_α will not measure any subsets of its critical point which are constructed in the model we are building after stage α. When we form an ultrapower using E_α, we shall generally form the ultrapower of the largest fragment of the model we can, the largest fragment containing only subsets of $\mathrm{crit}(E_\alpha)$ constructed before α. We need finestructure theory to do this properly, and so adding such "partial" extenders to our sequence may seem to complicate matters. In fact, it leads to dramatic simplifications. The idea is due to Stewart Baldwin and William Mitchell.

Definition 2.2. A structure $\mathfrak{M} = \langle M, \in, \vec{F}, G\rangle$ is called a *premouse*, if it is of the form $\langle \mathbf{J}_\alpha[\vec{E}], \in, E \restriction \alpha, E_\alpha\rangle$, where $\vec{E}$ is a good extender sequence.

If $\mathfrak{M} = \langle M, \in, \vec{F}, G\rangle$ is a premouse then we call $\vec{F}$ the *$\mathfrak{M}$–sequence*. We shall often refer to "extenders on the $\mathfrak{M}$–sequence".

Definition 2.3. If $\mathfrak{M} = \langle \mathbf{J}_\alpha[\vec{E}], \in, E \restriction \alpha, E_\alpha\rangle$ is a premouse and $\beta \le \alpha$ then set

$$\mathcal{J}_\beta^{\mathfrak{M}} := \langle \mathbf{J}_\beta[\vec{E}], \in, E \restriction \beta, E_\beta\rangle$$

Definition 2.4. $\mathfrak{M}$ is an *initial segment* of $\mathfrak{N}$ (in symbols $\mathfrak{M} \trianglelefteq \mathfrak{N}$) if there is a $\beta \le \mathrm{Ord}^{\mathfrak{N}}$ such that $\mathfrak{M} = \mathcal{J}_\beta^{\mathfrak{N}}$. We say that $\mathfrak{M}$ and $\mathfrak{N}$ *agree up to* α if $\mathcal{J}_\alpha^{\mathfrak{M}} = \mathcal{J}_\alpha^{\mathfrak{N}}$.

Definition 2.5. A structure $\mathfrak{M}$ is *k-sound* if for all $n \le k$

$$\mathfrak{M} = \mathrm{Hull}_n^{\mathfrak{M}}(\varrho_n \cup \{p_n\})$$

[16]More precisely, it states that every proper initial segment of E_α is either on $\vec{E}$ with index $< \alpha$, or "one ultrapower away" from being so.

[17]In [MiSt94], the authors take $\alpha = (\nu(E_\alpha)^+)^{\mathrm{Ult}(\mathbf{J}_\alpha[\vec{E}], E_\alpha)}$, where $\nu(E_\alpha)$ is the supremum of the "generators" of E_α. This difference affects only details we shall ignore here.

[18]This idea is due to Sy Friedman, and has been developed by Jensen. *Cf.* [Jen88].

[19]The comparability of inner model operators proved in [St82] is good evidence this *must* be the case.

where ϱ_n is the Σ_n–projectum of $\mathfrak{M}$ and p_n is the nth standard parameter of $\mathfrak{M}$. We say $\mathfrak{M}$ is *sound* just in case $\mathfrak{M}$ is k-sound for all $k < \omega$.[20]

All levels of the core models we shall construct will be sound premice. Nonetheless, we must study unsound premice as well, because iterates of sound premice may fail to be sound. This occurs when the critical point of one of the ultrapower embeddings lies above one of the projecta of the premouse whose ultrapower is being taken:

Proposition 2.6. Let $\mathfrak{M}$ be a premouse and E an extender on the $\mathfrak{M}$–sequence whose critical point is above the Σ_1–projectum ϱ of $\mathfrak{M}$. Then $\mathrm{Ult}(\mathfrak{M}, E)$ is not 1–sound.

Proof: A little finestructural analysis shows that in this case, the Σ_1–projectum of $\mathrm{Ult}(\mathfrak{M}, E)$ is still ϱ, and the 1st standard parameter is the image $i_E(p)$ of the 1st standard parameter p of $\mathfrak{M}$. It follows immediately that $\mathrm{Ult}(\mathfrak{M}, E)$ is not 1–sound; in fact, $\mathrm{Hull}_1^{\mathrm{Ult}(\mathfrak{M},E)}(\varrho \cup \{i_E(p)\})$ is just $\mathrm{ran}(i_E)$, and $\mathrm{crit}(E)$ is not a member of it. $\square$

But our finestructural ultrapowers satisfy enough of Łos's theorem that the property that all proper initial segments are sound is passed from the structure to its ultrapower. Since we shall make heavy use of this property[21], we make it part of the definition of "$\mathfrak{M}$ is a premouse" that all proper initial segments of $\mathfrak{M}$ are sound.[22]

Given an arbitrary premouse $\mathfrak{M}$, there is a natural way to try to extract from $\mathfrak{M}$ a sound premouse $\mathfrak{C}(\mathfrak{M})$ which is in some sense equivalent to $\mathfrak{M}$. The correct procedure involves some finestructural subtleties, but we can easily sketch the basic idea. Set $\mathfrak{C}_0 = \mathfrak{M}$, and

$$\mathfrak{C}_{n+1} = \mathrm{Hull}_{n+1}^{\mathfrak{C}_n}(\varrho_{n+1} \cup \{p_{n+1}\}),$$

where ϱ_{n+1} and p_{n+1} are the Σ_{n+1}–projectum and standard parameter of $\mathfrak{C}_n$.[23] If $\mathfrak{M}$ is well–behaved, then the projecta $\varrho_{n+1}^{\mathfrak{C}_n}$ are decreasing, and hence they

[20] For readers unfamiliar with these standard notions of finestructure theory of core models, [Dev84] or [Jen72] are good references. Roughly speaking, ϱ_n is the first ordinal α such that there is a new Σ_n subset of α which is not in $\mathfrak{M}$, and p_n is the least parameter in the order of construction from which you can define such a subset. These are not literally the concepts and definitions used in [MiSt94] for technical reasons which are beyond the resolution of the microscope we are using here.

[21] Or rather, we would do so if we were giving detailed proofs.

[22] This can be regarded as another clause in the elliptical Definition 2.1.

[23] It would be wrong to replace $\mathfrak{C}_n$ by $\mathfrak{M}$ in this definition. For example, $\mathfrak{M}$ might come from a sound mouse by iterating it above its Σ_1–projectum in such a way to encode some random information in a $\Sigma_2^{\mathfrak{M}}$ subset of ω. We want $\mathfrak{C}_2$ to be *canonical*, so we cannot take $\mathfrak{C}_2 = \mathrm{Hull}_2^{\mathfrak{M}}(\varrho_2 \cup \{p_2\})$. Instead, we first pass to $\mathfrak{C}_1 = \mathrm{Hull}_1^{\mathfrak{M}}(\varrho_1 \cup \{p_1\})$, which in effect undoes our iteration, then set $\mathfrak{C}_2 = \mathrm{Hull}_2^{\mathfrak{C}_1}(\varrho_2 \cup \{p_2\})$.

and the "cores" $\mathfrak{C}_n$ are eventually constant. We then set

$$\mathfrak{C}(\mathfrak{M}) = \text{ eventual value of } \mathfrak{C}_n,$$

and

$$\varrho_\omega^{\mathfrak{M}} = \text{ eventual value of } \varrho_{n+1}^{\mathfrak{C}_n},$$

as n approaches ω, and call $\mathfrak{C}(\mathfrak{M})$ and $\varrho_\omega^{\mathfrak{M}}$ the (ωth) *core* and *projectum* of $\mathfrak{M}$. We call a premouse $\mathfrak{M}$ *solid* if it is well–behaved in the way that guarantees that $\mathfrak{C}(\mathfrak{M})$ and $\varrho_\omega^{\mathfrak{M}}$ exist.[24]

One important consequence of the soundness of their proper initial segments is that premice satisfy a strong local form of GCH:

If there is a subset of κ in $\mathbf{J}_{\tau+1}^{\mathfrak{M}} \setminus \mathbf{J}_\tau^{\mathfrak{M}}$, then $\mathcal{J}_{\tau+1}^{\mathfrak{M}} \models |\tau| \leq \kappa$.

So for the models $\mathbf{L}[\vec{E}]$ we build, we have

$$\mathfrak{P}(\kappa) \cap \mathbf{L}[\vec{E}] \subseteq \mathbf{J}_{(\kappa^+)^{\mathbf{L}[\vec{E}]}}[\vec{E}],$$

and an immediate consequence of this is that our models will be models of GCH.

The levels of the usual hierarchy for $\mathbf{L}$ are all sound, but this is not true for the levels of the most obvious hierarchy on Silver's core model $\mathbf{L}[U]$, the universe of sets constructible from a normal ultrafilter U on a measurable cardinal.[25] Silver's model *is* one of the core models we study here, but we do so via a different hierarchy. The soundness of the levels of this new hierarchy is a direct consequence of the Baldwin–Mitchell idea of putting *partial* extenders on the coherent sequence from which we generate it. This new hierarchy has a substantially simpler finestructure, one which generalizes to arbitrary core models in a way the old finestructure theory of $\mathbf{L}[U]$ did not.[26]

2.3. **Iteration Trees.** We now move on to one of the main tools of core model theory, the Comparison Process.

The key to Kunen's theory of $\mathbf{L}[U]$ is the method of *iterated ultrapowers*. Given a structure $\mathfrak{M}_0 = \langle \mathbf{L}_\zeta[U], \in, U \rangle$ with appropriate ultrafilter U, one can form ultrapowers by U and its images under the canonical embeddings repeatedly, taking direct limits at limit ordinals. One obtains thereby structures $\mathfrak{M}_\alpha$ and embeddings $i_{\alpha,\beta}\colon \mathfrak{M}_\alpha \to \mathfrak{M}_\beta$ for $\alpha < \beta$. We call the structures $\mathfrak{M}_\alpha$ *iterates* of $\mathfrak{M}_0$, and say that $\mathfrak{M}_0$ is *iterable* just in case all its iterates are wellfounded. Kunen's key *comparison lemma* states that if $\mathfrak{M}_0$ and $\mathfrak{N}_0$ are

[24]We have only sketched an oversimplification of the correct definitions of the core and projectum of $\mathfrak{M}$, and the reader should see [MiSt94, Definition 2.8.1] for the true definitions. Solidity is defined in [MiSt94, Definition 2.8.2].

[25]The usual $\mathbf{L}[U]$ hierarchy is not 1–sound at $\kappa+1$, where κ is the measurable cardinal. For $0^\#$ is Σ_1–definable over $\langle \mathbf{J}_{\kappa+1}[U], \in, U \rangle$ but not a member of $\mathbf{J}_{\kappa+1}[U]$, so that the Σ_1–projectum of this structure is ω. Since $\langle \mathbf{J}_{\kappa+1}[U], \in, U \rangle$ is an uncountable structure, it cannot be the Σ_1–Skolem closure of any countable set.

[26]This finestructure is due to Solovay (unpublished) and [DoJen82].

two iterable structures of this form, then there are iterates $\mathfrak{M}_\alpha$ and $\mathfrak{N}_\alpha$ such that one of the two is an initial segment of the other.[27]

One can form iterated ultrapowers of an arbitrary premouse $\mathfrak{M}_0$ similarly. In this case, the $\mathfrak{M}_\alpha$–sequence may have more than one extender, and we are allowed to choose any one of them to continue. If E_α is the extender chosen, then we take $\mathfrak{M}_{\alpha+1}$ to be $\mathrm{Ult}(\mathfrak{M}_\alpha, E_\alpha)$.[28] At limit stages we form direct limits and continue. We call any such sequence $\langle\langle\mathfrak{M}_\alpha, E_\alpha\rangle : \alpha < \beta\rangle$ a *linear iteration* of $\mathfrak{M}_0$, and the structures $\mathfrak{M}_\alpha$ in it *linear iterates* of $\mathfrak{M}_0$. We say $\mathfrak{M}_0$ is *linearly iterable* just in case all its linear iterates are wellfounded.[29]

Given linearly iterable premice $\mathfrak{M}_0$ and $\mathfrak{N}_0$, there is a natural way to try to compare the two via linear iteration. Having reached $\mathfrak{M}_\alpha$ and $\mathfrak{N}_\alpha$, and supposing neither is an initial segment of the other (as otherwise our work is finished), we pick extenders E and F representing the least disagreement between $\mathfrak{M}_\alpha$ and $\mathfrak{N}_\alpha$, and use these to form $\mathfrak{M}_{\alpha+1}$ and $\mathfrak{N}_{\alpha+1}$.

If the extenders of the coherent sequence of $\mathfrak{M}_0$ do not overlap one another too much, and similarly for $\mathfrak{N}_0$, then this process must terminate with all disagreements between some $\mathfrak{M}_\alpha$ and $\mathfrak{N}_\alpha$ eliminated, so that one is an initial segment of the other. This is the key to core model theory at the level of strong cardinals.

At bottom, the reason this comparison process must terminate is the following: if E and F are the extenders used at a typical stage α, then there will be some a and sets $\overline{X}$ and $\tilde{X}$ such that $X = i_{\eta,\alpha}(\overline{X}) = j_{\xi,\alpha}(\tilde{X})$, and X is measured differently by E_a and F_a.[30]

But then $a \in i_{\alpha,\alpha+1}(X) \Leftrightarrow a \notin j_{\alpha,\alpha+1}(X)$, so $i_{\eta,\alpha+1}(\overline{X}) \neq j_{\xi,\alpha+1}(\tilde{X})$, and the images of $\overline{X}$ and $\tilde{X}$ do not participate in a disagreement at stage $\alpha+1$ the way they did at stage α. *If* all future extenders used in either iteration have critical point above $\sup(a)$, then $i_{\eta,\beta}(\overline{X}) \neq j_{\xi,\beta}(\tilde{X})$ for all β, so the images of $\overline{X}$ and $\tilde{X}$ never again participate in a disagreement, and we have made real progress at stage α. A simple Fodor argument shows that if we never

[27]This means that there is a filter F such that $\mathfrak{M}_\alpha$ and $\mathfrak{N}_\alpha$ are of the form

$$\mathfrak{M}_\alpha = \langle \mathbf{L}_\xi[F], \in, F\rangle$$

and

$$\mathfrak{N}_\alpha = \langle \mathbf{L}_\eta[F], \in, F\rangle$$

for some ξ and η. (Here and elsewhere we identify wellfounded, extensional structures with their transitive isomorphs.) In fact, in this simple case we can take α to be $\sup\{|\mathfrak{M}_0|, |\mathfrak{N}_0|\}^+$ and F to be the club filter on α.

[28]This must be qualified, since if E_α does not measure all subsets of its critical point in $\mathfrak{M}_\alpha$, then $\mathrm{Ult}(\mathfrak{M}_\alpha, E_\alpha)$ makes no sense. In this case we take the "largest" E_α–ultrapower of an initial segment of $\mathfrak{M}_\alpha$ we can in order to form $\mathfrak{M}_{\alpha+1}$. (See below.)

[29]In which case we identify these iterates with the premice to which they are isomorphic. Linear iterability should be taken to include the condition that no linear iteration of $\mathfrak{M}_0$ drops to proper initial segments infinitely often.

[30]We use i for the embeddings in the $\mathfrak{M}$–iteration, and j in the $\mathfrak{N}$–iteration.

"move generators" in one of our iterations,[31] then eventually all disagreements are removed.[32] The lack of overlaps in the sequences of mice below a strong cardinal means that this process of iterating away the least disagreement does not move generators, and hence terminates in a successful comparison.

However, beyond a strong cardinal this linear comparison process definitely will lead to moving generators. There are tricks for making do with linear iterations a bit beyond strong cardinals, but the right solution is to give up linearity. If the extender E_α from the $\mathfrak{M}_\alpha$–sequence we want to use has critical point less than $\mathrm{lh}(E_\beta)$ for some $\beta < \alpha$, then we apply E_α not to $\mathfrak{M}_\alpha$, but to $\mathfrak{M}_\beta$, for the least such β: *i.e.*, we set $\mathfrak{M}_{\alpha+1} = \mathrm{Ult}(\mathfrak{M}_\beta, E_\alpha)$, where β is least such that $\mathrm{crit}(E_\alpha) < \mathrm{lh}(E_\beta)$.[33] We have an embedding $i_{\beta,\alpha+1}\colon \mathfrak{M}_\beta \to \mathfrak{M}_{\alpha+1}$. Thus this new iteration process gives rise to a *tree* of models, with embeddings along each branch of the tree. Along each branch the generators of the extenders used are not moved by later embeddings, and this is good enough to show that if a comparison process involving the formation of such "iteration trees" goes on long enough, it must eventually succeed.

What one needs to keep the construction of an iteration tree going past some limit ordinal λ is a branch of the tree which has been visited cofinally often before λ and is such that the direct limit of the premice along the branch is wellfounded. Thus the iterability we need for comparison amounts to the existence of some method for choosing such branches. We can formalize this as the existence of a winning strategy in a certain game.

We shall now define this "iteration game" and introduce the important notions of the theory of iteration trees in terms of this game.

Let $\mathfrak{M}$ be a premouse and ϑ an ordinal. We shall define the iteration game on $\mathfrak{M}$ of length ϑ. In this game, players I and II cooperate to produce an iteration tree $\mathfrak{T}$. This system consists of a tree order[34] T on ϑ together with, for $\alpha, \beta < \vartheta$:

- a premouse $\mathfrak{M}_\alpha^{\mathfrak{T}}$,

[31] *I.e.*, if whenever E is used before E' in the $\mathfrak{M}$–iteration, then $\mathrm{lh}(E) \leq \mathrm{crit}(E')$, and similarly on the $\mathfrak{N}$–side.

[32] More precisely, there must be a stage $\alpha < \sup\{|\mathfrak{M}_0|, |\mathfrak{N}_0|\}^+$ at which $\mathfrak{M}_\alpha$ is an initial segment of $\mathfrak{N}_\alpha$, or *vice versa*.

[33] Again, if E_α fails to measure all sets in $\mathfrak{M}_\beta$, we take the ultrapower of the longest possible initial segment of $\mathfrak{M}_\beta$.

[34] An order T on an ordinal ϑ is called a *tree order* if

- For all $\alpha \neq 0$, we have $0T\alpha$,
- $\{\alpha : \alpha T\beta\}$ is wellordered by T for all $\beta < \vartheta$,
- successor ordinals are successors in the order T,
- if $\alpha T\beta$, then $\alpha < \beta$, and
- for any limit $\lambda < \vartheta$, $\{\alpha\colon \alpha T\lambda\}$ must be cofinal in λ.

We use the standard interval notation in tree orders, *i.e.* $[\alpha, \beta] := \{\gamma \leq \beta : \alpha T\gamma T\beta\} \cup \{\alpha, \beta\}$ and likewise for open and half–open intervals.

- if $\alpha + 1 < \vartheta$, an extender $F_\alpha^{\mathfrak{T}}$ on the $\mathfrak{M}_\alpha^{\mathfrak{T}}$–sequence,
- maps $i_{\alpha\beta}^{\mathfrak{T}} : \mathfrak{M}_\alpha^{\mathfrak{T}} \to \mathfrak{M}_\beta^{\mathfrak{T}}$ for $\alpha T \beta$ such that the branch of T leading from α to β has not dropped.[35]

The game is played as follows: To start, we set $\mathfrak{M}_0^{\mathfrak{T}} := \mathfrak{M}$. Then in move $\alpha + 1$, we suppose that we have already defined

- $F_\gamma^{\mathfrak{T}}$ for $\gamma < \alpha$,
- $\mathfrak{M}_\gamma^{\mathfrak{T}}$ for $\gamma \leq \alpha$,
- and the tree structure T up to α.

Now, player I picks some $F_\alpha^{\mathfrak{T}}$ from the $\mathfrak{M}_\alpha^{\mathfrak{T}}$–sequence such that $\mathrm{lh}(F_\alpha^{\mathfrak{T}}) > \mathrm{lh}(F_\gamma^{\mathfrak{T}})$ for all $\gamma < \alpha$. If he can't, he loses, and the game is over.

Let β be least such that $\mathrm{crit}(F_\alpha^{\mathfrak{T}}) < \mathrm{lh}(F_\beta^{\mathfrak{T}})$. Then set β to be the T–predecessor of $\alpha + 1$. We wish to take the $F_\alpha^{\mathfrak{T}}$–ultrapower of $\mathfrak{M}_\beta^{\mathfrak{T}}$, but (as we pointed out before) there is a problem in that it might be the case that $(\mathfrak{P}(\kappa))^{\mathfrak{M}_\beta^{\mathfrak{T}}} \not\subseteq \mathfrak{M}_\alpha^{\mathfrak{T}}$ for $\kappa = \mathrm{crit}(F_\alpha^{\mathfrak{T}})$ so that the ultrapower construction simply doesn't make sense.

So set $(\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*$ to be the longest initial segment $\mathfrak{R}$ of $\mathfrak{M}_\beta^{\mathfrak{T}}$ for which $\mathfrak{P}(\kappa)^{\mathfrak{R}} \subseteq \mathfrak{M}_\alpha^{\mathfrak{T}}$. One can show that $\mathrm{lh}(F_\beta^{\mathfrak{T}}) \leq \mathrm{Ord}^{(\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*}$ and $\mathfrak{P}(\kappa)^{\mathfrak{M}_\alpha^{\mathfrak{T}}} = \mathfrak{P}(\kappa)^{(\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*}$. Now let

$$\mathfrak{M}_{\alpha+1}^{\mathfrak{T}} := \mathrm{Ult}((\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*, F_\alpha^{\mathfrak{T}}).^{36}$$

We say the iteration tree $\mathfrak{T}$ *drops at* $\alpha + 1$ if $\mathfrak{M}_\beta^{\mathfrak{T}} \neq (\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*$ and we say a branch b *drops at* $\alpha + 1$ if $\mathfrak{T}$ drops at $\alpha + 1$ and $\alpha + 1 \in b$.

Obviously, we do not necessarily have embeddings from $\mathfrak{M}_\beta^{\mathfrak{T}}$ to the models attached to the successors of β in T, because of possible dropping to an initial segment. But if $\mathfrak{T}$ does not drop at $\alpha + 1$ then we have the ultrapower embedding, and we can take it to define $i_{\beta,\alpha+1}^{\mathfrak{T}}$, and then define $i_{\gamma,\alpha+1}^{\mathfrak{T}} := i_{\beta,\alpha+1}^{\mathfrak{T}} \circ i_{\gamma,\beta}^{\mathfrak{T}}$ for $\gamma T \beta$. If $\mathfrak{T}$ drops at $\alpha + 1$, we let $i_{\beta,\alpha+1}^{\mathfrak{T}}$ be undefined.

---

[35] We will say later what we mean by this.

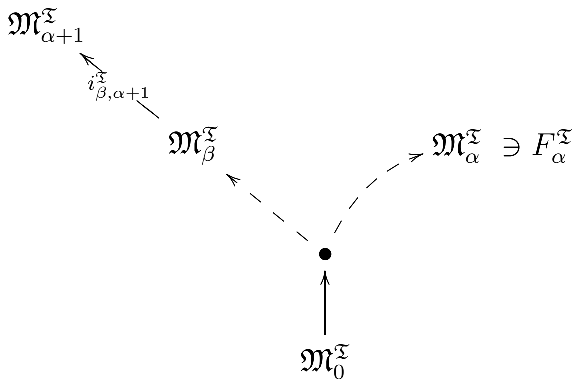
[36] To be more precise, let n be largest such that $\kappa < \varrho_n^{\mathfrak{M}^*_{\alpha+1}}$ and set

$$\mathfrak{M}_{\alpha+1}^{\mathfrak{T}} := \mathrm{Ult}_n((\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*, F_\alpha^{\mathfrak{T}}),$$

where the subscript n in Ult_n indicates that the ultrapower is to be formed using functions which are Σ_n over $(\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*$.

It is important in many contexts that the ultrapowers $\mathrm{Ult}_n((\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*, F_\alpha^{\mathfrak{T}})$ taken at successor steps in the iteration game be "as large as possible", but one needs basic finestructural facts about the Σ_n–projectum $\varrho_n^{(\mathfrak{M}_{\alpha+1}^{\mathfrak{T}})^*}$ in order to keep track precisely of the degree of elementarity of the maps $i_{\gamma,\delta}^{\mathfrak{T}}$.

This is important for example for the proof of the comparison lemma 2.10. On the other hand, the proofs of the basic facts about the Σ_n–projectum require the comparison lemma. As a consequence the full development of the theory we outlined, requires an induction on the Levy hierarchy of mice.



In a limit move λ, II picks a branch b of $\mathfrak{T}$ such that

- b is cofinal in λ,
- b drops at finitely many β, and
- the direct limit of the $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ under the $i^{\mathfrak{T}}_{\alpha,\beta}$ for sufficiently large $\alpha \in b$ is wellfounded.

If II fails to do so, then the game is over and he loses. We call such a branch b a *cofinal wellfounded branch* of $\mathfrak{T} \restriction \lambda$.

Set $\mathfrak{M}^{\mathfrak{T}}_{\lambda} := \mathrm{dirlim}_{\alpha \in b} \mathfrak{M}^{\mathfrak{T}}_{\alpha}$, set $\alpha T \lambda$ for all $\alpha \in b$ and take as $i^{\mathfrak{T}}_{\alpha,\lambda}$ the canonical embeddings for $\alpha T \lambda$ into the direct limit, if α is sufficiently large for the embedding to exist.

This completes the rules of the iteration game. If no one has lost after ϑ moves, player II wins. A tree $\mathfrak{T}$ constructed by playing this game where no player violates the rules is called an *iteration tree*.

Definition 2.7. $\mathfrak{M}$ is *ϑ–iterable* iff player II has a winning strategy in the iteration game of length ϑ. Such a strategy is called a *ϑ–iteration strategy* for $\mathfrak{M}$. $\mathfrak{M}$ is *iterable* iff $\mathfrak{M}$ is ϑ–iterable for all ϑ.

It is customary to call a premouse with useful iterability properties a *mouse*, and we shall adhere to this custom in informal discussion. We prefer, however, to give no formal definition of the term. There are many varieties of iterability beyond the one we have described formally above, and it is not clear which one should be enshrined in a formal definition of "mouse".

The possibility of dropping in an iteration tree adds some complexity, but it cannot be avoided. Indeed, even if we had demanded that all extenders on the sequence of a premouse be total on the premouse itself, the fact that in our nonlinear iterations we want to apply an extender E from the $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$–sequence to some $\mathfrak{M}^{\mathfrak{T}}_{\beta}$ for $\beta < \alpha$ commits us to dropping, for there is no reasonable way to insure that E will measure all sets in $\mathfrak{M}^{\mathfrak{T}}_{\beta}$. But once we are committed to dropping to an initial segment of $\mathfrak{M}^{\mathfrak{T}}_{\beta}$, we need finestructure to make sure we do not drop too far, that we keep every last quantifier's worth

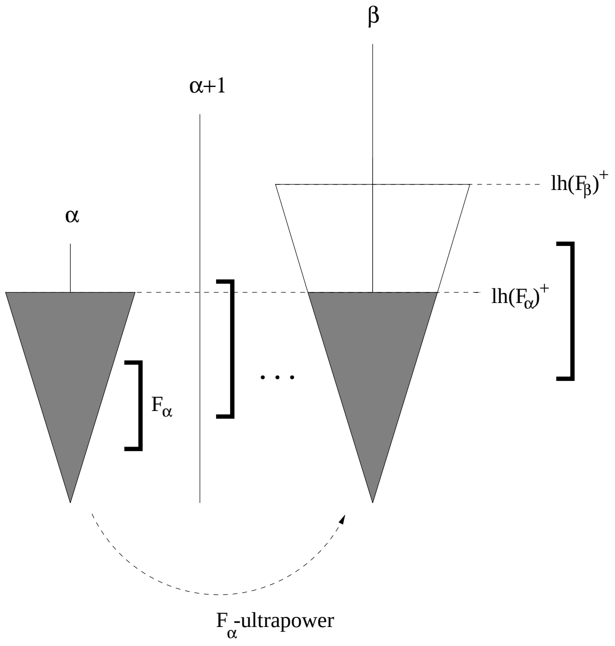


FIGURE 2. The agreement in iteration trees

of elementarity we can. This is why the Baldwin–Mitchell approach, which gives up the totality of the extenders on our coherent sequences in order to simplify finestructure, is an unadulterated gain once we enter the realm of nonlinear iterations.

One basic fact about iteration trees is the following agreement lemma:

Theorem 2.8. Let $\mathfrak{M}$ be a premouse, $\mathfrak{T}$ an iteration tree on $\mathfrak{M}$ with models

$$\langle \mathfrak{M}^{\mathfrak{T}}_{\gamma} : \gamma < \mathrm{lh}(\mathfrak{T}) \rangle,$$

let $\alpha < \beta < \mathrm{lh}(\mathfrak{T})$, and μ the index of $F^{\mathfrak{T}}_{\alpha}$ on the $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$–sequence (so that μ is the cardinal successor of $\mathrm{lh}(F^{\mathfrak{T}}_{\alpha})$ in $\mathrm{Ult}(\mathcal{J}^{\mathfrak{M}^{\mathfrak{T}}_{\alpha}}_{\mu}, F^{\mathfrak{T}}_{\alpha})$); then

(1) $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ and $\mathfrak{M}^{\mathfrak{T}}_{\beta}$ agree up to μ,
(2) μ is a cardinal in $\mathfrak{M}^{\mathfrak{T}}_{\beta}$, and
(3) $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ and $\mathfrak{M}^{\mathfrak{T}}_{\beta}$ do not agree below $\mu + 1$.

Proof : The proof is a routine induction using clause (1.) of Definition 2.1.
The agreement of $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ with later models passes through limit stages λ because if $\beta T \lambda$ and $\alpha < \beta$, then all extenders used in $[\beta, \lambda)_T$ have critical point above $\mathrm{lh}(F^{\mathfrak{T}}_{\alpha})$, so the agreement between $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ and $\mathfrak{M}^{\mathfrak{T}}_{\beta}$ gives the desired agreement between $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$ and $\mathfrak{M}^{\mathfrak{T}}_{\lambda}$.

For the successor step, it is enough to see that $\mathfrak{M}^{\mathfrak{T}}_{\beta}$ agrees with $\mathfrak{M}^{\mathfrak{T}}_{\beta+1}$ up to ν, where ν is the index of $F^{\mathfrak{T}}_{\beta}$ on the $\mathfrak{M}^{\mathfrak{T}}_{\beta}$–sequence. Let $\gamma = \mathrm{pred}_T(\beta + 1)$, and let $\kappa = \mathrm{crit}(F^{\mathfrak{T}}_{\beta})$. Since $\kappa < \mathrm{lh}(F^{\mathfrak{T}}_{\gamma})$, our induction hypothesis implies that $\mathcal{J}_\nu^{\mathfrak{M}_\beta^{\mathfrak{T}}}$ agrees with $\mathfrak{M}_\gamma^{\mathfrak{T}}$ below their common value for κ^+, and hence agrees similarly with $(\mathfrak{M}_{\beta+1}^{\mathfrak{T}})^*$. It follows that $\mathrm{Ult}(\mathcal{J}_\nu^{\mathfrak{M}_\beta^{\mathfrak{T}}}, F_\beta)$ agrees with $\mathrm{Ult}((\mathfrak{M}_{\beta+1}^{\mathfrak{T}})^*, F_\beta) = \mathfrak{M}_{\beta+1}^{\mathfrak{T}}$ below the common image of κ^+ in the two ultrapowers. By the coherence condition on good extender sequences, $\mathfrak{M}_\beta^{\mathfrak{T}}$ agrees with $\mathrm{Ult}(\mathcal{J}_\nu^{\mathfrak{M}_\beta^{\mathfrak{T}}}, F_\beta^{\mathfrak{T}})$ below ν, and since ν is less than the image of κ in this ultrapower, we have proved (1.).

The proof of (2.) is similar. For (3.), notice that by the second coherence condition in Definition 2.1, μ is the index of the empty "extender" on the sequence of $\mathrm{Ult}(\mathcal{J}_\mu^{\mathfrak{M}_\alpha^{\mathfrak{T}}}, F_\alpha^{\mathfrak{T}})$, and hence μ indexes the empty extender in $\mathfrak{M}_\beta^{\mathfrak{T}}$. Since $F_\alpha^{\mathfrak{T}} \neq \emptyset$, $\mathfrak{M}_\alpha^{\mathfrak{T}}$ disagrees with $\mathfrak{M}_\beta^{\mathfrak{T}}$ at μ. □

The fact that the index μ of $F_\alpha^{\mathfrak{T}}$ is a cardinal in all later models of $\mathfrak{T}$ contrasts with the fact that it is not a cardinal in $\mathfrak{M}_\alpha^{\mathfrak{T}}$ itself. This is because one can use the representation of μ in the $F_\alpha^{\mathfrak{T}}$–ultrapower to define a map from $\mathrm{lh}(F_\alpha^{\mathfrak{T}})$ onto μ.[37]

These observations yield the following useful corollary to Theorem 2.8. Let us say that two extenders are *compatible* if one is an initial segment of the other.

Corollary 2.9. Let $\mathfrak{T}$ be an iteration tree, and $F_\alpha^{\mathfrak{T}}$ and $F_\beta^{\mathfrak{T}}$ be extenders used in $\mathfrak{T}$ such that $\alpha \neq \beta$; then $F_\alpha^{\mathfrak{T}}$ is incompatible with $F_\beta^{\mathfrak{T}}$.

Proof: Say $\alpha < \beta$, so that $\mathrm{lh}(F_\alpha^{\mathfrak{T}}) < \mathrm{lh}(F_\beta^{\mathfrak{T}})$. If the two extenders are compatible, then $F_\alpha^{\mathfrak{T}}$ is a proper initial segment of $F_\beta^{\mathfrak{T}}$, and so by the initial segment condition of [MiSt94], $F_\alpha^{\mathfrak{T}} \in \mathfrak{M}_\beta^{\mathfrak{T}}$. By the observation above, this means the index of $F_\alpha^{\mathfrak{T}}$ on the $\mathfrak{M}_\alpha^{\mathfrak{T}}$–sequence is not a cardinal in $\mathfrak{M}_\beta^{\mathfrak{T}}$, contrary to Theorem 2.8. □

2.4. **The Comparison Process.** We come to the main tool of core model theory, the comparison process. We shall illustrate its use in the proof of the following "comparison lemma". There are many other uses for the comparison process.

Theorem 2.10 (Comparison Lemma)**.** Let Ω be a regular cardinal, and let $\mathfrak{M}$ and $\mathfrak{N}$ be $\Omega+1$–iterable premice of ordinal height $\leq \Omega$; then there are iteration trees $\mathfrak{T}$ and $\mathfrak{U}$ of length $\leq \Omega+1$ on $\mathfrak{M}$ and $\mathfrak{N}$ respectively, with last models $\mathfrak{P}$ and $\mathfrak{Q}$, such that either

(1) $\mathfrak{P} \trianglelefteq \mathfrak{Q}$, $\mathfrak{P}$ has ordinal height $\leq \Omega$, and the branch of $\mathfrak{T}$ from $\mathfrak{M}$ to $\mathfrak{P}$ does not drop, or
(2) $\mathfrak{Q} \trianglelefteq \mathfrak{P}$, $\mathfrak{Q}$ has ordinal height $\leq \Omega$, and the branch of $\mathfrak{U}$ from $\mathfrak{N}$ to $\mathfrak{Q}$ does not drop.

[37]It is possible that μ is the ordinal height of $\mathfrak{M}_\alpha^{\mathfrak{T}}$. In this case, the surjection of $\mathrm{lh}(F_\alpha^{\mathfrak{T}})$ onto μ is not a member of $\mathfrak{M}_\alpha^{\mathfrak{T}}$, but is Σ_1–definable over $\mathfrak{M}_\alpha^{\mathfrak{T}}$.

Moreover, if $\mathfrak{M}$ and $\mathfrak{N}$ both have ordinal height $< \Omega$, then $\mathfrak{T}$ and $\mathfrak{U}$ have length $< \Omega$.

Proof : Fix $\Omega+1$–iteration strategies Σ for $\mathfrak{M}$ and Γ for $\mathfrak{N}$. We construct $\mathfrak{T}$ and $\mathfrak{U}$ by induction, "iterating away the least disagreement" at successor steps and using Σ and Γ to pick branches at limit steps.

Let $\mathfrak{M}^{\mathfrak{T}}_0 := \mathfrak{M}$ and $\mathfrak{M}^{\mathfrak{U}}_0 := \mathfrak{N}$. At an arbitrary successor step our current approximations to $\mathfrak{T}$ and $\mathfrak{U}$ have last models $\mathfrak{M}^{\mathfrak{T}}_\alpha$ and $\mathfrak{M}^{\mathfrak{U}}_\beta$. If one is an initial segment of the other, we are done; otherwise let λ be least such that $\mathcal{J}_\lambda^{\mathfrak{M}^{\mathfrak{T}}_\alpha} \neq \mathcal{J}_\lambda^{\mathfrak{M}^{\mathfrak{U}}_\alpha}$. If $E_\lambda^{\mathfrak{M}^{\mathfrak{T}}_\alpha} = \emptyset$ then we do not extend $\mathfrak{T}$ at this step; otherwise, set $F^{\mathfrak{T}}_\alpha := E_\lambda^{\mathfrak{M}^{\mathfrak{T}}_\alpha}$. The rules of the iteration game now determine the T-predecessor of $\alpha+1$ (*i.e.* the model to which we apply $F^{\mathfrak{T}}_\alpha$), and $\mathfrak{M}^{\mathfrak{T}}_{\alpha+1}$, *etc.* Similarly, if $E_\lambda^{\mathfrak{M}^{\mathfrak{U}}_\beta} = \emptyset$ then we do not extend $\mathfrak{U}$ at this step; otherwise, set $F^{\mathfrak{U}}_\alpha := E_\lambda^{\mathfrak{M}^{\mathfrak{U}}_\alpha}$. Again, this choice of extender and the rules of the iteration game determine a one–model extension of $\mathfrak{U}$. Notice that our agreement lemma 2.8 implies that $F^{\mathfrak{T}}_\alpha$ and $F^{\mathfrak{U}}_\beta$ satisfy the increasing–length condition on the extenders in an iteration tree.

At an arbitrary limit step, one or both of our current approximations to $\mathfrak{T}$ and $\mathfrak{U}$ have limit length. If $\mathfrak{T}$ has limit length, we use Σ to produce a one–model extension of it; otherwise, we do not extend $\mathfrak{T}$. We proceed similarly with $\mathfrak{U}$.

Since Σ and Γ win the iteration game, we do not produce any illfounded ultrapowers at successor stages in the constructions of $\mathfrak{T}$ and $\mathfrak{U}$, and we always get cofinal wellfounded branches at limit stages.

If it terminates at some stage $< \Omega$, then one of the last models on $\mathfrak{T}$ and $\mathfrak{U}$ at that stage is an initial segment of the other, and by appealing to some finestructure theory we can verify the remaining clauses of the conclusion:

We must see that the side which comes out shorter does not drop, and here we use soundness in a crucial way. Suppose $\mathfrak{P}$ is the last model on $\mathfrak{T}$, and the branch from $\mathfrak{M}$ to $\mathfrak{P}$ drops. Consider the last drop along this branch. We dropped because the extender E we applied did not measure all sets in the model $\mathfrak{S}$ to which we wanted to apply it, and therefore we formed instead $\mathrm{Ult}_n(\mathfrak{R}, E)$ for the longest $\mathfrak{R} \trianglelefteq \mathfrak{S}$ and the largest n possible. The maximality of $\mathfrak{R}$ and n implies $\varrho^{\mathfrak{R}}_{n+1} \leq \mathrm{crit}(E)$, and therefore $\mathrm{Ult}_n(\mathfrak{R}, E)$ is not $n+1$–sound by the argument in the proof of Proposition 2.6. The extenders used on the branch from $\mathfrak{M}$ to $\mathfrak{P}$ after E have critical points $\geq \mathrm{lh}(E)$, and thus $\mathfrak{P}$ is not $n+1$–sound. It follows at once that $\mathfrak{P}$ is not a proper initial seqment of the last model $\mathfrak{Q}$ of $\mathfrak{U}$, for any proper initial segment of a premouse is fully sound. A somewhat more subtle finestructural argument shows that $\mathfrak{P} \neq \mathfrak{Q}$.

Suppose now that the process continues for Ω, and hence $\Omega+1$, many steps. Let $\mathfrak{P} = \mathfrak{M}^{\mathfrak{T}}_\Omega$ and $\mathfrak{Q} = \mathfrak{M}^{\mathfrak{T}}_\Omega$ be the respective last models. If one of these has ordinal height Ω, then by Theorem 2.8 it is an initial segment of the other, and we can finish the proof as before. The alternative is that each has ordinal height $> \Omega$. We shall show that this alternative leads to a contradiction, thereby completing the proof.

For this, the following general fact about the trees arising in a comparison process is useful: for any α and β, $F_\alpha^{\mathfrak{T}}$ is incompatible with $F_\beta^{\mathfrak{U}}$. The proof of this fact is very close to the proof of Corollary 2.9.

Now suppose $\Omega \in \mathrm{Ord}^{\mathfrak{P}} \cap \mathrm{Ord}^{\mathfrak{Q}}$, so that we have $\Omega \in \mathrm{ran}(i^{\mathfrak{T}}_{\xi,\Omega}) \cap \mathrm{ran}(i^{\mathfrak{U}}_{\tau,\Omega})$ for some $\xi \in [0,\Omega]_T$ and $\eta \in [0,\Omega]_U$. Now every branch of an iteration tree is closed below its supremum, and so $[0,\Omega]_T$ and $[0,\Omega]_U$ are closed unbounded in Ω. Using these facts it is easy to see that there is a closed unbounded set C in Ω such that:

- $C \subseteq [0,\Omega]_T \cap [0,\Omega]_U$,
- if $\kappa \in C$, then $\kappa = \mathrm{crit}(i^{\mathfrak{T}}_{\kappa,\Omega}) = \mathrm{crit}(i^{\mathfrak{U}}_{\kappa,\Omega})$, and $i^{\mathfrak{T}}_{\kappa,\Omega}(\kappa) = i^{\mathfrak{U}}_{\kappa,\Omega}(\kappa) = \Omega$.

Let $\kappa \in C$, and let E and F be the extenders with critical point κ used along $[0,\Omega]_T$ and $[0,\Omega]_U$ respectively. Because E and F are incompatible, we can find a set

$$A_\kappa \in \mathfrak{M}^{\mathfrak{T}}_\kappa \cap \mathfrak{M}^{\mathfrak{U}}_\kappa$$

such that for some $a \in \mathrm{lh}(E) \cap \mathrm{lh}(F)$,

$$A_\kappa \in E_a \Leftrightarrow A_\kappa \notin F_a,$$

so that if τ and σ are the next ordinals on $[0,\Omega]_T$ and $[0,\Omega]_U$ after κ respectively, then

$$a \in i^{\mathfrak{T}}_{\kappa,\tau}(A_\kappa) \Leftrightarrow a \in i^{\mathfrak{U}}_{\kappa,\sigma}(A_\kappa).$$

Because generators are not moved along the branches of an iteration tree, if $\nu > \kappa$ and $\nu \in C$, then

$$a \in i^{\mathfrak{T}}_{\kappa,\nu}(A_\kappa) \Leftrightarrow a \in i^{\mathfrak{U}}_{\kappa,\nu}(A_\kappa),$$

and therefore

$$i^{\mathfrak{T}}_{\kappa,\nu}(A_\kappa) \neq A_\nu \text{ or } i^{\mathfrak{U}}_{\kappa,\nu}(A_\kappa) \neq A_\nu.$$

On the other hand, each $\kappa \in C$ is a limit ordinal, so we can find $\xi_\kappa \in [0,\kappa)_T$ and $\eta_\kappa \in [0,\kappa]_U$ such that $A_\kappa \in \mathrm{ran}(i^{\mathfrak{T}}_{\xi_\kappa,\kappa}) \cap \mathrm{ran}(i^{\mathfrak{U}}_{\eta_\kappa,\kappa})$. By Fodor's lemma we can fix the value ξ of ξ_κ on a stationary set S_0, then we thin S_0 to a stationary S_1 on which the value η of η_κ is fixed, and then fix the pre–images of A_κ in $\mathfrak{M}^{\mathfrak{T}}_\xi$ and $\mathfrak{M}^{\mathfrak{U}}_\eta$ on a stationary $S_2 \subseteq S_1$. It follows that if $\kappa, \nu \in S_2$ and $\kappa < \nu$, then $i^{\mathfrak{T}}_{\kappa,\nu}(A_\kappa) = i^{\mathfrak{U}}_{\kappa,\nu}(A_\kappa) = A_\nu$, which is the desired contradiction. $\square$

We shall call the pair of iteration trees $\langle \mathfrak{T}, \mathfrak{U} \rangle$ produced in the proof of Theorem 2.10 the $\langle \Sigma, \Gamma \rangle$–*coiteration* of $\mathfrak{M}$ with $\mathfrak{N}$. We say that $\mathfrak{N}$ $\langle \Sigma, \Gamma \rangle$–*iterates past* $\mathfrak{M}$ if the first alternative in the conclusion of Theorem 2.10 holds, and say that $\mathfrak{M}$ $\langle \Sigma, \Gamma \rangle$–*iterates past* $\mathfrak{N}$ if the second alternative holds. If both alternatives hold, then we say $\mathfrak{M}$ and $\mathfrak{N}$ have a *common* $\langle \Sigma, \Gamma \rangle$–*iterate*. Later on we shall be dealing mostly with premice having unique iteration strategies, and in this case we shall drop reference to the strategies in these locutions.

The comparison process is used in a crucial way in the proof that sufficiently iterable premice have cores.[38]

Theorem 2.11. If $\mathfrak{N}$ is a premouse all of whose countable elementary substructures are ω_1+1-iterable, then the core $\mathfrak{C}(\mathfrak{N})$ exists, and agrees with $\mathfrak{N}$ below $(\varrho^+)^{\mathfrak{N}}$, where $\varrho = \varrho_\omega^{\mathfrak{N}}$ is the projectum of $\mathfrak{N}$.[39]

Theorem 2.11 is one of the central results of basic finestructure theory, and its proof is far from trivial. Many authors have contributed to the evolution of the theorem.[40]

Since we have avoided even the subtleties involved in correct definitions is this area, we shall not attempt to sketch a proof of Theorem 2.11. However, the reader can gain some appreciation of the problem as follows. Let $\mathfrak{C}_1 = \mathrm{Hull}_1^{\mathfrak{N}}(\varrho_1 \cup \{p_1\})$, where ϱ_1 is the Σ_1–projectum of $\mathfrak{N}$ and p_1 is its standard parameter. One thing we need to see is that ϱ_1 is also the Σ_1–projectum of $\mathfrak{C}_1$, and for this we need to see that every subset of ϱ_1 in $\mathfrak{N}$ is in $\mathfrak{C}_1$. The natural proof of this involves *comparing* $\mathfrak{N}$ with $\mathfrak{C}_1$. If the iteration maps $i\colon \mathfrak{N} \to \mathfrak{P}$ and $j\colon \mathfrak{C}_1 \to \mathfrak{Q}$ to the last models on the two sides have critical point $\geq \varrho_1$, then because both $\mathfrak{C}_1$ and $\mathfrak{N}$ define the same "new" Σ_1 subset of ϱ_1, we get that $\mathfrak{P} = \mathfrak{Q}$, and from this it is easy to see that $\mathfrak{N}$ and $\mathfrak{C}_1$ have the same subsets of ϱ_1.[41] Since $\mathfrak{N}$ and $\mathfrak{C}_1$ agree below ϱ_1, there is reason to hope that i and j have critical point $\geq \varrho_1$, but the existence of extenders E on the sequences of $\mathfrak{N}$ and $\mathfrak{C}_1$ overlapping ϱ_1 is a severe problem.[42] Why couldn't i or j use such an extender? The solution to this problem involves abandoning the simple coiteration above for a more complicated version, and making extensive use of the Dodd-Jensen Lemma 2.16. The idea of comparing $\mathfrak{N}$ and $\mathfrak{C}_1$ via iterations with critical point above ϱ_1 is still at the heart of it, however.

Mice which iterate past all other mice of no greater ordinal height will play an important rôle later on.

Definition 2.12. Let $\mathfrak{M}$ be a premouse whose ordinal height is a regular cardinal Ω. We say that $\mathfrak{M}$ is *Ω-universal* just in case there is an $\Omega+1$–iteration strategy Σ for $\mathfrak{M}$ such that whenever $\mathfrak{N}$ is a premouse of ordinal

[38]*Cf.* the discussion following Definition 2.5 for the definition of $\mathfrak{C}(\mathfrak{N})$.

[39]*Cf.* Footnote 20 for a short discussion of finestructural terminology.

If $(\varrho^+)^{\mathfrak{N}}$ does not exist, that is if $\varrho = \mathrm{Ord}^{\mathfrak{N}}$ or ϱ is the largest cardinal of $\mathfrak{N}$, then this should be taken to mean that $\mathfrak{C}(\mathfrak{N}) = \mathfrak{N}$.

[40]We shall name Tony Dodd, Ronald Jensen, Sy Friedman, Bill Mitchell, and Ernest Schimmerling (*cf.* [SchSt96]). The final version of Theorem 2.11 was proved in [NeeSt9?]; the argument in that paper rests heavily on [MiSt94, Theorem 8.1].

[41]For premice "below a strong cardinal", one can show that $\mathfrak{N}$ never moves in this coiteration, and so is an iterate of its core $\mathfrak{C}_1$ via an iteration with critical point above the projectum ϱ_1. More complicated mice, however, need not be iterates of their cores.

[42]*I.e.*, extenders such that $\mathrm{crit}(E) < \varrho_1 \leq \mathrm{lh}(E)$.

height $\leq \Omega$ and Γ is an $\Omega+1$–iteration strategy for $\mathfrak{N}$, then $\mathfrak{M}$ $\langle\Sigma,\Gamma\rangle$–iterates past $\mathfrak{N}$.

Covering theorems can be used to prove that there are universal mice. In a covering theorem one proves that if there is some kind of bound on the complexity of the mice in $\mathbf{V}$, then there is a mouse which is "close" to $\mathbf{V}$ in some sense. The closeness to $\mathbf{V}$ required for universality is that the mouse should compute many successor cardinals correctly; this sort of covering is called *weak covering*.[43] The following very useful lemma shows that a form of weak covering implies universality.

Lemma 2.13. Let Ω be weakly Mahlo, and let $\mathfrak{M}$ be an $\Omega+1$–iterable premouse such that for stationarily many regular cardinals $\alpha < \Omega$, $(\alpha^+)^W = \alpha^+$; then W is Ω–universal.

Proof : Let Σ be an $\Omega+1$–iteration strategy for W, and let $\mathfrak{N}$ be a premouse of ordinal height $\leq \Omega$, and Γ an $\Omega+1$–iteration strategy for $\mathfrak{N}$. Let $\langle\mathfrak{T},\mathfrak{U}\rangle$ be the $\langle\Sigma,\Gamma\rangle$–coiteration of W with $\mathfrak{N}$, and assume toward contradiction that W does not iterate past $\mathfrak{N}$. Thus the last model of $\mathfrak{T}$ is a proper initial segment of that of $\mathfrak{U}$, and the branch of $\mathfrak{T}$ leading to its last model does not drop. Since $\mathrm{Ord}^W = \Omega$, this means that the last model of $\mathfrak{U}$ has ordinal height $> \Omega$. Since $\mathrm{Ord}^{\mathfrak{N}} \leq \Omega$, we have $\mathrm{lh}(\mathfrak{U}) = \Omega+1$, and as in the proof of the comparison lemma 2.10 we have a closed unbounded set $C \subseteq \Omega$ such that for all $\kappa \in C$

$$\kappa = \mathrm{crit}(i^{\mathfrak{U}}_{\kappa,\Omega}) \text{ and } i^{\mathfrak{U}}_{\kappa,\Omega}(\kappa) = \Omega.$$

Now if $\kappa,\mu \in C$ and $\kappa < \mu$, then

$$i^{\mathfrak{U}}_{\kappa,\mu}((\kappa^+)^{\mathfrak{M}^{\mathfrak{U}}_\kappa}) = (\mu^+)^{\mathfrak{M}^{\mathfrak{U}}_\mu} = (\mu^+)^{\mathfrak{M}^{\mathfrak{U}}_\Omega},$$

where the second equality holds because $\mathrm{crit}(i^{\mathfrak{U}}_{\mu,\Omega}) = \mu$, so that $\mathfrak{M}^{\mathfrak{U}}_\mu$ and $\mathfrak{M}^{\mathfrak{U}}_\Omega$ have the same subsets of μ. Since all embeddings of an iteration tree are continuous at successor cardinals[44], this implies that

$$\sup(i^{\mathfrak{U}}_{\kappa,\mu}{}''(\kappa^+)^{\mathfrak{M}^{\mathfrak{U}}_\kappa}) = (\mu^+)^{\mathfrak{M}^{\mathfrak{U}}_\Omega} < \mu^+$$

for all but the least $\mu \in C$.

[43] Covering theorems begin, of course, with Jensen's great leap forward ([DevJen75]). One cannot prove the existence of mice with the stronger covering properties studied in [DevJen75] and [DoJen81] without assuming the non–existence of mice satisfying that there is an inaccessible limit of measurable cardinals, or at least something close to that, so these stronger covering properties are not very useful at the Woodin cardinal level. Mitchell ([Mi84]) first realized that the covering proof still gives weak covering in the more general situation, and that this is very useful in basic core model theory. Lemma 2.13 is due to him.

[44] This is an easy induction; discontinuities come only from the ultrapower construction at points of cofinality $\mathrm{crit}(E)$, where E is the extender used, and $\mathrm{crit}(E)$ is never a successor cardinal of the model to which E is applied.

Let's assume that $\mathrm{lh}(\mathfrak{T}) = \Omega+1$, the case that $\mathrm{lh}(\mathfrak{T}) < \Omega$ being similar but a bit simpler. Since $\mathrm{Ord}^{\mathfrak{M}^{\mathfrak{T}}_{\Omega}} = \Omega$ by the Comparison Lemma 2.10, $i^{\mathfrak{T}}_{0,\Omega}(\mu) < \Omega$ for all $\mu < \Omega$, and thus there is a closed unbounded set of $\mu \in [0,\Omega]_T$ such that

$$i^{\mathfrak{T}}_{0,\mu}{}''\mu \subseteq \mu.$$

Because Ω is Mahlo, stationarily many of these μ are regular, and for these

$$i^{\mathfrak{T}}_{0,\mu}(\mu) = \mu.$$

Stationarily many of these μ are such that $(\mu^+)^W = \mu^+$, and for these

$$(\mu^+)^{\mathfrak{M}^{\mathfrak{T}}_{\Omega}} = (\mu^+)^{\mathfrak{M}^{\mathfrak{T}}_{\mu}} = i^{\mathfrak{T}}_{0,\mu}(\mu^+) = \mu^+.$$

Since $\mathfrak{M}^{\mathfrak{T}}_{\Omega}$ is an initial segment of $\mathfrak{M}^{\mathfrak{U}}_{\Omega}$, we have that $(\mu^+)^{\mathfrak{M}^{\mathfrak{T}}_{\Omega}} \leq (\mu^+)^{\mathfrak{M}^{\mathfrak{U}}_{\Omega}}$ for all $\mu < \Omega$, so that $\mathfrak{M}^{\mathfrak{U}}_{\Omega}$ also computes stationarily many successor cardinals correctly, in contradiction to the previous paragraph. $\square$

The reader may well be upset by the appearance of a Mahlo cardinal in the hypotheses of Lemma 2.13. There are variants of the lemma which do not make use of Mahlo cardinals, but for us later it will be Lemma 2.13 which is useful.

2.5. **The Mouse Order and the Dodd–Jensen Lemma.** The comparison lemma 2.10 can be stated in a different language: It says that the mouse order $\leq^*$ is linear on iterable premice. Although we won't use the mouse order in any applications in this paper, it is an important concept in core model theory and thus we shall devote this section to its introduction.

Definition 2.14. Let $\mathfrak{M}$ and $\mathfrak{N}$ be iterable premice with iteration strategies Σ and Γ such that for every $\alpha \in \mathrm{Ord}$ the restriction of Σ and Γ to games of length α is the unique α–iteration strategy.
Then set $\mathfrak{M} \leq^* \mathfrak{N}$ if and only if $\mathfrak{M}$ $\langle\Sigma,\Gamma\rangle$–iterates past $\mathfrak{N}$.
This relation is called the *mouse order*.

The term "order" indicates that the relation is transitive, though this is not at all obvious from the definition. In this section we will use the comparison lemma 2.10 and the Dodd–Jensen lemma 2.16 to show that $\leq^*$ is a linear prewellordering.[45]

Proposition 2.15. The relation $\leq^*$ is linear.

[45]The assumption that the iteration strategies be unique is made mainly to suppress complications. The proofs will work with assumptions that are a lot weaker than that, *cf.* [St93, Theorem 3.2] and [NeeSt9?]. But in our situation below one Woodin cardinal iteration strategies are always unique (*cf.* [MiSt94, §6]), so the restriction is of no harm here.

Proof : Immediate from Theorem 2.10. □

For transitivity we need the Dodd–Jensen lemma, a theorem about minimality of iteration maps that has uses far beyond this little application and has already been mentioned before:

Theorem 2.16 (Dodd–Jensen Lemma)**.** Let $\mathfrak{M}$ be a premouse with a unique iteration strategy Σ and let $\mathfrak{T}$ be an iteration tree on $\mathfrak{M}$ according to Σ with last model $\mathfrak{M}_\alpha^\mathfrak{T}$.[46] Suppose there is an embedding $\pi : \mathfrak{M} \to \mathfrak{P}$ where $\mathfrak{P}$ is an initial segment of $\mathfrak{M}_\alpha^\mathfrak{T}$. Then:

(1) $\mathfrak{P} = \mathfrak{M}_\alpha^\mathfrak{T}$,
(2) $[0, \alpha]$ does not drop, and
(3) for all $\eta \in \mathrm{Ord}^\mathfrak{M}$ we have $i_{0\alpha}^\mathfrak{T}(\eta) \leq \pi(\alpha)$.

Proof : We shall show the third claim as the proofs of the other claims are similar.
The basic technique used in the proof is the technique of copying iteration trees. Given an iterable premouse $\mathfrak{Q}$, an iteration tree $\mathfrak{U}$ on $\mathfrak{Q}$ and an embedding $\sigma : \mathfrak{Q} \to \mathfrak{R}$, we can form an iteration tree $\sigma\mathfrak{U}$ on $\mathfrak{R}$ with embeddings $\sigma_\beta : \mathfrak{M}_\beta^\mathfrak{U} \to \mathfrak{M}_\beta^{\sigma\mathfrak{U}}$ for all $\beta < \mathrm{lh}(\mathfrak{U})$ such that

- if $i_{\alpha\beta}^\mathfrak{U}$ is defined then so is $i_{\alpha\beta}^{\sigma\mathfrak{U}}$ and $\pi_\beta \circ i_{\alpha\beta}^\mathfrak{U} = i_{\alpha\beta}^{\sigma\mathfrak{U}} \circ \sigma_\alpha$,
- $\mathfrak{U}$ and $\sigma\mathfrak{U}$ have the same underlying trees and they drop at the same points.

To prove that such a tree $\sigma\mathfrak{U}$ exists, we construct the embeddings σ_α inductively.[47] For the successor step, we use the so–called "shift lemma":

Lemma 2.17. Let $\mathfrak{M}$ and $\mathfrak{N}$ be premice, F the last extender of $\mathfrak{N}$, κ its critical point, and let $\pi : \mathfrak{M} \to \mathfrak{M}^*$ and $\psi : \mathfrak{N} \to \mathfrak{N}^*$ be embeddings such that

- $\mathcal{J}^\mathfrak{M}_{(\kappa^+)^\mathfrak{M}} = \mathcal{J}^\mathfrak{N}_{(\kappa^+)^\mathfrak{M}}$
- $\mathcal{J}^{\mathfrak{M}^*}_{(\pi(\kappa)^+)^{\mathfrak{M}^*}} = \mathcal{J}^{\mathfrak{N}^*}_{(\pi(\kappa)^+)^{\mathfrak{M}^*}}$
- $\pi \restriction (\kappa^+)^\mathfrak{M} = \psi \restriction (\kappa^+)^\mathfrak{M}$, and
- $\mathrm{Ult}(\mathfrak{M}, F)$ and $\mathrm{Ult}(\mathfrak{M}^*, F)$ are defined and well–founded.

  Then there is an embedding σ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{\quad\pi\quad} & \mathfrak{M}^* \\ \downarrow & & \downarrow \\ \mathrm{Ult}(\mathfrak{M}, F) & \xrightarrow{\sigma} & \mathrm{Ult}(\mathfrak{M}^*, F) \end{array}$$

[46] In fact, we need more iterability than we talked about up to now. In the proof we need that linear stacking of iteration trees gives iterations, which is not true in the context of our Section 2.3. The reader can find correct definitions for this generalized iteration game in [NeeSt9?] and [St9?a].

[47] In fact, it is not necessary that σ is fully elementary. It suffices that σ is a weak k–embedding, where k is the highest degree of elementarity occuring in the embeddings of non–dropping branches in $\mathfrak{U}$. For definitions, *cf.* [MiSt94, p. 52].

We shall say that the copying construction breaks down in the limit step if the copied branch is ill–founded. Otherwise we continue.

This process also gives us a method to pull back strategies from $\mathfrak{R}$ to $\mathfrak{Q}$:

If we have constructed $\sigma\mathfrak{U}$ up to λ and Σ is any strategy for player II on $\mathfrak{R}$, then Σ chooses a branch in $\sigma\mathfrak{U} \upharpoonright \lambda$. The appropriate strategy on $\mathfrak{Q}$ is called *the pullback of* Σ *via* σ and denoted by Σ^σ.

Because of the embeddings we are constructing, the property of being winning for player II also gets pulled back, so if Σ was an iteration strategy for $\mathfrak{R}$, Σ^σ will be an iteration strategy for $\mathfrak{Q}$.

Now we will use this technique to prove the Dodd–Jensen lemma:
We shall construct models $\mathfrak{M}_n$, "vertical" embeddings $\pi^n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$, "horizontal" embeddings $\sigma^n : \mathfrak{M}_n \to \mathfrak{M}_{n+1}$ and iteration trees $\mathfrak{T}_n$ of length α on $\mathfrak{M}_n$ with last models $\mathfrak{M}_{n+1}$ such that $\sigma^n = i_{0\alpha}^{\mathfrak{T}_n}$ inductively:

Set $\mathfrak{M}_0 := \mathfrak{M}$, $\pi^0 := \pi$ and $\sigma^0 := i_{0\alpha}^{\mathfrak{T}}$. Suppose the models have been constructed up to $n+1$ and the embeddings and iteration trees have been constructed up to n. Then we have

$$\begin{tikzcd}
\mathfrak{M}_{n+1} & \\
\mathfrak{M}_n \arrow[u, "\pi^n"] \arrow[r, "\sigma^n"] & \mathfrak{M}_{n+1}
\end{tikzcd}$$

We now copy $\mathfrak{T}_n$ to $\mathfrak{M}_{n+1}$ to get the tree $\mathfrak{T}_{n+1} := \pi^n\mathfrak{T}_n$. This process doesn't break down because of our uniqueness assumption: Suppose that at a given limit step λ, the copied branch of $\pi^n\mathfrak{T}_n$ is ill–founded. By iterability there is a $\lambda+1$–iteration strategy Σ that picks a wellfounded branch of $\pi^n\mathfrak{T}_n \upharpoonright \lambda$, but then the pullback strategy Σ^{π^n} is also an iteration strategy. Uniqueness gives us that Σ must already have picked the copied branch.

Our construction process gives us an embedding $\pi_\alpha^n : \mathfrak{M}_{n+1} = \mathfrak{M}_\alpha^{\mathfrak{T}_n} \to \mathfrak{M}_\alpha^{\pi^n\mathfrak{T}_n}$ and an embedding $i_{0\alpha}^{\pi^n\mathfrak{T}_n} : \mathfrak{M}_n \to \mathfrak{M}_\alpha^{\pi^n\mathfrak{T}_n}$ to complete the diagram to

$$\begin{tikzcd}
\mathfrak{M}_{n+1} \arrow[r, "i_{0\alpha}^{\pi^n\mathfrak{T}_n}"] & \mathfrak{M}_\alpha^{\pi^n\mathfrak{T}_n} \\
\mathfrak{M}_n \arrow[u, "\pi^n"] \arrow[r, "\sigma^n = i_{0\alpha}^{\mathfrak{T}_n}"] & \mathfrak{M}_\alpha^{\mathfrak{T}_n} \arrow[u, "\pi_\alpha^n"]
\end{tikzcd}$$

We set $\mathfrak{M}_{n+2} := \mathfrak{M}_\alpha^{\pi^n\mathfrak{T}_n}$, $\pi^{n+1} := \pi_\alpha^n$ and $\sigma^{n+1} := i_{0\alpha}^{\pi^n\mathfrak{T}_n}$ and receive the following diagram:

$$\begin{tikzcd}
\mathfrak{M}_0 \arrow[r, "\sigma^0 = i^{\mathfrak{T}}_{0\alpha}"] & \mathfrak{M}_1 \arrow[r, "\sigma^1"] & \mathfrak{M}_2 \arrow[r, "\sigma^2"] & \mathfrak{M}_3 \arrow[r, "\sigma^3"] & \mathfrak{M}_4 \arrow[r, "\sigma^4"] & \cdots \\
& \mathfrak{M}_0 \arrow[u, "\pi^0 = \pi"] \arrow[r, "\sigma^0"] & \mathfrak{M}_1 \arrow[u, "\pi^1"] \arrow[r, "\sigma^1"] & \mathfrak{M}_2 \arrow[u, "\pi^2"] \arrow[r, "\sigma^2"] & \mathfrak{M}_3 \arrow[u, "\pi^3"] \arrow[r, "\sigma^3"] & \cdots \\
& & \mathfrak{M}_0 \arrow[u, "\pi^0"] \arrow[r, "\sigma^0"] & \mathfrak{M}_1 \arrow[u, "\pi^1"] \arrow[r, "\sigma^1"] & \mathfrak{M}_2 \arrow[u, "\pi^2"] \arrow[r, "\sigma^2"] & \cdots \\
& & & \mathfrak{M}_0 \arrow[u, "\pi^0"] \arrow[r, "\sigma^0"] & \mathfrak{M}_1 \arrow[u, "\pi^1"] \arrow[r, "\sigma^1"] & \cdots \\
& & & & \mathfrak{M}_0 \arrow[u, "\pi^0"] \arrow[r, "\sigma^0"] & \cdots
\end{tikzcd}$$

We now suppose that there is an η such that $\pi(\eta) < i^{\mathfrak{T}}_{0\alpha}(\eta)$ and construct a descending sequence of ordinals in $\operatorname{dirlim}_{n\in\omega}\mathfrak{M}_n$. This is a contradiction to the iterability of $\mathfrak{M}$.[48] To prove the claim, we construct for every $n \in \omega$ descending sequences $\langle \eta^n_0, \ldots, \eta^n_n \rangle$ in $\mathfrak{M}_n$ such that $\sigma^n(\eta^n_i) = \eta^{n+1}_i$ for $i \le n$.

Set $\eta^1_0 := \pi(\eta)$ and $\eta^1_1 := i^{\mathfrak{T}}_{0\alpha}(\eta)$. If the sequences are constructed up to n, then set $\eta^{n+1}_i := \sigma^{n+1}(\eta^n_i)$ for $i \le n$ and $\eta^{n+1}_{n+1} := \pi^{n+1} \circ \cdots \circ \pi^0(\eta)$. We have to show that the new sequence is descending, *i.e.* that $\eta^{n+1}_{n+1} < \eta^{n+1}_n$. But by commutativity and our assumption, we have

$$\begin{aligned}
\eta^{n+1}_n = \sigma^{n+1}(\eta^n_n) &= \sigma^{n+1} \circ \pi^n \circ \cdots \circ \pi^0(\eta) \\
&= \pi^{n+1} \circ \sigma^n \circ \pi^{n-1} \circ \cdots \circ \pi^0(\eta) \\
&> \pi^{n+1} \circ \pi^n \circ \pi^{n-1} \circ \cdots \circ \pi^0(\eta) \\
&= \eta^{n+1}_{n+1}
\end{aligned}$$

This completes the proof. □

Corollary 2.18. If $\mathfrak{T}$ and $\mathfrak{U}$ are two iteration trees on $\mathfrak{M}$ with last model $\mathfrak{N}$, then the branch leading to $\mathfrak{N}$ is either in both trees a dropping branch or in both trees a non–dropping branch.

Proof : If one branch is non–dropping, it gives rise to an embedding from $\mathfrak{M}$ into $\mathfrak{N}$. The Dodd–Jensen lemma 2.16 then tells us that the other branch can't drop either. □

Proposition 2.19. The mouse order $\le^*$ is a linear prewellordering.

Proof : We only have to show well–foundedness, as reflexivity is trivial, linearity is Proposition 2.15 and transitivity follows abstractly from well–foundedness and linearity:

[48] This is the point where the notion of generalized iteration would come in if we had not suppressed this detail.

Take a counterexample $\mathfrak{M}_1 \leq^* \mathfrak{M}_2 \leq^* \mathfrak{M}_3$ to transitivity, *i.e.* $\mathfrak{M}_1 \not\leq^* \mathfrak{M}_3$. By linearity $\mathfrak{M}_3 \leq^* \mathfrak{M}_1$ and we can get a descending chain by concatenating ω copies of $\langle \mathfrak{M}_3, \mathfrak{M}_2, \mathfrak{M}_1 \rangle$, contradicting well–foundedness.

For the proof of well–foundedness, we take an infinite descending chain of models $\langle \mathfrak{M}_i : i \in \omega \rangle$. Let $\mathfrak{Q}_i^0$ be the witness that $\mathfrak{M}_i$ iterates past $\mathfrak{M}_{i+1}$, *i.e.* the coiteration of $\mathfrak{M}_i$ and $\mathfrak{M}_{i+1}$ results in a model $\mathfrak{N}$ on the $\mathfrak{M}_i$–side and the model $\mathfrak{Q}_i^0$ on the $\mathfrak{M}_{i+1}$–side where $\mathfrak{Q}_i^0 \trianglelefteq \mathfrak{N}$.

Now inductively suppose we had defined models $\langle \mathfrak{Q}_i^n : i \in \omega \rangle$. We compare $\mathfrak{Q}_i^n$ and $\mathfrak{Q}_{i+1}^n$ and call the common initial segment of the two iterates $\mathfrak{Q}_i^{n+1}$. This gives us the following diagram:

$$\begin{tikzcd}
\mathfrak{M}_0 \arrow[r] & \mathfrak{Q}_0^0 \arrow[r] & \mathfrak{Q}_0^1 \arrow[r] & \mathfrak{Q}_0^2 \arrow[r, Rightarrow] & \mathfrak{Q}_0^3 \arrow[r] & \mathfrak{Q}_0^4 \arrow[r] & \mathfrak{Q}_0^5 \arrow[r] & \cdots \\
& \mathfrak{M}_1 \arrow[u] \arrow[r] & \mathfrak{Q}_1^0 \arrow[u] \arrow[r] & \mathfrak{Q}_1^1 \arrow[u, Rightarrow] \arrow[r] & \mathfrak{Q}_1^2 \arrow[u, Rightarrow] \arrow[r] & \mathfrak{Q}_1^3 \arrow[u] \arrow[r] & \mathfrak{Q}_1^4 \arrow[u] \arrow[r] & \cdots \\
& & \mathfrak{M}_2 \arrow[u] \arrow[r] & \mathfrak{Q}_2^0 \arrow[u, Rightarrow] \arrow[r] & \mathfrak{Q}_2^1 \arrow[u, Rightarrow] \arrow[r] & \mathfrak{Q}_2^2 \arrow[u] \arrow[r] & \mathfrak{Q}_2^3 \arrow[u] \arrow[r] & \cdots \\
& & & \mathfrak{M}_3 \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \mathfrak{Q}_3^0 \arrow[u, Rightarrow] \arrow[r] & \mathfrak{Q}_3^1 \arrow[u] \arrow[r] & \mathfrak{Q}_3^2 \arrow[u] \arrow[r] & \cdots \\
& & & & \ddots \arrow[u] & \ddots \arrow[u] & \ddots \arrow[u] & \ddots
\end{tikzcd}$$

We now claim that the sequence $\mathfrak{M}_i \to \mathfrak{Q}_i^0 \to \mathfrak{Q}_i^1 \to \ldots$ is a *degenerate iteration* of $\mathfrak{M}_i$, *i.e.* an iteration with infinitely many drops. This would be a contradiction to the iterability of $\mathfrak{M}_i$, because player II plays according to the unique iteration strategy in every step.[49]

To see that this iteration drops infinitely many times, we claim that every iteration $\mathfrak{Q}_i^n \to \mathfrak{Q}_i^{n+1}$ drops: If $\mathfrak{Q}_i^n \to \mathfrak{Q}_i^{n+1}$ doesn't drop then

$$\mathfrak{M}_{n+i+1} \to \mathfrak{Q}_{n+i}^0 \to \cdots \to \mathfrak{Q}_i^n \to \mathfrak{Q}_i^{n+1}$$

is an iteration of $\mathfrak{M}_{n+i+1}$ without drops. But as $\mathfrak{M}_{n+i+1} \to \mathfrak{Q}_{n+i+1}^0$ drops by assumption, the iteration

$$\mathfrak{M}_{n+i+1} \to \mathfrak{Q}_{n+i+1}^0 \to \cdots \to \mathfrak{Q}_i^{n+1}$$

is a dropping iteration, contradicting Corollary 2.18.[50] $\square$

[49]Note that this argument requires the generalized notion of iteration we mentioned in Footnote 46.

[50]An example ($n = 2$ and $i = 0$) of this argument is depicted in the above diagram by the double arrows.

2.6. **The construction of $\mathbf{K}^c$.** We have been studying mice in the abstract, but we have yet to produce any! In this section we shall describe the construction of a core model $\mathbf{K}^c$. This construction is sufficiently cautious about adding extenders to the model that one gets an iterable model in the end,[51], yet sufficiently daring that if one carries it out in a universe satisfying certain statements of consistency strength at least that of "there is a Woodin cardinal" one gets a model satisfying "there is a Woodin cardinal".[52]

The natural idea is to construct a good extender sequence $\vec{E}$ by induction. Given $\vec{E} \restriction \alpha$, we set $E_\alpha = \emptyset$ unless there is a certified[53] extender F such that $(\vec{E} \restriction \alpha)^\frown F$ is still a good extender sequence; if there is such an F we may set $E_\alpha = F$ for some such F. Here "certified" means roughly that F is the restriction to $\mathbf{J}_\alpha[\vec{E} \restriction \alpha]$ of a "background extender" F^* which measures a broader collection of subsets of its critical point than does F, and whose ultrapower agrees with $\mathbf{V}$ a bit past $\mathrm{lh}(F)$. This background certificate demand is necessary in order to ensure that the premice we are constructing are iterable. Unfortunately, the background certificate demand conflicts with the demand that all levels of the model we are constructing be sound.[54] A $\mathbf{K}^c$–construction deals with this conflict by continually replacing the premouse $\mathfrak{N}_\alpha$ currently approximating the model being built by its core, the Skolem hull $\mathfrak{C}(\mathfrak{N}_\alpha)$.[55] Taking cores ensures soundness, while the background extenders one can resurrect by going back into the history of the construction ensure iterability.

This last claim must be qualified. We do not have a general proof of iterability for the premice $\mathfrak{N}_\alpha$ produced in a $\mathbf{K}^c$–construction. At the moment, in order to prove that such a premouse is appropriately iterable, we need to make an additional "smallness" assumption. One assumption that suffices, and which we shall spell out in more detail shortly, is that no initial segment of $\mathfrak{N}_\alpha$ satisfies "there is an extender E on my sequence such that $\mathrm{lh}(E)$ is a Woodin cardinal". We shall call this property of $\mathfrak{N}_\alpha$ *tameness*. Iterability is essential from the very beginning, for our proof that $\mathfrak{C}(\mathfrak{N}_\alpha)$ exists involves comparison arguments, and hence relies on the iterability of $\mathfrak{N}_\alpha$. Thus, for all we know, a $\mathbf{K}^c$–construction might simply break down by reaching a non–tame premouse $\mathfrak{N}_\alpha$ such that $\mathfrak{C}(\mathfrak{N}_\alpha)$ does not exist.

The following definitions describe our background certificate condition. They come from [St96, §1].

---

[51]This is something between a conjecture and a theorem; see below.

[52]Again, there are qualifications to come. $\mathbf{K}^c$ is not actually the core model in which we shall be most interested in the end, but a stepping–stone to it.

[53]Whence the "c" in $\mathbf{K}^c$.

[54]Part of the requirement on F^* is that it be countably complete, and so $\mathrm{crit}(F^*)$ must be uncountable; on the other hand, if α is least so that $E_\alpha \neq \emptyset$, then $\langle \mathbf{J}_\alpha[\vec{E} \restriction \alpha], \in, \vec{E} \restriction \alpha, E_\alpha \rangle$ has Σ_1–projectum ω, so that $\mathrm{crit}(E_\alpha)$ must be countable if this structure is even 1–sound.

[55]$\mathfrak{C}(\mathfrak{M})$ is "defined" in the discussion following 2.5.

Definition 2.20. Let $\mathfrak{M} = \langle \mathbf{J}_\eta[\vec{E}], \in, \vec{E}, F\rangle$ be a premouse such that $F \neq \emptyset$, let $\kappa = \mathrm{crit}(F)$, and let $\nu = \mathrm{lh}(F)$. Let $\mathcal{A} \subseteq \bigcup_{n<\omega} \mathfrak{P}([\kappa]^n)^{\mathfrak{M}}$; then an $\mathcal{A}$–*certificate for* $\mathfrak{M}$ is a pair $\langle N, G\rangle$ such that

(1) N is a transitive, power admissible set, $\mathbf{V}_\kappa \cup \mathcal{A} \subseteq N$, N is closed under ω–sequences, and G is an extender over N,
(2) $F \cap ([\nu]^{<\omega} \times \mathcal{A}) = G \cap ([\nu]^{<\omega} \times \mathcal{A})$,
(3) $\mathbf{V}_{\nu+1} \subseteq \mathrm{Ult}(N, G)$, and
(4) for all $\gamma < \eta$ we have $\mathbf{J}_\gamma^{\mathfrak{M}} = \mathbf{J}_\gamma^{i(\mathbf{J}_\kappa^{\mathfrak{M}})}$, where i is the canonical embedding from N to $\mathrm{Ult}(N, G)$.

Definition 2.21. Let $\mathfrak{M}$ be a premouse whose last extender predicate is nonempty, and let κ be the critical point of this last extender. We say $\mathfrak{M}$ *is countably certified* iff for every countable $\mathcal{A} \subseteq \bigcup_{n<\omega} \mathfrak{P}([\kappa]^n)^{\mathfrak{M}}$, there is an $\mathcal{A}$–certificate for $\mathfrak{M}$. We say that $\mathfrak{M}$ *leaves gaps* iff κ is inaccessible, and either $(\kappa^+)^{\mathfrak{M}} < \kappa^+$ or $\{\mu < \kappa \colon (\mu^+)^{\mathfrak{M}} = \mu^+\}$ is stationary in κ.

In the situation described in Definition 2.20, we shall typically have $|N| = \kappa$, so that $\mathrm{Ord}^N < \mathrm{lh}(G)$. We are therefore not thinking of $\langle N, G\rangle$ as a structure to be iterated; N simply provides a reasonably large collection of sets to be measured by G. The conditions $\mathbf{V}_\kappa \subseteq N$ and $\mathbf{V}_{\nu+1} \subseteq \mathrm{Ult}(N, G)$ are crucial.

We are ready for one of the central definitions of this paper.

Definition 2.22. A $\mathbf{K}^c$*–construction* is a sequence $\langle \mathfrak{N}_\alpha \colon \alpha < \vartheta\rangle$ of premice such that

(1) $\mathfrak{N}_0 = \langle \mathbf{V}_\omega, \in, \emptyset, \emptyset\rangle$;
(2) if $\alpha + 1 < \vartheta$, then the core
$$\mathfrak{C}(\mathfrak{N}_\alpha) = \langle \mathbf{J}_\eta[\vec{E}], \in, \vec{E}, H\rangle,$$
exists, and either
 (a) $H = \emptyset$, $\mathfrak{N}_{\alpha+1}$ is a countably certified premouse which leaves gaps, and $\mathfrak{N}_{\alpha+1}$ is of the form
$$\mathfrak{N}_{\alpha+1} = \langle \mathbf{J}_\eta[\vec{E}], \in, \vec{E}, F\rangle,$$
 for some F such that $\mathrm{lh}(F)$ is as small as possible, or
 (b) $H \neq \emptyset$, or there is no countably certified premouse which leaves gaps of the form $\langle \mathbf{J}_\eta[\vec{E}], \in, \vec{E}, F\rangle$ and
$$\mathfrak{N}_{\alpha+1} = \langle \mathbf{J}_{\eta+1}[\vec{E}^\frown H], \in, \vec{E}^\frown H, \emptyset\rangle;$$
(3) if $\lambda < \vartheta$ is a limit ordinal, then $\mathfrak{N}_\lambda$ is the unique premouse $\mathfrak{P}$ such that
$$\mathrm{Ord}^{\mathfrak{P}} = \sup\{\omega\beta : \mathbf{J}_\beta^{\mathfrak{N}_\alpha} \text{ is defined and eventually constant as } \alpha \to \lambda\},$$
and for all β such that $\omega\beta < \mathrm{Ord}^{\mathfrak{P}}$,
$$\mathbf{J}_\beta^{\mathfrak{P}} = \text{ eventual value of } \mathbf{J}_\beta^{\mathfrak{N}_\alpha}, \text{ as } \alpha \to \lambda,$$

and the last extender predicate of $\mathfrak{P}$ is empty.

So at successor steps in a $\mathbf{K}^c$–construction one replaces the previous premouse with its core, and then either adds a countably certified extender to the resulting extender sequence or takes one step in its constructible closure. At limit steps one forms the natural limit of the stabilized initial segments of the previous premice.

We have required our $\mathbf{K}^c$–constructions to be maximal, in the sense that they must add to the coherent sequence a certified extender of minimal length whenever it is possible to do so. It would be reasonable to drop this condition, or perhaps relax it by requiring more of the background certificates.[56] However, we have no need for the more general kind of $\mathbf{K}^c$–construction.

To what extent are the $\mathbf{K}^c$–constructions canonical? Modulo a natural iterability conjecture, one can show that any two $\mathbf{K}^c$–constructions are compatible, in that one is an initial segment of the other (*cf.* [MiSt94, §9]). We shall have no use for this fact, however.

Because we replace $\mathfrak{N}_\alpha$ by its core at each step in a $\mathbf{K}^c$–construction, the models of the construction may not grow by end–extension, and we need a little argument to show, for example, that a construction of proper class length converges to a premouse of proper class size. Our Theorem 2.11 on the agreement of $\mathfrak{N}$ with $\mathfrak{C}(\mathfrak{N})$ is the key here.

Theorem 2.23. Let κ be an uncountable regular cardinal or $\kappa = \mathrm{Ord}$, and let $\langle \mathfrak{N}_\alpha \colon \alpha < \kappa \rangle$ be a $\mathbf{K}^c$–construction; then there is a unique premouse $\mathfrak{N}_\kappa$ of ordinal height κ such that $\langle \mathfrak{N}_\alpha \colon \alpha \leq \kappa \rangle$ is a $\mathbf{K}^c$–construction.

Proof : For any limit ordinal κ and $\mathbf{K}^c$–construction $\langle \mathfrak{N}_\alpha \colon \alpha < \kappa \rangle$, there is a unique premouse $\mathfrak{N}_\kappa$ satisfying the limit ordinal clause of Definition 2.22. We need only show that $\mathfrak{N}_\kappa$ has ordinal height κ in the case κ is an uncountable cardinal or $\kappa = \mathrm{Ord}$. It is clear that $|\mathfrak{N}_\alpha| < \kappa$ for all $\alpha < \kappa$, so $\mathfrak{N}_\kappa$ has ordinal height $\leq \kappa$.

For $\nu < \kappa$, let

$$\vartheta_\nu = \inf\{\varrho_\omega^{\mathfrak{N}_\alpha} \colon \nu \leq \alpha < \kappa\}.$$

So $\vartheta_0 = \omega$, and the ϑ's are nondecreasing. By Theorem 2.11, $\mathfrak{N}_\nu$ agrees with all later $\mathfrak{N}_\alpha$ below ϑ_ν, so if $\kappa = \sup\{\vartheta_\nu \colon \nu < \kappa\}$, we are done. Since κ is regular, the alternative is that the ϑ's are eventually constant; say $\vartheta_\nu = \varrho$ for all ν such that $\eta \leq \nu < \kappa$. Now notice that if $\eta \leq \nu < \kappa$ and $\varrho_\omega^{\mathfrak{N}_\nu} = \varrho$, then $\mathfrak{C}(\mathfrak{N}_\nu)$ is a proper initial segment of $\mathfrak{N}_{\nu+1}$.[57] Moreover, $\mathfrak{C}(\mathfrak{N}_\nu)$ has cardinality

[56] For example, one might require that the background extenders be total extenders over $\mathbf{V}$. This enables one to lift iteration trees on the $\mathfrak{N}_\alpha$'s to iteration trees on $\mathbf{V}$, and thereby simplifies the proof of iterability somewhat. This is the approach taken in [MiSt94]. Another sometimes useful device is to impose a lower bound on the critical points of the background extenders.

[57] Assume the last extender predicate of $\mathfrak{N}_\nu$ is empty here, as it obviously is for cofinally many such ν.

ϱ in $\mathfrak{N}_{\nu+1}$ by soundness. It follows from Theorem 2.11 that $\mathfrak{C}(\mathfrak{N}_\nu)$ is an initial segment of $\mathfrak{N}_\alpha$, for all $\alpha \geq \nu$. Since there are cofinally many $\nu < \kappa$ such that $\varrho = \varrho_\omega^{\mathfrak{N}_\nu}$, we again get that $\mathfrak{N}_\kappa$ has height κ. □

It is not hard to see that the ϑ_ν defined in the proof above are just the infinite cardinals of $\mathfrak{N}_\kappa$.

It would be natural at this point to fix some $\mathbf{K}^c$–construction of length $\mathrm{Ord}+1$, and define $\mathbf{K}^c$ itself to be the premouse in this construction indexed at Ord. We shall in effect eventually do this, but at certain points we shall need third order properties of Ord which go beyond third-order ZFC, and indeed come close to the assertion that Ord is a measurable cardinal.[58] So we shall eventually fix a measurable cardinal Ω, and let $\mathbf{K}^c$ be the Ωth model in some $\mathbf{K}^c$–construction.

2.7. **The iterability of $\mathbf{K}^c$.** It is clear by now that we have gotten nowhere unless we can prove that the premice we have constructed are sufficiently iterable. Here we encounter what is perhaps the central open problem of core model theory. We formulate it as a conjecture:

Conjecture 2.24. If $\mathfrak{N}$ occurs in a $\mathbf{K}^c$–construction, then every countable elementary substructure of $\mathfrak{N}$ is ω_1+1-iterable.

A proof of this conjecture would yield the basics of core model theory at the level of superstrong cardinals, and it would no doubt extend to do the same for supercompact cardinals.[59] In particular, we could apply Theorem 2.11 and Theorem 2.23 to see that $\mathbf{K}^c$–constructions cannot break down, and, if carried on for Ord stages, converge to proper class premice.

In general, iterability proofs break up into an *existence* proof and a *uniqueness* proof for "sufficiently good" branches in iteration trees on the premice under consideration. The existence proof breaks itself breaks into two parts, a direct existence argument in the countable case and a reflection argument in the uncountable case.

The direct existence argument applies to countable iteration trees on countable elementary submodels of the premice under consideration, and proceeds by using something like the countable completeness of the extenders involved in the iteration to transform an ill–behaved iteration into an infinite descending $\in$–chain. When coupled with the uniqueness proof, this

[58]The need for these assumptions is a defect in core model theory at the level of Woodin cardinals which we hope will some day be removed.

[59]A new problem arises between supercompact and huge cardinals, but this is not the place to go into it. We should also note that one really needs a sharper version of the conjecture which keeps track of the degree of elementarity of the substructures; we have suppressed this finestructure here.

shows that any countable elementary submodel of a premouse under consideration has an ω_1–iteration strategy, namely, the strategy of choosing the unique cofinal "sufficiently good" branch.[60]

The reflection argument extends this method of iterating by choosing sufficiently good branches to the uncountable: given an iteration tree $\mathfrak{T}$ on $\mathfrak{M}$, we go to $\mathbf{V}[G]$ where G is $\mathrm{Col}(\kappa, \omega)$–generic over $\mathbf{V}$ and κ is large enough that $\mathfrak{M}$ and $\mathfrak{T}$ have become countable, and find a sufficiently good branch there. This branch is unique, and hence by the homogeneity of the collapse it is in $\mathbf{V}$. In order to execute this argument[61] one needs a certain level of absoluteness between $\mathbf{V}$ and $\mathbf{V}[G]$. Once one gets past mice with Woodin cardinals, "sufficiently good" can no longer be taken simply to mean "wellfounded", and in fact "sufficiently good" is no longer a Σ^1_2–notion at all. Because of this, the generic absoluteness required by our reflection argument needs large cardinal/mouse existence principles which go beyond ZFC.[62]

The conjecture above overlaps slightly with the uncountable case because it is ω_1+1–iterability, rather than ω_1–iterability, which is at stake. One needs ω_1+1–iterability to guarantee the comparability of countable mice; the Fodor argument that shows coiterations terminate requires a wellfounded branch of length ω_1. Nevertheless, we believe that the conjecture is provable in ZFC.[63]

At present, we can only prove the conjecture for premice of limited complexity. We shall call these special premice "tame". Our direct existence argument in the countable case seems perfectly general, but our uniqueness results are less definitive, and it is here that we resort to the tameness assumption. We begin by stating the existence theorem in the countable case.

We shall say that b is a *maximal branch* of an iteration tree $\mathfrak{T}$ if b has limit order type but is not continued in $\mathfrak{T}$ (so a cofinal branch is always maximal). If b is a branch of $\mathfrak{T}$ and $\sup(b) = \lambda < \mathrm{lh}(\mathfrak{T})$, then b is maximal iff b is different from $[0, \lambda)_T$, the branch chosen by $\mathfrak{T}$. We call a system a *putative iteration tree* if it has all the properties of an ordinary iteration tree except that it may have a last, ill–founded model.[64]

[60] Of course a sufficiently good branch must be wellfounded, but in general more is required, for we want to be able to find cofinal wellfounded branches later in the iteration game as well.

[61] See Theorem 2.35 below for a concrete example of such an argument.

[62] For example, if it is consistent that there is a Woodin cardinal, then it is consistent that there is a premouse $\mathfrak{N}$ occurring on a $\mathbf{K}^c$–construction which is not fully iterable. *Cf.* Section 4.

[63] We suspect that if κ is strictly less than the infimum of the critical points of the background extenders, then the κ–iterability of the size κ elementary submodels of premice in a $\mathbf{K}^c$–construction is provable in ZFC.

[64] Thus a play of the iteration game at which player II has just lost at a successor step is a putative iteration tree, but not an iteration tree.

Theorem 2.25 (Branch Existence Theorem)**.** Let $\pi : \mathfrak{M} \to \mathfrak{N}_\alpha$ be an elementary embedding where $\mathfrak{M}$ is countable and $\mathfrak{N}_\alpha$ is a model of the $\mathbf{K}^c$–construction. Let $\mathfrak{T}$ be a countable putative iteration tree on $\mathfrak{M}$. Then either

(1) there is a maximal branch b of $\mathfrak{T}$ such that
  (a) b does not drop and there is a $\sigma : \mathfrak{M}_b^{\mathfrak{T}} \to \mathfrak{N}_\alpha$ such that

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\pi"] \arrow[dr, "i^{\mathfrak{T}}_{0b}"'] & \mathfrak{N}_\alpha \\ & \mathfrak{M}^{\mathfrak{T}}_b \arrow[u, "\sigma"'] \end{tikzcd}$$

  commutes, or
  (b) b drops, and there is a $\beta < \alpha$ with $\sigma : \mathfrak{M}_b^{\mathfrak{T}} \to \mathfrak{N}_\beta$,

or

(2) $\mathfrak{T}$ has a last model $\mathfrak{M}_\vartheta^{\mathfrak{T}}$ and
  (a) the branch $[0, \vartheta]_T$ does not drop, and there is $\sigma : \mathfrak{M}_\vartheta^{\mathfrak{T}} \to \mathfrak{N}_\alpha$ such that

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\pi"] \arrow[dr, "i^{\mathfrak{T}}_{0\vartheta}"'] & \mathfrak{N}_\alpha \\ & \mathfrak{M}^{\mathfrak{T}}_\vartheta \arrow[u, "\sigma"'] \end{tikzcd}$$

  commutes, or
  (b) the branch $[0, \vartheta]_T$ does drop, and there is a $\beta < \alpha$ with $\sigma : \mathfrak{M}_\vartheta^{\mathfrak{T}} \to \mathfrak{N}_\beta$

The proof of Theorem 2.25 is a direct construction which transforms a counterexample to the theorem into an infinite descending $\in$–chain. The details of the construction are rather complicated, and seem to shed little light on the rest of the theory, so we shall not attempt to describe them. They can be found in [MaSt94, §4], where the theorem was first proved for "coarse" mice occurring in a $\mathbf{K}^c$–construction using only full background extenders over $\mathbf{V}$, and in [St96, §§2,4], where the full result Theorem 2.25 was first proved.

Because the Branch Existence Theorem 2.25 asserts only the existence of *maximal* wellfounded branches, not necessarily *cofinal* ones, it does not even give the ω_1–iterability of countable elementary submodels of premice in a $\mathbf{K}^c$–construction. For that we need an accompanying uniqueness theorem for branches, and here our results are less definitive. What we can show, roughly, is that a failure of uniqueness yields a premouse with a Woodin cardinal. We now make this more precise.

Definition 2.26. Let $\kappa < \delta$ and $A \subseteq \mathbf{V}_\delta$, then κ is called *A–reflecting in δ* iff for all $\nu < \delta$ there is a $j : \mathbf{V} \to \mathfrak{M}$ such that $\mathrm{crit}(j) = \kappa$ and $j(A) \cap \mathbf{V}_\nu = A \cap \mathbf{V}_\nu$,

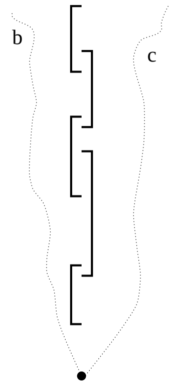


FIGURE 3. The overlapping pattern of two distinct well–founded branches

Definition 2.27. A cardinal δ is called a Woodin cardinal, iff for all $A \subseteq \mathbf{V}_\delta$ there is a $\kappa < \delta$ such that κ is A–reflecting in δ.

Woodin cardinals lie between strong and superstrong cardinals in the consistency strength hierarchy.[65] A Woodin cardinal need not itself be strong, or even measurable, but it is easy to see that if δ is Woodin, then δ is Mahlo, and δ is a limit of cardinals κ such that $\forall\alpha < \delta(\kappa \text{ is } \alpha\text{–strong})$.[66] Measurable Woodin cardinals are well beyond the reach of current core model theory.

The main result connecting Woodin cardinals with the uniqueness of cofinal wellfounded branches in iteration trees is the following theorem of [MaSt94].

Theorem 2.28 (Branch Uniqueness Theorem)**.** Let $\mathfrak{T}$ be an iteration tree of limit length λ, and let b and c be distinct cofinal wellfounded branches of $\mathfrak{T}$. Let

$$\delta := \sup\{\mathrm{lh}(F_\alpha^{\mathfrak{T}}) : \alpha < \lambda\},$$

and suppose $A \subseteq \mathbf{V}_\delta^{\mathfrak{M}_b^{\mathfrak{T}}} = \mathbf{V}_\delta^{\mathfrak{M}_c^{\mathfrak{T}}}$ and $A \in \mathfrak{M}_b^{\mathfrak{T}} \cap \mathfrak{M}_c^{\mathfrak{T}}$. Then

$$\mathfrak{M}_b^{\mathfrak{T}} \models \exists\kappa < \delta(\kappa \text{ is } A\text{–reflecting in } \delta)$$

Thus δ is Woodin in $\mathfrak{M}_b^{\mathfrak{T}} \cap \mathfrak{M}_c^{\mathfrak{T}}$.

Proof : The extenders used on b and c have an overlapping pattern pictured in Figure 3:

To see this, pick any successor ordinal

$$\alpha_0 + 1 \in b \setminus c,$$

and then let

$$\beta_n + 1 = \min\{\gamma \in c : \gamma > \alpha_n + 1\}$$

[65]For a diagram of the known large cardinals and their consistency strengths, *cf.* [Kan94, p. 471].

[66]Woodinness of δ is a Π_1–fact about $V_{\delta+1}$, so the least Woodin cardinal is not even weakly compact.

and

$$\alpha_{n+1}+1=\min\{\eta\in b:\eta>\beta_n+1\},$$

for all $n<\omega$. Now for any n, the T-predecessor of β_n+1 is on c and $\leq\alpha_n+1$, hence $\leq\alpha_n$, so by the rules of the iteration game,

$$\mathrm{crit}(F_{\beta_n})<\mathrm{lh}(F_{\alpha_n}).$$

Similarly, for any n,

$$\mathrm{crit}(F_{\alpha_{n+1}})<\mathrm{lh}(F_{\beta_n}).$$

Now extenders used along the same branch of an iteration tree do not overlap (*i.e.*, if E is used before F, then $\mathrm{lh}(E)<\mathrm{crit}(F)$), so we have

$$\begin{aligned}\mathrm{crit}(F_{\beta_n}) &< \mathrm{lh}(F_{\alpha_n})<\mathrm{crit}(F_{\alpha_{n+1}})<\mathrm{lh}(F_{\beta_n})\\ &< \mathrm{crit}(F_{\beta_{n+1}})<\mathrm{lh}(F_{\alpha_{n+1}})<\mathrm{crit}(F_{\alpha_{n+2}})\end{aligned}$$

which is the overlapping pattern pictured.

Now $\sup\{\alpha_n:n<\omega\}=\sup\{\beta_n:n<\omega\}$, and since branches of iteration trees are closed below their suprema in the order topology on Ord, the common supremum of the α_n and β_n is λ. Let us assume α_0 was chosen large enough that we have

$$A=i_{\beta_0+1,c}(A^*)=i_{\alpha_1+1,b}(A^{**})$$

for some A^* and A^{**}. Let

$$\kappa=\mathrm{crit}(F_{\beta_0})=\mathrm{crit}(i_{\beta_0+1,c});$$

we shall show that κ is A–reflecting in δ in the model $\mathfrak{M}_b$.

Let $E_0=F_{\beta_0}\restriction\mathrm{crit}(F_{\alpha_1})$. Because of the overlapping pattern, E_0 is a proper initial segment of F_{β_0}, and by initial segment condition on premice and the agreement of the models of an iteration tree, $E_0\in\mathfrak{M}_b$. Moreover, if $j\colon\mathfrak{M}_b\to\mathrm{Ult}(\mathfrak{M}_b,E_0)$ is the canonical embedding, then because A and A^* agree below κ, $j(A)$ and $i_{\beta_0+1,c}(A^*)$ agree below $\mathrm{crit}(F_{\alpha_1})$. That is, $j(A)$ agrees with A below $\mathrm{crit}(F_{\alpha_1})$, and hence E_0 witnesses that κ is A–reflecting up to $\mathrm{crit}(F_{\alpha_1})$ in $\mathfrak{M}_b$.

To get A–reflection all the way up to δ, we set

$$E_{2n}=F_{\beta_n}\restriction\mathrm{crit}(F_{\alpha_{n+1}})\text{ and }E_{2n+1}=F_{\alpha_{n+1}}\restriction\mathrm{crit}(F_{\beta_{n+1}}),$$

for all n. Each of the E_n is in $\mathfrak{M}_b$ for the same reason E_0 is in $\mathfrak{M}_b$. Therefore the extender E which represents the embedding coming from "composing" the ultrapowers by the E_i for $0\leq i\leq 2n$, is in $\mathfrak{M}_b$. The argument above generalizes easily to show that E witnesses that κ is A–reflecting up to $\mathrm{crit}(F_{\alpha_{n+1}})$. Since $\mathrm{crit}(F_{\alpha_{n+1}})$ approaches δ as n approaches ω, κ is A-reflecting in δ in the model $\mathfrak{M}_b$. □

Definition 2.29. A premouse $\mathfrak{M}$ is 1–*small* iff whenever κ is the critical point of a (nonempty) extender on the $\mathfrak{M}$–sequence, then $\mathcal{J}_\kappa^{\mathfrak{M}}\models$ "There are no Woodin cardinals".

This means that a mouse is 1–small just in case it hasn't reached the sharp of a proper class model with a Woodin cardinal.

Definition 2.30. A premouse $\mathfrak{M}$ is *properly 1–small* iff $\mathfrak{M}$ is 1–small, and $\mathfrak{M} \models$ "There are no Woodin cardinals, and there is a largest cardinal".

The branch existence and uniqueness theorems rather easily yield $\omega_1 + 1$–iteration strategies for countable elementary submodels of the properly 1–small premice occurring in $\mathbf{K}^c$–constructions.[67] The strategy is just to choose at each limit stage $\lambda \leq \omega_1$ the unique cofinal branch of the iteration tree built so far. At countable stages λ, the Branch Existence Theorem 2.25 guarantees existence, modulo uniqueness at earlier stages, and the Branch Uniqueness Theorem 2.28 guarantees uniqueness.[68] At stage $\lambda = \omega_1$, we go to $\mathbf{V}[G]$, where G is $\mathrm{Col}(\omega_1, \omega)$–generic over $\mathbf{V}$. A strengthened form of Theorem 2.25 gives us a cofinal wellfounded branch in $\mathbf{V}[G]$ for the tree of length $\omega_1^{\mathbf{V}}$ (*cf.* [St93]). By Theorem 2.28 it is unique, hence definable in $\mathbf{V}[G]$ from the tree, and hence by the homogeneity of the collapse it is in $\mathbf{V}$.

One can push this sort of argument further. Notice that we have not used the full strength of the Branch Existence Theorem yet; the direct limit $\mathfrak{M}_b$ along the branch b it gives is not just wellfounded, but is itself embedded into a level of the same $\mathbf{K}^c$–construction. Thus if there is more than one cofinal branch of the tree built so far satisfying the conclusion of Theorem 2.25, we can simply pick one such branch b and start all over. As long as from this point on our opponent in the iteration game plays extenders which can be interpreted as forming a tree on $\mathfrak{M}_b$, this will work. We run into trouble, however, if at some later point our opponent plays an extender E which, according to the rules of the iteration game, should be applied to some model reached before we reached $\mathfrak{M}_b$. If we set things up correctly, however, this extender E will overlap the local Woodin cardinal $\delta := \sup\{\mathrm{lh}(F_\alpha) \colon \alpha \in b\}$.

Definition 2.31. A premouse $\mathfrak{M}$ is *tame* iff whenever E is an extender on the $\mathfrak{M}$–sequence, say the last extender of $\mathcal{J}_\lambda^{\mathfrak{M}}$, then

$$\mathcal{J}_\lambda^{\mathfrak{M}} \models \forall \delta \geq \mathrm{crit}(E)(\delta \text{ is not Woodin})$$

Otherwise $\mathfrak{M}$ is called *wild*.

Proposition 2.32. If there is a wild mouse, then there is a model with a proper class of Woodin cardinals.

Proof : Let $\mathfrak{M}$ be the wild mouse with a $\langle \kappa, \lambda \rangle$–extender E overlapping a Woodin cardinal δ. Let

$$\Psi_{\alpha,\beta} :\equiv \exists \gamma(\alpha < \gamma < \beta(\gamma \text{ is Woodin}))$$

[67] These constitute an initial segment of the construction.

[68] The uniqueness argument is a bit more subtle than might at first appear, because of the possibility that one of the cofinal branches might drop. Finestructure to the rescue!

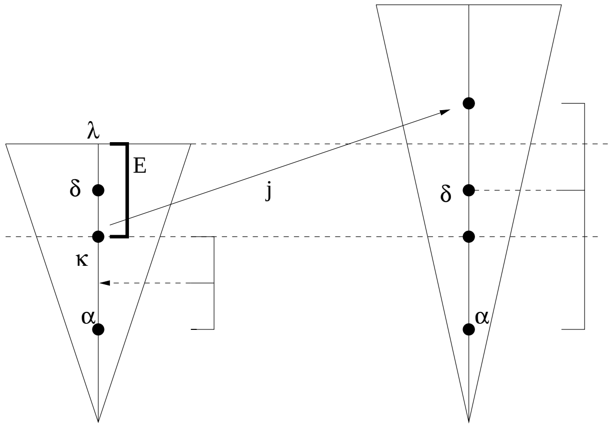


FIGURE 4. Reflection of Woodin cardinals in wild mice

Now we take the ultrapower by E, and we have for all $\alpha < \kappa$ that $\mathrm{Ult}(\mathfrak{M}, E) \models \Psi_{\alpha, j_E(\kappa)}$, as δ is still Woodin in the ultrapower.[69] But then we can transport these sentences back to $\mathfrak{M}$ giving $\mathfrak{M} \models \Psi_{\alpha,\kappa}$ for all $\alpha < \kappa$. This reflection is pictured in Figure 4.

Then $\mathbf{V}_\kappa^{\mathfrak{M}}$ is a model with a proper class of Woodin cardinals. □

Proposition 2.33. If $\mathfrak{M}$ is a tame premouse, then it cannot satisfy "there is a limit of Woodin cardinals which is itself a Woodin cardinal".

Proof : Working in $\mathfrak{M}$, suppose that there is a sequence $\langle \delta_\alpha : \alpha < \delta \rangle$ with limit δ which is itself a Woodin cardinal. (Note that δ is regular and hence the sequence has length δ.) As we noted in the short discussion of Woodin cardinals after Definition 2.27, δ is the limit of $< \delta$–strong cardinals $\langle \kappa_\alpha : \alpha < \delta \rangle$. Then there is some $\alpha < \delta$ such that $\kappa_0 < \delta_\alpha$ and κ_0 is $\delta_{\alpha+1}$–strong. Hence the extender witnessing this overlaps δ_α, and $\mathfrak{M}$ can't be tame. □

We mention without proof that a tame mouse can satisfy "there is a limit of Woodin cardinals which is a strong cardinal".

The argument we sketched very vaguely above can be turned into a proof of the following theorem:

Theorem 2.34. Every countable elementary substructure of a tame premouse occurring in a $\mathbf{K}^c$–construction is $\omega_1 + 1$–iterable.

This theorem is proved in [St93] and [SchSt96].

Although the $\omega_1 + 1$–iterability of countable elementary submodels of premice in a $\mathbf{K}^c$–construction suffices for many basic applications of iterability,

[69]This is the case because the strength of the extender is greater than δ; here we use our assumption that the extender overlaps the Woodin cardinal.

there are important contexts in which we need more.[70] The standard argument here once again involves collapsing. By restricting the complexity of the mice under consideration, one gets for each countable elementary submodel $\mathfrak{M}$ of $\mathfrak{N}$ an ω_1–iteration strategy $\Sigma(\mathfrak{M})$ which is uniformly definable from $\mathfrak{M}$ by some formula φ. We then define a putative ϑ–iteration strategy on $\mathfrak{N}$, for arbitrary ϑ, by going to $\mathbf{V}[G]$ where G is $\mathrm{Col}(\sup\{|\mathfrak{N}|, \vartheta\}, \omega)$–generic over $\mathbf{V}$, and using the prescription defined by φ from $\mathfrak{N}$ there. *If* we have enough generic absoluteness, this prescription will pick a unique branch in $\mathbf{V}[G]$, and by homogeneity of the collapse, this branch will be in $\mathbf{V}$.

The generic absoluteness required is closely related to the complexity of the formula φ defining the iteration strategies $\Sigma(\mathfrak{M})$, and thus to the complexity bound we imposed on $\mathfrak{M}$. If we want to make do with Shoenfield absoluteness, we must work with something like properly 1–small $\mathfrak{M}$. To handle mice beyond that, we need forms of generic absoluteness which are not provable in ZFC, but are themselves equivalent to the existence of nontrivial mice over arbitrary sets. We then need to construct these mice, and that leads the whole theory into an inductive proof that the universe is closed under building various mice over arbitrary sets. This process is called the *Core Model Induction.* Since it would take us too far afield to describe this further, we shall stick here to the iterability one can get using Shoenfield absoluteness.

Theorem 2.35. Suppose that there is no proper class premouse satisfying "There is a Woodin cardinal". Let $\mathfrak{N}$ be a premouse occurring in a $\mathbf{K}^c$–construction, and let $\vartheta \in \mathrm{Ord}$; then $\mathfrak{N}$ is ϑ–iterable.

Proof : If $\mathfrak{N}$ is not 1–small, then we can linearly iterate the last extender of its first non–1–small level Ord times, and thereby produce a proper class premouse with a Woodin cardinal. So we may assume $\mathfrak{N}$ is 1–small. Similarly, we can extend $\mathfrak{N}$ by adding ordinals on top (*i.e.* without adding new extenders) until we reach a premouse occurring in a $\mathbf{K}^c$–construction which satisfies "there are no Woodin cardinals". We may as well assume $\mathfrak{N}$ is this structure.[71]

Because of its smallness properties, Theorems 2.25 and 2.28 imply that every countable iteration tree of limit length on a countable elementary submodel of $\mathfrak{N}$ has a unique cofinal wellfounded branch. This gives ω_1–iteration strategies for the countable elementary submodels of $\mathfrak{N}$. We attempt to construct a similar strategy applying to arbitrary trees on $\mathfrak{N}$. For this it is enough to show

[70] We can handle trees and premice of size less than the critical points of the background extenders used in the construction by collapsing their size to be countable, as in the argument sketched above, but in important contexts one needs more than this.

[71] Any iteration strategy for this structure induces one for $\mathfrak{N}$. However, if $\mathfrak{N}$ itself satisfies "there is a Woodin cardinal", it may have iteration strategies which do not lift to this structure.

Claim : If $\mathfrak{T}$ is an iteration tree on $\mathfrak{N}$ of limit length, then $\mathfrak{T}$ has a unique cofinal wellfounded branch.

We prove the claim: Let $\delta := \sup\{\mathrm{lh}(F_\alpha^{\mathfrak{T}})\colon \alpha < \mathrm{lh}(\mathfrak{T})\}$, and let $\mathfrak{M}$ be the premouse of ordinal height δ to which the models of $\mathfrak{T}$ converge.[72] By adding ordinals on top of $\mathfrak{M}$ we reach a premouse $\mathfrak{Q}$ extending $\mathfrak{M}$ which satisfies "δ is not Woodin", as otherwise we have a proper class premouse with a Woodin cardinal.

Now let $\pi\colon R \to \mathbf{V}_\eta$ where η is sufficiently large, R is countable and transitive, and

$$\pi(\langle\overline{\mathfrak{N}}, \overline{\mathfrak{M}}, \overline{\mathfrak{Q}}, \overline{\mathfrak{T}}\rangle) = \langle\mathfrak{N}, \mathfrak{M}, \mathfrak{Q}, \mathfrak{T}\rangle.$$

We want to see that $\mathfrak{T}$ has a unique cofinal wellfounded branch, so it is enough to see that R satisfies "$\overline{\mathfrak{T}}$ has a unique cofinal wellfounded branch". By our branch existence and uniqueness theorems 2.25 and 2.28, $\overline{\mathfrak{T}}$ does indeed have a unique cofinal wellfounded branch b, so it is enough to show that $b \in R$.

Now $\mathfrak{M}_b^{\overline{\mathfrak{T}}} \models$ "$\overline{\delta}$ is not Woodin", by our smallness assumption on $\mathfrak{N}$ and the elementarity of $i_{0,b}$. It follows from this and Theorem 2.28 that b is the unique cofinal branch c of $\overline{\mathfrak{T}}$ such that $\overline{\mathfrak{Q}}$ is an initial segment of $\mathfrak{M}_c^{\overline{\mathfrak{T}}}$. Letting G be $\mathrm{Col}(|\overline{\mathfrak{Q}}|, \omega)$-generic over R, we get by Σ_1^1–absoluteness that $b \in R[G]$. This is true for all such G, so $b \in R$, and we are done with the claim, and thus with the proof.[73] □

The proof of Theorem 2.35 can be generalized to situations in which one does not have the very strong smallness assumption that there is no proper class model with a Woodin cardinal. What plays the rôle of the "branch oracle" structure $\mathfrak{Q}$ in the proof then involves extenders beyond those of the "lined up part" $\mathfrak{M}$ of the models of $\mathfrak{T}$. For example, if there is no proper class model with two Woodin cardinals, and over every premouse $\mathfrak{M}$ we can build an iterable–above–$\mathrm{Ord}^{\mathfrak{M}}$ proper class mouse with one Woodin cardinal, then the proof of Theorem 2.35 shows that every premouse on a $\mathbf{K}^c$–construction is ϑ–iterable for all $\vartheta \in \mathrm{Ord}$.

3. The core model K

3.1. **The Universality of $\mathbf{K}^c$.** For the rest of this paper, we shall assume that there is a measurable cardinal. This assumption plays no rôle in core model theory at the level of strong cardinals and below, but it is not known how to do without it at the level of Woodin cardinals. We therefore fix for

[72] So for $\alpha < \delta$, $\mathcal{J}_\alpha^{\mathfrak{M}}$ is the common value of $\mathcal{J}_\alpha^{\mathfrak{M}_\beta^{\mathfrak{T}}}$ for all β such that $\alpha < \mathrm{lh}(F_\beta^{\mathfrak{T}})$.

[73] Modulo the usual missing finestructure. The main point there is that δ may actually be Woodin with respect to all $A \in \mathfrak{Q}$; the A witnessing non–Woodinness may only be definable over $\mathfrak{Q}$. Similarly, b may drop, and thus $\overline{\delta}$ may only be definably non–Woodin over $\mathfrak{M}_b^{\overline{\mathfrak{T}}}$.

the rest of this paper a measurable cardinal Ω, and a normal measure μ_0 on Ω.

Let us also assume henceforth that *every premouse occurring in a* $\mathbf{K}^c$*–construction is tame.*[74] It follows from Theorem 2.34 and Theorem 2.23 that there is a $\mathbf{K}^c$–construction of length $\Omega + 1$. Let us fix such a construction $\langle \mathfrak{N}_\alpha : \alpha \leq \Omega \rangle$.[75]

Definition 3.1. $\mathbf{K}^c$ is the Ωth model of our designated $\mathbf{K}^c$–construction, that is, $\mathbf{K}^c = \mathfrak{N}_\Omega$.

We shall not need to refer to premice of ordinal height $> \Omega$, and so from now on we let "premouse" stand for "premouse of ordinal height $\leq \Omega$". We call premice which are as long as possible "weasels":

Definition 3.2. A *weasel* is a premouse of ordinal height Ω. A weasel is *universal* just in case it is Ω–universal (*cf.* Definition 2.12).

Our next result, which is sometimes called "cheapo covering", shows that we have been sufficiently liberal about putting extenders on the $\mathbf{K}^c$–sequence. The evidence of this is that we get a weasel which is close enough to $\mathbf{V}$ to compute many successor cardinals correctly, and therefore by Lemma 2.13 is universal.

Theorem 3.3. Suppose every premouse occurring in the $\mathbf{K}^c$ construction is tame; then for μ_0–almost every α we have $(\alpha^+)^{\mathbf{K}^c} = \alpha^+$.

Proof : We use the ultrapower embedding $j : \mathbf{V} \to \mathrm{Ult}(\mathbf{V}, \mu_0)$ that we get from the measurability of Ω. If cheapo covering is false, then $\{\alpha : (\alpha^+)^{\mathbf{K}^c} = \alpha^+\}$ has measure zero, and hence by Łos's theorem $(\Omega^+)^{j(\mathbf{K}^c)} < \Omega^+$. Because $\mathbf{K}^c \models \mathsf{GCH}$ this means that the power set of Ω in $j(\mathbf{K}^c)$ has cardinality Ω.

Let E_j be the extender coding j. If we restrict E_j to $j(\mathbf{K}^c)$, then the resulting extender is in $\mathrm{Ult}(\mathbf{V}, \mu_0)$: By the assumption, we can enumerate $\mathfrak{P}(\Omega) \cap j(\mathbf{K}^c)$ as $\{M_\alpha : \alpha < \Omega\}$. Then $\langle j(M_\alpha) : \alpha < \Omega \rangle$ lies in the ultrapower and therefore we can define the restricted extender by

$$M_\alpha \in (E_j)_a :\Longleftrightarrow a \in j(M_\alpha)$$

in the ultrapower.

We can now show that every proper initial segment of this extender is on the $j(\mathbf{K}^c)$–sequence. These initial segments witness that Ω is Woodin in $j(\mathbf{K}^c)$. It follows by reflection that Ω is a limit of Woodins in $\mathbf{K}^c$, and thus $j(\Omega)$ is a limit of Woodins in $j(\mathbf{K}^c)$. Hence one of the initial segments of E_j overlaps a local Woodin cardinal, and there is a wild mouse on the $j(\mathbf{K}^c)$–construction. Therefore there is a wild mouse on the $\mathbf{K}^c$–construction. $\Box$

[74] We shall strengthen this smallness assumption dramatically in a moment.

[75] As we said before, one can show that there is exactly one such construction.

The proof of Theorem 3.3 works in greater generality: if we generalize the $\mathbf{K}^c$–construction so as to allow extenders representing *e.g.* superstrong embeddings, then (granted sufficient iterability[76]), either $\mathbf{K}^c$ satisfies "there is a superstrong cardinal", or $\mathbf{K}^c$ computes successor cardinals correctly μ_0–almost everywhere.

Other evidence that we have put enough extenders into $\mathbf{K}^c$ comes from the fact that various large cardinal properties of $\mathbf{V}$ are inherited by $\mathbf{K}^c$. For example, if one modifies the $\mathbf{K}^c$–construction by omitting the requirement that $\mathfrak{N}_{\alpha+1}$ leave gaps, then either this modified construction reaches a wild mouse, or it converges to a weasel W such that for any Woodin cardinal δ, $W \models$"δ is Woodin".

If $\mathbf{K}^c$ is $\Omega + 1$–iterable, then Theorem 3.3 and Lemma 2.13 immediately imply that $\mathbf{K}^c$ is universal. We have obtained the $\Omega + 1$–iterability of $\mathbf{K}^c$ by restricting ourselves to the situation in which there is no proper class model with a Woodin cardinal (*cf.* Theorem 2.35). So we have

Theorem 3.4. If there is no proper class model with a Woodin cardinal, then $\mathbf{K}^c$ is a universal weasel.

In fact, if there is no proper class model with a Woodin cardinal, then a weasel W is universal if and only if $(\alpha^+)^W = \alpha^+$ for stationarily many $\alpha < \Omega$. This can be seen by coiterating W with $\mathbf{K}^c$.

3.2. **The true core model K.** Any iterate of a universal weasel is itself universal, so there are many universal weasels if there are any. $\mathbf{K}^c$ has some claim to distinction among universal weasels, but it depends too heavily on the universe in which it is constructed to serve some purposes; that is, its definition is not sufficiently generically absolute. This is easy to see:

Lemma 3.5. The first measurable cardinal in $\mathbf{K}^c$, if it exists, is inaccessible in $\mathbf{V}$.

Proof : This is immediate from the fact that the levels where we added new extenders in our $\mathbf{K}^c$–construction were required to leave gaps, and one of the conditions for that was that the critical point of the last extender is inaccessible (*cf.* Definition 2.21). □

Proposition 3.6. Suppose that there is a measurable cardinal in $\mathbf{K}^c$. Then there is a forcing extension $\mathbf{V}[G]$ of $\mathbf{V}$ such that

$$(\mathbf{K}^c)^{\mathbf{V}[G]} \neq (\mathbf{K}^c)^{\mathbf{V}}.$$

Proof : Let κ be the first measurable in $\mathbf{K}^c$, and let G be $\mathrm{Col}(\kappa, \omega)$–generic for $\mathbf{V}$. Then in $\mathbf{V}[G]$, κ is countable and hence by Lemma 3.5, κ can't be the first measurable in $(\mathbf{K}^c)^{\mathbf{V}[G]}$. Hence $(\mathbf{K}^c)^{\mathbf{V}[G]} \neq (\mathbf{K}^c)^{\mathbf{V}}$. □

[76]We would need $\omega_1 + 1$–iterability for countable elementary substructures.

If there is no proper class model with a strong cardinal, then there is a privileged universal weasel, *i.e.* one which generates all other universal weasels by iteration. This canonical universal weasel $\mathbf{K}$ is *maximal*, in the sense that every extender which could be added to its sequence (subject to an iterability constraint) is already on its sequence, and *generically absolute*, in that it has a definition which defines it not just in $\mathbf{V}$, but in every set generic extension of $\mathbf{V}$.[77]

One can construct such a canonical universal weasel under the weaker hypothesis that there is no proper class model with a Woodin cardinal. In fact, the construction we shall present here is much more general than that. We use the assumption that there is no proper class model with a Woodin cardinal only to obtain the $\Omega + 1$–iterability of $\mathbf{K}^c$ via Theorem 2.35. As we remarked after presenting it, the proof of Theorem 2.35 goes through under less restrictive hypotheses, and under such hypotheses our construction of $\mathbf{K}$ will go through as well. To keep the exposition as simple as possible, we shall work here, and for the rest of the paper, under the hypothesis that *there is no proper class model with a Woodin cardinal*. Like the measurability of Ω, we shall not always display this hypothesis in the statements of the results to come which use it.

The idea of our construction is that $\mathbf{K}^c$ is an iterate of $\mathbf{K}$, and therefore we can obtain $\mathbf{K}$ from $\mathbf{K}^c$ by taking a Skolem hull of $\mathbf{K}^c$ which "undoes" the iteration.[78] We now introduce some concepts useful in isolating this Skolem hull.

Definition 3.7. Let

$$\begin{aligned} A_0 := \{\alpha < \Omega \ :\ & \alpha \text{ is inaccessible}, (\alpha^+)^{\mathbf{K}^c} = \alpha^+, \\ & \{\beta < \alpha : \beta \text{ is inaccessible}, (\beta^+)^{\mathbf{K}^c} = \beta^+\} \text{ is not} \\ & \text{stationary in } \alpha\}. \end{aligned}$$

Proposition 3.8. A_0 is stationary in Ω; moreover, for any $\alpha \in A_0$, α is not the critical point of an extender on the $\mathbf{K}^c$–sequence which is total on $\mathbf{K}^c$.

Proof : The first condition follows directly from Theorem 3.3, since A_0 is the "first Mahlo derivative" of a set which has μ_0-measure one, and is therefore stationary. For the second condition let $\kappa = \text{crit}(E)$ for some E on the $\mathbf{K}^c$–sequence which is total on $\mathbf{K}^c$. At some step $\mathfrak{N}_{\alpha+1}$ in the construction an extender F was added to the sequence, and eventually, perhaps through taking some cores, F collapsed to E. Since κ is a cardinal of $\mathbf{K}^c$, we get $\text{crit}(F) = \kappa$. Since E is total on $\mathbf{K}^c$, $\mathfrak{N}_{\alpha+1}$ and $\mathbf{K}^c$ have the same

[77] These are results of Dodd, Jensen, and Mitchell; *cf.* [Ze9?] or [St96, §8].

[78] Many of the ideas and arguments behind this construction are due to Mitchell, and can be found in an informally circulated paper which was to be a successor to [Mi84], but was never published. Mitchell's work was transformed into the form it takes here by the second author in [St96].

subsets of κ. But the requirement that $\mathfrak{N}_{\alpha+1}$ leave gaps gives us at once that $\kappa \notin A_0$.[79] $\square$

Definition 3.9. Let W be a weasel and let $\Gamma \subseteq W$; then Γ is *thick* in W iff for all but nonstationarily many $\alpha \in A_0$

- $\alpha \in \Gamma$,
- Γ contains a set which is α–club in α^+, and
- α is not the critical point of an extender from the W–sequence which is total on W.

So by Proposition 3.8

Lemma 3.10. Ω is thick in $\mathbf{K}^c$.

The following elementary lemmas are easy to prove:

Lemma 3.11. If Ω is thick in W, then the class of sets which are thick in W is an Ω-complete filter on Ω.

Lemma 3.12. Let $\pi\colon H \to W$ be elementary, where W is a weasel and $\operatorname{ran}(\pi)$ is thick in W; then $\{\alpha\colon \pi(\alpha) = \alpha\}$ is thick in both H and W.

Lemma 3.13. Let Ω be thick in W, and suppose $\mathfrak{T}$ is an iteration tree on W. Let $\lambda \leq \Omega$, and suppose there is no dropping in $\mathfrak{T}$ along $[0, \lambda]_T$ (and $\lambda < \operatorname{lh}(\mathfrak{T})$), so that $i^{\mathfrak{T}}_{0,\lambda}$ is defined. Suppose $i^{\mathfrak{T}}_{0,\lambda}{}''\Omega \subseteq \Omega$; then $\{\alpha < \Omega\colon i^{\mathfrak{T}}_{0,\lambda}(\alpha) = \alpha\}$ is thick in both W and $\mathfrak{M}^{\mathfrak{T}}_{\lambda}$.

Proof : This is clear if $\lambda < \Omega$, so let $\lambda = \Omega$. Then $[0, \lambda]_T$ is club in Ω, so for the typical $\alpha \in A_0$, $\alpha \in [0, \lambda]_T$ and $i_{0,\alpha}{}''\alpha \subseteq \alpha$. As α is inaccessible, $i_{0,\alpha}(\alpha) = \alpha$. As α is not the critical point of an extender from the W–sequence which is total on W, α is not the critical point of an extender from the $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$–sequence which is total on $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$. Thus α is not the critical point of an extender from the $\mathfrak{M}^{\mathfrak{T}}_{\beta}$–sequence which is total on $\mathfrak{M}^{\mathfrak{T}}_{\alpha}$, for any $\beta \geq \alpha$. Thus $\alpha^+ < \operatorname{crit}(i_{\alpha,\lambda})$, so that $i_{0,\lambda}(\alpha) = \alpha$ and $\{\gamma\colon i_{0,\lambda}(\gamma) = \gamma\}$ contains an α–club subset of α^+. $\square$

We shall define $\mathbf{K}$ to be the transitive collapse of the intersection of all thick hulls of $\mathbf{K}^c$. At the moment, it is not even clear that this intersection is a weasel, much less that it is universal.[80] The following concepts are the key to showing $\mathbf{K}$ is universal.

[79] This was the entire purpose of the leaving gaps requirement in the $\mathbf{K}^c$–construction.

[80] Every thick hull of $\mathbf{K}^c$ is universal, and hence contains all the reals of $\mathbf{K}^c$, so the intersection is nonempty.

Definition 3.14. Let Ω be thick in M; then M has the *definability property at* α iff for all Γ such that Γ is thick in M, we have

$$\alpha \in \mathrm{Hull}^M(\alpha \cup \Gamma),^{81}$$

i.e. α is first–order definable over M with parameters from Γ and ordinals $\beta < \alpha$.

Definition 3.15. Let Ω be thick in M; then M has *the hull property at* α iff for all Γ such that Γ is thick in M, we have

$$\mathfrak{P}(\alpha)^M \subseteq \mathcal{H}^M(\alpha \cup \Gamma)$$

where $\mathcal{H}^M(X)$ is the transitive collapse of $\mathrm{Hull}^M(X)$.

Lemma 3.16. There is an $\Omega + 1$–iterable weasel M such that Ω is thick in M and M has the hull property at all $\alpha < \Omega$.

Proof : Let W be any $\Omega+1$–iterable weasel such that Ω is thick in W; for example, we could take $W = \mathbf{K}^c$. We shall construct inductively decreasing classes $N_\alpha \prec W$ such that N_α is thick in W. The idea is to proceed along the cardinals of the hull of W we are determining, successively discarding all counterexamples to the hull property, so that the transitive collapse of the limit class $N_\Omega := \bigcap_{\xi\in\Omega} N_\xi$ will be the desired weasel M.

We start with $N_0 := W$, and for limit ordinals λ we set $N_\lambda := \bigcap_{\xi\in\lambda} N_\xi$. Now let N_α be given, let

$$\pi\colon N \to N_\alpha$$

be the transitive collapse, and let κ be the αth cardinal of N. By Lemma 3.12, Ω is thick in N. We now discard all counterexamples to the hull property at κ for N: for each $A \subseteq \kappa$ such that $A \in N$ pick a thick class Γ_A such that

$$A \notin \text{ transitive collapse of } \mathrm{Hull}^N(\kappa \cup \Gamma_A),$$

if there is such a thick class, and set $\Gamma_A = \Omega$ otherwise; then set

$$N_{\alpha+1} := \pi\text{''} \bigcap_A \mathrm{Hull}^N(\kappa \cup \Gamma_A).$$

We only have to check that N_Ω is thick in W. By Fodor's lemma, for all but nonstationarily many $\lambda \in A_0$, λ is the λth cardinal of N_λ, and for such λ, $\lambda \in N_{\lambda+1} \Leftrightarrow \lambda \in N_\Omega$. Thus is suffices to show that for all limit ordinals λ, $N_\lambda = N_{\lambda+1}$.

The proof of this illustrates well a typical use of the hull property. Let N be the transitive collapse of N_λ, and let κ be the λth cardinal of N. We must show that N has the hull property at κ. Our construction guarantees that κ is a limit cardinal of N, and that N has the hull property at all $\alpha < \kappa$. Let H be the transitive collapse of $\mathrm{Hull}^N(\kappa \cup \Gamma)$ for some Γ thick in N; it will be enough to show that $\mathfrak{P}(\kappa) \cap N \subseteq H$. For this we compare H with N; let $\mathfrak{T}$ on

[81] Here $\mathrm{Hull}^M(X)$ denotes the uncollapsed hull of X inside M.

H and $\mathfrak{U}$ on N be the resulting iteration trees. Since H and N are universal, $\mathfrak{T}$ and $\mathfrak{U}$ have a common last model Q, which is itself a universal weasel. Let

$$i\colon H \to Q \text{ and } j\colon N \to Q$$

be the iteration maps given by $\mathfrak{T}$ and $\mathfrak{U}$. It will be enough to show that $\mathrm{crit}(j) \geq \kappa$, for then if $A \in \mathfrak{P}(\kappa) \cap N$, we get $A = j(A) \cap \kappa \in Q$, which implies $A \in H$. (For the last assertion, notice that H and N agree below κ, so all extenders used in $\mathfrak{T}$ and $\mathfrak{U}$ have length $> \kappa$, and in particular, Q agrees with H below $(\kappa^+)^Q$.)

Suppose then that $\mathrm{crit}(j) < \kappa$. Let E be the first extender used along the branch from N to Q of $\mathfrak{U}$, so that $\mathrm{crit}(E) = \mathrm{crit}(j) := \mu < \kappa$. One can easily show using Lemma 3.13 that Q retains the hull property at μ. The sets $E \restriction \nu$ for $\mu < \nu < \kappa$ witness that Q does not have the hull property at any ν such that $\mu < \nu < \kappa$.[82] But then, by considering how the hull property is passed from H to Q, we see that μ is also the critical point of i, and thus of the first extender F used on the branch from H to Q in $\mathfrak{U}$.

We claim that E is compatible with F, contrary to the fact that they arose in a coiteration. For if $A \subseteq \mu$ and $A \in N$, then by Lemma 3.13 and Lemma 3.11 we can find a finite set $a \subseteq \mu$, a finite set b of common fixed points of i and j, and a Skolem term τ such that $A = \tau^N[a,b] \cap \mu$. Then $j(A) = \tau^Q[a,b] \cap j(\mu)$, and since $A = j(A) \cap \mu$, this implies that $A = \tau^H[a,b] \cap \mu$, and hence $i(A) = \tau^Q[a,b] \cap i(\mu)$. Thus $j(A)$ and $i(A)$ agree below $\inf\{j(\mu), i(\mu)\}$. Because generators are not moved along the branches of an iteration tree, this implies that E is compatible with F, the desired contradiction. □

From Lemma 3.16 we easily get

Theorem 3.17. Let W be an $\Omega+1$–iterable weasel such that Ω is thick in W; then for μ_0–almost every $\alpha < \Omega$, W has the hull property at α.

Proof : Let M be a weasel having the hull property everywhere, as provided by Lemma 3.16. We compare M with W, and obtain iteration maps $i\colon M \to Q$ and $j\colon W \to Q$ into a common iterate. For μ_0–almost every α, $i``\alpha \subseteq \alpha$ and $j``\alpha \subseteq \alpha$. Using this and Lemma 3.13 we can transfer the hull property from M to Q at μ_0–almost every α, and then pull it back from Q to W at μ_0-almost every α (*cf.* [St96, Lemma 4.6]). □

Corollary 3.18. $\mathbf{K}^c$ has the hull property at μ_0–almost every $\alpha < \Omega$.

Proof : By Proposition 3.8 and Theorem 3.17. □

We turn now to the definability property.

[82] *Cf.* [St96, Example 4.3] for a proof of this.

Theorem 3.19. For μ_0–almost every $\alpha < \Omega$, $\mathbf{K}^c$ has the definability property at α.

Proof : The proof proceeds along the lines of the proof of Theorem 3.3. We assume that the set of α such that $\mathbf{K}^c$ has the definability property at α has measure zero. Let $j\colon \mathbf{V} \to M = \mathrm{Ult}(\mathbf{V}, \mu_0)$ be the canonical embedding. Since $j(\mathbf{K}^c)$ does not have the definability property at Ω in M, we can in M find a thick hull of $j(\mathbf{K}^c)$ omitting Ω, and hence a map $\pi\colon H \to j(\mathbf{K}^c)$ such that $\mathrm{ran}(\pi)$ is thick in $j(\mathbf{K}^c)$. With some care, one can arrange that E_π is compatible with E_j, so that the fragments of E_j which are in M can be used as background extenders to certify E_π for addition to the $j(\mathbf{K}^c)$–sequence in M. Notice that E_π is total on $j(\mathbf{K}^c)$ by the hull property of $j(\mathbf{K}^c)$ at Ω. But then, as in the proof of Theorem 3.3, Ω is Woodin in $j(\mathbf{K}^c)$[83], contrary to our assumption that there is no proper class model with a Woodin cardinal. $\square$

We proceed to the definition of $\mathbf{K}$.

Definition 3.20. $\mathbf{K}$ is the transitive collapse of

$$D_{\mathbf{K}^c} := \{x : \forall\Gamma(\Gamma \text{ is thick in } \mathbf{K}^c \to x \in \mathrm{Hull}^{\mathbf{K}^c}(\Gamma))\}.$$

Theorem 3.21. $\mathbf{K}$ is a universal weasel, and $(\alpha^+)^{\mathbf{K}} = \alpha^+$ for μ_0–almost every $\alpha < \Omega$.

Proof : We have to show that $D_{\mathbf{K}^c}$ is unbounded in Ω (this shows that $\mathbf{K}$ is a weasel) and that $D_{\mathbf{K}^c}$ is unbounded in $(\nu^+)^{\mathbf{K}^c}$ for a μ–measure one set of $\nu < \Omega$ (together with Theorem 3.3 this shows Cheapo Covering for $\mathbf{K}$).

Both parts are similar, the unboundedness in Ω uses the definability property of $\mathbf{K}^c$ and the second part uses the hull property. We shall sketch the first part.

If $D_{\mathbf{K}^c}$ is bounded then we easily construct a decreasing sequence of thick classes Γ_ξ such that, letting b_ξ be the least ordinal in $\mathrm{Hull}^{\mathbf{K}^c}(\Gamma_\xi) \setminus D_{\mathbf{K}^c}$, the sequence $\langle b_\xi : \xi < \Omega\rangle$ is strictly increasing and $D_{\mathbf{K}^c} \subseteq b_0$. Now by Theorem 3.19 we can choose ν such that $\nu = \sup\{b_\xi : \xi < \nu\}$ and $\mathbf{K}^c$ has the definability property at ν. Since $\Gamma_{\nu+1}$ is thick, we can find a finite subset a of ν, a finite subset d of $\Gamma_{\nu+1}$, and a Skolem term τ such that $\nu = \tau^{\mathbf{K}^c}[a, d]$. Let $\xi < \nu$ be such that $a \subseteq b_\xi$. We have

$$\mathbf{K}^c \models \exists a \in [b_\xi]^{<\omega}(b_\xi < \tau[a, d] < b_{\nu+1}),$$

and since the parameters $b_\xi, d, b_{\nu+1}$ in this statement all belong to $\mathrm{Hull}^{\mathbf{K}^c}(\Gamma_\xi)$, we can find a^* in $\mathrm{Hull}^{\mathbf{K}^c}(\Gamma_\xi)$ which is a witness to its existential quantifier. By the definition of b_ξ, this means $a^* \in D_{\mathbf{K}^c}$. But that implies

$$\tau^{\mathbf{K}^c}[a^*, d] \in \mathrm{Hull}^{\mathbf{K}^c}(\Gamma_{\nu+1}) \text{ and } b_\xi < \tau^{\mathbf{K}^c}[a^*, d] < b_{\nu+1},$$

which contradicts the definition of $b_{\nu+1}$. $\square$

[83] In fact, the proof would give a wild mouse.

We lack the space to go much further into the pure theory of $\mathbf{K}$ here, but we wish to state some basic theorems. First and foremost, there is a full weak covering theorem for $\mathbf{K}$.

Theorem 3.22. (Mitchell, Schimmerling) Let $\mu < \Omega$ be a singular cardinal; then $(\mu^+)^{\mathbf{K}} = \mu^+$.

This theorem is definitely no cheapo. The central idea is due to Mitchell, and was brought to fruition in the case $\alpha < \mu \Rightarrow \alpha^\omega < \mu$ in [MiSchSt97]. The full theorem was then proved by Mitchell and Schimmerling in [MiSch95].

Concerning embeddings of $\mathbf{K}$, the situation is a bit more complicated than it was below a strong cardinal, in that it is consistent with our assumptions[84] that there is an $\Omega+1$-iterable universal weasel which is not an iterate of $\mathbf{K}$ (*cf.* [St96, §8]). However, it is shown in [St96] that $\mathbf{K}$ elementarily embeds in every $\Omega + 1$–iterable universal weasel, and that there is no nontrivial embedding of $\mathbf{K}$ into itself. This leads to a nice characterization of $\mathbf{K}$:

Theorem 3.23. $\mathbf{K}$ is the unique $\Omega + 1$–iterable universal weasel which elementarily embeds into every $\Omega + 1$–iterable universal weasel.

This characterization is Theorem 8.10 of [St96].

Although there may be iterable universal weasels which are not iterates of $\mathbf{K}$, it turns out that $\mathbf{K}^c$ itself *is* an iterate of $\mathbf{K}$. This is proved in [SchSt9?]. A key ingredient in the proof is a natural *maximality* property of $\mathbf{K}$ and its iterates. For $\mathbf{K}$ itself, this result states that every countably certified extender which "could be added" to the $\mathbf{K}$–sequence (in that the resulting structure would be a premouse) is already on the $\mathbf{K}$–sequence.

Finally, Jensen's Σ^1_3–correctness theorem (*cf.* [Do82]) has an extension to our situation.

Theorem 3.24. Suppose there is a measurable cardinal $\mu < \Omega$; then the Martin-Solovay tree at μ is in $\mathbf{K}$, and therefore $\mathbf{K}$ is Σ^1_3–correct.

This is Theorem 7.9 of [St96]. It is very probably only a provisional result, as the second measurable cardinal μ should not be necessary.

3.3. **The definability of K and generic absoluteness.** We now sketch the application we promised in the introduction. For this we need some further basic results about $\mathbf{K}$. We wish to produce a formula defining $\mathbf{K}$ whose logical form is as simple as possible. We shall then show that the class defined by this formula interpreted in $\mathbf{V}$ is the same as the class it defines when interpreted in $\mathbf{V}[G]$, for any G which is $\mathbb{P}$–generic over $\mathbf{V}$ where $\mathbb{P} \in \mathbf{V}_\Omega$.

We have already defined $\mathbf{K}$; the only parameter entering into its definition is Ω. This definition involves quantification over $\mathbf{V}_{\Omega+1}$, however, and is therefore too complicated for our purposes. Moreover, it is not clear that this

[84] *I.e.*, that Ω is measurable and there is no proper class model with a Woodin cardinal.

definition is absolute, as we used $\mathbf{K}^c$ to define $\mathbf{K}$, and $(\mathbf{K}^c)^{\mathbf{V}[G]} \neq (\mathbf{K}^c)^{\mathbf{V}}$ in general. We show now that, nevertheless, our definition of $\mathbf{K}$ as the intersection of all thick hulls of $\mathbf{K}^c$ is generically absolute.

Definition 3.25. Let M be a weasel, and $S, \Gamma \subseteq M$. We say that Γ *is S–thick in M* iff

(1) S is stationary in Ω, and
(2) for all but nonstationarily many $\alpha \in S$:
  (a) α is inaccessible, $(\alpha^+)^M = \alpha^+$, and α is not the critical point of an extender from the M–sequence which is total on M, and
  (b) $\alpha \in \Gamma$, and $\Gamma \cap \alpha^+$ contains an α–club.

So Ω is A_0–thick in $\mathbf{K}^c$. We can now relativise our definitions of the hull and definability properties to an arbitrary S in the obvious way: for example, W has the S–definability property at κ iff Ω is S–thick in W, and $\kappa \in \mathrm{Hull}^W(\kappa \cup \Gamma)$ for all Γ which are S–thick in W.

Definition 3.26. A premouse $\mathfrak{M}$ is called *S–very sound*, if there is an $\Omega+1$–iterable weasel W such that $\mathfrak{M} \trianglelefteq W$, Ω is S–thick in W and W has the S–definability property at all $\beta \in \mathrm{Ord} \cap \mathfrak{M}$. We say $\mathfrak{M}$ is *very sound* just in case it is A_0–very sound.

It is easy to see that every proper initial segment of $\mathbf{K}$ is very sound. By the following lemma, this fact characterizes $\mathbf{K}$.

Lemma 3.27. Let $\mathfrak{M}$ be S–very sound and let $\mathfrak{N}$ be T–very sound; then $\mathfrak{M} \trianglelefteq \mathfrak{N}$ or $\mathfrak{N} \trianglelefteq \mathfrak{M}$.

Proof : Let W and R be weasels witnessing the soundness of $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Assume without loss of generality that $\mathrm{Ord}^{\mathfrak{M}} \leq \mathrm{Ord}^{\mathfrak{N}}$.

The proof of Lemma 2.13 shows that for all but nonstationarily many $\alpha \in S \cup T$, $(\alpha^+)^W = (\alpha^+)^R = \alpha^+$. Now let W^* be the (linear) iterate of W obtained by taking an ultrapower by the order zero total measure on α from W, for each $\alpha \in T \setminus \mathrm{Ord}^{\mathfrak{M}}$ such that $W \models$ "α is measurable". Similarly, let R^* be obtained from R by taking an ultrapower by the order zero measure on α at each $\alpha \in S \setminus \mathrm{Ord}^{\mathfrak{N}}$ such that $R \models \alpha$ is measurable. Then W^* and R^* still witness the S and T soundness of $\mathfrak{M}$ and $\mathfrak{N}$, respectively; moreover, Ω is $S \cup T$ thick in each of W^* and R^*.

Let $i\colon W^* \to Q$ and $j\colon R^* \to Q$ come from coiteration, and let

$$\kappa = \inf\{\mathrm{crit}(i), \mathrm{crit}(j)\}.$$

It is enough to show $\mathrm{Ord}^{\mathfrak{M}} \leq \kappa$, for then $\mathfrak{M} \trianglelefteq \mathfrak{N}$ as desired, so assume that $\kappa < \mathrm{Ord}^{\mathfrak{M}}$.

Suppose that $\kappa = \mathrm{crit}(i) < \mathrm{crit}(j)$. Since Ω is T thick in R^* and W^*, and R^* has the T-definability preoperty at κ, we can find a finite set a of common fixed points of i and j such that $\kappa = \tau^{R^*}[a]$ for some term τ. But then $\kappa = j(\kappa) = \tau^Q[a] = i(\tau^{W^*}[a])$, so $\kappa \in \mathrm{ran}(i)$, a contradiction. Similarly, we get $\mathrm{crit}(i) \leq \mathrm{crit}(j)$, and hence $\mathrm{crit}(i) = \mathrm{crit}(j) = \kappa$.

An argument using the S and T hull properties at κ now shows that the first extenders used along the branches giving rise to i and j in our coiteration are compatible with one another, a contradiction. This argument is essentially the same as the "typical use of the hull property" at the end of the proof of Lemma 3.16. $\square$

Corollary 3.28. $\mathfrak{M}$ is a proper initial segment of $\mathbf{K}$ if and only if $\mathfrak{M}$ is very sound if and only if $\mathfrak{M}$ is S–very sound for some S.

These results imply at once that our Definition 3.20 of $\mathbf{K}$ using using thick hulls is generically absolute.

Theorem 3.29. Let G be $\mathbb{P}$–generic over $\mathbf{V}$, where $\mathbb{P} \in \mathbf{V}_\Omega$; then $\mathbf{K}^{\mathbf{V}} = \mathbf{K}^{\mathbf{V}[G]}$.

Proof : We claim that if W is a weasel witnessing that $\mathfrak{M}$ is S–very sound in $\mathbf{V}$, then W continues to witness that $\mathfrak{M}$ is S–very sound in $\mathbf{V}[G]$. For it is clear that S remains stationary in $\mathbf{V}[G]$, and that every S–thick class in $\mathbf{V}[G]$ contains an S-thick class in $\mathbf{V}$. It is less obvious that W remains $\Omega + 1$–iterable in $\mathbf{V}[G]$, but this is provable (*cf.* [St96, Lemma 5.12]).

Thus all the proper initial segments of $\mathbf{K}^{\mathbf{V}}$ remain $A_0^{\mathbf{V}}$–very sound in $\mathbf{V}[G]$, as witnessed by the appropriate hulls of $(\mathbf{K}^c)^{\mathbf{V}}$. By Lemma 3.27 we then have $\mathbf{K}^{\mathbf{V}} = \mathbf{K}^{\mathbf{V}[G]}$. $\square$

Unfortunately, the generically absolute definition of $\mathbf{K}$ we have just given is logically too complicated for our application. We now sketch an inductive definition of $\mathbf{K}$ which has the optimal logical complexity.[85]

Definition 3.30. A premouse $\mathfrak{M}$ is α–strong if there is a weasel W witnessing that $\mathcal{J}_\alpha^{\mathfrak{M}}$ is very sound, and an iteration tree $\mathfrak{T}$ on W played according to its unique $\Omega + 1$–iteration strategy which uses only extenders with length $\geq \alpha$, and an elementary embedding from $\mathfrak{M}$ to an initial segment of the last model of $\mathfrak{T}$ such that $\pi \upharpoonright \alpha$ is the identity.

This definition of α–strongness above involves quantification over $\mathbf{V}_{\Omega+1}$. We now give a definition by induction on α which is of essentially optimal complexity. The key is a certain iterability property: the α–strong mice are those which are "jointly iterable" with all mice which are β–strong for all $\beta < \alpha$.[86] We now explain this property further.

[85]Hugh Woodin has shown that there are universes which satisfy "there is a measurable cardinal and no proper class model with a Woodin cardinal" within which $\mathbf{K}$ has no definition logically simpler than the one we shall sketch.

[86]This idea traces back to Dodd's proof that GCH holds in the models of [Do81].

Definition 3.31. A triple $\langle\mathfrak{M},\mathfrak{N},\alpha\rangle$ is called a *phalanx* if

- $\mathfrak{M}$ and $\mathfrak{N}$ are premice with $\mathcal{J}^{\mathfrak{M}}_\alpha = \mathcal{J}^{\mathfrak{N}}_\alpha$,
- α is a cardinal in both models $\mathfrak{M}$ and $\mathfrak{N}$, and
- the projecta $\varrho(\mathfrak{M})$ and $\varrho(\mathfrak{N})$ are $\geq \alpha$

In building an iteration tree on a phalanx $\langle\mathfrak{M},\mathfrak{N},\alpha\rangle$, we pretend that the phalanx is already an iteration tree of length 1, where $\mathfrak{M}^{\mathfrak{T}}_0 = \mathfrak{M}$, $\mathfrak{M}^{\mathfrak{T}}_1 = \mathfrak{N}$ and α is interpreted, for the purpose of deciding which model to take an ultrapower of, as the length of a "virtual extender" used to get from $\mathfrak{M}$ to $\mathfrak{N}$. Note that $\mathfrak{N}$ need not be an ultrapower of $\mathfrak{M}$ — we simply play the iteration game as if it were.

The notions of ϑ–iterability and iterability for phalanxes are easily derived from this description.

The following theorem yields an inductive characterization of strongness.

Theorem 3.32. Let $\mathfrak{M}$ be a premouse, and suppose that α is a cardinal in $\mathbf{K}$ such that $\mathcal{J}^{\mathbf{K}}_\alpha = \mathcal{J}^{\mathfrak{M}}_\alpha$; then the following are equivalent:

(1) $\mathfrak{M}$ is α–strong,
(2) If $\mathfrak{N}$ is β–strong for all $\mathbf{K}$–cardinals $\beta < \alpha$, then $\langle\mathfrak{N},\mathfrak{M},\alpha\rangle$ is $\Omega+1$–iterable.
(3) If $\mathfrak{N}$ has cardinality $\leq \alpha$, and $\mathfrak{N}$ is β–strong for all $\mathbf{K}$–cardinals $\beta < \alpha$, then $\langle\mathfrak{N},\mathfrak{M},\alpha\rangle$ is $\Omega+1$–iterable.

Proof: For the direction "(2.)⇒(1.)" let W witness that $\mathcal{J}^{\mathfrak{M}}_\alpha$ is very sound. We will show that W has an iterate as demanded in the definition of α–strong. Clearly, W is β–strong for all $\beta < \alpha$, so by (2.) the phalanx $\langle W,\mathfrak{M},\alpha\rangle$ is $\Omega+1$–iterable. We now compare this phalanx with W in the obvious way, and obtain thereby iteration trees $\mathfrak{T}$ on $\langle W,\mathfrak{M},\alpha\rangle$ and $\mathfrak{U}$ on W with last models $\mathfrak{R}$ and $\mathfrak{S}$ respectively.[87] The key observation is that $\mathfrak{R}$ is above $\mathfrak{M}$ in the tree $\mathfrak{T}$. This is because otherwise it is above W, in which case we get $\mathfrak{R} = \mathfrak{S}$ by universality, and can argue using the hull and definability properties as in Lemma 3.27 that the first extenders used on the branches from W to $\mathfrak{R}$ and from W to $\mathfrak{S}$ of the $\mathfrak{T}$ and $\mathfrak{U}$ are compatible. It is now easy to see that $\mathfrak{R}$ is an initial segment of $\mathfrak{S}$, all extenders used in $\mathfrak{U}$ have length $> \alpha$, and the embedding $\pi\colon \mathfrak{M} \to \mathfrak{R}$ given by $\mathfrak{T}$ is the identity below α.

For "(1.)⇒(2.)", we use the embeddings given by the fact that our premice are strong at various α and β to lift iteration trees on $\langle\mathfrak{N},\mathfrak{M},\alpha\rangle$ to something like iteration trees on $\mathbf{K}^c$, and then use some simple generalizations of our iterability results for $\mathbf{K}^c$.

The equivalence of (2.) with (3.) is a simple Löwenheim–Skolem argument. □

[87] The comparison proceeds by iterating the least disagreement between the current last models as usual, beginning with $\mathfrak{M}$ *versus* W. The rules of iteration trees on premice and phalanxes respectively determine the rest.

We can now use Theorem 3.32 to give an inductive definition of $\mathbf{K}$. First note the following simple consequence of the definition of α–strength.

Proposition 3.33. Let α be a cardinal of $\mathbf{K}$; then the following are equivalent:

(1) $\mathfrak{N}$ is some $\mathcal{J}^{\mathbf{K}}_{\beta}$ for $\beta < (\alpha^+)^{\mathbf{K}}$
(2) There is an α–strong premouse $\mathfrak{M}$ with projectum α such that $\mathfrak{N} = \mathcal{J}^{\mathfrak{M}}_{\beta}$ for some $\beta < (\alpha^+)^{\mathfrak{M}}$.

We can now define the class of $\mathbf{K}$–cardinals α, α–strength for $\mathbf{K}$–cardinals α, and $\mathcal{J}^{\mathbf{K}}_{\alpha}$ for $\mathbf{K}$–cardinals α, by induction on α. Given a $\mathbf{K}$–cardinal α, given $\mathcal{J}^{\mathbf{K}}_{\alpha}$, and given the class of α–strong premice, let $\mathfrak{M}$ be the union of all α–strong mice projecting to α; then the ordinal height of $\mathfrak{M}$ is $(\alpha^+)^{\mathbf{K}}$ and $\mathfrak{M}$ is $\mathcal{J}^{\mathbf{K}}_{(\alpha^+)^{\mathbf{K}}}$. We can then use (3.) of Theorem 3.32 to identify the class of $(\alpha^+)^{\mathbf{K}}$–strong mice. The limit steps in the induction are easy, using (3.) of Theorem 3.32 again to determine the α–strong mice from $\mathcal{J}^{\mathbf{K}}_{\alpha}$.

For our application, we want to measure the complexity of this inductive definition descriptive–set–theoretically, so we shall restrict ourselves to the definition of $\mathcal{J}^{\mathbf{K}}_{\alpha}$, for $\alpha < \omega_1^{\mathbf{V}}$. In this case the mice over which one quantifies in (3.) of Theorem 3.32 are all countable, so one easily obtains

Theorem 3.34. There is a formula $\varphi(v_0, v_1)$ such that the following are equivalent for all $\alpha < \omega_1$:

(1) $\mathbf{L}_{\alpha+1}(\mathbb{R}) \models \varphi[x, y]$
(2) x is a code for some $\delta < \alpha$ and y codes $\mathcal{J}^{\mathbf{K}}_{\delta}$

As $\mathbf{K}$ is invariant under forcing of size $< \Omega$, the formula φ defines $\mathbf{K}$ up to $\omega_1^{\mathbf{V}[G]}$ in $\mathbf{V}[G]$ whenever G is $\mathbb{P}$–generic for a poset $\mathbb{P}$ of size $< \Omega$.

We can now give the application we promised in the introduction:

Theorem 3.35. Suppose Ω is measurable, and for all partial orderings $\mathbb{P} \in \mathbf{V}_{\Omega}$ and all G $\mathbb{P}$–generic over $\mathbf{V}$ we have

$$(\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}} \equiv (\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}[G]};$$

then there is a proper class model satisfying "there is a Woodin cardinal".

Proof : We suppose toward contradiction that there is no such proper class model. This supposition puts the theory of $\mathbf{K}$ we have developed at our disposal. In particular, we can use the formula from Theorem 3.34 get a sentence saying that $\omega_1^{\mathbf{V}}$ is a successor cardinal in $\mathbf{K}$. We shall call this sentence σ:

$$\mathbf{L}_{\omega_1}(\mathbb{R}) \models \sigma \iff \exists\alpha < (\omega_1)^{\mathbf{V}}((\alpha^+)^{\mathbf{K}} = (\omega_1)^{\mathbf{V}})$$

Our hypotheses guarantee that $\mathbf{K}$ computes some successor correctly, *i.e.* there is some β such that $\beta^+ = (\beta^+)^{\mathbf{K}}$. Because Ω is measurable, we can

assume that β is inaccessible.
We have two different cases:
Case 1 : σ is true in $\mathbf{V}$, *i.e.* ω_1 is a successor in $\mathbf{K}$. Let G be $\mathrm{Col}(\omega, <\beta)$–generic for $\mathbf{V}$. (This is the Levy collapse for collapsing β to ω_1.) Then the generic extension can't satisfy σ anymore.

Case 2 : σ is false in $\mathbf{V}$, *i.e.* ω_1 is a limit in $\mathbf{K}$. Then we collapse β to be countable. By Theorem 3.29 we know that in the generic extension, the successor of β is still computed correctly. But the successor is $(\omega_1)^{\mathbf{V}[G]}$, so $\mathbf{V}[G] \models \sigma$.

In either case, $(\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}}$ and $(\mathbf{L}_{\omega_1}(\mathbb{R}))^{\mathbf{V}[G]}$ disagree as to the truth of σ, a contradiction.[88] $\square$

4. Beyond one Woodin cardinal

We close with some brief remarks on the progress which has been made in extending the theory we have described to core models satisfying stronger large cardinal hypotheses.

The most important obstacle is that we have no proof of the $\omega_1 + 1$–iterability of the countable elementary submodels of the premice occurring in $\mathbf{K}^c$–constructions. Andretta, Neeman, and the second author (unpublished) have been able to extend the proof for tame mice (Theorem 2.34) to slightly stronger mice, mice strong enough that their existence yields an inner model with a cardinal λ which is both a limit of Woodin cardinals and a limit of cardinals which are $< \lambda$–strong.[89] They obtain thereby

Theorem 4.1. If there is a measurable cardinal and the proper forcing axiom PFA holds, then there is an inner model with a cardinal λ which is both a limit of Woodin cardinals and a limit of cardinals which are $< \lambda$–strong.

This extends earlier work of Schimmerling, who showed that the same hypotheses imply the existence of a wild mouse (*cf.* [Sch95] and [Sch9?]).

It is important in the proof of Theorem 4.1 that one can make do with $\mathbf{K}^c$ in the relative consistency proof; one does not need a generically absolute core model like $\mathbf{K}$. This is important because the full $\Omega + 1$–iterability of $\mathbf{K}^c$, on which the theory of $\mathbf{K}$ rests, is another kettle of fish. We obtained full iterability of $\mathbf{K}^c$ under the hypothesis that there is no proper class model with a Woodin cardinal in Theorem 2.35. The argument used collapsing and absoluteness to reflect a failure of iterability to the countable; the level of generic absoluteness required is closely connected to that of the complexity

[88] This argument is due to Woodin, and comes from [Woo82].

[89] This large cardinal hypothesis is known as the $\mathsf{AD}_{\mathbb{R}}$–*hypothesis*, because Woodin (unpublished) has shown that its consistency implies the consistency of $\mathsf{AD}_{\mathbb{R}}$.

of the ω_1–iteration strategies for countable elementary submodels of $\mathbf{K}^c$.[90] Once our mice are no longer 1–small, this level of generic absoluteness is not provable in ZFC, and consequently, not every model of ZFC is a suitable environment in which to construct $\mathbf{K}$.

Here are some examples which illustrate this. The first is due to Jensen (*cf.* [He97]). Let us call a weasel W a *strongly local core model* just in case there is a formula Φ which locally defines W both in W and in all its set generic extensions in the sense that for any (possibly trivial) G which is $\mathbb{P}$–generic over W for some $\mathbb{P} \in W$, and any inaccessible cardinal κ of $W[G]$,

$$W \cap \mathbf{V}_\kappa = \{x \colon W[G] \cap \mathbf{V}_\kappa \models \Phi[x]\}.$$

Then there is no strongly local core model W which satisfies "there is a Woodin cardinal". For let κ be Woodin in W, and let

$$j \colon W \to M \subseteq W[G], \text{ with } \operatorname{crit}(j) = \omega_1^W$$

be a generic elementary embedding coming from Woodin's full stationary tower forcing.[91] Since $\operatorname{crit}(j) = \omega_1$, there is a real $x \in W[G] \setminus W$. But κ remains inaccessible in $W[G]$, so if Φ is our formula locally defining W,

$$W[G] \cap \mathbf{V}_\kappa \models \neg\Phi[x].$$

On the other hand, since j is elementary

$$M \cap \mathbf{V}_\kappa \models \Phi[x].$$

But Woodin's forcing has the property that $M \cap \mathbf{V}_\kappa = W[G] \cap \mathbf{V}_\kappa$, a contradiction.

The second example is due to Woodin. Suppose W is the minimal $\Omega + 1$–iterable weasel satisfying "there is a Woodin cardinal".[92] Let κ be the Woodin cardinal of W; then W satisfies "I am not $\kappa^+ + 1$–iterable".[93] Thus, although W really is fully iterable, it doesn't know how to iterate itself.

The third example is the following: let W be the minimal $\Omega + 1$–iterable weasel such that for some α, $\mathcal{J}_\alpha^W$ is not 1–small. It is easy to see that $W = \mathbf{L}[\mathfrak{M}]$, where $\mathfrak{M}$ is essentially the sharp of the minimal fully iterable model with one Woodin. Then $\mathfrak{M}$ is not ordinal definable in W, so that we have a core model which fails to satisfy $\mathbf{V} = \mathrm{HOD}$. As in the previous example, the problem here is that although $\mathfrak{M}$ is iterable, W does not know how to iterate it; in fact, $\mathfrak{M}$ is not even $\omega_1 + 1$–iterable in W.

These examples show that the absoluteness of the definition of $\mathbf{K}$, and what underlies that, the absoluteness of iterability, no longer hold good in the setting of arbitrary models of ZFC once one gets past the 1–small mice.

---

[90] We used in Theorem 2.35 the fact that every properly 1–small countable mouse $\mathfrak{M}$ has a $\Delta_2^1(\mathfrak{M})$ ω_1–iteration strategy. For mice which are not 1–small, the iteration strategies are necessarily more complicated.

[91] For a reference, *cf.* the forthcoming monograph [Ma9?].

[92] Reasonable large cardinal hypotheses imply that there is such a weasel.

[93] Neeman sharpened the argument to show that W satisfies "I am not $\kappa + 1$–iterable".

Nevertheless, one can get a useful theory of $\mathbf{K}$ generalizing the one presented in Section 3 to stronger mice by working only in a sufficiently good background universe, a universe closed under building the "lower level" mice over arbitrary sets. In a relative consistency application of this theory of $\mathbf{K}$, one may then have to construct this sufficiently good background universe, and hence the lower level mice, as part of an induction. Here is one theorem proved by such a technique:

Theorem 4.2 (Woodin)**.** Suppose that for every G which is set generic over $\mathbf{V}$, $\mathbf{L}(\mathbb{R}) \equiv (\mathbf{L}(\mathbb{R}))^{\mathbf{V}[\mathbb{G}]}$; then there is an inner model with ω Woodin cardinals.[94]

This result strengthens an earlier result of the second author which gave the same conclusion under the stronger hypothesis that every set of reals in $\mathbf{L}(\mathbb{R})$ is weakly homogeneous.

It is not known how to extend the bootstrapping technique which yields the theorem above, and others like it which involve a full theory of $\mathbf{K}$, so as to obtain models with more than a proper class of Woodin cardinals. Thus there are still basic open problems concerning the iterabilty of $\mathbf{K}^c$ for uncountable trees within the tame mice. These are connected to a number of very interesting potential relative consistency strength applications. The reader who started with only what we claimed were the prerequisites for this paper and has now made it to the end should have no trouble conquering these problems, although perhaps he would like to take a brief rest first.

References

[Bal83] Stewart **Baldwin**, Generalizing in the Mahlo hierarchy, with applications to the Mitchell models, Annals of Pure and Applied Logic 25 (1983), p. 103–127

[Bal86] Stewart **Baldwin**, Between strong and superstrong. Journal of Symbolic Logic 51 (1986), p. 547–559

[Dev84] Keith **Devlin**, Constructibility, Berlin 1984 [Perspectives of Mathematical Logic]

[DevJen75] Keith **Devlin**, Ronald B. **Jensen**, Marginalia to a theorem of Silver, *in :* Gert H. Müller, Arnold Oberschelp, Klaus Potthoff (*eds.*), ISILC Logic Conference, Proceedings of the International Summer Institute and Logic Colloquium, Kiel, 1974, Berlin 1975 [Lecture Notes in Mathematics 499], p. 115–142

[Do81] Anthony **Dodd**, Strong Cardinals, *handwritten notes* 1981

[Do82] Anthony **Dodd**, The Core Model, Cambridge 1982 [London Mathematical Society Lecture Notes Series 61]

[DoJen81] Anthony **Dodd**, Ronald B. **Jensen**, The Core Model, Annals of Mathematical Logic 20 (1981), p. 43–75

[DoJen82] Anthony **Dodd**, Ronald B. **Jensen**, The covering lemma for $\mathbf{L}[U]$, Annals of Mathematical Logic 22 (1982), p. 127–135

[Gö40] Kurt F. **Gödel**, The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory, Annals of Mathematical Studies # 3, Princeton 1940

[94]This result is unpublished. A slightly weaker version of it will appear in [St9?b].

[Gö47] Kurt F. **Gödel**, What is Cantor's Continuum Problem, American Mathematical Monthly 54 (1947), p. 515-525

[He97] Annekathrin **Hegewald**, Generische Ultraprodukte und Woodin–Forcing, Diplomarbeit, Technische Universität Berlin, Berlin 1997

[Jen72] Ronald B. **Jensen**, The Fine Structure of the Constructible Hierarchy, Annals of Mathematical Logic 4 (1972), p. 229–308

[Jen88] Ronald B. **Jensen**, Measures of order zero, *handwritten notes* 1988

[Jen90] Ronald B. **Jensen**, Nonoverlapping Extenders, *handwritten notes without date*, around 1990

[Jen95] Ronald B. **Jensen**, Inner Models and Large Cardinals, Bulletin of Symbolic Logic 1 (1995), p. 393-407

[Kan94] Akihiro **Kanamori**, The Higher Infinite, Berlin 1994 [Perspectives in Mathematical Logic]

[Ku70] Kenneth **Kunen**, Some applications of iterated ultrapowers in set theory, Annals of Mathematical Logic 1 (1970), p. 179–227

[Ku71] Kenneth **Kunen**, Elementary Embeddings and infinitary combinatorics, Journal of Symbolic Logic 36 (1971), p. 407–413

[Ma9?] Donald A. **Martin**, Determinacy, *monograph in preparation*

[MaSt94] Donald A. **Martin**, John R. **Steel**, Iteration Trees, Journal of the American Mathematical Society 7 (1994), p. 1–73

[Mi74] William J. **Mitchell**, Sets constructible from sequences of ultrafilters, Journal of Symbolic Logic 39 (1974), p. 57–66

[Mi78] William J. **Mitchell**, Hypermeasurable Cardinals, *in :* Maurice Boffa, Dirk van Dalen, Kenneth McAloon (*eds.*), Logic Colloquium '78, Proceedings of the Colloquium held in Mons, August 1978, Amsterdam 1979, p. 303–316

[Mi84] William J. **Mitchell**, The core model for sequences of measures I, Mathematical Proceedings of the Cambridge Philosophical Society 95 (1984), p. 229–260

[MiSch95] William J. **Mitchell**, Ernest **Schimmerling**, Weak covering without countable closure, Mathematical Research Letters 2 (1995), p. 595–609

[MiSchSt97] William J. **Mitchell**, Ernest **Schimmerling**, John R. **Steel**, The weak covering lemma up to a Woodin cardinal, Annals of Pure and Applied Logic 84 (1997), p. 219-255

[MiSt94] William J. **Mitchell**, John R. **Steel**, Fine Structure and Iteration Trees, Berlin 1994 [Lecture Notes in Logic 3]

[Nee98] Itay **Neeman**, Games of countable length, *this volume*

[NeeSt9?] Itay **Neeman**, John R. **Steel**, A Weak Dodd–Jensen Lemma, *submitted to* Journal of Symbolic Logic

[Si71a] Jack H. **Silver**, The consistency of the GCH with the existence of a measurable cardinal, in: Dana S. Scott (*ed.*), Axiomatic Set Theory, Proceedings of Symposia in Pure Mathematics of the American Mathematical Society 13.1 (1971), p. 391–395

[Si71b] Jack H. **Silver**, Measurable cardinals and Δ^1_3 well–orderings, Annals of Mathematics 94 (1971), p. 414–446

[Sch95] Ernest **Schimmerling**, Combinatorial Principles in the core model for one Woodin cardinal, Annals of Pure and Applied Logic 74 (1995), p. 153–201

[Sch9?] Ernest **Schimmerling**, A finite family weak square principle, *to appear in* Journal of Symbolic Logic

[SchSt96] Ernest **Schimmerling**, John R. **Steel**, Fine structure for tame inner models, Journal of Symbolic Logic 61 (1996), p. 621–639

[SchSt9?] Ernest **Schimmerling**, John R. **Steel**, The maximality of the core model, *to appear in* Transactions of the American Mathematical Society

[St82] John R. **Steel**, A classification of jump operators, Journal of Symbolic Logic 47 (1982), p. 347–358

[St93] John R. **Steel**, Inner models with many Woodin cardinals, Annals of Pure and Applied Logic 65 (1993), p. 185–209

[St96] John R. **Steel**, The Core Model Iterability Problem, Berlin 1996 [Lecture Notes in Logic 8]

[St9?a] John R. **Steel**, An Outline of Inner Model Theory, *to appear in :* Matthew Foreman, Akihiro Kanamori, Menachem Magidor (*eds.*), Handbook of Set Theory, *in preparation*

[St9?b] John R. **Steel**, Core models with more Woodin cardinals, *in preparation*

[Woo82] W.Hugh **Woodin**, On the consistency strength of projective absoluteness, *in:* Jacques Stern (*ed.*), Proceedings of the Herbrand Symposium, Logic Colloquium '81, held in Marseille, France, July 1981, Amsterdam 1982 [Studies in Logic and the Foundations of Mathematics 107], p. 365–384

[Ze9?] Martin **Zeman**, Inner Models and Strong Cardinals, *monograph in preparation*

Department of Mathematics, University of California, Berkeley CA 94720, USA

E-mail address, Benedikt Löwe: `loewe@math.berkeley.edu`

E-mail address, John Steel: `steel@math.berkeley.edu`